Engineering Mathematics
for Semesters I and II

About the Authors

C B Gupta is presently working as Professor in the Department of Mathematics, Birla Institute of Technology and Science, Pilani (Rajasthan). With over 25 years of teaching and research experience, he is a recipient of numerous awards like the Shiksha Rattan Puraskar 2011, Best Citizens of India Award 2011, Glory of India Award 2013, and Mother Teresa Award 2013. He was listed in *Marquis' Who's Who in Science and Technology in the World 2010* and *2013*, and in top 100 scientists of the world in 2012. He obtained his master's degree in Mathematical Statistics and PhD in Operations Research from Kurukshetra University, Kurukshetra. His fields of specialization are Applied Statistics, Optimization, and Operations Research. A number of students have submitted their theses/dissertations on these topics under his supervision. He has published a large number of research papers on these topics in peer-reviewed national and international journals of repute. He has authored/co-authored 12 books on Probability and Statistics, Quantitative Methods, Optimization in Operations Research, Advance Discrete Mathematics, Engineering Mathematics I–III, Advanced Mathematics, and the like. He is also on the editorial board and a reviewer of many national and international journals. Dr. Gupta is a member of various academic and management committees of many institutes/universities. He has participated in more than 30 national and international conferences in which he has delivered invited talks and chaired technical sessions. He has been a member of Rajasthan Board of School Education, Ajmer, and also a member of various committees of RPSC Ajmer, UPSC, New Delhi, and AICTE, New Delhi.

S R Singh is presently working as Associate Professor in the Department of Mathematics at Chaudhary Charan Singh University, Meerut (Uttar Pradesh) and has an experience of 20 years in academics and research. His areas of specialization are Inventory Control, Supply-Chain Management, and Fuzzy Set Theory. He has attended various seminars/conferences. Fifteen students have been awarded PhD under his supervision. He has published more than hundred research papers in reputed national and international journals. His research papers have been published in *International Journal of System Sciences*, *Asia Pacific Journal of Operational Research*, *Control and Cybernetics*, *Opsearch*, *International Journal of Operational Research*, *Fuzzy Sets and Systems*, and *International Journal of Operations and Quantitative Management*. He has authored/co-authored nine books.

Mukesh Kumar is presently working as Associate Professor in the Department of Mathematics at Graphic Era University, Dehradun (Uttarakhand). He received an MPhil in Mathematics from the Indian Institute of Technology, Roorkee, and a PhD in Mathematics (Operations Research) from HNB Garhwal Central University, Srinagar. He has a teaching experience of more than 10 years and his fields of specialization are Inventory Control, Supply-Chain Management, and Operations Research. He has published many research papers in national and international reputed journals. He has authored a book, *Mathematical Foundations in Computer Science*. He is on the editorial board and is a reviewer of many national and international journals.

Engineering Mathematics
for Semesters I and II

C B Gupta
Professor
Department of Mathematics
Birla Institute of Technology and Science (BITS)
Pilani, Rajasthan

S R Singh
Associate Professor
Department of Mathematics
Chaudhary Charan Singh University
Meerut, Uttar Pradesh

Mukesh Kumar
Associate Professor
Department of Mathematics
Graphic Era University
Dehradun, Uttarakhand

McGraw Hill Education (India) Private Limited
NEW DELHI

McGraw Hill Education Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

McGraw Hill Education (India) Private Limited

Published by McGraw Hill Education (India) Private Limited
P-24, Green Park Extension, New Delhi 110 016

Engineering Mathematics for Semesters I and II

Copyright © 2015, by McGraw Hill Education (India) Private Limited.
No part of this publication may be reproduced or distributed in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise or stored in a database or retrieval system without the prior written permission of the publishers. The program listing (if any) may be entered, stored and executed in a computer system, but they may not be reproduced for publication.

This edition can be exported from India only by the publishers,
McGraw Hill Education (India) Private Limited.

Print Edition
ISBN 13: 978-93-392-1964-2
ISBN 10: 93-392-1964-3

EBook Edition
ISBN 13: 978-93-392-1965-9
ISBN 10: 93-392-1965-1

Managing Director: *Kaushik Bellani*
Head—Products (Higher Education and Professional): *Vibha Mahajan*

Assistant Sponsoring Editor: *Koyel Ghosh*
Senior Editorial Researcher: *Sachin Kumar*

Manager—Production Systems: *Satinder S Baveja*
Assistant Manager—Editorial Services: *Sohini Mukherjee*
Senior Production Executive: *Suhaib Ali*
Senior Graphic Designer—Cover: *Meenu Raghav*

Senior Publishing Manager (SEM & Tech. Ed.): *Shalini Jha*
Assistant Product Manager (SEM & Tech. Ed.): *Tina Jajoriya*

General Manager—Production: *Rajender P Ghansela*
Manager—Production: *Reji Kumar*

Information contained in this work has been obtained by McGraw Hill Education (India), from sources believed to be reliable. However, neither McGraw Hill Education (India) nor its authors guarantee the accuracy or completeness of any information published herein, and neither McGraw Hill Education (India) nor its authors shall be responsible for any errors, omissions, or damages arising out of use of this information. This work is published with the understanding that McGraw Hill Education (India) and its authors are supplying information but are not attempting to render engineering or other professional services. If such services are required, the assistance of an appropriate professional should be sought.

Typeset at Text-o-Graphics, B-1/56, Aravali Apartment, Sector-34, Noida 201 301, and printed at

Cover Printer:

Visit us at: www.mheducation.co.in

Contents

Preface *xvii*

1. Matrix Algebra **1.1–1.73**

1.1 Introduction *1.1*
1.2 Notation and Terminology *1.1*
1.3 Special Types of Matrices *1.2*
1.4 Equality of Two Matrices *1.5*
1.5 Properties of Matrices *1.5*
1.6 Properties of Matrix Multiplication *1.7*
1.7 Transpose of a Matrix *1.8*
1.8 Symmetric and Skew-Symmetric Matrices *1.8*
1.9 Transposed Conjugate, Hermitian, and Skew-Hermitian Matrices *1.9*
1.10 Elementary Transformation or Elementary Operations *1.13*
1.11 The Inverse of a Matrix *1.13*
Exercise 1.1 *1.16*
Answers *1.17*
1.12 Echelon Form of a Matrix *1.17*
1.13 Rank of a Matrix *1.17*
1.14 Canonical Form (or Normal Form) of a Matrix *1.18*
Exercise 1.2 *1.25*
Answers *1.25*
1.15 Linear Systems of Equations *1.26*
1.16 Homogeneous Systems of Linear Equations *1.27*
1.17 Systems of Linear Non-Homogeneous Equations *1.27*
1.18 Condition for Consistency Theorem *1.28*
1.19 Condition for Inconsistent Solution *1.28*
1.20 Characteristic Roots and Vectors (or Eigenvalues and Eigenvectors) *1.33*
1.21 Some Important Theorems on Characteristic Roots and Characteristic Vector *1.34*
1.22 Nature of the Characteristic Roots *1.34*
1.23 The Cayley–Hamilton Theorem *1.40*
1.24 Similarity of Matrices *1.43*
1.25 Diagonalization Matrix *1.43*

	Exercise 1.3	*1.46*
	Answers	*1.48*
1.26	Quadratic Forms	*1.48*
1.27	Complex Quadratic Form	*1.50*
1.28	Canonical Form	*1.51*
1.29	Positive Definite Quadratic and Hermitian Forms	*1.51*
1.30	Some Important Remark's	*1.51*
	Exercise 1.4	*1.54*
	Answers	*1.54*
1.31	Applications of Matrices	*1.54*
	Summary	*1.59*
	Objective-Type Questions	*1.63*
	Answers	*1.73*
2.	**Differential Calculus**	**2.1–2.18**
2.1	Introduction	*2.1*
2.2	Differentiation	*2.1*
2.3	Geometrical Meaning of Derivative at a Point	*2.1*
2.4	Successive Differentiation	*2.2*
2.5	Calculation of n^{th} Order Differential Coefficients	*2.3*
	Exercise 2.1	*2.8*
	Answers	*2.9*
2.6	Leibnitz's Theorem	*2.9*
	Exercise 2.2	*2.16*
	Answers	*2.16*
	Summary	*2.17*
	Objective-Type Questions	*2.18*
	Answers	*2.18*
3.	**Partial Differentiation**	**3.1–3.54**
3.1	Introduction	*3.1*
3.2	Partial Derivatives of First Order	*3.1*
3.3	Geometric Interpretation of Partial Derivatives	*3.2*
3.4	Partial Derivatives of Higher Orders	*3.2*
	Exercise 3.1	*3.10*
3.5	Homogeneous Function	*3.11*
3.6	Euler's Theorem on Homogeneous Functions	*3.12*
3.7	Relations Between Second-Order Derivatives of Homogeneous Functions	*3.13*
3.8	Deduction from Euler's Theorem	*3.14*
	Exercise 3.2	*3.20*
3.9	Composite Function	*3.21*
	Exercise 3.3	*3.26*
	Answers	*3.26*
3.10	Jacobian	*3.27*
3.11	Important Properties of Jacobians	*3.28*
3.12	Theorems on Jacobian (Without Proof)	*3.30*

3.13 Jacobians of Implicit Functions *3.30*
3.14 Functional Dependence *3.30*
Exercise 3.4 *3.40*
Answers *3.40*
3.15 Expansion of Functions of Several Variables *3.41*
3.16 Expansion of Functions of Two Variables *3.41*
3.17 Taylor's and Maclaurin's Theorems for Three Variables *3.43*
Exercise 3.5 *3.48*
Answers *3.48*
Summary *3.49*
Objective-Type Questions *3.51*
Answers *3.54*

4. Maxima and Minima **4.1–4.37**
4.1 Introduction *4.1*
4.2 Maxima and Minima of Functions of Two Independent Variables *4.1*
4.3 Necessary Conditions for the Existence of Maxima or Minima of $f(x, y)$ at the Point (a, b) *4.1*
4.4 Sufficient Conditions for Maxima and Minima (Lagrange's Condition for Two Independent Variables) *4.2*
4.5 Maximum and Minimum Values for a Function $f(x, y, z)$ *4.10*
4.6 Lagrange's Method of Multipliers *4.12*
Exercise 4.1 *4.16*
Answers *4.17*
4.7 Convexity, Concavity, and Point of Inflection *4.18*
4.8 Asymptotes to a Curve *4.19*
4.9 Curve Tracing *4.21*
Exercise 4.2 *4.32*
Answers *4.33*
Summary *4.35*
Objective-Type Questions *4.36*
Answers *4.37*

5. Integral Calculus **5.1–5.106**
5.1 Introduction *5.1*
5.2 Indefinite Integral *5.1*
5.3 Some Standard Results on Integration *5.1*
5.4 Definite Integral *5.2*
5.5 Geometrical Interpretation of Definite Integral *5.2*
5.6 Leibnitz's Rule of Differentiation under the Sign of Integration *5.2*
5.7 Reduction Formula for the Integrals *5.3*
Exercise 5.1 *5.8*
Answers *5.9*
5.8 Areas of Curves *5.9*
Exercise 5.2 *5.16*
Answers *5.17*

5.9	Area of Closed Curves	*5.17*
	Exercise 5.3	*5.19*
	Answers	*5.19*
5.10	Rectification	*5.19*
	Exercise 5.4	*5.26*
	Answers	*5.27*
5.11	Intrinsic Equations	*5.27*
	Exercise 5.5	*5.30*
	Answers	*5.31*
5.12	Volumes and Surfaces of Solids of Revolution	*5.31*
	Exercise 5.6	*5.38*
	Answers	*5.38*
5.13	Surfaces of Solids of Revolution	*5.39*
	Exercise 5.7	*5.42*
	Answers	*5.43*
5.14	Applications of Integral Calculus	*5.43*
	Exercise 5.8	*5.50*
	Answers	*5.50*
	Exercise 5.9	*5.54*
	Answers	*5.54*
5.15	Improper Integrals	*5.54*
	Exercise 5.10	*5.63*
	Answers	*5.63*
5.16	Multiple Integrals	*5.63*
	Exercise 5.11	*5.74*
	Answers	*5.75*
	Exercise 5.12	*5.80*
	Answers	*5.81*
	Exercise 5.13	*5.91*
	Answers	*5.92*
	Exercise 5.14	*5.100*
	Answers	*5.101*
	Summary	*5.102*
	Objective-Type Questions	*5.104*
	Answers	*5.106*
6.	**Special Functions**	**6.1–6.47**
6.1	Introduction	*6.1*
6.2	Bessel's Equation	*6.1*
6.3	Solution of Bessel's Differential Equation	*6.1*
6.4	Recurrence Formulae/Relations of Bessel's Equation	*6.3*
6.5	Generating Function for $J_n(x)$	*6.6*
6.6	Integral Form of Bessel's Function	*6.8*
	Exercise 6.1	*6.11*
6.7	Legendre Polynomials	*6.13*
6.8	Solution of Legendre's Equations	*6.13*

6.9 Generating Function of Legendre's Polynomials *6.16*
6.10 Rodrigues' Formula *6.17*
6.11 Laplace Definite Integral for $P_n(x)$ *6.18*
6.12 Orthogonal Properties of Legendre's Polynomial *6.20*
6.13 Recurrence Formulae for $P_n(x)$ *6.22*
Exercise 6.2 *6.27*
Answers *6.29*
6.14 Beta Function *6.29*
6.15 Gamma Function *6.31*
6.16 Relation between Beta and Gamma Functions *6.32*
6.17 Duplication Formula *6.35*
Exercise 6.3 *6.42*
Answers *6.43*
Summary *6.44*
Objective-Type Questions *6.46*
Answers *6.47*

7. Vector Differential and Integral Calculus 7.1–7.51

7.1 Introduction *7.1*
7.2 Parametric Representation of Vector Functions *7.1*
7.3 Limit, Continuity, and Differentiability of a Vector Function *7.2*
7.4 Gradient, Divergence, and Curl *7.3*
7.5 Physical Interpretation of Curl *7.6*
7.6 Important Vector Identities *7.10*
Exercise 7.1 *7.11*
Answers *7.11*
7.7 Vector Integration *7.12*
7.8 Line Integrals *7.12*
7.9 Conservative Field and Scalar Potential *7.15*
7.10 Surface Integrals: Surface Area and Flux *7.17*
7.11 Volume Integrals *7.21*
Exercise 7.2 *7.23*
Answers *7.24*
7.12 Green's Theorem in the Plane: Transformation between Line and Double Integral *7.24*
7.13 Gauss's Divergence Theorem (Relation between Volume and Surface Integers) *7.28*
7.14 Stokes' Theorem (Relation between Line and Surface Integrals) *7.36*
Exercise 7.3 *7.43*
Answers *7.44*
Summary *7.45*
Objective-Type Questions *7.47*
Answers *7.51*

8. Infinite Series 8.1–8.41

8.1 Sequence *8.1*
8.2 The Range *8.1*
8.3 Bounds of a Sequence *8.1*

8.4 Convergence of a Sequence *8.2*
8.5 Monotonic Sequence *8.2*
8.6 Infinite Series *8.2*
Exercise 8.1 *8.5*
Answers *8.6*
8.7 Geometric Series *8.6*
Exercise 8.2 *8.7*
Answers *8.7*
8.8 Alternating Series *8.7*
8.9 Leibnitz Test *8.7*
Exercise 8.3 *8.10*
Answers *8.10*
8.10 Positive-Term Series *8.10*
8.11 *p*-series Test *8.11*
8.12 Comparison Test *8.11*
Exercise 8.4 *8.16*
Answers *8.16*
8.13 D'Alembert's Ratio Test *8.16*
Exercise 8.5 *8.20*
Answers *8.20*
8.14 Cauchy's Root (or Radical) Test *8.21*
Exercise 8.6 *8.24*
Answers *8.24*
8.15 Raabe's Test *8.24*
Exercise 8.7 *8.26*
Answers *8.27*
8.16 Absolute Convergence and Conditional Convergence *8.27*
8.17 Test for Absolute Convergence *8.27*
Exercise 8.8 *8.30*
Answers *8.30*
8.18 Power Series *8.31*
8.19 Uniform Convergence *8.32*
8.20 Binomial, Exponential, and Logarithmic Series *8.35*
Exercise 8.9 *8.37*
Answers *8.37*
Summary *8.37*
Objective-Type Questions *8.40*
Answers *8.41*

9. Fourier Series **9.1–9.46**
9.1 Introduction *9.1*
9.2 Periodic Function *9.1*
9.3 Fourier Series *9.2*
9.4 Euler's Formulae *9.3*
9.5 Dirichlet's Conditions for a Fourier Series *9.4*
9.6 Fourier Series for Discontinuous Functions *9.4*

9.7	Fourier Series for Even and Odd Functions	*9.12*
	Exercise 9.1	*9.19*
	Answers	*9.20*
9.8	Change of Interval	*9.20*
9.9	Fourier Half-Range Series	*9.21*
	Exercise 9.2	*9.28*
	Answers	*9.31*
9.10	More on Fourier Series	*9.32*
9.11	Special Waveforms	*9.37*
9.12	Harmonic Analysis and its Applications	*9.39*
	Summary	*9.42*
	Objective-Type Questions	*9.44*
	Answers	*9.46*

10. Ordinary Differential Equations: First Order and First Degree **10.1–10.60**

10.1	Introduction	*10.1*
10.2	Basic Definitions	*10.1*
10.3	Formation of an Ordinary Differential Equation	*10.2*
10.4	First-Order and First-Degree Differential Equations	*10.3*
	Exercise 10.1	*10.6*
	Answers	*10.6*
	Exercise 10.2	*10.9*
	Answers	*10.10*
	Exercise 10.3	*10.15*
	Answers	*10.16*
	Exercise 10.4	*10.21*
	Answers	*10.21*
	Exercise 10.5	*10.28*
	Answers	*10.28*
	Exercise 10.6	*10.31*
	Answers	*10.32*
	Exercise 10.7	*10.39*
	Answers	*10.40*
10.5	Physical Applications	*10.41*
10.6	Rate of Growth or Decay	*10.43*
10.7	Newton's Law of Cooling	*10.43*
10.8	Chemical Reactions and Solutions	*10.46*
10.9	Simple Electric Circuits	*10.47*
10.10	Orthogonal Trajectories and Geometrical Applications	*10.48*
10.11	Velocity of Escape from the Earth	*10.52*
	Exercise 10.8	*10.53*
	Answers	*10.55*
	Summary	*10.55*
	Objective-Type Questions	*10.57*
	Answers	*10.60*

11. Linear Differential Equations of Higher Order with Constant Coefficients **11.1–11.34**

11.1 Introduction *11.1*
11.2 The Differential Operator D *11.1*
11.3 Solution of Higher Order Homogeneous Linear Differential Equations with Constant Coefficient *11.2*
Exercise 11.1 *11.5*
Answers *11.5*
11.4 Solution of Higher Order Non-homogeneous Linear Differential Equation with Constant Coefficients *11.5*
11.5 General Methods of Finding Particular Integrals (PI) *11.6*
11.6 Short Methods of Finding the Particular Integral When 'R' is of a Certain Special Forms *11.7*
Exercise 11.2 *11.19*
Answers *11.20*
11.7 Solutions of Simultaneous Linear Differential equations *11.21*
Exercise 11.3 *11.28*
Answers *11.29*
Summary *11.29*
Objective-Type Questions *11.31*
Answers *11.34*

12. Solutions of Second-Order Linear Differential Equations with Variable Coefficients **12.1–12.34**

12.1 Introduction *12.1*
12.2 Complete Solutions of $y'' + Py' + Qy = R$ in Terms of One Known Solution Belonging to the CF *12.1*
12.3 Rules for Finding an Integral (Solution) Belonging to Complementary Function (CF), i.e., Solution of $y'' + P(x)y + Q(x)y = 0$ *12.3*
Exercise 12.1 *12.10*
Answers *12.10*
12.4 Removal of the First Derivative: Reduction to Normal form *12.10*
12.5 Working Rule for Solving Problems by using Normal Form *12.12*
Exercise 12.2 *12.15*
Answers *12.16*
12.6 Transformation of the Equation by Changing the Independent Variable *12.16*
12.7 Working Rule for Solving Equations by Changing the Independent Variable *12.17*
Exercise 12.3 *12.23*
Answers *12.23*
12.8 Method of Variation of Parameters *12.23*
12.9 Working Rule for Solving Second Order LDE by the Method of Variation of Parameters *12.24*
12.10 Solution of Third-order LDE by Method of Variation of Parameters *12.25*
Exercise 12.4 *12.30*
Answers *12.30*

Summary *12.31*
Objective-Type Questions *12.33*
Answers *12.34*

13. Series Solutions **13.1–13.32**
13.1 Introduction *13.1*
13.2 Classification of Singularities *13.1*
13.3 Ordinary and Singular Points *13.2*
Exercise 13.1 *13.3*
13.4 Power Series *13.3*
13.5 Power-Series Solution about the Ordinary Point $x = x_0$ *13.4*
Exercise 13.2 *13.11*
Answers *13.11*
13.6 Frobenius Method *13.12*
Exercise 13.3 *13.19*
Answers *13.19*
Exercise 13.4 *13.24*
Answers *13.24*
Exercise 13.5 *13.27*
Answers *13.27*
Exercise 13.6 *13.29*
Answers *13.30*
Summary *13.30*

14. Partial Differential Equations (PDE's) **14.1–14.66**
14.1 Introduction *14.1*
14.2 Linear Partial Differential Equation *14.1*
14.3 Classification of Partial Differential Equations of Order One *14.2*
14.4 Formation of Partial Differential Equations *14.2*
Exercise 14.1 *14.9*
Answers *14.9*
14.5 Lagrange's Method of Solving the Linear Partial Differential Equations of First Order *14.10*
Exercise 14.2 *14.14*
Answers *14.15*
14.6 Nonlinear Partial Differential Equations of First Order *14.15*
Exercise 14.3 *14.20*
Answers *14.21*
14.7 Clairaut's Equation *14.21*
Exercise 14.4 *14.23*
Answers *14.23*
14.8 Linear Partial Differential Equation with Constant Coefficients *14.23*
Exercise 14.5 *14.28*
Answers *14.28*
14.9 Method of Finding the Particular Integral (PI or Z_P) of the Linear Homogeneous PDE with Constant Coefficients *14.29*

14.10 Nonhomogeneous Linear Partial Differential Equations with Constant Coefficients *14.38*
14.11 Equation Reducible to Linear Equations with Constant Coefficients *14.44*
Exercise 14.6 *14.47*
Answers *14.48*
14.12 Classification of Partial Differential Equations of Second Order *14.48*
Exercise 14.7 *14.50*
Answers *14.50*
14.13 Charpit's Method *14.50*
Exercise 14.8 *14.53*
Answers *14.53*
14.14 Nonlinear Partial Differential Equations of Second Order: (Monge's Method) *14.53*
Exercise 14.9 *14.57*
Answers *14.57*
14.15 Monge's Method of Integrating *14.57*
Exercise 14.10 *14.60*
Answers *14.61*
Summary *14.61*
Objective-Type Questions *14.65*
Answers *14.66*

15. Applications of Partial Differential Equations **15.1–15.59**

15.1 Introduction *15.1*
15.2 Method of Separation of Variables *15.1*
Exercise 15.1 *15.5*
Answers *15.5*
15.3 Solution of One-dimensional Wave Equation *15.5*
Exercise 15.2 *15.16*
Answers *15.17*
15.4 One-Dimensional Heat Equation *15.17*
15.5 Solution of One-dimensional Heat Equation *15.18*
Exercise 15.3 *15.25*
Answers *15.26*
15.6 Vibrating Membrane—Two-Dimensional Wave Equation *15.26*
15.7 Solution of Two-Dimensional Wave Equation *15.27*
Exercise 15.4 *15.32*
Answers *15.32*
15.8 Two-Dimensional Heat Flow *15.33*
15.9 Solution of Two-Dimensional Heat Equation by the Method of Separation of Variables *15.34*
15.10 Solution of Two-Dimensional Laplace's Equation by the Method of Separation of Variables *15.37*
Exercise 15.5 *15.48*
Answers *15.49*
15.11 Transmission-Line Equations *15.49*
Exercise 15.6 *15.54*
Answers *15.55*

Summary *15.55*
Objective-Type Questions *15.59*
Answers *15.59*

16. Laplace Transform **16.1–16.80**
16.1 Introduction *16.1*
16.2 Definition of Laplace Transform *16.1*
16.3 Laplace Transforms of Elementary Functions *16.3*
16.4 Linearity Property of Laplace Transforms *16.6*
16.5 A Function of Class A *16.11*
16.6 Sectionally Continuous *16.11*
16.7 Laplace Transforms of Derivatives *16.13*
16.8 Differentiation of Transforms *16.13*
16.9 Laplace Transform of the Integral of a function *16.14*
16.10 Integration of Transform! (Division by t) *16.15*
16.11 Heaviside's Unit Function (Unit-Step Function) *16.17*
16.12 Diracs Delta Function (or Unit-Impulse Function) *16.17*
16.13 Laplace Transforms of Periodic Functions *16.19*
16.14 The Error Function *16.21*
16.15 Laplace Transform of Bessel's Functions $J_0(t)$ and $J_1(t)$ *16.22*
16.16 Initial- and Final-Value Theorems *16.23*
16.17 Laplace Transform of the Laplace Transform *16.24*
Exercise 16.1 *16.31*
Answers *16.32*
16.18 The Inverse Laplace Transform (ILT) *16.33*
16.19 Null Function *16.33*
16.20 Uniqueness of Inverse Laplace Transform *16.33*
16.21 Use of Partial Fractions to Find ILT *16.45*
16.22 Convolution *16.48*
16.23 The Heaviside Expansion Formula *16.51*
16.24 Method of Finding Residues *16.52*
16.25 Inversion Formula for the Laplace Transform *16.52*
Exercise 16.2 *16.54*
Answers *16.56*
16.26 Applications of Laplace Transform *16.56*
Exercise 16.3 *16.61*
Answers *16.62*
16.27 Solution of Ordinary Differential Equations with Variable Coefficients *16.62*
Exercise 16.4 *16.63*
Answers *16.63*
16.28 Simultaneous Ordinary Differential Equations *16.64*
Exercise 16.5 *16.66*
Answers *16.66*
16.29 Solution of Partial Differential Equations (PDE) *16.66*
Exercise 16.6 *16.69*
Answers *16.69*

Summary 16.70
Objective-Type Questions 16.76
Answers 16.80

Appendix: Basic Formulae and Concepts **A.1–A.10**

Index *I.1–I.5*

Preface

Engineering mathematics (also called mathematical engineering) is a branch of applied mathematics concerning mathematical models (mathematical methods and techniques) that are typically used in engineering and industry. Along with fields like engineering physics and engineering geology—both of which may belong to the wider category, i.e., engineering science—engineering mathematics is an interdisciplinary subject motivated by engineers' needs. These needs may be practical, theoretical, or other considerations together with their specializations, and deal with constraints effective in engineering work.

Historically, engineering mathematics consisted mostly of mathematical analysis (applied analysis), most notably differential equations, real analysis, and complex analysis including vector analysis, numerical analysis, Fourier analysis, as well as linear-algebra applied probability, outside of analysis.

Salient Features

- Complete coverage of the foundational topics in Engineering Mathematics
- 360° coverage of subject matter: **Introduction - History – Pedagogy - Applications**
- Engrossing problem sets based on real life situations
- 626 solved problems with detailed procedure and solutions
- 397 MCQs with answers derived from important competitive examinations
- Appendix includes chapter-wise list of formulae
- Other pedagogical aids include
    - Drill and Practice Problems: 993
    - Illustrations: 130

Chapter Organization

The book is divided into sixteen chapters. In **Chapter 1**, we have discussed matrix algebra which includes basic terminology of a matrix, matrix inverse, rank of a matrix, solutions of homogeneous and non-homogeneous simultaneous equations, characteristic roots and vectors, quadratic forms, and applications of matrices. **Chapter 2** deals with successive differentiation and Leibnitz's theorem, while **Chapter 3** discusses partial derivatives of higher orders, homogeneous functions including Euler's theorem, Jacobian and its properties, and Taylor's series. **Chapter 4** covers Lagrange's multipliers method to find extreme points of two and more variables, convexity, concavity, and point of inflection. Asymptotes of the curve and curve tracing in Cartesian, polar, and parametric coordinates are discussed.

In **Chapter 5**, we present areas, volumes, and surfaces of solids of revolution of curves in Cartesian, polar, and parametric coordinates; moment of inertia; improper and multiple integrals; and Dirichlet's integral. In **Chapter 6**, special functions which include Bessel's equation, Legendre's polynomials, Beta and Gamma functions, along with their properties, including orthogonal properties, are discussed. **Chapter 7** covers differential and integral calculus which includes parametric representation of vector functions, gradient of a scalar field, divergence of a vector field, curl of a vector function, Green's theorem, Gauss's theorem, and Stokes' theorem. In **Chapter 8**, infinite series and sequences are discussed.

Chapter 9 deals with Fourier series which includes periodic functions, even and odd functions, Euler's formulae, Fourier series for discontinuous functions, even and odd functions, and Fourier sine and cosine series. **Chapters 10, 11, and 12** cover the basics of ordinary differential equations, integrating factors, exact differential equations and linear differential equations of higher orders with constant coefficients and the methods to solve these equations. Second-order differential equations and various methods to solve them are discussed in **Chapter 13**. Series solutions which deal with analytic functions, ordinary and singular points, power series and its solution including Frobenius method is also covered.

Chapters 14 and 15 deal with partial differential equations. In Chapter 14, methods to solve homogeneous and nonhomogeneous linear partial differential equations, Clairaut's equation, Charpit's method, and Monge's method, along with classifications of partial differential equations, are discussed. Chapter 15 deals with applications of partial differential equations including wave and heat equations. Finally, **Chapter 16** presents Laplace and inverse Laplace transformations with different properties and theorems and applications of the Laplace transform. Also, *Summary* and *Objective-Type Questions* are discussed at the end of every chapter.

Online Learning Center

The Online Learning Center can be accessed at *https://www.mhhe.com/gupta/em1/2* and contains the *Instructor Elements*: Solutions Manual.

Acknowledgements

The authors are thankful to all who have directly or indirectly helped them during the preparation of this book. We are also thankful to all our family members for their encouragement, patience, and all possible help provided by them while we were engaged in writing the manuscript. We would also like to thank the following reviewers for their feedback and comments:

Rama Bhargava	*Indian Institute of Technology (IIT) Roorkee, Uttarakhand*
Kuldeep Sharma, Sachin Kumar, V P Gupta	*Krishna Institute of Engineering and Technology (KIET) Ghaziabad, Uttar Pradesh*
Vatsala Mathur	*Malaviya National Institute of Technology, Jaipur, Rajasthan*
Chirag Barasara	*Atmiya Institute of Technology and Science, Rajkot, Gujarat*
Bikash Bhattacharjya	*Indian Institute of Technology (IIT) Guwahati, Assam*
Bimal Kumar Mishra	*Birla Institute of Technology (BIT), Mesra, Jharkhand*
Debdas Mishra	*C V Raman College of Engineering, Bhubaneswar, Odisha*
S K Abdur Rauf	*Sikkim Manipal Institute of Technology, Majitar, Sikkim*

Mahesh A Yeolekar	*Amiraj College of Engineering and Technology, Sanand, Gujarat*
S Sriram	*Patrician College of Arts and Science, Chennai, Tamil Nadu*
R Sekar	*Pondicherry Engineering College, Pondicherry*

We wish to express our appreciation for the support provided by the staff at McGraw Hill Education (India) during the publication of this book.

Feedback Request

We shall be grateful to acknowledge any constructive comments/suggestions from the readers for further improvement of the book.

C B Gupta
S R Singh
Mukesh Kumar

Publisher's Note

McGraw Hill Education (India) invites suggestions and comments from you, all of which can be sent to *info.india@mheducation.com* (kindly mention the title and author name in the subject line).

Piracy-related issues may also be reported.

1 Matrix Algebra

1.1 INTRODUCTION

In many applications in physics, pure and applied mathematics, and engineering, it is useful to represent and manipulate data in tabular or array form. A rectangular array which obeys certain algebraic rules of operation is called a *matrix*. The aim of this chapter is to study the algebra of matrices and algebraic structures along with its application to the study of systems of linear equations.

1.2 NOTATION AND TERMINOLOGY

A matrix is a rectangular array of numbers (may be real or complex); its order is the number of rows and columns that define the array.

Thus, the matrices

$$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -2 \\ 3 & 2 & 5 \\ 7 & -1 & 0 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}, [1, 1-i, 1], [0]$$

have orders 2×2, 2×3, 3×3, 3×1, 1×3, and 1×1, respectively (the order 2×2 is read "two by two").

In general, the matrix A, defined by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & & & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}$$

is of order $p \times q$. The numbers a_{ij} ($i = 1, 2, 3, \ldots p, j = 1, 2, 3, \ldots q$) are called the *entries* or *elements* of A; the first subscript defines its row position and the second, its column position.

In general, we will use bold letters to represent the matrices, but sometimes it is convenient to explicitly mention the order of A or display a typical element by use of the notations A_{pxq} and (a_{ij}),

$$A_{3\times3} = (i^j) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{bmatrix}_{3\times3} \quad (i = 1, 2, 3, j = 1, 2, 3)$$

$$A_{2\times4} = (i - j) = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \end{bmatrix}_{2\times4} \quad (i = 1, 2, j = 1, 2, 3, 4)$$

Consider the system of simultaneous equations:

$$3x_1 - 2x_2 + 3x_3 - x_4 = 1$$

$$x_1 - 2x_3 = 2$$

$$x_1 + x_2 + x_4 = -1$$

Matrix $A = \begin{bmatrix} 3 & -2 & 3 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ is the coefficient matrix, and $C = \left[\begin{array}{cccc|c} 3 & -2 & 3 & -1 & 1 \\ 1 & 0 & -2 & 0 & 2 \\ 1 & 1 & 0 & 1 & -1 \end{array}\right]$ is the augmented matrix. The augmented matrix is the coefficient matrix with an extra column containing the right-hand side constant matrix.

1.3 SPECIAL TYPES OF MATRICES

(i) Square Matrix

A matrix A in which the number of rows is equal to the number of columns is called a square matrix. Thus, for the elements a_{ij} of a square matrix $A = [a_{ij}]_{n\times n}$ for which $i = j$, the elements $a_{11}, a_{22}, \dots a_{nn}$ are called the diagonal elements and the line along which they lie is called the principal diagonal of the matrix.

Example: $A = \begin{bmatrix} 0 & 5 & 4 \\ 3 & -1 & 7 \\ 4 & 3 & 1 \end{bmatrix}_{3\times3}$ is a *square matrix* of order 3. The elements 0, –1, 1 constitute the principal diagonal of the matrix A.

(ii) Row Matrix–Column Matrix

Any $1 \times n$ matrix which has only one row and n column is called a *row matrix* or a *row vector*. Similarly, any $m \times 1$ matrix which has m rows and only one column is called a *column matrix* or a *column vector*.

Example: $A = [3, 4, 5, 6]_{1\times4}$ is a row matrix of the type 1×4, while $B = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 5 \end{bmatrix}_{4\times1}$ is a column matrix of the type 4×1.

(iii) Unit Matrix or Identity Matrix

A square matrix in which each diagonal element is one and each non-diagonal elements is equal to zero is called a *unit matrix* or an *identity matrix* and is denoted by I. It will denote a unit matrix of order n.

Thus, a square matrix $A = [a_{ij}]$ is a unit matrix if

$$a_{ij} = 1 \text{ if } i = j \text{ and } a_{ij} = 0 \text{ if } i \neq j.$$

Example: $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

(iv) Null Matrix or Zero Matrix

The $m \times n$ matrix whose elements are all zero is called the *null matrix* or *zero matrix* of the type $m \times n$. It is denoted by $O_{m\times n}$

Example: $O_{3\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$ and $O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are matrices of the type 3×4 and 3×3.

(v) Diagonal Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ whose elements above and below the principal diagonal are all zero, i.e., $a_{ij} = 0 \ \forall \ i \neq j$, is called a *diagonal matrix*. Thus, a diagonal matrix has both upper and lower triangular matrices.

Example: $A_{3\times 3} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ and $B_{3\times 3} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are diagonal matrices.

(vi) Scalar Matrix

A diagonal matrix whose diagonal elements are all equal to a scalar is called a *scalar matrix*.

Example: $A_{4\times 4} = \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{bmatrix}$ is a scalar matrix each of whose diagonal elements is equal to K.

(vii) Upper and Lower Triangular Matrices

A square matrix $A = [a_{ij}]$ is called an *upper triangular matrix* if $a_{ij} = 0$ whenever $i > j$. Thus, in an upper triangular matrix, all the elements below the principal diagonal are zero. Similarly, a square matrix $A = [a_{ij}]$ is called a *lower triangular matrix* if $a_{ij} = 0$ whenever $i < j$. Thus, in a lower triangular matrix all the elements above the principal diagonal are zero.

Example: $A = \begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}_{4\times 4}$

are upper triangular matrices.

$P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 7 \end{bmatrix}_{3\times 3}$ and $Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 4 & 3 & 6 & 5 \\ 6 & -1 & 0 & 8 \end{bmatrix}_{4\times 4}$ are lower triangular matrices.

(viii) Orthogonal Matrix

A square matrix A is said to be orthogonal if $A^T A = I$

Example: $A = \dfrac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}_{3\times 3}$

(ix) Idempotent Matrix

A matrix A is said to be idempotent if $A^2 = A$

Example: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(x) Involuntary Matrix

A matrix A is said to be involuntary if $A^2 = I$, where I is the identity matrix.

Example: $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

(xi) Nilpotent Matrix

A matrix A is said to be nilpotent if $A^K = 0$ (null matrix) where K is a positive integer. However, if K is a least positive integer for which $A^K = 0$ then K is called the index of the nilpotent matrix.

Example: $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

(xii) Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements of A lying along the principal diagonal is called the trace of the matrix A. Trace of matrix A is denoted as tr A.

Thus, if $A = \left[a_{ij}\right]_{n\times n}$ then

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

Note: Let A and B be two square matrices of order n and λ be a scalar then

(i) $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr} A$

(ii) $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$

(iii) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

1.4 EQUALITY OF TWO MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of the same size and the elements in the corresponding elements of the two matrices are the same, i.e., $a_{ij} = b_{ij}\ \forall\ i, j$.

Thus, if two matrices A and B are equal, we write $A = B$. If two matrices A and B are not equal, we write $A \neq B$. If two matrices are not of the same size, they cannot be equal.

Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2\times 2}$

Then $A = B$ iff $a = e$, $b = f$, $c = g$ and $d = h$.

Example 1 Find the values of a, b, c and d so that the matrices A and B may be equal, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ B = \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix}$$

Solution The matrices A and B are of the same size, 2×2. If $A = B$ then the corresponding elements of A and B must be equal.

$\therefore$ if $a = 1$, $b = 3$, $c = 0$ and $d = -5$ then we will have $A = B$

1.5 PROPERTIES OF MATRICES

1.5.1 Addition and Subtraction of Two Matrices

Two matrices A and B are said to be comparable for addition and subtraction, if they are of the same order.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be the two matrices.

Then the addition of the matrices A and B is defined by

$$\begin{aligned} C = [c_{ij}] = A + B &= [a_{ij}] + [b_{ij}] \\ &= [a_{ij} + b_{ij}] \end{aligned}$$

Thus, $$c_{ij} = a_{ij} + b_{ij};\ i = 1, 2, 3, \ldots m$$
$$j = 1, 2, 3, \ldots n$$

The order of the new matrix C is same as that of A and B.

Similarly, $C = A - B = [a_{ij}] - [b_{ij}]$

$C = [a_{ij} - b_{ij}]$

Thus, $c_{ij} = a_{ij} - b_{ij}$; $i = 1, 2, 3, \dots m$

$j = 1, 2, 3, \dots n$

Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $c = [c_{ij}]$ be $m \times n$ matrices with entries from the complex numbers. Then the following properties hold:

(i) Commutative law for addition, i.e., $A + B = B + A$.

(ii) Associative law for addition, i.e., $(A + B) + C = A + (B + C)$.

(iii) Existence of additive identity, i.e., $A + O = O + A = A$
'O' is the additive identity.

(iv) Existence of inverse, i.e., $A + (-A) = O = (-A) + A$
'$-A$' is the additive inverse of A.

1.5.2 Multiplication of Matrices

Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are said to comparable for the product AB, if the number of columns in the matrix A is equal to the number of rows in the matrix B. Then the matrix multiplication exists.

Let $A = [a_{ij}]_{m\times n}$ and $[b_{ij}]_{n\times p}$ be two matrices.

Then, the product AB is the matrix $C = [c_{ij}]_{m\times p}$ such that

$$C = AB$$

$$C_{ij} = [a_{ij}]\,[b_{ij}]$$

$$C_{ij} = a_{i1}\,b_{1j} + a_{i2}\,b_{2j} + \dots + a_{1n}\,b_{nj}$$

$$= \sum_{r=1}^{n} a_{ir}\,b_{rj} \text{ for } \begin{pmatrix} i = 1, 2, 3, \cdots m \\ j = 1, 2, 3, \cdots n \end{pmatrix}$$

Example 2 If $A = \begin{bmatrix} 3 & 5 \\ 6 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, find $A + B$ and $A - B$

Solution

$$A + B = \begin{bmatrix} 3 & 5 \\ 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2 & 5+3 \\ 6+1 & -1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 \\ 7 & -1 \end{bmatrix}$$

Now, $$A - B = \begin{bmatrix} 3 & 5 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 5-3 \\ 6-1 & -1-0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix}$$

Example 3 Find the product of matrices A and B, where

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 & 7 \\ -2 & -4 & 6 \\ 4 & 2 & 6 \end{bmatrix}$$

Example 4 Give an example to show that the product of two non-zero matrices may be a zero matrix.

Solution Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then A and B are both 2×2 matrices. Hence, they are conformable for product. Now,

$$A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

1.5.3 Multiplication of a Matrix by a Scalar

Let λ be a scalar (real or complex) and A be a given matrix. Then the multiplication of $A = [a_{ij}]$ by a scalar λ is defined by

$$\alpha A = \alpha[a_{ij}] = [\alpha\, a_{ij}].$$

Thus, each element of the matrix A is multiplied by the scalar λ. The size of the matrix so obtained will be the same as that of the given matrix A.

Example: $$2\begin{bmatrix} 1 & 3 & 5 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2\times 1 & 2\times 3 & 2\times 5 \\ 2\times 6 & 2\times 1 & 2\times 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 12 & 2 & 0 \end{bmatrix}$$

1.6 PROPERTIES OF MATRIX MULTIPLICATION

If $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$ and $C = [c_{kl}]_{p\times q}$ are three matrices with entries from the set of complex numbers then

(i) **Associative Law**

$(AB)\, C = A(BC)$

(ii) **Distributive Law**
$A(B + C) = AB + AC$

(iii) $AB \neq BA$, in general,
Thus, the **Commutative Law** is not true for matrix multiplication.

1.7 TRANSPOSE OF A MATRIX

If $A = [a_{ij}]_{m \times n}$ matrix then the transpose of a matrix A is denoted by A' or A^T and defined as

$$A' \text{ or } A^T = [a_{ji}]_{n \times m}$$

Thus, a matrix obtained by interchanging the corresponding rows and columns of a matrix A is called the transpose matrix of A.

Example: If $A = \begin{bmatrix} 2 & 0 & 7 \\ 2 & 5 & 8 \\ 2 & 1 & 7 \end{bmatrix}_{3 \times 3}$, the transpose of the matrix A is $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 5 & 1 \\ 7 & 8 & 7 \end{bmatrix}_{3 \times 3}$

Further,

(a) The transpose of a column matrix is a row matrix.

Example: If $A = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 8 \end{bmatrix}$, then $A^T = [3\ 4\ 5\ 8]$.

(b) The transpose of a row matrix is a column matrix.

Example: If $A = [3\ 4\ 5\ 8]$ then

$$A^T = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 8 \end{bmatrix},$$

(c) If A is $p \times q$ matrix then A^T is an $q \times p$ matrix. Therefore, the products of AA^T, A^TA are both defined and are of order $p \times p$ and $q \times q$, respectively.

(d) If A^T and B^T denote the transpose of A and B respectively then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(AB)^T = B^TA^T$

1.8 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

(i) Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$.
Thus, for a symmetric matrix $a_{ij} = a_{ji}\ \forall\ i, j$.

Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix of order 3×3.

(ii) Skew-Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $A^T = -A$. Thus, for a skew-symmetric matrix $a_{ij} = -a_{ji} \; \forall \; i, j$.

For diagonal elements $i = j$,

$\therefore \quad a_{ii} = -a_{ii}$ or $2a_{ii} = 0$

or $\quad a_{ii} = 0$

Thus, the diagonal elements are all zero.

Example: $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is a skew-symmetric matrix.

1.9 TRANSPOSED CONJUGATE, HERMITIAN, AND SKEW-HERMITIAN MATRICES

(i) Transposed Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called the *transposed conjugate* of A and it is denoted by $\boldsymbol{A}^\theta$ or by $\boldsymbol{A}^*$.

Thus, the conjugate of the transpose of A is the same as the transpose of the conjugate of A, i.e.,

$$(\overline{A})^T = (\overline{A^T}) = A^\theta$$

If $\quad A = [a_{ij}]_{m \times n}$ then

$$A^\theta = [b_{ji}]_{n \times m}, \text{ where } b_{ji} = \left[\overline{a}_{ij}\right]$$

Example: If
$$A = \begin{bmatrix} 1+2i & 2-4i & 2+5i \\ 4-5i & 7+2i & 7+3i \\ 8 & 5+6i & 7 \end{bmatrix}$$

Then
$$A^T = \begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-4i & 7+2i & 5+6i \\ 2+5i & 7+3i & 7 \end{bmatrix}$$

and
$$A^\theta = (\overline{A^T}) = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+4i & 7-2i & 5-6i \\ 2-5i & 7-3i & 7 \end{bmatrix}$$

Theorem 1 If A^θ and B^θ be the transposed conjugate of A and B respectively then

(i) $(A^\theta)^\theta = A$

(ii) $(A + B)^\theta = A^\theta + B^\theta$, A and B being of the same order

(iii) $(\lambda A)^\theta = \bar{\lambda} A^\theta$, λ being any complex number

(iv) $(AB)^\theta = B^\theta A^\theta$, A and B being conformable to multiplication

(ii) Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be Hermitian if $A^\theta = A$.

Thus, for a Hermitian matrix $a_{ij} = \bar{a}_{ij} = \bar{a}_{ji}\ \forall\ i, j$.

If A is a Hermitian matrix then

$$a_{ii} = \bar{a}_{ii}\ \forall\ i,\ \text{by definition}$$

Therefore, a_{ii} is real for all i. Thus, every diagonal element of a Hermitian matrix must be real.

Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & 2+i & 3-4i \\ 2-i & 0 & 5-4i \\ 3+4i & 5+4i & 3 \end{bmatrix}$$ are Hermitian matrices.

(iii) Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be skew-Hermitian if $A^\theta = -A$.

Thus, a matrix is skew-Hermitian if $a_{ij} = -\bar{a}_{ji}\ \forall\ i, j$.

If A is a skew-Hermitian matrix then

$$a_{ii} = \bar{a}_{ii}\ \forall\ i$$

$$\therefore \quad a_{ii} + \bar{a}_{ii} = 0$$

Thus, the diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zero.

Example: $A = \begin{bmatrix} 0 & -3-2i \\ 3-2i & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -2i & 3+5i \\ -3+5i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

We observe the following notes:

1. If A is a symmetric (skew-symmetric) matrix then kA is also a symmetric (skew-symmetric) matrix, where k is any constant.
2. If A is a Hermitian matrix then iA is a skew-Hermitian matrix.
3. If A is a skew-Hermitian matrix then iA is a Hermitian matrix.
4. If A and B are symmetric (skew-symmetric) then $(A + B)$ is also asymmetric (skew-symmetric) matrix.
5. If A be any square matrix then $A + A^T$ is symmetric and $A - A^T$ is a skew-symmetric matrix.
6. If A be any square matrix then $A + A^\theta$, AA^θ, $A^\theta A$ are all Hermitian and $A - A^\theta$ is a skew-Hermitian matrix.
7. Every real symmetric matrix is Hermitian.

Example 5 Give an example of a matrix which is skew-symmetric but not skew-Hermitian.

Solution Let $A = \begin{bmatrix} 0 & 2+3i \\ -2-3i & 0 \end{bmatrix}$ be a square matrix of order 2×2.

Then $$A^T = \begin{bmatrix} 0 & -2-3i \\ 2+3i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2+3i \\ -2-3i & 0 \end{bmatrix} = -A$$

Thus, the matrix A is skew-symmetric.

Again, $$A^{\theta} = (\overline{A^T}) = \begin{bmatrix} 0 & -2+3i \\ 2-3i & 0 \end{bmatrix} \neq -A$$

So that, the matrix A is not skew-Hermitian.

Example 6 Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution Let A be any square matrix we can write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$A = P + Q \text{ (say)}$$

where $$P = \frac{1}{2}(A + A^T) \text{ and } Q = \frac{1}{2}(A - A^T)$$

Now, $$P^T = \left[\frac{1}{2}(A + A^T)\right]^T$$

$$= \frac{1}{2}(A + A^T)^T \qquad \left[\because (\lambda A)^T = \lambda A^T\right]$$

$$P^T = \frac{1}{2}\left[A^T + (A^T)^T\right] \qquad \left[\because (A+B)^T = A^{T+} B^T\right]$$

$$= \frac{1}{2}(A^T + A)$$

$$= \frac{1}{2}(A + A^T) = P$$

Thus, P is a symmetric matrix.

Again, $$Q^T = \left[\frac{1}{2}(A - A^T)\right]^T$$

$$= \frac{1}{2}(A - A^T)^T = \frac{1}{2}\left[A^T - (A^T)^T\right]$$

$$= \frac{1}{2}[A^T - A]$$

$$= -\frac{1}{2}[A - A^T]$$

$$Q^T = -Q$$

Therefore, Q is a skew-symmetric matrix.

Thus, we have expressed the square matrix A as the sum of a symmetric and skew-symmetric matrix.

Now, to prove that the representation is unique, let $A = R + S$ be another such representation of A, where R is symmetric and S is skew-symmetric. Then to prove that $R = P$ and $S = Q$,

we have $A^T = (R + S)^T$

$$= R^T + S^T$$

$$= R - S(Q\ R^T = R \text{ and } S^T = -S)$$

Therefore, $A + A^T = 2R$ and $A - A^T = 2\,S$

This implies that $R = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$.

Thus, $R = P$ and $S = Q$.

Therefore, the representation is unique.

Example 7 Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Solution If A is any square matrix then $A + A^\theta$ is a Hermitian matrix and $A - A^\theta$ is a skew-Hermitian matrix.

Therefore, $\frac{1}{2}(A + A^\theta)$ is a Hermitian and $\frac{1}{2}(A - A^\theta)$ is a skew-Hermitian matrix.

Now, $$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$$

$$A = P + Q \text{ (say)}$$

where $P = \frac{1}{2}(A + A^\theta)$ is Hermitian and $\theta = \frac{1}{2}(A - A^\theta)$ is a skew-Hermitian matrix.

Thus, every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Now, to prove that $A = R + S$ be the another representation is unique, where R is Hermitian and S is skew-Hermitian.

Therefore, $A^\theta = (R + S)^\theta$

$$= R^\theta + S^\theta$$

$$= R - S \qquad (\because R^\theta = R \text{ and } S^\theta = -S)$$

$\therefore$ $$R = \frac{1}{2}(A + A^\theta) = P \text{ and } S = \frac{1}{2}(A - A^\theta) = Q$$

Thus, the representation is unique.

1.10 ELEMENTARY TRANSFORMATION OR ELEMENTARY OPERATIONS

The following transformations are called elementary transformations of a matrix.

(i) Interchanging of rows (columns)
(ii) Multiplication of a row (column) by a non-zero scalar
(iii) Adding/subtracting K multiples of a row (column) to another row (column)

Notation

The following row (column) transformations will be denoted by the following symbols.

(i) $R_i \Leftrightarrow R_j$ ($C_i \Leftrightarrow C_j$) for the interchange of the i^{th} and j^{th} row (column)
(ii) $R_i \to KR_i$ ($C_i \to KC_i$) for multiplication of the i^{th} row (column) by K
(iii) $R_i \to R_i + \alpha R_j$ ($C_i \to C_i + \alpha C_j$) for addition to the i^{th} row (column), α times the j^{th} row (column)

1.11 THE INVERSE OF A MATRIX

The inverse of an $n \times n$ matrix $A = [A_{ij}]$ is denoted by A^{-1} and is an $n \times n$ matrix such that

$$AA^{-1} = A^{-1}A = I_n$$

where I is the $n \times n$ unit matrix.

If A has an inverse, then A is called a non-singular matrix.

If A has no inverse, then A is called a singular matrix.

1.11.1 The Inverse of a Square Matrix is Unique

Proof Let B and C are inverse of A then

$$AB = BA = I_n \tag{1}$$

And $$AC = CA = I_n \tag{2}$$

From (1), we have $AB = I_n$

Premultiplication with C gives

$$C(AB) = CI_n$$

$$C(AB) = C \tag{3}$$

From (2), we have $CA = I_n$

Post-multiplication with B, gives

$$(CA)B = I_n\,B = B \tag{4}$$

Since $$C(AB) = (CA)B$$

Therefore, from (3) and (4), we have $B = C$.

Hence, the inverse of a square matrix is unique.

1.11.2 Some Special Points on Inverse of a Matrix

(i) If A be any n-rowed square matrix, then

$$(\text{Adj } A)A = A(\text{Adj } A) = |A|\, I_n,$$

where I_n is the n-rowed unit matrix.

(ii) The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

(iii) If A be an $n \times n$ non-singular matrix then

$$(A')^{-1} = (A^{-1})', \text{ where } (\,'\,) \text{ (desh) denote, the transpose.}$$

(iv) If A be an $n \times n$ non-singular matrix then

$$(A^{-1})^{\theta} = (A^{\theta})^{-1}$$

(v) If A, B be two n-rowed non-singular matrices then AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(vi) If A is a non-singular matrix then $\det(A^{-1}) = (\det A)^{-1}$.

(vii) If the matrices A and B commute then A^{-1} and B^{-1} also commute.

(viii) If A, B, C be three matrices conformable for multiplication then $(ABC)^{-1} = C^{-1} \cdot B^{-1} \cdot A^{-1}$.

(ix) If the product of two non-zero square matrices is a zero matrix then both must be singular matrices.

(x) If A be an $n \times n$ matrix then

$$|\text{adj } A| = |A|^{n-1}$$

(xi) If A is a non-singular matrix then

$$\text{adj adj } A = |A|^{n-2} A$$

(xii) If A and B are square matrices of the same order then adj (AB) = adj B. adj A

1.11.3 Method of Finding the Inverse by Elementary Operations

The elementary row transformations which reduce a given square matrix A to the unit matrix I, when we applied the above transformations to the unit matrix I. Give the inverse of matrix A, i.e., A^{-1}.

To find A^{-1}, write the matrix A and I side by side and then applying the same row operations on both sides. We get A is reduced to I, the other matrix represents A^{-1}.

Example 8 Using row elementary operation, find the inverse of the matrix A, where

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}_{3\times 3}$$

Solution Writing the given matrix A side by side with the unit matrix of the same order as A, we have

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 4R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right] = \left[I_3 \mid A^{-1}\right]$$

Hence, $$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Example 9 Using row elementary operations, find the inverse of the given matrix A, where

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}_{4\times 4}$$

Solution

$$[A \mid I_4] = \left[\begin{array}{cccc|cccc} 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{array} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 4 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - R_3 \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 4 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 4 & -2 & -1 & 2 & 1 \end{array}\right]$$

$$R_4 \to R_4 - R_3 \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_3 \to R_3 - 3R_4 \\ R_1 \to R_1 + 2R_4 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & -2 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -5 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 + R_3 \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -5 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -3 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -5 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right] = \left[I_4 \middle| A^{-1}\right]$$

Hence, $$A^{-1} = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

EXERCISE 1.1

1. Find the inverses of the following matrices by elementary transformation:

(i) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 13 & 17 \\ 5 & 7 & 1 \\ 8 & 3 & 11 \end{bmatrix}$

Answers

(i) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (ii) $\dfrac{1}{5}\begin{bmatrix} 2 & -2 & -3 \\ -2 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$ (vi) $\dfrac{1}{1086}\begin{bmatrix} -74 & 92 & 106 \\ 47 & 103 & -82 \\ 41 & -95 & 44 \end{bmatrix}$

1.12 ECHELON FORM OF A MATRIX

A matrix A is said to be in **Echelon form** if the following hold:

(i) Every row of matrix A, which has all its entries zero occurs below every row which has a non-zero entry.
(ii) The first non-zero entry in each non-zero row is equal to one.
(iii) the number of zeros preceding the first non-zero element in a row is less than the number of such zero in the succeeding row.

Example: The matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & 6 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is an Echelon form.}$$

1.13 RANK OF A MATRIX

The rank of a matrix A is said to be 'r' if it possesses the following two properties:

(i) There is at least one non-zero minor of order 'r' whose determinant is not equal to zero.
(ii) If the matrix A contains any minor of order $(r + 1)$ then the determinant of every minor of A of order $r + 1$, should be zero.

Thus, the rank of a matrix is the largest order of a non-zero minor of the matrix

The rank of a matrix A is denoted by $\rho(A)$.

The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix, i.e.,

$\rho(A)$ = Number of non-zeros in the Echelon form of the matrix.

Some Important Results

(i) Rank of A and A^T is same.
(ii) Rank of a null matrix is zero
(iii) Rank of a non-singular matrix A of order n is n.
(iv) Rank of an identity matrix of order n is n.
(v) For a rectangular matrix A of order $m \times n$, rank of $A \le \min(m, n)$, i.e., rank cannot exceed the smaller of m and n.

(vi) For an n-square matrix A, if $\rho(A) = n$ then $|A| \neq 0$, i.e., matrix A is non-singular.
(vii) For any square matrix, if $\rho(A) < n$ then $|A| = 0$, i.e., matrix A is singular.
(viii) The rank of a product of two matrices cannot exceed the rank of either matrix.

1.14 CANONICAL FORM (OR NORMAL FORM) OF A MATRIX

The normal form of a matrix A of order $m \times n$ of rank 'r' is one of the forms

$$[I_r], \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

where I_r is an identity matrix of order r. By the application of a number of elementary operations, a matrix of rank r can be reduced to normal form. Then the rank of matrix A is r.

Example 10 Reduce the matrix A to echelon form and, hence, find its rank, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \\ \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_3 \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in echelon form. The number of non-zero rows in this matrix is 3.
Therefore, $\rho(A)$ = number of non-zero rows in echelon form of the matrix

$$\rho(A) = 3$$

Example 11 Determine the rank of the given matrix.

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

$$\begin{array}{r} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \\ \end{array} \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\begin{array}{r} R_2 \leftrightarrow R_3 \\ \\ \end{array} \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in echelon form. The number of non-zero rows in this matrix is 2. Therefore, $\rho(A) = 2$

Example 12 Determine the values of K such that the rank of the matrix A is 3, where.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$$

Solution We have

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$$

$$\begin{array}{r} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 9R_1 \\ \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ K-2 & 0 & 4 & 2 \\ 0 & 0 & K+9 & 3 \end{bmatrix}$$

$$\begin{array}{r} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 3R_2 \\ \\ \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ K-2 & 0 & 0 & -2 \\ 0 & 0 & K+6 & 0 \end{bmatrix}$$

$$\begin{array}{r} R_4 \leftrightarrow R_3 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & K+6 & 0 \\ K-2 & 0 & 0 & -2 \end{bmatrix}$$

(i) If $K = 2$, $|A| = 1.0.8. - 2 = 0$, the rank of the matrix $A = 3$.
(ii) If $K = -6$, number of non-zero rows is 3, the rank of matrix A is 3.

Example 13 Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical (normal) form. Hence, find the rank of A.

Solution

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$$\begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 8 & 5 & 0 \\ 1 & -2 & 1 & -8 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$$C_2 \to \frac{1}{8}C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -\frac{1}{4} & 1 & -8 \end{bmatrix}$$

$$C_3 \to C_3 - 5C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{9}{4} & -8 \end{bmatrix}$$

$$R_3 \to 4R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 9 & -32 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

$$C_3 \to \frac{1}{9}C_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -32 \end{bmatrix}$$

$$C_4 \to C_4 + 32C_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left[I_3 \vdots 0\right]$$

Hence, the rank of the matrix $A = 3$ (number of non-zero rows in I_3).

Example 14 Reduce the matrix $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and, hence, find its rank.

Solution The given matrix is $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$$\begin{matrix} C_2 \to C_2 + C_1 \\ C_3 \to C_3 - 2C_1 \\ C_4 \to C_4 + 3C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix}$$

$$C_4 \to C_4 - 2C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix}$$

$$\begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - 5R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}$$

$$C_3 \leftrightarrow C_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix}$$

$$\begin{array}{l} C_3 \leftrightarrow \frac{1}{3}C_3 \\ C_4 \to -\frac{1}{8}C_4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$R_4 \leftrightarrow R_4 + 2R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim [I_4]$$

which is in normal form.

Hence, the rank of the matrix A is 4.

Example 15 Find two non-singular matrices P and Q such that PAQ is in the normal form, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution We write $A = I_3AI_3$, i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pre factor

Post factor

Now, we apply the elementary operations on the matrix A (left-hand side of the above equation). Until it is reduced to the normal form, every elementary row operation will also be applied to the prefactor and every elementary column operation to the post-factors of the above equation.

Performing $R_2 \to R_2 - R_1$, $R_2 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to -\frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 - C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, the rank of matrix A is 2.

Example 16 Find the two non-singular matrices P and Q such that the normal form of A is PAQ where $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}$. Hence, find its rank.

Solution Consider

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1,\ C_3 \to C_3 - 6C_1,\ C_4 \to C_4 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2,\ C_4 \to C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix},$$

where $$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ the rank of the matrix A is 2.

EXERCISE 1.2

1. Find the ranks of the following matrices:

(i) $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4\times 4}$

(v) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}_{3\times 4}$ (vi) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}_{4\times 5}$

2. Find the non-singular matrices P and Q such that PAQ is in the normal form for A. Hence, find the rank of A.

(i) $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

3. Reduce the matrix $A = \begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$ to normal form and find its rank.

4. Reduce the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ to normal form and find its rank.

Answers

1. (i) 3 (ii) 2 (iii) 2 (iv) 4 (v) 2 (vi) 4

2. (i) $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & -1/3 & 1/6 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & \frac{4}{7} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$, $\rho(A) = 2$

(ii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$; $\rho(A) = 3$

(iii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$

(iv) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\rho(A) = 2$

3. $\rho(A) = 3$
4. $\rho(A) = 2$

1.15 LINEAR SYSTEMS OF EQUATIONS

Matrices play a very important role in the solution of linear systems of equations, which appear frequently as models of various problems, for instance, in electrical networks, traffic flow, production and consumption, assignment of jobs to workers, population of growth, statistics, and many others. In this section we shall the study of the nature of solutions of linear systems of equations. We shall first consider systems of homogeneous linear equations and proceed to discuss systems of nonhomogeneous linear equations.

1.16 HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Consider

$$\left.\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n &= 0, \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n &= 0 \\ &\vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n &= 0 \end{aligned}\right\} \tag{5}$$

is a system of m homogeneous equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Then, the system (5) can be written in the matrix form

$$AX = O \tag{6}$$

The matrix A is called the *coefficient matrix* of the system of (5).

The system (5) has the trivial (zero) solution if the rank of the coefficient matrix A in the echelon form of the matrix is equal to the number of unknown variables (n), i.e.,

$$\rho(A) = n$$

The system (5) has infinitely many solutions if the rank of coefficient matrix is less than the number of unknown variables, i.e., $\rho(A) < n$.

Remark I: The number of linearly independent solutions of m homogeneous linear equations in n variables, $AX = O$, is $(n - r)$, where r is the rank of the matrix A.

Remark II: A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

1.17 SYSTEMS OF LINEAR NON-HOMOGENEOUS EQUATIONS

Suppose a system of m non-homogeneous linear equations with n unknown variables is of the form

$$\left.\begin{aligned} &a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n = b_1 \\ &a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n = b_2 \\ &\vdots \\ &a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n = b_m \end{aligned}\right\} \tag{7}$$

If we write $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

Then, the above system (7) can be written in the form of a single matrix equation $AX = B$.

The matrix $[A \mid B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called the ***augmented matrix*** of the given system of equations. Any set of values which simultaneously satisfy all these equations is called a solution of the given system (7). When the system of equations has one or more solution then the given system is consistent, otherwise it is inconsistent.

1.18 CONDITION FOR CONSISTENCY THEOREM

The system of equations $AX = B$ is consistent, i.e., possesses a solution, if and only if the coefficient matrix A and the augmented matrix $[A \mid B]$ are of the same rank. Now, two cases arise.

Case 1 If the rank of the coefficient matrix and the rank of the augmented matrix are equal to the number of unknown variables, i.e.,

$\rho(A) = \rho(A : B) = n$ (number of unknown variables) then the system has a unique solution.

Case 2 If the rank of the coefficient matrix and the rank of the augmented matrix are equal but less than the number of unknown variables, i.e., $\rho(A) = \rho(A : B) < n$ then the given system has infinitely many solution.

1.19 CONDITION FOR INCONSISTENT SOLUTION

The system of equations $AX = B$ is inconsistent, i.e., possesses no solution if the rank of coefficient matrix A is not equal to the rank of augmented matrix $[A \mid B]$ i.e. $\rho(A) \neq \rho[A \mid B]$

Example 17 Show that the given system of equations

$x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$ are consistent and solve them.

Solution The given systems of equations can be written in matrix form $AX = B$ i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Now, the augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ \\ \end{array} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above is the echelon form of the matrix $[A \mid B]$.

Here, $\rho[A \mid B]$ = number of non-zero rows = 2 and $\rho(A) = 2$.

$$\rho(A) = 2$$

$\therefore \quad \rho[A \mid B] = 2 = \rho(A).$

Hence, the system is consistent.
Now, the number of unknown variables is 3.
Since $\rho(A) < 3$, the given system will have an infinite number of solutions.
The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

Or $\quad x + y + z = 6$

$$y + 2z = 8$$

Let $z = k$ so that $y = 8 - 2k$ and $x = 6 - 8 + 2k - k = k - 2$
where k is an arbitrary constant.

Example 18 Investigate for what values of λ and μ, the simultaneous equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution, and (iii) an infinite number of solutions.

Solution The given system can be written in matrix form.

$AX = B$, i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The augmented matrix

$$[A \mid B] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) If $\lambda = 3$ and $\mu \neq 10$ then $\rho(A|B) = 3$ and $\rho(A) = 2$.
Thus, $\rho(A \mid B) \neq \rho(A)$
The given system is inconsistent.
Hence, the system has no solution.

(ii) If $\lambda \neq 3$ and μ have any value then
$\rho(A \mid B) = \rho(A) = 3$ = number of unknown variables
Hence, the system has a unique solution.

(iii) If $\lambda = 3$ and $\mu = 10$ then $\rho(A \mid B) = 2 = \rho(A)$
In this case, the system is consistent. Here, the number of unknown variables is 3.
$\therefore \quad \rho(A \mid B) = r(A) < 3$
Hence, the system of equations possesses an infinite number of solutions.

Example 19 Solve the following system of linear equations:

$x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$ and $x - y + z = -1$.

Solution The given system of equations can be written in matrix form $AX = B$, i.e.,

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

The augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$$R_2 \to R_2 - R_3 \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$$\begin{array}{l} R_3 \to R_3 - 6R_2 \\ R_4 \to R_4 - 3R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

$$\begin{array}{l} R_3 \to \frac{1}{5} R_3 \\ \\ \\ R_4 \to \frac{1}{2} R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$R_4 \to R_4 - R_3 \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The augmented matrix $[A \mid B]$ has been reduced to echelon form.

$\rho[A \mid B]$ = number of non-zero rows in echelon form = 3.

Also, $\rho(A) = 3$

$\therefore$ $\rho[A \mid B] = 3 = \rho(A)$ = number of unknown variables

Hence, the given system of equations has a unique solution.

Now,
$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

$$x + 2y - z = 3, \ -y = -4, \ z = 4$$

$$\therefore \quad y = 4, \ z = 4, \ x = -1$$

Example 20 Discuss for all values of K for the system of equations

$$x + y + 4z = 6, \ x + 2y - 2z = 6, \ kx + y + z = 6$$

as regards existence and nature of solutions.

Solution The given system of equations can be written in matrix form $AX = B$;

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ K & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

The given set of equations will have a unique solution if and only if the coefficient matrix A is non-singular.

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - KR_1 \end{matrix} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-K & 1-4K \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6-6K \end{bmatrix}$$

$$R_3 \to R_3 - (1-K)R_2 \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 7-10K \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6-6K \end{bmatrix} \tag{1}$$

$\therefore$ The coefficient matrix A in (1) will be a non-singular iff

$$7 - 10K \neq 0, \text{ i.e., } K \neq \frac{7}{10}$$

Hence, the given system of equations will have a unique solution if $K \neq \frac{7}{10}$

In case $K = \frac{7}{10}$ then (1) becomes

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

The above system is not consistent if $K = \frac{7}{10}$.

Example 21 Solve $x + 3y - 2z = 0$, $2x - y + 4z = 0$ and $x - 11y + 14z = 0$.

Solution The given system of equations can be written in matrix form as $AX = 0$;

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 30 - 72 + 42 = 0$$

$\therefore$ the matrix A is singular, i.e., $\rho(A) < n$.

Thus, the given system has a nontrivial solution and will have an infinite number of solutions.

Now, the given system is

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ \\ \end{array}\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

And so we have

$$x + 3y - 2z = 0$$

$$-7y + 8z = 0$$

Let $z = K$, $y = \frac{8}{7}K$, $x = -\frac{10}{7}K$ where K is any arbitrary constant.

$$\therefore \quad x = -\frac{10}{7}K,\ y = \frac{8}{7}K \text{ and } z = K.$$

Thus, the system has an infinite number of solutions.

1.20 CHARACTERISTIC ROOTS AND VECTORS (OR EIGENVALUES AND EIGENVECTORS)

Let A be a square matrix of order n, λ is a scalar and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ a column vector.

Consider the equation

$$AX = \lambda X \tag{8}$$

Clearly, $X = 0$ is a solution of (8) for any value λ.

Now, let us see whether there exist scalars λ and non-zero vectors X which satisfy (8). This problem is known as characteristic value problem. If I_n is unit matrix of order n then (8) may be written in the form

$$(A - \lambda I_n)X = 0 \tag{9}$$

Equation (9) is the matrix form of a system of n homogeneous linear equations in n unknowns. This system will have a nontrivial solution if and only if the determinant of the coefficient matrix $A - \lambda I_n$ vanishes, i.e.,

If $$|A - \lambda I_n| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

The expansion of this determinant yields a polynomial of degree n in λ, called *the characteristic polynomial* of the matrix A.

The equation $|A - \lambda I_n| = 0$ is called the ***characteristic equation*** or *secular equation* of the matrix A.

The n^{th} roots of the characteristic equation of a matrix A of order n are called the *characteristic roots*, *characteristic values*, *proper values*, *eigenvalues*, or *latent roots* of the matrix A.

The set of the eigenvalues of a matrix A is called the spectrum of A.

If λ is a characteristic root of an $n \times n$ matrix A then a non-zero vector X such that

$$AX = \lambda X$$

is called a ***characteristic vector***, *eigenvector*, *proper vector* or *latent vector* of the matrix A corresponding to the characteristic root λ.

1.21 SOME IMPORTANT THEOREMS ON CHARACTERISTIC ROOTS AND CHARACTERISTIC VECTOR

Theorem 2

λ is a characteristic root of a matrix A if and only if there exists a non-zero vector X such that $AX = \lambda X$.

Theorem 3

If X is a characteristic vector of A then X cannot correspond to more than one characteristic value of A.

Theorem 4

The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Theorem 5

The characteristic roots of a Hermitian matrix are real.

1.22 NATURE OF THE CHARACTERISTIC ROOTS

(i) The characteristic roots of a real symmetric matrix are all real.

(ii) The characteristic roots of a skew-Hermitian matrix are either pure imaginary or zero.

(iii) The characteristic roots of a real symmetric matrix are either pure imaginary or zero, for every such matrix is skew-Hermitian.

(iv) The characteristic roots of a unitary matrix are of unit modulus.

(v) The characteristic roots of an orthogonal matrix are of unit modulus.

(vi) The sum of the eigenvalues of a matrix A is equal to the trace of the matrix also equal to the sum of the elements of the principal diagonal.

(vii) The product of the eigenvalues of A is equal to the determinant of A.

(viii) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then the eigenvalues of (a) A^k are $\lambda_1{}^k, \lambda_2{}^k, \ldots, \lambda_n{}^k$, and (b) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

Example 22 Find the characteristic roots and the corresponding characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

i.e.,
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

or $(8 - \lambda) \{(7 - \lambda) (3 - \lambda) - 16\} + 6\{-6(3 - \lambda) + 8\} + 2\{24 - 2(7 - \lambda)\} = 0$

or $\lambda^3 - 18 \lambda^2 + 45 \lambda = 0$

or $\lambda(\lambda - 3) (\lambda - 15) = 0$

or $\lambda = 0, 3, 15.$

The characteristic roots of A are 0, 3, 15

The eigenvector of A corresponding to the eigenvalue 0 is given by

$$(A - 0I) X = O$$

or
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \leftrightarrow R_3$$

or
$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

or
$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + 2R_2$$

The coefficient matrix is of rank 2.

Therefore, these equations have $n - r = 3 - 2 = 1$ linearly independent solutions.

The above equations are

$$2x_1 - 4x_2 + 3x_3 = 0$$
$$-5x_2 + 5x_3 = 0$$

From the last equation, we get $x_2 = x_3$.

Choose $x_2 = k$, $x_3 = k$; then the first equation gives $x_1 = \frac{k}{2}$, where k is any scalar.

$\therefore\ X_1 = k\left[\frac{1}{2}, 1, 1\right]' = k[1, 2, 2]'$ is an eigenvector of A corresponding to the eigenvalue 0.

The eigenvector of A corresponding to the eigenvalue 3 are given by $(A - 3I)X = 0$.

or $$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \to R_1 + R_2$$

or $$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } \begin{array}{l} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

or $$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + \frac{1}{2}R_2$$

The coefficient matrix of these equations is of rank 2.
Therefore, this equation is of rank 2.
Therefore, these equations have $n - r = 3 - 2 = 1$ linearly independent solutions.
The above equations are

$$-x_1 - 2x_2 - 2x_3 = 0$$
$$16x_2 + 8x_3 = 0$$

From the last equation, we get $x_2 = -\frac{1}{2}x_3$.

Choose $x_3 = 4k$, $x_2 = -2k$; then from the first equation. $X_1 = -4\,k$, where k is any scalar.
$\therefore\quad X_2 = k[-4, -2, 4]'$ is an eigenvector of A corresponding to the eigenvalue 3.
Now, the eigenvector of A corresponding to the eigenvalue 15 is given by $(A - 15\,I)X = 0$

or $$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \to R_1 - R_2.$$

or $$\begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } \begin{array}{l} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

The coefficient matrix of these equations is of rank 2.

$\therefore$ these equations have $n - r = 3 - 2 = 1$ linearly independent solutions.

The above equations are

$$-x_1 + 2x_2 + 6x_3 = 0$$
$$-20\,x_2 - 40\,x_3 = 0$$

From the last equation, we get $x_2 = -2x_3$

Choose $x_3 = k, x_2 = -2\,k.$

Then from the first equation, we get $x_1 = 2\,k$, where k is any scalar.

$\therefore$ $x_3 = k\,[2, -2, 1]'$ is an eigenvector of A corresponding to the eigenvalue 15.

Method 1

Example 23 Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution The characteristic equation of A is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 2-\lambda \\ 2 & -1 & 2-\lambda \end{vmatrix}_2 = 0 \quad \text{by} \quad C_3 \to C_3 + C_2$$

or $$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

or $$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 0 \\ -4 & 4-\lambda & 0 \\ 2 & -1 & 1 \end{vmatrix} = 0, \quad \text{by} \quad R_2 \to R_3 - R_3$$

or $$(2-\lambda)\big[(6-\lambda)\quad(4-\lambda)\quad -8\big] = 0$$

or $$(2-\lambda)\,(\lambda-2)\,(\lambda-8) = 0$$

or $$\lambda = 2, 2, 8$$

$\therefore$ the characteristic roots of A are 2, 2, 8.

Now, the eigenvectors of the matrix A corresponding to the eigenvalue 2 are given by non-zero solutions of the equation

$$(A - 2I)\, X = 0$$

or
$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \leftrightarrow R_2$$

or
$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

The coefficient matrix of these equations is of rank 1.

$\therefore$ there are $n - r = 3 - 1 = 2$ linearly independent solutions.

The above equation is

$$-2\, x_1 + x_2 - x_3 = 0$$

Clearly,

$X_1 = [-1, 0, 2]'$ and $X_2 = [1, 2, 0]'$ are two linearly independent vectors.

Then X_1 and X_2 are two linearly independent eigenvectors of A, linearly independent vector corresponding to the eigenvalue 2.

The eigenvectors of A corresponding to the eigenvalue 8 are given by the non-zero solutions of the equation

$$(A - 8I)X = 0$$

or
$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

or
$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - R_2$$

The coefficient matrix of these equations is of rank 2.

$\therefore$ there are $n - r = 3 - 2 = 1$ linearly independent solutions.

The above equations are

$$-2x_1 - 2\, x_2 + 2\, x_3 = 0$$
$$-3\, x_2 - 3x_3 = 0$$

The last equation gives $x_2 = -x_3$

Choose $x_3 = 1$, $x_2 = -1$; then the first equation gives $x_1 = 2$. Hence, $X_3 = [2, -1, 1]$ is an eigenvector of A corresponding to the eigenvalue 8.

Method 2

Example 24 Find the eigenvalues and eigenvectors of the given matrix A, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution The characteristic equation for A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or $(2-\lambda)\,[(2-\lambda)^2 - 1] + 1[-2 + \lambda + 1] + 1[1 - 2 + \lambda] = 0$

or $$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

or $\lambda = 1, 1, 4$

The eigenvalues are 1, 1, 4.

Let $X_1 = (x_1, x_2, x_3)^T$ be the eigenvectors corresponding to the eigenvalues $\lambda = 4$.

$$[A - \lambda I]\, X_1 = 0$$

or $$[A - 4I]\, X_1 = 0$$

or $$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & -3/2 & -3/2 \\ 0 & -3/2 & -3/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 + \frac{1}{2}R_1 \end{matrix}$$

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & -3/2 & -3/2 \\ 0 & 0 & -3/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_3 \to R_3 - R_2$$

or $$-2x_1 - x_2 + x_3 = 0 \text{ or } 2x_1 + x_2 - x_3 = 0$$

$$-\frac{3}{2}x_2 - \frac{3}{2}x_3 = 0 \text{ or } x_2 + x_3 = 0$$

Let $x_3 = k_1$, $x_2 = -k_1$; then $x_1 = k_1$

$\therefore$ $$X_1 = k_1[1, -1, 1]^T \text{ or } X_1 = [1, -1, 1]^T$$

Let $x_2 = [x_1, x_2, x_3]^T$ be the eigenvectors corresponding to eigenvalue $\lambda = 1$

$$[A - \lambda I]\, X_2 = 0 \text{ or } [A - I]\, X_2 = 0$$

or
$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

or
$$x_1 - x_2 + x_3 = 0$$

Let $x_1 = k_1$ and $x_2 = k_2$; then $x_3 = k_2 - k_1$

$\therefore$ $X_2 = [k_1, k_2, k_2 - k_1]^T$ or $X_2 = [1, 1, 0]^T$. Put $k_1 = 1$ and $k_2 = 1$

The given matrix A is symmetric with repeated eigenvalues.

So we consider the third eigenvector $X_3 = [l, m, n]^T$.

Since, the given matrix is symmetric, the vector X_3 is orthogonal to X_1, i.e., $X_1X_3{}^T = 0$

$$[1, -1, 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \text{ or } l - m + n = 0 \tag{1}$$

Now, X_3 is orthogonal to X_2, i.e., $X_2X_3{}^T = 0$

$$[1, -1, 0]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \text{ or } l + m = 0 \tag{2}$$

Solving (1) and (2), we get $l = -1$, $m = 1$ and $n = 2$.

$\therefore$ $X_3 = [-1, 1, 2]^T$.

Thus, the eigenvectors X_1, X_2, X_3 corresponding to eigenvalues $\lambda = 4, 1, 1$ are given by

$$X_1 = [1, -1, 1]^T,\ X_2 = [1, 1, 0]^T \text{ and } X_3 = [-1, 1, 2]^T$$

1.23 THE CAYLEY–HAMILTON THEOREM

Statement

Every square matrix satisfies its own characteristic equation, i.e., if for a square matrix A of order n,

$$|A - \lambda I_n| = (1)^n\, [\lambda^n + a_1\, \lambda^{n-1} + a_n\, \lambda^{n-2} + \ldots + a_n]$$

then the matrix equation

$$X^n + a_1X^{n-1} + a^2\, X^{n-2} + \ldots + a^n\, I_n = 0$$

is satisfied by $X = A$

i.e., $$A^n + a_1\, A^{n-1} + a_2\, A^{n-2} + \ldots + a^n\, I_n = 0.$$

Proof The characteristic matrix of A is $A - \lambda\, I_n$.

Since the elements of $A - \lambda\, I_n$ are at most of the first degree in λ, the elements of Adj $(A - \lambda\, I_n)$ are ordinary polynomials in λ of degree $(n - 1)$ or less.

Therefore, Adj $(A - \lambda I_n)$ can be written as a matrix polynomial in λ, given by

$$\text{Adj } (A - \lambda I_n) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \ldots + B^{\lambda}{}_{n-2} + B_{n-1}$$

where $B_0, B_1, B_2, \ldots, B_{n-2}, B_{n-1}$ are matrices of order $n \times n$ whose elements are functions of a_{ij}'s.

Now,

$$(A - \lambda I_n) \text{ Adj } (A - \lambda I_n) = |A - \lambda I_n| \cdot I_n \qquad [\because A \text{ Adj } A = |A| \cdot I_n]$$

$$\therefore \quad (A - \lambda I_n)(B_0 \lambda^{n-1} + B1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1})$$
$$= (-1)^n [\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n] I_n.$$

Equating the coefficients of like powers of λ on both sides, we get

$$-I_n B_0 = (-1)^n I_n$$
$$A B_0 - I_n B_1 = (-1)^n a_1 I_n$$
$$\ldots$$
$$AB_{n-1} = (-1)^n a_2 I_n$$

Multiplying these successively by $A^n, A^{n-1}, \ldots, I_n$ and adding, we get

$$0 = (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I_n]$$

Thus, $\boldsymbol{A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I_n = 0}$

Corollary The matrix A is non-singular, i.e., $|A| \neq 0$.

Also, $|A| = (-1)\, a_n$ and $a_n \neq 0$.

$$A^{-1} = \left(-\frac{1}{a_n}\right)\left[A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I_n\right]$$

Example 25 Verify the Cayley–Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \text{ and, hence, find } A^{-1}.$$

Solution The characteristic equation of the matrix A is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or $\quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

or $\quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To verify Cayley–Hamilton theorem, we have to show that

$$A^3 - 6A^2 + 9A - 4I = 0 \qquad (1)$$

We have $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix},$$

and $$A^3 = A^2 \cdot A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now,

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence, theorem is verified.

Further, premultiplying (1) by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

or $$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 26 Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley–Hamilton theorem for the matrix A. Find the inverse of the matrix A and also express

$A^5 - 4A^4 - 7A^3 + 11\,A^2 - A - 10\,I$ as a linear polynomial in A.

Solution The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 4\lambda - 5 = 0 \tag{1}$$

or $$(\lambda - 5)(\lambda + 1) = 0 \text{ or } \lambda = 5, -1$$

Thus, the eigenvalues of A are 5, –1.

By Cayley–Hamilton theorem, the matrix A must satisfy its characteristic equation (1).

We have,

$$A^2 - 4A - 5I = 0 \tag{2}$$

We have,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now,

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, the theorem is verified

Now, premultiplying (2) by A^{-1}, we get

$$A - 4I - 5A^{-1} = 0$$

or

$$A^{-1} = \frac{1}{5}[A - 4I]$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation of A is $\lambda^2 - 4\lambda - 5 = 0$. Dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$ by the polynomial $\lambda^2 - 4\lambda - 5 = 0$, we get

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$\therefore \quad A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$$

$$= (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I.$$

$$= A + 5I \qquad [\because A^2 - 4A - 5I = 0]$$

which is a linear polynomial in A.

1.24 SIMILARITY OF MATRICES

Let A and B be the two square matrices of order n. Then B is said to be similar to A if there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

If B is similar to A then

$$|B| = |P^{-1}AP|$$

$$= |P^{-1}||A||P|$$

$$= |P^{-1}||P||A|$$

$$= |P^{-1}P||A|$$

$$= |I||A|$$

$$= |A|$$

Thus, the similar matrices have the same determinant

1.25 DIAGONALIZATION MATRIX

A matrix A is said to be diagonalizable if it is similar to a diagonal matrix.

Thus, the matrix A is diagonalizable if $\exists$ an invertible matrix P such that

$D = P^{-1}AP$, where D is a diagonal matrix.

Theorem 6

A matrix of order n is diagonalizable if and only if it possesses n linearly independent eigenvectors.

Proof Suppose first that A is diagonalizable. Then A is similar to a diagonal matrix

$$D = \text{Diag. } [\lambda_1, \lambda_2, \ldots, \lambda_n].$$

$\therefore$ there exists an invertible matrix $P = [X_1, X_2, \ldots, X_n]$, such that

$$P^{-1} AP = D$$

i.e., $$AP = PD$$

or $$A[X_1, X_2, \ldots , X_n] = [X_1, X_2, \ldots, X_n]D = [X_1, X_2, \ldots, X_n] \text{ diag. } [\lambda_1, \lambda_2, \ldots, \lambda_n]$$

or $$[A X_1, A X_2, \ldots, AX_n] = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$$

Hence, $$A X_1 = \lambda_1 X_1, A X_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n$$

Thus, X_1, X_2, …, X_n are eigenvectors of A corresponding to the eigenvalues λ_1, λ_2, …, λ_n respectively.

Since the matrix P is non-singular, its column vectors

$X_1, X_2, \ldots, X_n$ are linearly independent.

Hence, A has n linearly independent eigenvectors.

Conversely, suppose that A possesses n linearly independent eigenvectors $X_1, X_2, \ldots, X_n$ and let λ_1, λ_2, …, λ_n be the corresponding eigenvalues.

Then $$A X_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n.$$

Let $$P = [X_1, X_2, \ldots, X_n] \text{ and } D = \text{Diag. } [\lambda_1, \lambda_2, \ldots, \lambda_n]$$

Then
$$\begin{aligned} AP &= A[X_1, X_2, \ldots, X_n] \\ &= [A X_1, A X_2, \ldots, AX_n] \\ &= [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n] \\ &= [X_1, X_2, \ldots, X_n] \text{ diag. } [\lambda_1, \lambda_2, \ldots, \lambda_n] \\ &= PD \end{aligned}$$

Since the column vectors $X_1, X_2, \ldots, X_n$ of the matrix P are linearly independent, so P is invertible and P^{-1} exists.

$\therefore$ $AP = PD$

$\Rightarrow$ $P^{-1} AP = P^{-1} PD$

$\Rightarrow$ $P^{-1} AP = D$ [$\because$ $P^{-1} P = I$]

$\Rightarrow$ A is similar to D

$\Rightarrow$ A is diagonalizable.

Theorem 7

If the eigenvalues of a matrix of order $n \times n$ are all distinct then it is always similar to a diagonal matrix.

Proof Suppose A be a square matrix of order n and let matrix A have n distinct eigenvalues λ_1, λ_2, …, λ_n. We know that eigenvectors of a matrix corresponding to distinct eigenvalues are linearly independent.

$\therefore$ A has n linearly independent eigenvectors and so it is similar to a diagonal matrix D = Diag. $[\lambda_1, \lambda_2, \ldots, \lambda_n]$.

Corollary The two matrices of order n with the same set of n distinct eigenvalues are similar.

Example 27 Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also, find the diagonal form and a diagonalizing matrix P.

Solution The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix} = 0, \text{ by } C_1 \to C_1 + C_2 + C_3$$

or $$-(1+\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix} = 0$$

or $$-(1+\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0, \quad \text{by} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

or $$(1+\lambda)(1+\lambda)(3-\lambda) = 0$$

or $$\lambda = -1, -1, 3$$

$\therefore$ the eigenvectors of the matrix A corresponding to the eigenvalue -1 are given by the equation

$$[A - (-1)I]\,X = 0 \text{ or } (A + I)\,X = 0$$

or $$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ -0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{by} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

The coefficient matrix of these equations has the rank 1.

$\therefore$ there are $n - r = 3 - 1 = 2$ linearly independent solutions.

The above equation

$$-8\,x_1 + 4\,x_2 + 4\,x_3 = 0$$

or $\quad -2x_1 + x_2 + x_3 = 0$

Clearly,

$X_1 = [1, 1, 1]'$ and $X_2 = [0, 1, -1]'$ are two linearly independent solutions.

$\therefore$ X_1 and X_2 are two linearly independent eigenvectors of A corresponding to the eigenvalue –1.

Now, the eigenvectors of A corresponding to the eigenvalue 3 are given by $(A - 3I)X = 0$

or
$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

or
$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_2$$

The coefficient matrix of these equations has the rank 2.

$\therefore$ there are $n - r = 3 - 2 = 1$ linearly independent solutions.

The above equations are

$$-12x_1 + 4x_2 + 4x_3 = 0$$
$$4x_1 - 4x_2 = 0$$

The last equation gives $x_2 = x_1$

Choose $x_1 = 1$, so $x_2 = 1$, then from the first equation, we have $x_3 = 2$

$\therefore$ $X_3 = [1, 1, 2]'$ is an eigenvector of A corresponding to the eigenvalue 3.

Now, the modal matrix

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

The columns of P are linearly independent eigenvectors of A corresponding to the eigenvalues –1, –1, 3 respectively. The matrix P will transform A to the diagonal form D which is given by the relation.

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

EXERCISE 1.3

1. Test for consistency and solve the following systems of equations.
   (i) $2x + 6y + 11 = 0$, $6x + 20y + 6z = -3$, $6y - 18z = -1$
   (ii) $2x - y + 3z = 8$, $-x + 2y + z = 4$, $3x + y - 4z = 0$

2. Find the values of α and β for which the equations
$x + 2y + 3z = 4$, $x + 3y + 4z = 5$, $x + 3y + \alpha z = \beta$ have (i) no solution, (ii) a unique solution, and (iii) an infinite number of solutions.
3. Solve:
$x + y + z = 0$, $2x + 5y + 7z = 0$, $2x - 5y + 3z = 0$
4. Show that the only real value of λ for which the following equations have a non-zero solution is 6:
$x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda z$.
5. For what values of do the equations
$x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$, have a solution and solve them completely in each case.
6. Show that the three equations
$-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ have no solutions unless $a + b + c = 0$, in which case they have infinitely many solutions. Find these solutions when $a = 1$, $b = 1$, and $c = -2$.
7. Find the eigenvalues and eigenvectors of the matrix
$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$
8. Find the eigenvalues and eigenvectors of the matrix
$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$
9. Verify the Cayley–Hamilton theorem for the matrix
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$
Hence, or otherwise, evaluate A^{-1}.
10. Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its own characteristic equation. Is it true of every square matrix? State the theorem that applies here.
11. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.
12. Show that the following matrices are not similar to diagonal matrices:
(i) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

13. Show that the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Find the diagonalizing matrix P.

14. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Answers

1. (i) Not consistent (ii) consistent, $x = 2, y = 2, z = 2$.
2. (i) $\alpha = 4, \beta \neq 5$ (ii) $\alpha \neq 4$, (iii) $\alpha = 4, \beta = 5$.
3. $x = 0 = y = z$.
5. for $\lambda = 1, x = 1 + 2c, y = -3c, z = c$, where c is any arbitrary constant. For $\lambda = 2, x = 2K, y = 1 - 3K, z = K$, where K is any arbitrary constant.
6. $x = c - 1, y = c - 1, z = c$, where c is any arbitrary constant.
7. $\lambda = 6, 1, X_1 = [4, 1]', X_2 = [1, -1]'$
8. $\lambda = 5, -3, -3, X_1 = [1, 2, -1]', X_2 = [-2, 1, 0]', X_3 = [3, 0, 1]'$.
9. $A^{-1} = \dfrac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$.
11. $-4A + 5I$
13. $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$, diag. $[-1, -1, 3]$
14. $D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$

1.26 QUADRATIC FORMS

Let $X = [x_1, x_2, x_3,\ldots, x_n]^T$ be an n-vector in the vector space V_n over a field F, and let $A = [a_{ij}]$ be an n-square matrix over F. A real quadratic form is a homogeneous expression of the form.

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_i x_j \tag{10}$$

in which the power of each term is 2.

Now, Eq. (10) can be written as

$$Q_1 = a_{11}x^2 + (a_{12} + a_{21})x_1x_2 + \cdots + (a_{1n} + a_{n1})x_1x_n + a_{22}x_2^2 + (a_{23} + a_{32})x_2x_3 + \cdots + (a_{2n} + a_{n2})x_2x_n + \cdots + a_{nn}x_n^2$$

$$Q = X^T AX \tag{11}$$

Using the definition of matrix multiplication, let $b_{ij} = \dfrac{(a_{ij} + a_{ji})}{2}$ and the matrix $B = [b_{ij}]$ be symmetric since $b_{ij} = b_{ji}$. Further, $b_{ij} + b_{ji} = a_{ij} + a_{ji}$. Then Eq. (11) becomes $Q = X^T BX$, where B is a symmetric matrix and $b_{ij} = (a_{ij} + a_{ji})/2$.

Example 28 Obtain the matrix of the quadratic form $Q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3$.

Solution $a_{11} = 1, a_{22} = 2, a_{33} = -7$

$$a_{12} = \frac{1}{2}(\text{coefficient of } x_1x_2) = \frac{1}{2}(-4) = -2$$

$$a_{13} = \frac{1}{2}(\text{coefficient of } x_1x_3) = \frac{1}{2}(8) = 4$$

$$a_{23} = \frac{1}{2}(\text{coefficient of } x_2x_3) = \frac{1}{2}(5) = 5/2$$

Then the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix} \qquad [\because \; a_{12} = a_{21}, a_{23} = a_{32}]$$

which is a symmetric matrix.

Example 29 Find the matrix of the quadratic form $Q = 2x_1^2 + 3x_2^2 + x_3^2 - 3x_1x_2 + 2x_1x_3 + 4x_2x_3$.

Solution $a_{11} = 2, a_{22} = 3, a_{33} = 1$

$$a_{12} = \frac{1}{2}(\text{coefficient of } x_1x_2) = \frac{1}{2}(-3) = -\frac{3}{2} = a_{21}$$

$$a_{13} = \frac{1}{2}(\text{coefficient of } x_1x_3) = \frac{1}{2}(2) = 1 = a_{31}$$

$$a_{23} = \frac{1}{2}(\text{coefficient of } x_2x_3) = \frac{1}{2}(4) = 2 = a_{32}$$

$$\therefore \qquad \text{Matrix } A = \begin{bmatrix} 2 & -3/2 & 1 \\ -3/2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

which is a symmetric matrix.

Example 30 Obtain the symmetric matrix B for the quadratic form $Q = x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_2^3$.

Solution $a_{11} = 1, a_{22} = -5, a_{33} = 4$

$\therefore$ $b_{11} = a_{11} = 1, b_{22} = a_{22} = -5, b_{33} = a_{33} = 4$ and

$$b_{12} = b_{21} = \frac{1}{2}(a_{12} + a_{21}) = \frac{1}{2}(2) = 1$$

$$b_{13} = b_{31} = \frac{1}{2}(a_{13} + b_{31}) = \frac{1}{2}(-4) = -2$$

$$b_{23} = b_{32} = \frac{1}{2}(a_{23} + a_{32}) = \frac{1}{2}(6) = 3$$

Hence, the symmetric matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

1.27 COMPLEX QUADRATIC FORM

Let A be a complex matrix.

Then the quadratic form is defined as

$$Q = \sum_{i,j=1}^{n} a_{ij}\bar{x}_i x_j = \bar{X}^T AX \tag{12}$$

where $X = [x_1, x_2, \ldots, x_n]^T$ be a vector in C.

Complex quadratic form is defined for Hermitian matrix and it is called a ***Hermitian form*** and is always real.

Example 31 Let the Hermitian matrix $A = \begin{bmatrix} 1 & 2+i \\ 2-i & 3 \end{bmatrix}$.

The quadratic form.

$$Q = \bar{X}^T AX = [\bar{x}_1, \bar{x}_2]\begin{bmatrix} 1 & 2+i \\ 2-i & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= |x_1|^2 + (2+i)\bar{x}_1x_2 + (2-i)x_1\bar{x}_2 + 3|x_2|^2$$

$$= |x_1|^2 + 2(\bar{x}_1x_2 + x_1\bar{x}_2) + i(\bar{x}_1x_2 - x_1\bar{x}_2) + 3|x_2|^2$$

Since $\bar{x}_1x_2 + x_1\bar{x}_2$ is real and $\bar{x}_1x_2 - x_1\bar{x}_2$ is an imaginary,

$\therefore$ $Q = |x_1|^2 + \text{Real} + 3|x^2| = \text{Real}$

Hence, the Hermitian matrix A is always real.

1.28 CANONICAL FORM

The sum of the square form of a real quadrate form $Q = X^T AX$ is

$$Y^T DY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2 \tag{13}$$

Equation (13) is formed with the help of the orthogonal transformation $X = PY$
where P is the modal matrix and D is a diagonal matrix or a spectral matrix whose diagonal elements are the eigenvalues of the matrix A.

Consider the rank of a matrix A as r and let n be the number of variables in quadratic form.

Index

The number of positive terms in the canonical form of a quadratic form is called the *index* and it is denoted by P.

Signature

The *signature* of a quadratic form is the difference between the number of positive terms and the negative terms.

1.29 POSITIVE DEFINITE QUADRATIC AND HERMITIAN FORMS

Let a real quadratic or Hermitian form $Q(x) = X^T AX$ or $\overline{X}^T HX$ over a real field R or a complex field C. Then a real quadratic or Hermitian form $Q(x)$ is said to be

(i) positive definite if $Q(x) > 0$ when $X \neq 0$
(ii) negative definite if $Q(x) < 0$ when $X \neq 0$
(iii) positive semidefinite if $Q(x) \geq 0$ when $X \neq 0$
(iv) negative semidefinite if $Q(x) \leq 0$ when $X \neq 0$

A real symmetric matrix A (or a Hermitian matrix H) is said to be a positive definite matrix (or a +ve definite Hermitian matrix) iff $Q = X^T AX$ (or $\overline{X}^T AX$) is +ve definite and is defined as $A > 0$ (or $H > 0$). Similarly

- for a negative definite matrix, $A < 0$ (or $H < 0$)
- for a positive semidefinite matrix, $A \geq 0$ (or $H \geq 0$)
- for a negative semidefinite matrix, $A \leq 0$ (or $H \leq 0$)

1.30 SOME IMPORTANT REMARK'S

(*R*–1)

Let $Q(X) = X^T AX$ be a real quadratic form of order (n), rank (r) and index (P). Then $Q(X)$ is

(i) Positive Definite (PD) iff $r = p = n$ or if all the eigenvalues of A are positive.
(ii) Positive SemiDefinite (PSD) iff $r = p < n$ or all the eigenvalues of A are ≥ 0.
(iii) Negative Definite (ND) iff $r = -p = n$ or all the eigenvalues of A are negative.
(iv) Negative SemiDefinite (NSD) iff $r = -p < n$ or all the eigenvalues of A are ≤ 0.

(*R*–2)

A real quadratic form $Q(x) = X^TAX$ is PD, if the

(i) $\det\cdot(A) > 0$
(ii) every principal minor of A is positive
(iii) $a_{ii} > 0$, $i = 1, 2, \ldots, n$, where $A = [a_{ij}]$

(*R*–3)

If a real quadratic form $Q(X) = X^T AX$ is PSD then

(i) $\det(A) = 0$

(ii) every principal minor of A is positive

(iii) $a_{ii} > 0$ if x_i^2 appears in $Q(X)$

(*R*–4)

A real quadratic form $Q(X) = X^T AX$ is ND iff all the principal minors of A of even order are positive and those of odd order are negative.

(*R*–5)

A quadratic form $Q(X) = X^T AX$ is NSD iff A is singular and all principal minors of even order of A are non-negative while those of odd order are negative.

(*R*–6)

A real symmetric matrix A is indefinite iff at least one of the following conditions is satisfied.

(i) A has a –ve principal minor of even order

(ii) A has a +ve principal minor and odd order and a –ve principal minor of odd order

***Note*:** All the above remarks are same for Hermitian form.

Theorem 8: Sylvester Criterion for Positive Definiteness

A quadratic form $Q(X) = X^T AX = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ (14)

is positive definite iff all the leading principal minors of A are positive.

and (14) is negative definite iff $a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0$ and so on ….

Example 32 Determine the nature, index, and signature of the quadratic form

$$Q(X) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = X^T AX$$

Solution Here, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characteristic matrix for the matrix A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 3\lambda - 2 = 0$

or $\lambda = 2 -1, -1$

Therefore, some eigenvalues are +ve and some are –ve. Hence, the $Q(X)$ is indefinite.
The index of $Q(X)$ is 1 and signature = 1 – 2 = –1

Example 33 Examine whether the quadratic form $Q(X)$ is positive definite, where $Q(X) = X^T AX = 3x_1^2 + 3x_1x_2 + 4x_2^2$.

Solution Here, $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$; then the eigenvalues of A are 2 and 5 both are positive and the leading minor, $|3| = 3 > 0$ and $\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12 - 2 = 10 > 0$

Hence, $Q(X)$ is positive definite.

Example 34 Determine the nature, index, and signature of the quadratic form

$$Q(X) = x_1^2 + 4x_3^2 + 4x_1x_2 + 10x_1x_3 + 6x_2x_3$$

Solution Here, $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$

The characteristic equation for A is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} (1-\lambda) & 2 & 5 \\ 2 & -\lambda & 3 \\ 5 & 3 & (4-\lambda) \end{vmatrix} = 0$$

or $(1 - \lambda)[\lambda(\lambda - 4) - 9] - 2[2(4 - \lambda) - 15] + 5[6 + 5\lambda] = 0$

or $$\lambda^3 - \lambda^2 - 38\lambda - 36 = 0$$

or $$\lambda = -1,\ 1 \pm \sqrt{37}$$

$\therefore$ some of the eigenvalues are positive and some are negative. Hence, $Q(X)$ is indefinite.
Now, index = 1, signature = 1 – 2 = –1.

Example 35 Determine the nature of the quadratic form

$$Q(X) = \bar{X}^T \begin{bmatrix} 3 & -2i \\ 2i & 4 \end{bmatrix} X$$

Solution
$$Q(X) = [\bar{x}_1, \bar{x}_2] \begin{bmatrix} 3 & -2i \\ 2i & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\bar{x}_1, \bar{x}_2] \begin{bmatrix} 3x_1 - 2ix_2 \\ 2ix_1 + 4x_2 \end{bmatrix}$$

$$= 3x_1\bar{x}_1 - 2i(\bar{x}_1x_2 - x_1\bar{x}_2) + 4x_2\bar{x}_2 > 0 \text{ and real.}$$

Since $\bar{x}_1x_2$ and $x_1\bar{x}_2$ both are imaginary. Hence, $Q(x)$ is a positive definite.

EXERCISE 1.4

Determine the nature, index and signature of the following quadratic forms:

1. $Q(X) = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$
2. $Q(X) = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$
3. $Q(X) = 6x_1^2 - 4x_1x_2 + 3x_2^2 - 2x_2x_3 + 3x_3^2 + 4x_1x_3$
4. $Q(X) = 5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_1x_3 + 6x_1x_2$
5. $Q(X) = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$
6. $Q(X) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3$

Answers

1. Positive definite, index = 3, signature = 3
2. Positive definite, index = 3, signature = 3
3. Positive definite, index = 3, signature = 3
4. Positive semi-definite, index = 3, signature = 3
5. Negative definite, index = 0, signature = –3
6. Indefinite, index = 2, signature = 1

1.31 APPLICATIONS OF MATRICES

(i) Differentiation and Integration of a Matrix

The elements of a matrix A may be functions of a variable, say, t. This functional dependence of A on t is shown by writing A and its elements as $A(t)$ and $a_{ij}(t)$, defined as

$$A = A(t) = [a_{ij}(t)]$$

Thus, the differential coefficient of A w.r.t. t is defined as

$$\mathring{A} = \frac{d}{dt}(A) = \left[\frac{d}{dt}a_{ij}(t)\right]$$

$$= \begin{bmatrix} \frac{d}{dt}a_{11} & \frac{d}{dt}a_{12} & \cdots & \frac{d}{dt}a_{1n} \\ \frac{d}{dt}a_{21} & \frac{d}{dt}a_{22} & \cdots & \frac{d}{dt}a_{2n} \\ \vdots & & & \vdots \\ \frac{d}{dt}a_{n1} & \frac{d}{dt}a_{n2} & \cdots & \frac{d}{dt}a_{nn} \end{bmatrix}$$

The integral of the matrix A is defined as $\int A(t)dt = \left[\int a_{ij}(t)\,dt\right]$, assuming the elements in $A(t)$ to be integrable.

Thus, the integral of A is obtained by integrating each element of A.

Example 36 Prove that $\frac{d}{dt}(e^{\alpha t}) = \alpha e^{\alpha t}$.

Solution We know that

$$e^{\alpha t} = 1+\frac{\alpha t}{1!}+\frac{(\alpha t)^2}{2!}+\frac{(\alpha t)^3}{3!}+\cdots$$

$$\therefore \qquad \frac{d}{dt}(e^{\alpha t}) = \frac{d}{dt}\left[1+\frac{\alpha t}{1!}+\frac{(\alpha t)^2}{2!}+\frac{(\alpha t)^3}{3!}+\cdots\right]$$

$$= \frac{d}{dt}\left[1+\frac{\alpha}{1!}t+\frac{\alpha^2}{2!}t^2+\frac{\alpha^3}{3!}t^3+\cdots\right]$$

$$= \frac{d}{dt}(1)+\frac{\alpha}{1!}\frac{d}{dt}(t)+\frac{\alpha^2}{2!}\frac{d}{dt}(t^2)+\frac{\alpha^3}{3!}\frac{d}{dt}(t^3)+\cdots$$

$$= 0+\alpha+\frac{\alpha^2}{2!}2t+\frac{\alpha^3}{3!}3t^2+\cdots$$

$$= \alpha\left[1+\frac{\alpha}{1!}t+\frac{\alpha^2}{2!}t^2+\cdots\right]$$

$$= \alpha\, e^{\alpha t}$$ **Proved.**

Example 37 Solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}-12y=0$ (1)

$y(0)$, $y'(0) = 8$ by matrix method.

Solution Suppose $y = y_1$ and $\frac{dy_1}{dt} = y_2$ (2)

Equation (1) becomes

$$\frac{d}{dt}\left(\frac{dy_1}{dt}\right) = -4\frac{dy_1}{dt}+12y_1 \qquad (3)$$

$$\frac{dy_2}{dt} = 12y_1 - 4y_2$$

Equations (2) and (3) written in matrix form are

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 12 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad (4)$$

RHS of Eq. (4) gives the eigenvector. The characteristic equation

$$\begin{vmatrix} 0-\lambda & 1 \\ 12 & -4-\lambda \end{vmatrix} = 0$$

or $\qquad -\lambda(-4-\lambda) - 12 = 0$

or $\lambda^2 + 4\lambda + 12 = 0$

or $(\lambda - 2)(\lambda + 6) = 0$

or $\lambda = 2, -6$

For $\lambda = 2$ and $\lambda = -6$, eigenvectors are $[1, 2]^T$ and $[1, -6]^T$

Matrix of eigenvectors is

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}, P^{-1} = \frac{1}{8}\begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix}$$

Now, $$P\, e^{\lambda t} P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-6t} \end{bmatrix}\frac{1}{8}\begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} e^{2t} & e^{-6t} \\ 2e^{2t} & -6e^{-6t} \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 6e^{2t} + 2e^{-6t} & e^{2t} - e^{-6t} \\ 12e^{2t} - 12e^{-6t} & 2e^{2t} + 6e^{-6t} \end{bmatrix}$$

Using the initial conditions,

$y(0) = 0$ and $y'(0) = 8$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 6e^{2t} + 2e^{-6t} & e^{2t} - e^{-6t} \\ 12e^{2t} - 12e^{-6t} & 2e^{2t} + 6e^{-6t} \end{bmatrix}\begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} - e^{-6t} \\ 2e^{2t} + 6e^{-6t} \end{bmatrix}$$

$\therefore$ $$y_1 = y = e^{2t} - e^{-6t} \text{ and } y_2 = \frac{dy}{dt} = 2e^{2t} + 6e^{-6t}$$

(ii) Use of Matrices in Graph Theory

Matrices play an important role in the field of graph theory, the first application of graph theory, which shows its face in communications, transportation, sciences, and many more fields. There are two important matrices associated with graphs.

(a) Adjacency matrix (b) Incidence matrix.

(a) Adjacency Matrix In this matrix, vertices (nodes) are written as rows and columns also. This matrix is the symmetric matrix

$$A = [a_{ij}]$$

where $$a_{ij} = \begin{cases} 1, & \text{if } (V_i, V_j) \text{ is an edge, i.e., if } V_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

(b) Incidence Matrix In this matrix, vertices are as rows and edges are as columns. If any graph consists of m vertices and n edges then the incidence matrix is $m \times n$.

$A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if vertex } V_i \text{ is incident on the edge } e_j \\ 0, & \text{otherwise} \end{cases}$$

Example 38 Find the adjacency matrix $A = [a_{ij}]$ of the given graph G.

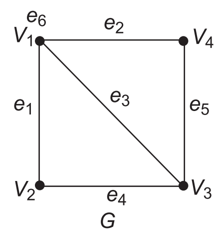


Fig. 1.1

Solution The given graph G has 4 vertices V_1, V_2, V_3, and V_4. The adjacency matrix contains 4 rows and 4 columns. By the definition of adjacency matrix,

$$A = [a_{ij}] = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times 4} \end{array}$$

Example 39 Find the incidence matrix of the graph G in the above example.

Solution The graph G has 4 vertices and 6 edges. The incidence matrix contain 4 rows and 6 columns. By the definition of incidence matrix.

$$A = [a_{ij}] = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}_{4\times 6} \end{array}$$

(iii) Use of Matrices in Characteristic Values and Characteristic Vectors

Matrices play an important role in the field of vibrational problems, the fundamental frequencies are the characteristic values and the fundamental modes of vibration are characteristic vectors. Thus, they are very important in the study of quantum mechanics, population dynamics, genetics, and vibrations of beams.

(iv) Use of Matrices in Electric Circuits

Matrices play an important role in the study of electric circuits by using the following laws:

(a) Ohm's Law $R = \frac{V}{i}$, where V is the potential difference in volts, i is the current in amperes, and R is the resistance in ohms.

(b) Voltage Law The algebraic sum of the voltage drop around a closed circuit is equal to the resultant electromotive force in the circuit.

(c) Kirchhoff's Law Closed circuit is equal to the resultant electromotive force to the circuit.

(d) Current Law At a node or junction, current coming is equal to the current going.

(v) Use of Matrices in Mechanical Equilibrium

Consider mass m_1 suspended from spring (1) with spring constant k_1 and the displacement is y_1

$\therefore$ Restoring force = Spring constant × Displacement

$$= k_1 y_1$$

Now, the spring (2) with the spring constant k_2 is connected with the spring (1). Let mass m_2 be suspended on the spring (2) and the displacement be y_2.

Then Restoring force = Spring constant × Displacement

$$= k_2(y_1 - y_2)$$

At equilibrium condition,

Downward force = Restoring force

$$m_1 \frac{d^2 y_1}{dt^2} = -k_1 y_1 - k_2(y_1 - y_2) \qquad (15)$$

$$m_1 \frac{d^2 y_1}{dt^2} = -(k_1 + k_2) y_1 + k_2 y_2$$

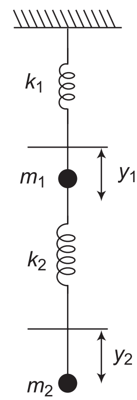


Fig. 1.2

Similarly, the equation of motion of mass m_2.

$$m_2 = \frac{d^2 y_2}{dt^2} = -k_2(y_2 - y_1) \qquad (16)$$

$$= -k_2 y_2 + k_2 y_1$$

The downward force is taken as positive and the negative restoring force act to restore the masses to their original position upwards. Then (15) and (16) can be written in matrix form.

$$AY = m\ddot{y}, \text{ where } \ddot{y} = \frac{d^2 y}{dt^2}$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} m_1 \dfrac{d^2 y_1}{dt^2} \\ m_2 \dfrac{d^2 y_2}{dt^2} \end{bmatrix}$$

SUMMARY

1. Elementary Transformation or Elementary Operations

The following transformations are called elementary transformations of a matrix.

(i) Interchanging of rows (columns).
(ii) Multiplication of a row (column) by a non-zero scalar.
(iii) Adding/subtracting K multiple of a row (column) to another row (column).

2. The Inverse of a Matrix

The inverse of an $n \times n$ matrix $A = [a_{ij}]$ is denoted by A^{-1}, such that

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the $n \times n$ unit matrix.

Some Special Points on Inverse of a Matrix

(i) If A be any n-rowed square matrix, then
$$(\text{Adj } A)A = A(\text{Adj } A) = |A|\, I_n$$
where I_n is the n-rowed unit matrix.
(ii) The necessary and sufficient condition for a square matrix A to possess the inverse, that is, $|A| \neq 0$.
(iii) If A be an $n \times n$ non-singular matrix then $(A')^{-1} = (A^{-1})'$, where ($'$) (dash) denote the transpose.
(iv) If A be an $n \times n$ non-singular matrix then
$$(A^{-1})^{\theta} = (A^{\theta})^{-1}$$
(v) If A, B be two n-rowed non-singular matrices then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.
(vi) If A is a non-singular matrix then $\det(A^{-1}) = (\det A)^{-1}$.
(vii) If the matrices A and B commute then A^{-1} and B^{-1} also commute.
(viii) If A, B, C be three matrices conformable for multiplication then $(ABC)^{-1} = C^{-1}\, B^{-1}\, A^{-1}$.
(ix) If the product of two non-zero square matrices is a zero matrix then both of them must be singular matrices.
(x) If A be an $n \times n$ matrix then $|\text{adj } A| = |A|^{n-1}$.
(xi) If A is a non-singular matrix then adj adj $A = |A|^{n-2} A$.
(xii) If A and B are square matrices of the same order then adj (AB) = adj B adj A.

3. Echelon Form of a Matrix

A matrix A is said to be in Echelon form if

(i) Every row of the matrix A, which has all its entries zero occurs below every row which has a non-zero entry.
(ii) The first non-zero entry in each non-zero row is equal to one.
(iii) The number of zeros preceding the first non-zero element in a row is less than the number of such zero in the succeeding row.

4. Rank of a Matrix

The rank of a matrix A is said to be 'r' if it possesses the following two properties:

(i) There is at least one non-zero minor of order 'r' whose determinant is not equal to zero.
(ii) If the matrix A contains any minor of order $(r + 1)$ then the determinant of every minor of A of order $r + 1$, should be zero.

Thus, the rank of a matrix is the largest order of a non-zero minor of the matrix.

The rank of the matrix A is denoted by $\rho(A)$.

The rank of a matrix is equal to the number of non-zero rows of in the Echelon form of the matrix. i.e., $\rho(A)$ = number of non-zeros in Echelon form of a matrix.

Some Important Results

(i) Rank of A and A^T is same.
(ii) Rank of a null matrix is zero.
(iii) Rank of a non-singular matrix A of order n is n.
(iv) Rank of an identity matrix of order n is n.
(v) For a rectangular matrix A of order $m \times n$, rank of $A \leq \min(m, n)$, i.e., rank cannot exceed the smaller of m and n.
(vi) For n-square matrix A, if $\rho(A) = n$ then $|A| = \neq 0$, i.e., the matrix A is non-singular.
(vii) For any square matrix, if $\rho(A) < n$ then $|A| = 0$, i.e., the matrix A is singular.
(viii) The rank of a product of two matrices cannot exceed the rank of either matrix.

5. Canonical Form (or Normal Form) of a Matrix

The normal form of a matrix A of order $m \times n$ of rank 'r' is one of the following forms

$$[I_r], \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

where I_r is an identity matrix of order 'r'.

6. Homogeneous Systems of Linear Equations

Consider

$$\left.\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n &= 0, \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n &= 0 \\ &\vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n &= 0 \end{aligned}\right\} \tag{1}$$

This is a system of m homogeneous equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Then, the system (1) can write in the matrix form:

$$AX = O \tag{2}$$

The matrix A is called the coefficient matrix of the system of (1).

The system (1) has the trivial (zero) solution if the rank of the coefficient matrix is equal to the number of unknown variables (n), i.e.,

$$\rho(A) = n$$

The system (1) has infinitely many solutions if the rank of the coefficient matrix is less than the number of unknown variables, i.e., $\rho(A) < n$.

7. Systems of Linear Nonhomogeneous Equations

Suppose a system of 'm' non-homogeneous linear equations with n unknown variables is of the form

$$\left.\begin{array}{l} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n = b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n = b_2 \\ \vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n = b_m \end{array}\right\} \tag{3}$$

If we write $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

Then, the above system (3) can be written in the form of a single matrix equation $AX = B$.

The matrix $[A \,\vdots\, B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$ is called the augmented matrix of the given system of equations. Any set of values which simultaneously satisfies all these equations is called a solution of the given system (3). When the system of equations has one or more solutions then the given system is consistent, otherwise it is inconsistent.

(a) Condition for Consistency Theorem

The system of equations $AX = B$ is consistent, i.e., possesses a solution if and only if the coefficient matrix A and the augmented matrix $[A \,\vdots\, B]$ are of the same rank.

Now, there two case arise.

Case I: If the rank of the coefficient matrix and the rank of the augmented matrix are equal to the number of unknown variables, i.e., $\rho(A) = \rho(A : B) = n$ (number of unknown variables), then the system has a unique solution.

Case II: If the rank of the coefficient matrix and rank of the augmented matrix are equal but less than the number of unknown variables, i.e., $\rho(A) = \rho(A : B) < n$, then the given system has infinitely many solutions.

(b) Condition for Inconsistent Solution

The system of equations $AX = B$ is inconsistent, i.e., possesses no solution, if the rank of the coefficient matrix A is not equal to the rank of the augmented matrix $[A \,\vdots\, B]$, i.e., $\rho(A) \neq \rho[A \,\vdots\, B]$

8. Characteristic Roots and Vectors

Let A be a square matrix of order n, λ is a scalar and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ a column vector.

Consider the equation

$$AX = \lambda X \tag{4}$$

Clearly, $X = 0$ is a solution of (4) for any value λ.

Now let us see whether there exist scalars λ and non-zero vectors X which satisfy (4). This problem is known as the characteristic value problem. If I_n is a unit matrix of order n then (4) may be written in the form

$$(A - \lambda I_n)X = 0 \tag{2}$$

Equation (5) is the matrix form of a system of n homogeneous linear equations in n unknowns. This system will have a non-trivial solution if and only if the determinant of the coefficient matrix $A - \lambda I_n$ vanishes, i.e.,

$$|A - \lambda I_n| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

The equation $|A - \lambda I_n| = 0$ is called the characteristic equation or secular equation of the matrix A.

The n^{th} roots of the characteristic equation of a matrix A of order n are called the characteristic roots, Characteristic values, proper values, eigenvalues, or latent roots of the matrix A.

The set of the eigenvalues of a matrix A is called the spectrum of A.

If λ is a characteristic root of an $n \times n$ matrix A then a non-zero vector X such that

$$AX = \lambda X$$

is called a characteristic vector, eigenvector, proper vector or latent vector of the matrix A corresponding to the characteristic root λ.

9. The Cayley-Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation, i.e., if for a square matrix A of order n,

$|A - \lambda In| = (1)^n\ [\lambda^n + a_1\ \lambda^{n-1} + a_n\ \lambda^{n-2} + \ldots + a_n]$ then the matrix equation

$$X^n + a_1 X^{n-1} + a^2\ X^{n-2} + \ldots + a^n\ I_n = 0$$

is satisfied by $X = A$

i.e., $\quad A^n + a_1\ A^{n-1} + a_2\ A^{n-2} + \ldots + a^n\ I_n = 0.$

10. Diagonalization Matrix

A matrix A is said to be diagonalizable if it is similar to a diagonal matrix.

Thus, the matrix A is diagonalizable if $\exists$ and the invertible matrix P such that

$D = P^{-1}\ AP$, where D is a diagonal matrix.

OBJECTIVE-TYPE QUESTIONS

1. An arbitrary vector X is an eigenvector of the matrix $A=\begin{bmatrix}1&0&0\\0&a&0\\0&0&b\end{bmatrix}$, if $(a,b)=$

(a) (0, 0) (b) (1, 1)
(c) (0, 1) (d) (1, 2)

2. The inverse of a matrix $\begin{bmatrix}ab&0\\0&1\end{bmatrix}$ is

(a) $\begin{bmatrix}a&0\\0&a\end{bmatrix}$ (b) $\begin{bmatrix}b&0\\a&b\end{bmatrix}$
(c) $\begin{bmatrix}1/a&0\\0&1/b\end{bmatrix}$ (d) $\begin{bmatrix}a&0\\0&1/b\end{bmatrix}$

3. The eigenvalues of $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ are

(a) 0, 0, 0 (b) 0, 0, 1
(c) 0, 0, 3 (d) 1, 1, 1

4. The eigenvectors of the matrix $\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$ are

(a) (1, 0) (2, 3) (b) (0, 1) (1, 2)
(c) (1, 1) (0, 1) (d) (1, –1) (1, 1)

5. The inverse of the matrix $\begin{bmatrix}-3&5\\2&1\end{bmatrix}$ is

(a) $\begin{bmatrix}\frac{3}{13}&-\frac{1}{13}\\\frac{2}{13}&\frac{3}{13}\end{bmatrix}$ (b) $\begin{bmatrix}\frac{2}{13}&\frac{3}{13}\\-\frac{1}{13}&\frac{3}{13}\end{bmatrix}$
(c) $\begin{bmatrix}-\frac{1}{13}&\frac{5}{13}\\\frac{2}{13}&\frac{3}{13}\end{bmatrix}$ (d) $\begin{bmatrix}\frac{1}{13}&-\frac{3}{13}\\\frac{2}{13}&\frac{3}{13}\end{bmatrix}$

6. The real symmetric matrix C corresponding to the quadratic form $Q=4x_1x_2-5x_{22}$ is

(a) $\begin{bmatrix}1&2\\2&-5\end{bmatrix}$ (b) $\begin{bmatrix}2&0\\0&-5\end{bmatrix}$
(c) $\begin{bmatrix}1&1\\1&-2\end{bmatrix}$ (d) $\begin{bmatrix}0&2\\1&-5\end{bmatrix}$

[GATE (CE) 1998]

7. If A is a real symmetric matrix then AA^T is
(a) unsymmetric
(b) always symmetric
(c) skew symmetric
(d) sometimes symmetric **[GATE 1998]**

8. The vector $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ is an eigenvector of $A=\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}$ if one of the given eigenvalues of A is

(a) 1 (b) 2
(c) 5 (d) –1

[GATE (EE) 1998]

9. $A=\begin{bmatrix}2&0&0&-1\\0&1&0&0\\0&0&3&0\\-1&0&0&4\end{bmatrix}$. The sum of the eigenvalues of the matrix A is

(a) 10 (b) –10
(c) 24 (d) 22

[GATE (EE) 1998]

10. $A=\begin{bmatrix}5&0&2\\0&3&0\\2&0&1\end{bmatrix}$, the inverse of A is

(a) $\begin{bmatrix}1&0&-2\\0&1/3&0\\-2&0&5\end{bmatrix}$

(b) $\begin{bmatrix}5&0&2\\0&-1/3&0\\2&0&1\end{bmatrix}$

(c) $\begin{bmatrix} 1/5 & 0 & 1/2 \\ 0 & 1/3 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1/5 & 0 & -1/2 \\ 0 & 1/3 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$ **[GATE (EE) 1998]**

11. For a given matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$, one of the eigenvectors is 3. The other two eigenvectors are

(a) 7, –5 (b) 3, –5
(c) 2, 5 (d) 3, 5

[GATE (EE) 1996]

12. If $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ then eigenvalues of the matrix $I + A + A^2$, where I denotes the identity matrix is

(a) 3, 7, 11 (b) 3, 7, 12
(c) 3, 7, 13 (d) 3, 9, 16

13. The eigenvector of a real symmetric matrices corresponding to different eigenvalues are

(a) orthogonal (b) singular
(c) non-singular (d) none of these

14. The eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are

(a) 1, 1 (b) –1, –1
(c) $i, -i$ (d) 1, –1

[GATE (ECE) 1998]

15. Rank of the matrix $\begin{bmatrix} 3 & 2 & -9 \\ -6 & -4 & 18 \\ 12 & 8 & -36 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) 0

[GATE (ME) 1999]

16. The eigenvalues of the matrix $\begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$ are

(a) 6, –4 (b) 5, –4
(c) –3, –4 (d) –4, 4

[GATE (ME) 1999]

17. The eigenvalues of the matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ are

(a) 2, –2, 1, –1 (b) 2, 3, –2, 4
(c) 2, 3, 1, 4 (d) none of these

[GATE (ECE) 2000]

18. The three characteristic roots of the following matrix A, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ are

(a) 1, 2, 3 (b) 1, 2, 2
(c) 1, 0, 0 (d) 0, 2, 3

[GATE (ME) 2000]

19. If A, B, C are square matrices of the same order, $(ABC)^{-1}$ is equal to

(a) $C^{-1} A^{-1} B^{-1}$ (b) $C^{-1} B^{-1} A^{-1}$
(c) $A^{-1} B^{-1} C^{-1}$ (d) $A^{-1} C^{-1} B^{-1}$

[GATE (CE) 2000]

20. Consider the following two statements:

I. The maximum number of LI column vectors of a matrix A is called the rank of A.
II. If A is an $n \times n$ square matrix, A will be non-singular of rank $A = n$.

With reference to the above statements which of the following are applicable?

(a) Both statements are false.
(b) Both statements are true.
(c) I is true but II is false.
(d) I is false but II is true.

[GATE (CE) 2000]

21. If $A = \begin{bmatrix} a & 1 \\ a & 1 \end{bmatrix}$ then eigenvalues of A are

(a) $(a + 1)$, 0 (b) a, 0
(c) $(a - 1)$, 0 (d) 0, 0

[GATE (EE) 1994]

22. A 5 × 7 matrix has all its entries equal to –1. The rank of the matrix is

(a) 7 (b) 5
(c) 1 (d) 0

[GATE (EE) 1994]

23. The rank of a 3 × 3 matrix $C(= AB)$, found by multiplying a non-zero column matrix A of size 3 × 1 and a non-zero row matrix B of size 1 × 3 is

(a) 0 (b) 1
(c) 2 (d) 3

[GATE (ME) 2001]

24. The product $[P]\ [Q]^T$ of the following two matrices $[P]$ and $[Q]$ is

$$[P] = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}; Q = \begin{bmatrix} 4 & 8 \\ 9 & 2 \end{bmatrix}$$

(a) $\begin{bmatrix} 32 & 24 \\ 56 & 46 \end{bmatrix}$ (b) $\begin{bmatrix} 46 & 56 \\ 24 & 32 \end{bmatrix}$

(c) $\begin{bmatrix} 55 & 22 \\ 61 & 42 \end{bmatrix}$ (d) $\begin{bmatrix} 32 & 56 \\ 24 & 46 \end{bmatrix}$

[GATE (CE) 2001]

25. Consider the system of equations given below:

$$x + y = 2, 2x + 2y = 5$$

The system has

(a) one solution (b) no solution
(c) infinite solution (d) four solutions

[GATE (ME) 2001]

26. The determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$ is

(a) 100 (b) 200
(c) 1 (d) zero

[GATE (EE) 2002]

27. Consider the n eigenvalues of a n by n real value matrix. Which one of the following is necessarily true?

(a) The eigenvalues are n distinct real numbers
(b) The eigenvalues are n distinct complex numbers
(c) If any eigenvalue is the real number p then $-p$ is also an eigenvalue
(d) If any eigenvalue is the complex number $p + iq$ then $p - iq$ is also an eigenvalue

[GATE (IPE) 2003]

28. Given matrix $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(a) 4 (b) 3
(c) 2 (d) 1

[GATE (CE) 2003]

29. Consider the system of simultaneous equations $x + 2y + z = b$, $2x + y + 2z = 6$, $x + y + z = 5$. Then the system has

(a) a unique solution
(b) an infinite number of solutions
(c) no solution
(d) exactly two solutions

[GATE (ME) 2003]

30. A system of equations represented by $AX = 0$, where X is a column vector of unknown elements and A is a matrix containing coefficients has a non-trivial solution, when A is

(a) non-singular (b) singular
(c) symmetric (d) Hermitians

[GATE (IE) 2003]

31. For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, the eigenvalues are

(a) 3 and –3 (b) –3 and –5
(c) 3 and 5 (d) 5 and 0

32. The sum of the eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) 5 (b) 7
(c) 9 (d) 18

[GATE (ME) 2004]

33. For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

(a) 4 (b) 6
(c) 8 (d) 12

[GATE (ME) 2004]

34. The system of equations is

$$4x + 6y + 3z = 20, x + 2y + 3z = 26,$$
$$6x + 10y + 9z = 72$$

The system of equations

(a) has no solution
(b) defines a line in (x, y, z) space
(c) has a unique solution
(d) defines a plane in (x, y, z) space

[GATE (IPE) 2004]

35. Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigenvector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

[GATE (ECE) 2005]

36. Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$. Then $(a + b) =$

(a) $\dfrac{7}{20}$ (b) $\dfrac{3}{20}$

(c) $\dfrac{19}{60}$ (d) $\dfrac{11}{20}$

[GATE (ECE) 2005]

37. Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}; (AA^T)^{-1} \text{ is}$$

(a) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$

[GATE (ECE) 2005]

38. Consider the matrices $X_{4 \times 3}$, $Y_{4 \times 3}$ and $P_{2 \times 3}$. The order of $(P(X^T Y)^{-1} P^T)^T$ will be

(a) 2×2 (b) 3×3
(c) 4×3 (d) 3×4

[GATE (CE) 2005]

39. Consider a non-homogeneous system of linear equations representing mathematically an over determined system. Such system will be

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

[GATE (CE) 2005]

40. A is a 3×4 real matrix and $AX = b$ is an inconsistent system of equations. The highest possible rank of A is

(a) 1 (b) 2
(c) 3 (d) 4

[GATE (ME) 2005]

41. Which one of the following is an eigenvector of the matrix

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

[GATE (ME) 2005]

42. Let A be a 3×3 matrix with rank 2; then $AX = 0$ has

(a) only the trivial solution $X = 0$
(b) it has one independent solution
(c) it has two independent solutions
(d) it has three independent solutions

[GATE (IE) 2005]

43. Identify which one of the following is an eigenvector of the matrix $A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$

(a) $[-1 \quad 1]^T$ (b) $[3 \quad -1]^T$
(c) $[1 \quad -1]^T$ (d) $[-2 \quad 1]^T$

[GATE (IE) 2005]

44. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigenvalues is equal to –2. Which of the following is an eigenvector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

[GATE (EE) 2005]

45. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, the top row of R^{-1} is

(a) $[5 \quad 6 \quad 4]$ (b) $[5 \quad -3 \quad 1]$
(c) $[2 \quad 0 \quad -1]$ (d) $\left[2 - 1\frac{1}{2}\right]$

[GATE (EE) 2005]

46. Multiplication of matrices E and F is G. Matrices E and G are

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[GATE (ME) 2006]

47. For a given 2 × 2 matrix A, it is observed that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Then the matrix A is

(a) $A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$ **[GATE (IE) 2006]**

48. Eigenvalues of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1. What are the eigenvalues of the matrix S^2?

(a) 1 and 25 (b) 6 and 4
(c) 5 and 1 (d) 2 and 10

[GATE (ME) 2006]

49. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 5

[GATE (ECE) 2006]

50. Solution for the system defined by the set of equations $4y + 3z = 8$, $2x - z = 2$ and $3x + 2y = 5$ is

(a) $x=0,\ y=1,\ z=\frac{4}{3}$

(b) $x=0,\ y=\frac{1}{2},\ z=2$

(c) $x=3,\ y=\frac{1}{2},\ z=4$

(d) no solution exists **[GATE (CE) 2006]**

51. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(a) 2 (b) 4
(c) 6 (d) 8

[GATE (ECE) 2006]

52. The inverse of the 2 × 2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is

(a) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

[GATE (CE) 2007]

53. For what values of α and β do the following simultaneous equation have an infinite number of solutions?

$x + y + z = 5;\ x + 3y + 3z = 9;$
$x + 2y + \alpha z = \beta$

(a) 2, 7 (b) 3, 8
(c) 8, 3 (d) 7, 2

[GATE (CE) 2007]

54. The minimum and maximum eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are –2 and 6 respectively. What is the other eigenvalue?

(a) 5 (b) 3
(c) 1 (d) –1

[GATE (CE) 2007]

55. Let A be an $n \times n$ real matrix such that $A^2 = I$ and y be an n-dimensional vector. Then the linear system of equations $Ax = y$ has

(a) no solution
(b) a unique solution
(c) more than one but finitely many independent solutions
(d) infinitely many independent solutions **[GATE (IE) 2007]**

56. Let $A = [a_{ij}]$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i.j$ then the rank of A is

(a) 0 (b) 1
(c) $n - 1$ (d) n

[GATE (IE) 2007]

57. For what value of a if any will the following system of equations in x, y and z have a solution

$2x + 3y = 4,\ x + y + z = 4,\ x + 2y - z = a$

(a) any real number (b) 0
(c) 1
(d) there is no such value

[GATE (ME) 2008]

58. The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigenvalue equal to 3. The sum of the other two eigenvalues is

(a) p (b) $p - 1$
(c) $p - 2$ (d) $p - 3$

[GATE (ME) 2008]

59. The eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) 2

[GATE (ME) 2008]

60. For a matrix $[M] = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$, the value of x is given by

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

[GATE (ME) 2008]

61. Eigenvalues of the matrix $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ are

(a) 1, 1, 1 (b) 1, 1, 2
(c) 1, 1, 4 (d) 1, 2, 4

62. The largest eigenvalue of the matrix $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) 1 (b) 4
(c) 2 (d) 3

63. The system of linear equations $4x + 2y = 7$, $2x + y = 6$ has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) exactly two distinct solutions

[GATE (ECE) 2008]

64. Consider the matrix $p = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, the value of e^p is

(a) $\begin{bmatrix} 2e^{-2} - 3e^{-1} & e^{-1} - e^{-2} \\ 2e^{-2} - 2e^{-1} & 5e^{-2} - e^{-1} \end{bmatrix}$

(b) $\begin{bmatrix} e^{-1} + e^{-2} & 2e^{-2} - e^{-1} \\ 2e^{-1} - 4e^{-2} & 3e^{-1} + 2e^{-2} \end{bmatrix}$

(c) $\begin{bmatrix} 5e^{-2} - e^{-1} & 3e^{-1} - e^{-2} \\ 2e^{-2} - 6e^{-1} & 4e^{-2} + e^{-1} \end{bmatrix}$

(d) $\begin{bmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{bmatrix}$

[GATE (ECE) 2008]

65. The product of matrices $(PQ)^{-1} P$ is
(a) P^{-1} (b) Q^{-1}
(c) $P^{-1} Q^{-1}$ (d) PQP^{-1}

[GATE (CE) 2008]

66. The eigenvalues of the matrix $[P] = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$ are
(a) –7 and 8 (b) –6 and 5
(c) 3 and 4 (d) 1 and 2

[GATE (CE) 2008]

67. The following simultaneous equations

$x + y + z = 3; x + 2y + 3z = 4;$
$x + 4y + kz = 6$

will not have a unique solution for k equal to
(a) 0 (b) 30
(c) 6 (d) 7

[GATE (CE) 2008]

68. All the four entries of the 2×2 matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero, and one of its eigenvalues is zero, which of the following statements is true?
(a) $p_{11}p_{22} - p_{12}p_{21} = 1$
(b) $p_{11}p_{22} - p_{12}p_{21} = -1$
(c) $p_{11}p_{22} - p_{12}p_{21} = 0$
(d) $p_{11}p_{22} + p_{12}p_{21} = 0$ **[GATE (ECE) 2008]**

69. The characteristic equation of a 3×3 matrix P is defined as

$$\alpha(\lambda) = [\lambda I - P] = \lambda^3 + \lambda^2 + 2\lambda + I = 0$$

If I denotes the identity matrix then the inverse of matrix P will be
(a) $P^2 + P + 2I$ (b) $P^2 + P + I$
(c) $-(P^2 + P + I)$ (d) $-(P^2 + P + 2I)$

[GATE (EE) 2008]

70. If the rank of a 5×6 matrix Q is 4, then which one of the following statements is correct?
(a) Q will have four LI rows and four LI columns.
(b) Q will have four LI rows and five LI vectors.
(c) QQ^T will be invertible.
(d) Q^TQ will be invertible.

[GATE (EE) 2008]

71. The value of the determinant $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$ is
(a) –28 (b) –24
(c) 32 (d) 36

[GATE (IPE) 2009]

72. The value of x_3 obtained by solving the following system of linear equations is

$x_1 + 2x_2 - 2x_3 = 4, 2x_1 + x_2 + x_3 = -2,$
$-x_1 + x_2 - x_3 = 2$

(a) –12 (b) –2
(c) 0 (d) 12

[GATE (IPE) 2009]

73. A square matrix B is skew-symmetric, if

(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

[GATE (CE) 2009]

74. Which of the following is a Hermitian matrix?

(a) $\begin{bmatrix} 2 & 3+i \\ 3-i & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3+i \\ -3+i & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3+i & 2 \\ 1 & 3-i \end{bmatrix}$ (d) $\begin{bmatrix} 3+i & 2 \\ 1 & -3+i \end{bmatrix}$

75. The eigenvalues of the following matrix

$$\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \text{ are}$$

(a) $3, 3+5i, 6-i$ (b) $-6+5i, 3+i, 3-i$
(c) $3+i, 3-i, 5+i$ (d) $3, -1+3i, -1-3i$

[GATE (IE) 2009]

76. A square matrix B is skew-symmetric, if

(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

[GATE (CE) 2009]

77. The system of equations

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$

has no solution for values of λ and μ given by

(a) $\lambda = 6, \mu = 20$ (b) $\lambda = 6, \mu \neq 20$
(c) $\lambda \neq 6, \mu = 20$ (d) $\lambda \neq 6, \mu \neq 20$

[GATE (EC) 2011]

78. Given that $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the value of A^3 is

(a) $15A + 12I$ (b) $19A + 30I$
(c) $17A + 15I$ (d) $17A + 21I$

79. The system of algebraic equations given below has

$$x + 2y + z = 4$$
$$2x + y + 2z = 5$$
$$x - y + z = 1$$

(a) a unique solution of $x = 1, y = 1$, and $z = 1$
(b) only the two solutions of ($x = 1$, $y = 1$, and $z = 1$) and ($x = 2$, $y = 1$, and $z = 0$)
(c) infinite number of solutions
(d) no feasible solution

[GATE (ME) 2012]

80. For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, one of the normalized eigenvectors is given as

(a) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(c) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

[GATE (ME) 2012]

81. One of the eigenvalue of $P = \begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$ is

(a) 2 (b) 4
(c) 6 (d) 8

[GATE (BT) 2013]

82. If $P = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$ and $R = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$, which one of the following statements is TRUE?

(a) $PQ = PR$ (b) $QR = RP$
(c) $QP = RP$ (d) $PQ = QR$

[GATE (BT) 2013]

83. The solution of the following set of equations is

$$x + 2y + 3z = 20$$
$$7x + 3y + z = 13$$
$$x + 6y + 2z = 0$$

(a) $x = -2, y = 2, z = 8$
(b) $x = 2, y = -3, z = 8$
(c) $x = 2, y = 3, z = -8$
(d) $x = 8, y = 2, z = -3$ **[GATE (BT) 2013]**

84. Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that determinant $(I_m + AB)$ = determinant $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(a) 2 (b) 5
(c) 8 (d) 16

[GATE (EC) 2013]

85. The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) no solution
(b) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
(c) non-zero unique solution
(d) multiple solutions **[GATE (EE) 2013]**

86. A matrix has eigenvalues –1 and –2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

[GATE (EE) 2013]

87. The dimension of the null space of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[GATE (IN) 2013]

88. The minimum eigenvalue of the following matrix is

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

(a) 2 (b) 1
(c) 2 (d) 3

[GATE (EC) 2013]

89. The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ________ .

[GATE (CE) 2014]

90. The sum of eigenvalues of the matrix, $[M]$ is

where $[M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$

(a) 915 (b) 1355
(c) 1640 (d) 2180

91. Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is ________. **[GATE (CE) 2014]**

92. The determinant of the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is ________. **[GATE (CE) 2014]**

93. If the matrix A is such that

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} [1 \ 9 \ 5]$$

Then the determinant of A is equal to ________. **[GATE (CS) 2014]**

94. The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ________. **[GATE (EC) 2014]**

95. Given a system of equations:

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$

Which of the following is true regarding its solutions?

(a) The system has a unique solution for any given b_1 and b_2.
(b) The system will have infinitely many solutions for any given b_1 and b_2.

(c) Whether or not a solution exists depends on the given b_1 and b_2.
(d) The system would have no solution for any values of b_1 and b_2.

96. A system matrix is given as follows:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The absolute value of the ratio of the maximum eigenvalue to the minimum eigenvalue is ________. **[GATE (EE) 2014]**

97. The system of linear equations

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 14 \end{pmatrix} \text{ has}$$

(a) a unique solution
(b) infinitely many solutions
(c) no solution
(d) exactly two solutions

[GATE (EC) 2014]

98. Consider the matrix

$$J = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

which is obtained by reversing the order of the columns of the identity matrix I_6.
Let $P = I_6 + \alpha J_6$, where α is a non-negative real number. The value of α for which det(P) = 0 is ________. **[GATE (EC) 2014]**

99. For matrices of same dimension M, N, and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$
(b) $(cM^T)^T = c(M)^T$
(c) $(M + N)^T = M^T + N^T$
(d) $MN = NM$ **[GATE (EC) 2014]**

100. A real (4×4) matrix A satisfies the equation $A^2 = I$, where I is the (4×4) identity matrix. The positive eigenvalue of A is ________. **[GATE (EC) 2014]**

101. Given that determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is –12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is

(a) –96 (b) –24
(c) 24 (d) 96

[GATE (ME) 2014]

102. Which one of the following statements is TRUE for all real symmetric matrices?

(a) All the eigenvalues are real.
(b) All the eigenvalues are positive.
(c) All the eigenvalues are distinct.
(d) Sum of all the eigenvalues is zero.

[GATE (EE) 2014]

103. The matrix form of the linear system $\frac{dx}{dt} = 3x - 5y$ and $\frac{dy}{dt} = 4x + 8y$ is

(a) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(b) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(c) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(d) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

[GATE (ME) 2014]

104. One of the eigenvectors of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is

(a) $\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ (b) $\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

(c) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

105. Consider a 3×3 real symmetric matrix S such that two of its eigenvalues are $a \neq 0$, $b \neq 0$

with respective eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

If $a \neq b$ then $x_1y_1 + x_2y_2 + x_2y_2$ equals

(a) a (b) b
(c) ab (d) 0

[GATE (ME) 2014]

106. The state equation of a second-order linear system is given by

$$\dot{x}(t) = Ax(t), x(0) = x_0$$

For $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ and for

$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x(t) = \begin{bmatrix} e^{-t} & -e^{-2t} \\ -e^{-t} & +2e^{-2t} \end{bmatrix}$

When $x_0 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $x(t)$ is

(a) $\begin{bmatrix} -8e^{-t} & +11e^{-2t} \\ 8e^{-t} & -22e^{-2t} \end{bmatrix}$

(b) $\begin{bmatrix} 11e^{-t} & -8e^{-2t} \\ -11e^{-t} & +16e^{-2t} \end{bmatrix}$

(c) $\begin{bmatrix} 3e^{-t} & -5e^{-2t} \\ -3e^{-t} & +10e^{-2t} \end{bmatrix}$

(d) $\begin{bmatrix} 5e^{-t} & -3e^{-2t} \\ -5e^{-t} & +6e^{-2t} \end{bmatrix}$ **[GATE (EC) 2014]**

107. For the matrix A satisfying the equation given below, the eigenvalues are

$$[A]\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) $(1, -j, j)$ (b) $(1, 1, 0)$
(c) $(1, 1, -1)$ (d) $(1, 0, 0)$

108. Which one of the following statements is NOT true for a square matrix?

(a) If A is upper triangular, the eigenvalues of A are the diagonal elements of it
(b) If A is real symmetric, the eigenvalues of A are always real and positive
(c) If A is real, the eigenvalues of A and A^T are always the same
(d) If all the principal minors of A are positive, all the eigenvalues of A are also positive

109. The determinant of the matrix A is 5 and the determinant of the matrix B is 40. The determinant of the matrix AB is ________.

[GATE (EC) 2014]

ANSWERS

1. (b)	2. (c)	3. (c)	4. (d)	5. (c)	6. (a)	7. (b)	8. (c)	9. (a)	10. (a)
11. (b)	12. (c)	13. (a)	14. (a)	15. (a)	16. (a)	17. (b)	18. (b)	19. (b)	20. (b)
21. (a)	22. (c)	23. (b)	24. (a)	25. (b)	26. (c)	27. (d)	28. (b)	29. (c)	30. (b)
31. (c)	32. (b)	33. (a)	34. (d)	35. (c)	36. (a)	37. (c)	38. (a)	39. (a)	40. (c)
41. (a)	42. (c)	43. (b)	44. (d)	45. (b)	46. (a)	47. (b)	48. (a)	49. (c)	50. (d)
51. (c)	52. (a)	53. (a)	54. (b)	55. (b)	56. (b)	57. (b)	58. (c)	59. (c)	60. (a)
61. (c)	62. (b)	63. (b)	64. (d)	65. (b)	66. (b)	67. (d)	68. (c)	69. (d)	70. (a)
71. (b)	72. (b)	73. (a)	74. (a)	75. (b)	76. (a)	77. (b)	78. (b)	79. (c)	80. (b)
81. (c)	82. (a)	83. (b)	84. (b)	85. (d)	86. (d)	87. (b)	88. (a)	89. (2)	90. (a)
91. (23)	92. (88)	93. (0)	94. (49)	95. (b)	96. (1/3)	97. (b)	98. (1)	99. (d)	100. (1)
101. (a)	102. (a)	103. (a)	104. (d)	105. (d)	106. (b)	107. (c)	108. (b)	109. (200)	

2 Differential Calculus

2.1 INTRODUCTION

The derivative is an important and a fundamental tool of calculus for studying the behaviour of functions of the real variable. The derivative measures the sensitivity to change of one quantity (a function or a dependent variable) to another quantity (the independent variable).

The concept of derivative is at the core of calculus and modern mathematics. The definition of the derivative can be approached in two different ways, one is the *geometrical* (as a slope of the curve) and the other, *physical* (as a rate of change).

2.2 DIFFERENTIATION

Let $f(x)$ be a differentiable function on $[a, b]$. Then corresponding to each point $c \in [a, b]$, we obtain a unique real number equal to the derivative of $f(x)$ at $x = c$. This correspondence between the points in $[a, b]$ and derivatives at these points defines a new real-valued function with domain $[a, b]$ and range, a subset of R. The set of real numbers, such that the image of x in $[a, b]$ is the value of the derivative or differentiation of $f(x)$ with respect to x and it is denoted by $f'(x)$ or $Df(x)$ or $\frac{d}{dx} f(x)$.

Thus,

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

or

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \tag{2}$$

2.3 GEOMETRICAL MEANING OF DERIVATIVE AT A POINT

Let the curve $y = f(x)$. Suppose $f(x)$ be differentiable at $x = c$. Let $P(c, f(c))$ be a point on the curve and let $Q(x, f(x))$ be a neighbouring point on the curve.

Then, the slope of the chord $PQ = \frac{f(x) - f(c)}{x - c}$.

Taking limit $Q \to P$, i.e., $x \to c$, we get

$$\lim_{Q \to P} (\text{slope of the chord } PQ) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{3}$$

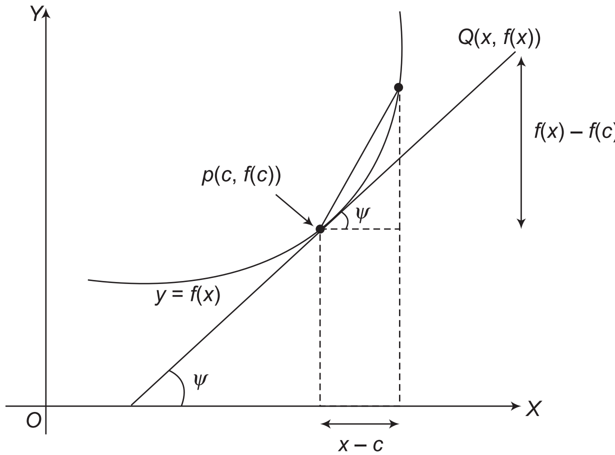


Fig. 2.1

As $Q \to P$, the chord PQ becomes the tangent at P.
$\therefore$ from Eq. (3), we have

$$\text{Slope of the tangent at} \quad P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to 0} \left(\frac{d}{dx} f(x) \right) \text{or } f'(c)$$

Thus, the derivative of a function at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$. Also, a function is not differentiable at $x = c$ only if the point $(c, f(c))$ is a corner point of the curve $y = f(x)$, i.e., the curve suddenly changes its direction at the point $(c, f(c))$.

2.4 SUCCESSIVE DIFFERENTIATION

Let $y = f(x)$ be a differentiable (or derivable) function of x. Then $\frac{dy}{dx} = f'(x)$ is called the *first differential coefficient of y with respect to x*. If $f'(x)$ is again derivable then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$ is called the *second differential coefficient of y with respect to x*. If $f''(x)$ is further differentiable then, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x)$ is called the *third differential coefficient of y with respect of x*. In general, the nth order differential coefficient of y with respect of x is $\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x)$.

The process of finding the differential coefficient of a function again and again is called *successive differentiation*.

Thus, if $y = f(x)$ then the successive differential coefficients of y with respect to x are $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$.

or $y', y'', y''', \cdots, y^{(n)}$

or $y_1, y_2, y_3, \cdots y_n$

or $Dy, D^2y, D^3y, \cdots D^ny$

or $f'(x), f''(x), f'''(x), \cdots, f^{(n)}(x)$.

The differential operator $\dfrac{d}{dx}$ is also denoted by D. As such, the n^{th} order derivative of y is denoted by D^n.

2.5 CALCULATION OF n^{th} ORDER DIFFERENTIAL COEFFICIENTS

(i) Find the n^{th} order differential coefficient of e^{ax+b}.

Let $y = e^{ax+b}$

Therefore,

$$y_1 = \frac{d}{dx} y = ae^{ax+b}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = a^2 e^{ax+b}$$

$$y_3 = a^3 e^{ax+b}$$

and so on.

Therefore, the n^{th} order derivative D^ny is

$$\boldsymbol{y_n = a^n e^{ax+b}}$$

(ii) Find the n^{th} order derivative of $(ax+b)^m$; $m > n$.

Let $y = (ax+b)^m$

Therefore, $y_1 = m(ax+b)^{m-1} \cdot a = ma(ax+b)^{m-1}$

$$y_2 = m(m-1)a^2 (ax+b)^{m-2}$$

$$y_3 = m(m-1)(m-2)\, a^3(ax+b)^{m-3}$$

and so on. Therefore,

$$y_n = m(m-1)(m-2)(m-3) \ldots (m-n+1)a^n (ax+b)^{m-n}$$

Remark I: If m is a positive integer then

$$\boldsymbol{y_n = \frac{m!}{(m-n)!} a^n (an+b)^{m-n}}$$

Remark II: If $m = n$ then

$$\boldsymbol{y_m = m! \cdot a^m}$$

(iii) Find the n^{th} order derivative of $\left(\dfrac{1}{(ax+b)}\right)$; $x \neq -\dfrac{b}{a}$.

Let $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

Therefore, $y_1 = (-1)\,(ax+b)^{-2} \cdot a$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2 = (-1)^2 \cdot 2! \cdot a^2 (ax+b)^{-3}$$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^3 = (-1)^3 \cdot 3!\, a^3(ax+b)^{-4}$$

and so on.

Therefore, $y_n = (-1)^n \cdot n!\, a^n(ax+b)^{-(n+1)}$

or $$\boldsymbol{y_n = \frac{(-1)^n\ n!\, a^n}{(ax+b)^{n+1}}}$$

(iv) Find the n^{th} order derivative of log ($ax + b$).

Let $y = \log(ax+b)$

$$y_1 = \frac{1}{ax+b} \cdot a = a \cdot (ax+b)^{-1}$$

$$y_2 = a^2(-1)\,(ax+b)^{-2}$$

$$y_3 = a^3\,(-1)\,(-2)\,(ax+b)^{-3} = a^3(-1)^2 \cdot 2!\,(ax+b)^{-3}$$

$$y_4 = a^4(-1)(-2)(-3)(ax+b)^{-4} = a^4(-1)^3 \cdot 3!\,(ax+b)^{-4}$$

and so on.

Therefore, $y_n = a^n(-1)^{n-1} \cdot (n-1)!\,(ax+b)^{-n}$

or $$\boldsymbol{y_n = \frac{a^n(-1)^{n-1} \times (n-1)!}{(ax+b)^n}}$$

(v) Find the n^{th} order differential coefficient of a^{mx}.

Let $y = a^{mx}$

Therefore, $y_1 = m \cdot a^{mx} \cdot (\log a)$.

$$y_2 = m^2 \cdot a^{mx} \cdot (\log a)^2$$

$$y_3 = m^3 \,.\, a^{mx} \cdot (\log a)^3$$

and so on.

Therefore,

$$\boldsymbol{y_n = m^n \cdot a^{mx} \cdot (\log a)^n}$$

(vi) Find the n^{th} order derivative of x^m.

Let $y = x^m$

Therefore, $y_1 = m \cdot x^{m-1}$

$$y_2 = m(m-1) \cdot x^{m-2}$$

$$y_3 = m(m-1)(m-2) \cdot x^{m-3}$$

and so on.

Therefore, $\boldsymbol{y_n = m(m-1)(m-2)(m-3) \ldots (m-n+1) \cdot x^{m-n}}$, where $m > n$

Remark: If m is a positive integer and $n = m$ then

$$y_m = m(m-1)\,(m-2)(m-3) \ldots (m-m+1) \cdot x^{m-m}$$

$$y_m = m(m-1)(m-2)\,(m-3) \ldots 1$$

$$\boldsymbol{y_m = m!}$$

(vii) Find the n^{th} order derivative of sin($ax + b$).

Let $y = \sin(ax+b)$

$\therefore$ $y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$ $\left[\because \sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta\right]$

$$y_2 = a^2 \cos\left(ax + b + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + 2.\frac{\pi}{2}\right)$$

$$y_3 = a^3 \cos\left(ax + b + n \cdot \frac{\pi}{2}\right) = a^3 \sin\left(ax + b + 3\frac{\pi}{2}\right)$$

and so on. Hence, in general,

$$\boldsymbol{y_n = a^n \sin\left(ax + b + n \cdot \frac{p}{2}\right)}$$

Similarly, the n^{th} order derivative of $\cos(ax + b)$ is

$$\boldsymbol{y_n = a^n \cos\left(ax + b + n \cdot \frac{p}{2}\right)}$$

(viii) Find the n^{th} order derivative of $e^{ax} \sin(bx + c)$.

Let $\quad y = e^{ax} \sin(bx + c)$

$\therefore \quad y_1 = ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)$

$$y_1 = e^{ax}[a \sin(bx + c) + b \cos(bx + c)] \tag{4}$$

Let us choose $a = r\cos\theta$ and $b = r\sin\theta$, then

$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(b/a) \text{ and (4) reduces to}$$

$$y_1 = re^{ax} \sin(bx + c + \theta)$$

Similarly, repeating the above argument, we obtain

$$y_2 = r^2 e^{ax} \sin(bx + c + 2\theta)$$

$$y_3 = r^3 e^{ax} \sin(bx + c + 3\theta)$$

and so on.

Hence, in general,

$$\boldsymbol{y_n = r^n e^{ax} \sin(bx + c + n\theta)}$$

where $\quad r = \sqrt{(a^2 + b^2)}$ and $\theta = \tan^{-1}(b/a)$

In a similar manner, we can find the n^{th} order derivative of $e^{ax} \cos(bx + c)$. Then

$$\boldsymbol{y_n = r^n e^{ax} \cos(bx + c + n\theta)}$$

where $\quad r = \sqrt{(a^2 + b^2)}$ and $\theta = \tan^{-1}(b/a)$

Example 1 Find the n^{th} derivative of $y = \sin 2x \cdot \sin 4x$.

Solution

Given $\quad y = \sin 2x \cdot \sin 4x = \frac{1}{2}[2 \sin 2x \sin 4x]$

$$y = \frac{1}{2}[\cos 2x - \cos 6x] \tag{1}$$

$[\because 2 \sin A \cos B = \cos(A - B) - \cos(A + B)]$

Differentiating n times, both sides on (1), we get

$$y_n = \frac{1}{2} D^n[\cos 2x - \cos 6x]$$

$$y_n = \frac{1}{2}[D^n \cos x - D^n \cos 6x]$$

$$y_n = \frac{1}{2}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) - 6^n \cos\left(6x + \frac{n\pi}{2}\right)\right]$$

Example 2 Find the n^{th} derivative of $\log(ax + x^2)$.

Solution

Let $\quad y = \log(ax + x^2) = \log x(a + x)$

$$y = \log x + \log(a + x) \tag{1}$$

Differentiating n times both sides on (1), we get

$$D^n y = D^n[\log x + \log(x + a)] \qquad \text{(By art. (iv))}$$

or

$$y_n = D^n \log x + D^n \log(x + a)$$

$$= \frac{(-1)^{n-1}(n-1)!\,(1)^n}{x^n} + \frac{(-1)^{n-1}\cdot(n-1)!\,(1)^n}{(x+a)^n}$$

$$= (-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right]$$

Example 3 Find the n^{th} derivative of $\dfrac{1}{6x^2 - 5x + 1}$.

Solution

Let
$$y = \frac{1}{6x^2 - 5x + 1}$$

$$y = \frac{1}{(2x-1)(3x-1)} = \frac{2}{(2x-1)} - \frac{3}{(3x-1)} \quad \text{(By partial fraction)}$$

Differentiating n times, both sides, we get

$$y_n = 2D^n\left(\frac{1}{2x-1}\right) - 3D^n\left(\frac{1}{3x-1}\right)$$

$$y_n = 2\left[\frac{(-1)^n\cdot n!\,(2)^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n\cdot n!\,(3)^n}{(3x-1)^{n+1}}\right]$$

$$= (-1)^n\cdot n!\left[\frac{(2)^{n+1}}{(2x-1)^{n+1}} - \frac{(3)^{n+1}}{(3x-1)^{n+1}}\right]$$

Example 4 Find the n^{th} derivative of $\dfrac{1}{x^2 - 6x + 8}$.

Solution Do same as Example 3.

$$y_n = \frac{(-1)^n\cdot n!}{2}\left[\frac{1}{(x-4)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right]$$

Example 5 Find the n$^{\text{th}}$ derivative of $\tan^{-1}\left(\frac{x}{a}\right)$.

Solution Let $y = \tan^{-1}\left(\frac{x}{a}\right)$. (1)

Differentiating (1) both sides w.r.t. 'x', we get

$$y_1 = \frac{a}{x^2 + a^2} = \frac{a}{(x+ia)(x-ia)} = \frac{1}{2i}\left[\frac{1}{(x-ia)} - \frac{1}{(x+ia)}\right] \tag{2}$$

Differentiating (2) $(n - 1)$ times, we get

$$y_n = D^{n-1}\left[\frac{1}{2i}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)\right]$$

$$= \frac{1}{2i} D^{n-1}\left(\frac{1}{x-ia}\right) - \frac{1}{2i} D^{n-1}\left(\frac{1}{x+ia}\right)$$

$$= \frac{1}{2i}\frac{(-1)^{n-1}\,(n-1)!\,(1)^{n-1}}{(x-ia)^n} - \frac{1}{2i}\frac{(-1)^{n-1}\,(n-1)!\cdot(1)^{n-1}}{(x+ia)^n}$$

$$y_n = \frac{1}{2i}(-1)^{n-1}\cdot(n-1)!\left[\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right] \tag{3}$$

Putting $x = r\cos\theta$ and $a = r\sin\theta$ so that

$$r^2 = x^2 + a^2 \text{ and } \theta = \tan^{-1}\left(\frac{a}{x}\right)$$

Thus, $$y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\ [(r\cos\theta - ir\sin\theta)^{-n} - (r\cos\theta + ir\sin\theta)^{-n}]$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!}{2i\times r^n}[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}]$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!}{2i\times r^n}\left[\cancel{\cos n\theta} + i\sin n\theta - \cancel{\cos n\theta} + i\sin n\theta\right]$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!}{2i\times r^n}[2i\sin n\theta]$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!\sin n\theta}{r^n}\text{; where } r = \sqrt{x^2 + a^2} \text{ and } \theta = \tan^{-1}\left(\frac{a}{x}\right)$$

Example 6 Find the n^{th} derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Solution Do same as Example 5

Ans $\Rightarrow y_n = 2(-1)^{n-1}\cdot(n-1)!\cdot\sin n\theta\cdot\sin^n\theta$, where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

Example 7 If $y = x \log\left(\frac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2} \cdot (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

Solution

$$y = x \log\left(\frac{x-1}{x+1}\right)$$

$$y = x[\log (x-1) - \log (x+1)] \tag{1}$$

Differentiating (1) both sides w.r.t. 'x', we get

$$y_1 = x\left[\frac{1}{x-1} - \frac{1}{x+1}\right] + \log (x-1) - \log(x+1)$$

$$= \left(\frac{x}{x-1} - \frac{x}{x+1}\right) + \log (x-1) - \log(x+1)$$

$$= \left(\frac{x-1+1}{x-1} - \frac{x+1-1}{x+1}\right) + \log (x-1) - \log(x+1)$$

$$= \left(1 + \frac{1}{x-1}\right) - \left(1 - \frac{1}{x+1}\right) \log (x-1) - \log(x+1)$$

$$y_1 = \frac{1}{x-1} + \frac{1}{x+1} + \log (x-1) - \log (x+1) \tag{2}$$

Again, differentiating (2), $(n-1)$ times w.r.t. 'x', we get

$$y_n = \frac{(-1)^{n-1} \cdot (n-1)!}{(x-1)^n} + \frac{(-1)^{n-1} \cdot (n-1)!}{(x+1)^n} + \frac{(-1)^{n-2} \cdot (n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2} \cdot (n-2)!}{(x+1)^{n-1}}$$

or $$y_n = (-1)^{n-2} \cdot (n-2)! \left[\frac{(-1)(n-1)}{(x-1)^n} + \frac{(-1)(n-1)}{(x+1)^n} + \frac{x-1}{(x+1)^n} - \frac{x-1}{(x+1)^n}\right]$$

or $$y_n = (-1)^{n-2} \cdot (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

EXERCISE 2.1

1. Find the n^{th} derivative of the following functions:
   (i) $y = \cos^2 x \cdot \sin^3 x$
   (ii) $y = e^{2x} \sin^3 x$
   (iii) $y = \log 3x$
   (iv) $y = \log (x^2 - a^2)$
   (v) $y = \dfrac{1}{x^2 + a^2}$
2. If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, prove that $y_n = (-1)^{n-1} \cdot (n-1)! \sin^n \theta \sin n\theta$, where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$.

3. Prove that the value n^{th} derivative of $\dfrac{x^3}{x^2-1}$ for $x = 0$ is 0 if n is even and $(-n)!$ if n is odd and greater than 1.

4. If $y = \cos^{-1}\left(\dfrac{x-1/x}{x+1/x}\right)$, prove that $y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$; where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

5. Find the n^{th} derivative of $\dfrac{x^2}{(x+2)(2x+3)}$.

6. Find the n^{th} derivative of $e^x\sin^2 x$.

7. If $I_n = D^n(x^n\log x)$; $D \equiv \dfrac{d}{dx}$, prove that $I_n = n\,I_{n-1} + (n-1)!$

8. If $y = x\log(1+x)$, prove that $y_n = \dfrac{(-1)^{n-1}(n-2)!\,(x+n)}{(1+x)^n}$

Answers

1. (i) $\dfrac{1}{16}\left[2\sin\left(x+\dfrac{n\pi}{2}\right)+3^n\sin\left(3x+\dfrac{n\pi}{2}\right)-5^n\sin\left(5x+\dfrac{n\pi}{2}\right)\right]$

   (ii) $\dfrac{3}{4}\left[\sqrt{2^2+1}\right]^n e^{2x}\cdot\sin\left[x+n\tan^{-1}\left(\dfrac{1}{2}\right)\right]-\dfrac{1}{4}\left[\sqrt{2^2+3^2}\right]^n e^{2x}\times\sin\left[2x+n\tan^{-1}\left(\dfrac{3}{2}\right)\right]$

   (iii) $\dfrac{(-1)^{n-1}\cdot(n-1)!}{x^n}$

   (iv) $(-1)^{n-1}\cdot(n-1)!\left[\dfrac{1}{(x+a)^n}+\dfrac{1}{(x-a)^n}\right]$

   (v) $\dfrac{(-1)^n\cdot n!\sin(n+1)\,\theta\cdot\sin^{n+1}\theta}{a^{n+2}}$

5. $\dfrac{(-1)^n\cdot n!}{2}\left[\dfrac{9\times 2^n}{(2x+3)^{n+1}}-\dfrac{8}{(x+2)^{n+1}}\right]$

6. $\dfrac{e^x}{2}\left[1-5^{n/2}\cos(2x+n\tan^{-1}2\right]$

2.6 LEIBNITZ'S THEOREM

This theorem is applicable for finding the n^{th} order derivative of the product of two differentiable functions.

Let u and v be the two functions of x such that n^{th} derivatives $D^n u$ and $D^n v$ exist. Then the n^{th} derivative, or differential coefficient, of their product is given by

$$D^n(u\cdot v) = {}^nC_0\,(D^n u)\cdot v + {}^nC_1\,(D^{n-1}u)Dv + {}^nC_2\,(D^{n-2}u)D^2v + \cdots + {}^nC_r\,(D^{n-r}u)\,D^r v + \cdots + {}^nC_n\,u(D^n v).$$

$$= {}^nC_0\,u_n\cdot v + {}^nC_1\,u_{n-1}\cdot v_1 + {}^nC_2\,u_{n-2}u^{n-2}v_2 + \cdots + {}^nC_r\,u_{n-r}\cdot v_r + \cdots + {}^nC_n\,u\cdot v_n.$$

Gottfried Wilhelm Leibniz (1 July 1646–14 November 1716), born in Leipzig, Saxony (now Germany) was a most versatile genius. He wrote on mathematics, natural science, history, politics, jurisprudence, economics, philosophy, theology, and philology. His father, Friedrich, was Vice Chairman of the Faculty of Philosophy and Professor of Moral Philosophy in the University of Leipzig. He offered his service for noblemen, statesmen, and members of royal families because he believed that they had the power to reform the country. However, it was his destiny not to be able to roam freely but to be a tool for his masters. During his service to the Elector of Mainz, as part of his duties, he travelled extensively through England, France, German, Italy, Poland, and other countries. During these trips, he met or got acquainted with many prominent scholars of his time, such as Huygens, Boole, Spinoza, and Newton, who greatly enriched his knowledge. He died in Hannover, Hanover (now Germany).

Proof: We shall prove the theorem using mathematical induction.

Step I: For $n = 1$, we have

$$D(uv) = (Du)v + u \cdot (Dv) = u_1 \cdot v + u \cdot v_1$$

Thus, the theorem is true for $n = 1$.

Step II: For $n = 2$, we have

$$D^2(uv) = (D^2u)v + {}^2C_1(Du) \cdot (Dv) + {}^2C_2\, u \cdot (D^2 \cdot v)$$

$$= u^2v + 2u_1\, v_1 + u \cdot v_2$$

Thus, the theorem is true for $n = 2$

Step III: Let us assume that the theorem holds for $n = k$. Thus,

$$D^k(uv) = (D^ku)\, v + {}^kC_1\, D^{k-1}u \cdot Dv + {}^kC_2\, D^{k-2}\, u \cdot D^2v + \cdots + {}^kC_r\, D^{k-r}\, u\, D^rv + \cdots + {}^kC_k\, u\, D^kv$$

Differentiating both sides with respect to 'x', we get

$$\begin{aligned} D^{k+1}(uv) &= \{(D^{k+1}\, u)\, v + D^ku \cdot Dv] + {}^kC_1[D^ku\, Dv + D^{k-1}\, u\, D^2v] \\ &\quad + {}^kC_2[D^{k-1}\, u \cdot D^2v + D^{k-2}\, u \cdot D^3v] + \cdots \\ &\quad + {}^kC_r[D^{k-r+1}\, u \cdot D^rv + D^{k-r}\, u \cdot D^{r+1}v] + \cdots + [Du.\, D^kv + uD^{k+1}\, v] \\ &= (D^{k+1}\, u)v + ({}^kC_0 + {}^kC_1)\, D^ku \cdot D_v + ({}^kC_1 + {}^kC_2)\, D^{k-1}\, u \cdot D^2v \\ &\quad + \cdots + ({}^kC_r + {}^kC_{r+1})\, D^{k-r}\, u \cdot D^{r+1}\, v + \cdots + u\, D^{k+1}\, v \\ &= (D^{k+1}\, u)v + {}^{k+1}C_1\, D^ku \cdot Dv + {}^{k+1}C_2\, D^{k-1}\, u \cdot D^2v \\ &\quad + \cdots + {}^{k+1}C_{r+1}\, D^{k-r}\, u \cdot D^{r+1}\, v + \cdots + u \cdot D^{k+1} \cdot v \qquad [\because {}^kC_{r-1} + {}^kC_r = {}^{k+1}C_r] \end{aligned}$$

Thus, the theorem is true for $n = k + 1$, i.e., it is also true for the next higher integral value of k. Hence, by mathematical induction, the theorem holds for all positive integral values of n.

Example 8 If $y = x^{n-1} \cdot \log x$, prove that $y_n = \dfrac{(n-1)!}{x}$.

Solution We have $y = x^{n-1} \cdot \log x$ (1)

Differentiating (1) both sides w.r.t. 'x', we get

$$y_1 = x^{n-1} \cdot \frac{1}{x} + (n-1)\, x^{n-2} \cdot \log x$$

or $\quad x\, y_1 = x^{n-1} + (n-1)x^{n-1} \cdot \log x$

or $\quad xy_1 = x^{n-1} + (n-1)y \qquad (2) \quad$ [By Eq. (1)]

Differentiating (2), $(n-1)$ times, we get

$$D^{n-1}(xy_1) = D^{n-1}\, x^{n-1} + (n-1)\, D^{n-1}\, y$$

$$(D^{n-1} \cdot y_1)x + {}^{n-1}C_1\, (D^{n-2}y_1)\, Dx = (n-1)! + (n-1)\, y_{n-1}$$

or $\quad xy_n\, ((n-1)\, y_{n-1} \cdot 1) = (n-1)! + (n-1)\, y_{n-1}$

or $\quad xy_n = (n-1)!$

or $\quad y_n = \dfrac{(n-1)!}{x}$ **Hence, proved.**

Example 9 If $x = \tan(\log y)$, show that

$$(1 + x^2)y_{n+2} + [2(n+1)x - 1]\, y_{n+1} + n(n+1)y_n = 0$$

Solution Given $x = \tan(\log y)$

or $\quad y = e^{\tan^{-1}x} \qquad (1)$

Differentiating (1) both sides, w.r.t. 'x', we get

$$y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2} = \frac{y}{1+x^2}$$

or $\quad (1 + x^2)\, y_1 = y \qquad (2)$

Differentiating (2), $(n+1)$ times both sides by Leibnitz's theorem, we get

$$D^{n+1}[(1+x^2)y_1] = D^{n+1}\, y \text{ or } (D^{n+1}\, y_1)\,(1+x^2) + {}^{n+1}C_1\,(D^n y_1)\, D(1+x^2) + {}^{n+1}C_2(D^{n-1}\, y_1)D^2(1+x^2) = y_{n+1}$$

or $\quad (1 + x^2)y_{n+2} + {}^{(n+1)}C_1\, y_{n+1}(2x) + {}^{n+1}C_2\, y_n \cdot 2 = y_{n+1}$

or $\quad (1 + x^2)\, y_{n+2} + [2(n+1)x - 1]\, y_{n+1} + n(n+1)y_n = 0$ **Hence, proved.**

Example 10 If $y = \sin(m \sin^{-1} x)$, prove that

$$(1 - x^2)y_{n+2} - (2n+1)x \cdot y_{n+1} + (m^2 - n^2)\, y_n = 0.$$

Solution Given $y = \sin(m \sin^{-1} x) \qquad (1)$

Differentiating (1) both sides w.r.t. 'x', we get

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

or $\quad y_1 = \dfrac{m}{\sqrt{1-x^2}} \cdot \cos(m \sin^{-1} x)$

or $\quad \sqrt{1-x^2} \cdot y_1 = m \cos(m \sin^{-1} x)$

Squaring both sides, we have.

$$\begin{aligned}(1 - x^2)y_1^{\,2} &= m^2 \cos^2(m \sin^{-1} x)\\ &= m^2[1 - \sin^2(m \sin^{-1} x)]\\ &= m^2[1 - y^2] \qquad \text{[Using (1)]}\end{aligned}$$

or $\quad (1-x^2)y_1^{\,2} = m^2(1-y^2)$ (2)

Again differentiating (2) both sides, we get

$$(1-x^2)\,2y_1y_2 - 2xy_1^{\,2} = m^2(-2y\,y_1)$$

or $\quad (1-x^2)y_2 - xy_1 + m^2y = 0$ (3)

Differentiating n times, by Leibnitz's theorem, we get

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n\right] - \left[xy_{n+1} + n(1)y_n\right] + m^2y_n = 0$$

$$[(1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} + (m^2 - n^2 + n - n)y_n = 0$$

or $\quad (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)\,y_n = 0$ **Hence, proved.**

Example 11 If $y = x^n \cdot \log x$, prove that $y_{n+1} = \dfrac{n!}{x}$.

Solution Given $y = x^n \cdot \log x$ (1)

Differentiating (1) both sides w.r.t. 'x', we get

$$y_1 = nx^{n-1}\cdot \log x + x^n \cdot \frac{1}{x}$$

or $\quad xy_1 = n\,x^n \log x + x^n$

or $\quad xy_1 = ny + x^n$ (2) [Using (1)]

Differentiating (2), n times by Leibnitz's theorem, we get

$x \cdot y_{n+1} + n(1)\cdot y_n = n\,y_n + n!$

or $\quad x \cdot y_{n+1} = n!$

or $\quad y_{n+1} = \dfrac{n!}{x}$ **Hence, proved.**

Example 12 If $y = a\cos(\log x) + b\sin(\log x)$ then show that

(i) $x^2y_2 + xy_1 + y = 0$

(ii) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Solution Given $y = a\cos(\log x) + b\sin(\log x)$ (1)

Differentiating (1) both sides w.r.t. 'x', we get.

$$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

or $\quad xy_1 = -a\sin(\log x) + b\cos(\log x)$ (2)

Again, differentiating (2) w.r.t. 'x', we get

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

or $\quad x(xy_2 + y_1) = -[a\cos(\log x) + b\sin(\log x)]$

or $\quad x^2y_2 + xy_1 = -y$ [Using (1)]

or $\quad x^2y_2 + xy_1 + y = 0$ (3)

Differentiating (3), n times by Leibnitz's theorem, we get

$$[x^2y_{n+2} + {}^nC_1\,y_{n+1}\,(2x) + {}^nC_2\,y_n\,(2)] + [xy_{n+1} + {}^nC_1\,y_n(1)] + y_n = 0$$

or $$x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + ny_{n+1} + ny_n + y_n = 0$$

or $$x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - n + n + 1)y_n = 0$$

or $$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)\,y_n = 0$$ **Hence, proved.**

Example 13 If $y = \cos(m\sin^{-1} x)$, prove that

$$(1-x^2)y_{n+2} - (2n+1)\,x\,y_{n+1} + (m^2 - n^2)y_n = 0$$

Solution Do same as Example 10.

Example 14 If $y = (x^2-1)^n$, prove that

$$(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0.$$

Solution Given $y = (x^2-1)^n$ (1)

Differentiating (1) both sides with respect to 'x', we get

$$y_1 = n(x^2-1)^{n-1}\cdot 2x$$

or $$y_1 = 2nx\,(x^2-1)^{n-1} \tag{2}$$

Multiplying (2) both sides by (x^2-1), we get

$$(x^2-1)y_1 = 2nx\,y(x^2-1)^n$$

$$(x^2-1)y_1 = 2nx\,y$$

Again, differentiating both sides w.r.t. 'x', we get

$$(x^2-1)y_2 + 2xy_1 = 2nx\,y_1 + 2ny \tag{3}$$

Differentiating (3) both sides n times, by Leibnitz's theorem, we have

$$(x^2-1)\,y_{n+2} + n\cdot(2x)\,y_{n+1} + \frac{n(n-1)}{2!}(2)\,y_n + 2x\,y_{n+1} + 2n(1)y_n$$
$$= 2nxy_{n+1} + (2n)\,n(1)y_n + 2ny_n$$

or $$(x^2-1)y_{n+2} + 2xy_{n+1} + (n^2 - n + 2n - 2n^2 - 2n)y_n = 0$$

or $$(x^2-1)y_{n+2} + 2xy_{n+1} - (n^2+n)y_n = 0$$

or $$(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0$$ **Hence, proved.**

Example 15 Find the n^{th} differential coefficient of $e^x\cdot\log x$.

Solution Let $u(x) = e^x$ and $v(x) = \log x$

$$D^n u = u_n = e^x$$

$$u_{n-1} = e^x$$

$$u_{n-2} = u_{n-3} = \ldots = u_2 = u_1 = e^x$$

and $$Dv = v_1 = \frac{1}{x}$$

$$D^2v = v_2 = -\frac{1}{x^2},\quad v_3 = \frac{2}{x^3}$$

......

......

......

$$D^n v = v_n = \frac{(-1)^{n-1}\cdot(n-1)!}{x^n}$$

Now, by Leibnitz's theorem, we have

$$D^n(e^x\cdot\log x) = {}^nC_0\, e^x\cdot\log x + {}^nC_1\, e^x\cdot\left(\frac{1}{x}\right) + {}^nC_2\, e^x\left(\frac{-1}{x^2}\right) + \cdots + {}^nC_n\,\frac{e^x\cdot(-1)^{n-1}\,(n-1)!}{x^n}$$

$$D^n(e^x\cdot\log x) = e^x\left[\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

Example 16 If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)\,y_n = 0.$$

Solution Given $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ (1)

Putting $z = y^{\frac{1}{m}}$; then (1), becomes

$$z + \frac{1}{z} = 2x$$

or $z^2 - 2xz + 1 = 0$ (2)

$$z = \frac{2x \pm\sqrt{4x^2-4}}{2} = x \pm \sqrt{(x^2-1)}$$

i.e. $y^{1/m} = x \pm \sqrt{(x^2-1)} \Rightarrow y = \left[x \pm \sqrt{x^2-1}\right]^m$

Let $y = \left[x + \sqrt{x^2-1}\right]^m$ (3)

Differentiating (3) w.r.t.'x', we get

$$y_1 = m\left[x+\sqrt{x^2-1}\right]^{m-1}\cdot\left(1+\frac{1}{2}\frac{2x}{\sqrt{x^2-1}}\right)$$

or $$y_1 = m\left[x+\sqrt{x^2-1}\right]^{m-1}\cdot\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right) = \frac{m\left[x+\sqrt{x^2-1}\right]^m}{\sqrt{x^2-1}}$$

$$y_1 = \frac{my}{\sqrt{x^2-1}} \qquad \text{[Using (3)]}$$

or $\left[\sqrt{x^2-1}\right]\cdot y_1 = my$ or $(x^2-1)\,{y_1}^2 = m^2y^2$

or $(x_2-1)\,{y_1}^2 = m^2y^2$ (4)

Similarly, if $y = \left[x - \sqrt{x^2 - 1}\right]^m$, then

$$(x^2 - 1)y_1^{\,2} = m^2y^2 \tag{5}$$

Thus, (4) and (5) are the same.

Differentiating (4) or (5) w.r.t. 'x', we get

$$(x^2 - 1)\, 2y_1y_2 + y_1^{\,2} \cdot 2x = m^2 \cdot 2y\, y_1$$

or $$(x^2 - 1)y_2 + xy_1 - m^2y = 0 \tag{6}$$

Differentiating (6) n times by Leibnitz's theorem, we get

$$\left[(y_{n+2})(x^2 - 1) + n\, y_{n+1} \cdot 2x + \frac{n(n-1)}{2!}\, y_n \cdot 2\right] + [y_{n+1} \cdot x + n\, y_n \cdot 1] - m^2 y_n = 0$$

or $$(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - n + n - m^2)y_n = 0$$

or $$(x^2 - 1)y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$$ **Hence, proved.**

Example 17 If $y = (\sin^{-1} x)^2$, prove that

(i) $(1 - x^2)\, y_2 - xy_1 - 2 = 0$

(ii) $(1 - x^2)\, y_{n+2} - (2n + 1)\, x\, y_{n+1} - n^2y_n = 0.$

Also find the value of n^{th} derivative of y for $x = 0$.

Solution Given $y = (\sin^{-1} x)^2$ (1)

Differentiating (1) both sides w.r.t. 'x', we get

$$y_1 = 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1 - x^2}} \tag{2}$$

Squaring both sides of (2), we obtain

$$(1 - x^2)y_1^{\,2} = 4(\sin^{-1} x)^2 = 4y$$

or $$(1 - x^2)y_1^{\,2} = 4y$$

Again, differentiating,

$$2(1 - x^2)\, y_1y_2 - 2x\, y_1^{\,2} = 4y_1$$

Dividing both sides by $2y_1$, we get

$$(1 - x^2)y_2 - x\, y_1 = 2$$

or $$(1 - x^2)y_2 - xy_1 - 2 = 0 \tag{3}$$

Differentiating (3) n times, by Leibnitz's theorem,

$$(1 - x^2)y_{n+2} + {}^nC_1\, y_{n+1}\, (-2x) + {}^nC_2\, y_n(-2) - xy_{n+1} - {}^nC_1\, y_n = 0$$

or $$(1 - x^2)y_{n+2} - (2n + 1)\, xy_{n+1} - n^2y_n = 0 \tag{4}$$

Putting $x = 0$ in (2), (3) and (4); then

$$y_1(0) = 0 \text{ and } y_2(0) = 2$$

and $$y_{n+2}(0) = n^2y_n(0) \tag{5}$$

Putting $n = 1, 2, 3, 4, \ldots$ in (5), we get

$$y_3(0) = 1^2 y_1(0) = 0$$
$$y_4(0) = 2^2 y_2(0) = 2.2^2$$
$$y_5(0) = 3^3 y_3(0) = 0$$
$$y_6(0) = 4^2 y_4(0) = 2.2^2 \cdot 4^2$$
$$y_7(0) = 5^2 y_5(0) = 0$$
$$y_8(0) = 6^2 y_6(0) = 2.2^2 \cdot 4^2 \cdot 6^2$$
$$\ldots$$
$$\ldots$$
$$\ldots$$

In general,

$y_n(0) = 0$; when n is odd

and $y_n(0) = 2.2^2 \cdot 4^2 \cdot 6^2 \ldots (n-2)^2$; when n is even.

EXERCISE 2.2

1. Apply Leibnitz's theorem to find y_n.
   (i) $x^{n-1} \cdot \log x$. (ii) $x^3 \log x$
2. If $x = \sin\left(\dfrac{\log y}{a}\right)$, prove that
   $(1 - x^2)y_{n+2} - (2n + 1)\, x\, y_{n+1} - (n^2 + a^2)\, y_n = 0.$
3. If $y = x \cos(\log x)$, prove that
   $x^2 y_{n+2} + (2n - 1)\, xy_{n+1} + (n^2 - 2n + 2)\, y_n = 0$
4. If $y = \cos(m \sin^{-1} x)$, prove that
   $(1 - x^2)y_{n+2} - (2n + 1)\, x\, y_{n+1} + (m^2 - n^2)\, y_n = 0$
5. If $y = x \log(1 + x)$, prove that
   $$y_n = \frac{\left[(-1)^{n-2}\,(n-2)!\,(x+n)\right]}{(1+x)^n}$$
6. If $x = \tan y$, prove that $(1 + x^2)\, y_{n+1} + 2nxy_n + n(n - 1)\, y_{n-1} = 0.$
7. If $y = \tan^{-1} x$, prove that $(1 + x^2)\, y_{n+2} + 2(n + 1)\, xy_{n+1} + n(n + 1)y_n = 0$
   Hence, find $y_n(0)$.
8. If $y = \sin(m \sin^{-1} x)$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$ and, hence, find $y_n(0)$.
9. If $y = \sin^{-1}x$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$ and, hence, find $y_n(0)$.

Answers

1. (i) $\dfrac{(n-1)!}{x}$ (ii) $\dfrac{(-1)^{n-1} \cdot n!}{x^{n-3}}\left[\dfrac{1}{n} - \dfrac{3}{n-1} + \dfrac{3}{n-2} - \dfrac{1}{n-3}\right]$

7. $y_n(0) = (-1)^{\frac{(n-1)}{2}} (n-1)!$
8. $y_n(0) = 0$; when n is even and $y_n(0) = m(1^2 - m^2)(3^2 - m^2) \ldots [(n-2)^2 - m^2]$; when n is odd.
9. $y_n(0) = 0$; when n is even and $y_n(0) = (n-2)^2 (n-4)^2 \ldots 3^2\, 1^2$; when n is odd.

SUMMARY

1. Successive Differentiation

The process of finding the differential coefficient of a function again and again is called successive differentiation. If $y = f(x)$ be a function then the successive differential coefficients of y with respect to x are

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \cdots, \frac{d^ny}{dx^n} \quad \text{or} \quad y', y'', y''', \ldots, y^{(n)} \quad \text{or} \quad y_1, y_2, y_3, \ldots, y^n$$

or $\quad D_y, D^2y, D^3y, \ldots, D^ny$ or $f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x)$.

The differential operator $\dfrac{d}{dx}$ is also denoted by D. As such, the n^{th} order derivative of y is denoted by D^n.

2. The n^{th} Order Differential Coefficients

(i) $D^n (e^{ax+b}) = a^n e^{ax+b}$

(ii) $D^n (ax+b)^m = m(m-1)(m-2)(m-3) \ldots (m-n+1)a^n (ax+b)^{m-n}$; $m > n$.

Remark I: If m is a positive integer then

$$y_n = \frac{m!}{(m-n)!} a^n (an+b)^{m-n}$$

Remark II: If $m = n$, then

$$y_m = m! \cdot a^m$$

(iii) $D^n\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}; x \neq -\dfrac{b}{a}$

(iv) $D^n \log(ax+b) = \dfrac{a^n (-1)^{n-1} \cdot (n-1)!}{(ax+b)^n}$

(v) $D^n a^{mx} = m^n \cdot a^{mx} \cdot (\log a)^n$

(vi) $D^n x^m = m(m-1)(m-2)(m-3) \ldots (m-n+1) \cdot x^{m-n}$, where $m > n$

Remark III: If m is a positive integer and $n = m$ then $y_m = m!$

(vii) $D^n \sin(ax+b) = a^n \sin\left(ax + b + n \cdot \dfrac{\pi}{2}\right)$

(viii) $D^n \cos(ax+b) = a^n \cos\left(ax + b + n \cdot \dfrac{\pi}{2}\right)$

(ix) $D^n e^{ax} \sin(ax+b) = r^n e^{ax} \sin(bx + c + n\theta)$, where $r = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1}(b/a)$.

(x) $D^n e^{ax} \cos(ax+b) = r^n e^{ax} \sin(bx + c + n\theta)$, where $r = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1}(b/a)$

3. Leibnitz's Theorem

This theorem is applicable for finding the n^{th} order derivative of the product of two differentiable functions. Let u and v be the two functions of x such that n^{th} derivatives $D^n u$ and $D^n v$ exist. Then the n^{th} derivative or differential coefficient of their product is given by

$$D^n(u.v) = n_{C_o}\,(D^n u).v + {}^nc_1\,(D^{n-1}\,u).Dv + {}^nC_2\,(D^{n-2}\,u).D^2 v + \ldots + {}^nC_r\,(D^{n-r}\,u)\,D^r v + \ldots + {}^nC_n\,u(D^n v).$$

$$= {}^nc_0\,u_n\,v + {}^nc_1\,u_{n-1}\,v_1 + {}^nc_2\,u_{n-2}\,v_2 + \ldots + {}^nc_r\,u_{n-r}\,v_r + \ldots + {}^nc_n\,u\,v_n.$$

OBJECTIVE-TYPE QUESTIONS

1. The differential coefficient of log sin x is

(a) 2 cos $2x$ (b) 2 cosec $2x$
(c) 2 sin $2x$ (d) 0

2. The derivative of $f(x) = \dfrac{\sin x^2}{x}$, when $x \neq 0$ and = 0; $x = 0$ is

(a) 0 (b) 2
(c) 1 (d) –1

3. If $y = \dfrac{1}{x}$ then $\dfrac{d^{10}y}{dx^{10}}$ is

(a) $-9!\,x^{10}$ (b) $-9!\,x^{-10}$
(c) $9!\,x^{-10}$ (d) $9!\,x^9$

4. If $y = \sin x$ then $\dfrac{d^n y}{dx^n}$ is

(a) $\sin\left(x + \dfrac{n\pi}{2}\right)$ (b) $\cos\left(x + \dfrac{n\pi}{2}\right)$

(c) $n\sin\left(x + \dfrac{\pi}{2}\right)$ (d) 0

5. The Leibnitz theorem for the n^{th} derivative is true, if n is

(a) positive integer (b) negative integer
(c) zero (d) all of above

6. The Liebnitz theorem is applied to find the n^{th} derivative, when

(a) product of the functions is given
(b) difference of two functions is given
(c) sum of two functions is given
(d) division of two functions is given

7. If $y = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$ then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{1}{1+x^2}$ (b) $\dfrac{-1}{1+x^2}$

(c) $\dfrac{1}{1+x}$ (d) $\dfrac{1}{1-x}$

8. The n^{th} derivative of $y = x^{n-1}\cdot\log x$ at $x = \dfrac{1}{2}$ is

(a) $(n-1)!$ (b) $2(n-1)!$
(c) $(n-2)!$ (d) $\dfrac{(n-1)!}{2}$

9. If $y = \log x^3$ then $\dfrac{d^n y}{dx^n}$ is equal to

(a) $(-1)^{n-1}\,n!\,x^{-n}$
(b) $3(n!)\,x^{-n}$
(c) $(-1)^{n-1}\,3(n-1)\,x^{-n}$
(d) none of the above

10. The n^{th} derivative of log $(x + 1)$ is

(a) $\dfrac{(-1)^{n-1}(n-1)!}{(x+1)^n}$ (b) $\dfrac{(-1)^{n-2}(n-2)!}{(x+1)^n}$

(c) $\dfrac{(-1)^n\,n!}{(x+1)^{n+1}}$ (d) $\dfrac{(-1)^{n-1}(n-1)!}{(x+1)^{-n+1}}$

ANSWERS

1. (b)	2. (c)	3. (b)	4. (a)	5. (a)	6. (a)	7.(b)	8. (b)	9. (c)	10. (a)

3 Partial Differentiation

3.1 INTRODUCTION

The partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant.

For example, in the package on introductory differentiation, rates of change of functions were shown to be measured by the derivative. Many applications require functions with more than one variable.

The ideal gas law is

$$PV = kT \tag{1}$$

where P is the pressure, V is the volume, T is the absolute temperature of gas, and k is a constant.

Equation (1) can be written as

$$P = \frac{kT}{V} \tag{2}$$

Equation (2) shows that P is a function of T and V.

If one of the variables, say T, is fixed and V changes then the derivative of P w.r.t. V measures the rate of change of pressure w.r.t. volume (V) and it is denoted by $\frac{\partial P}{\partial V}$.

Similarly, the rate of change of P w.r.t. T, keeping V as a constant is denoted by $\frac{\partial P}{\partial T}$.

Note: Partial derivatives are very useful in vector calculus and differential geometry.

3.2 PARTIAL DERIVATIVES OF FIRST ORDER

Consider $z = f(x, y)$ be a function of two independent variables, x and y.

The derivative of z, with respect to x, y is kept constant, is called the *partial derivative of z w.r.t. x* and is denoted by $\frac{\partial z}{\partial x}$ or z_x or f_x or $\frac{\partial f}{\partial x}$. It is defined as

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative of z, with respect to y, with x kept as a constant, is called the *partial derivative of z w.r.t. y* and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y or z_y. It is defined as

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Thus, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called first-order partial derivatives of z.

3.3 GEOMETRIC INTERPRETATION OF PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables which represents a surface in three-dimensional space. Calculate the partial derivatives z_x and z_y at the point (x_1, y_1). Then z_x is the slope (along the x-axis) of the line l_1, which is a tangent to the surface at the point $[x_1, y_1, f(x_1, y_1)]$, and z_y is the slope (along the y-axis) of the line l_2, which is a tangent to the surface at the point $[x_1, y_1, f(x_1, y_1)]$.

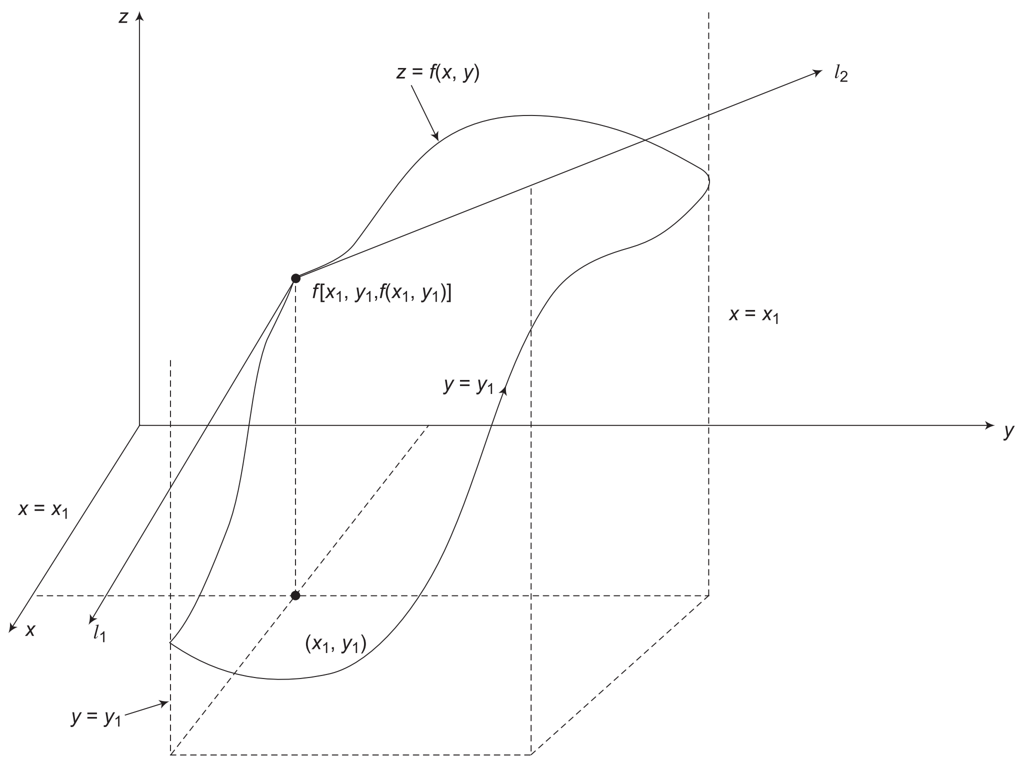


Fig. 3.1

3.4 PARTIAL DERIVATIVES OF HIGHER ORDERS

Let $z = f(x, y)$ be a function of x and y. If $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ exist then they can be further differentiated partially w.r.t. x or/and y. Thus,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx} \text{or } f_{xx} = \lim_{h\to 0}\frac{f_x(x+h,y)-f_x(x,y)}{h}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy} \text{or } f_y = \lim_{k\to 0}\frac{f_y(x,y+k)-f_y(x,y)}{k}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy} \text{or } f_{xy,} = \lim_{h\to 0}\frac{f_y(x+h,y)-f_y(x,y)}{h}$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{yx} \text{or } f_{yx,} = \lim_{k\to 0}\frac{f_x(x,y+k)-f_x(x,y)}{k}$$

Remark I: If $f(x,y)$ and its partial derivatives are continuous then

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}, \text{ i.e., } f_{xy} = f_{yx}$$

Example 1 Find the first-order partial derivatives of the function $u = \log(x^2 + y^2)$

Solution Given $u = \log(x^2 + y^2)$ (1)

Differentiating (1) partially w.r.t. 'x' treating 'y' as a constant, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}\cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$$

Again differentiating (1) partially w.r.t. 'y' treating x as a constant, we get

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

Example 2 If $z = e^{ax+by}\cdot f(ax - by)$, show that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Solution Given $Z = e^{ax+by}\cdot f(ax - by)$ (1)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial z}{\partial x} = e^{ax+by}\cdot f'(ax-by)\cdot a + f(ax-by)e^{ax+by}\cdot a$$

$$\frac{\partial z}{\partial x} = a\, e^{ax+by}[f'(ax-by)+f(ax-by)]$$

$$\therefore \quad b\frac{\partial z}{\partial x} = abe^{ax+by}[f'(ax-by)+f(ax-by)] \tag{2}$$

Again differentiating (1) partially, w.r.t. 'y', we get

$$\frac{\partial z}{\partial y} = e^{ax+by} \cdot f'(ax-by)\cdot(-b) + f(ax-b)]\cdot e^{ax+by}\cdot b$$

$$\frac{\partial z}{\partial y} = b\, e^{ax+by}[f'(ax-by)+f(ax-by)]$$

$$\therefore \qquad a\frac{\partial z}{\partial y} = abe^{ax+by}[-f'(ax-by)+f(ax-by)] \qquad (3)$$

Adding equations (2) and (3), we get

$$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abz.$$

Example 3 If $u = \sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$ then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Solution

$$u = \sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right) \qquad (1)$$

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{1}{y}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

$$\therefore \qquad x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2} \qquad (2)$$

Again differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = -\frac{x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

$$\therefore \qquad y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2} \qquad (3)$$

Adding (2) and (3), we get

$$x\frac{\partial y}{\partial x}+y\frac{\partial y}{\partial y}=0$$

Example 4 Prove that $y = f(x + at) + g(x - at)$ satisfies $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, where f and g are assumed to be at least twice differentiable and a is any constant.

Solution $y = f(x + at) + g(x - at)$ (1)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial y}{\partial x} = f'(x + at) + g'(x - at)$$

and $$\frac{\partial^2 y}{\partial x^2} = f''(x + at) + g''(x - at) \tag{2}$$

Again differentiating (1) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t} = f'(x + at) \cdot a + g'(x - at) \cdot (-a)$$

and $$\frac{\partial^2 y}{\partial t^2} = f''(x + at)\,(a)^2 + g''(x - at) \cdot (-a)^2$$

$$\frac{\partial^2 y}{\partial t^2} = a^2[f''(x + at) + g''(x - at)] \tag{3}$$

From (2) and (3),

$$\frac{\partial^2 y}{\partial t^2} = a^2 \cdot \frac{\partial^2 y}{\partial x^2}$$

Hence, proved.

Example 5 If $u = e^{xyz,}$ then find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

Solution $u = e^{xyz}$ (1)

Differentiating (1) partially, w.r.t. z, treating x and y as constants, we get

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot (xy)$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(e^{xyz} \cdot xy)$$

$$= e^{xyz} \cdot x + e^{xyz}(xy) \cdot (xz)$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}(x + x^2 yz)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left[e^{xyz} \cdot yz(x + x^2 yz)\right]$$

$$= e^{xyz} \cdot (1 + 2xyz) + e^{xyz} \cdot yz(x + x^2 yz)$$

$$= e^{xyz}[1 + 2xyz + xyz + x^2 y^2 z^2]$$

$$\frac{\partial^3 z}{\partial x \partial y \partial z} = e^{xyz}[1 + 3xyz + x^2y^2z^2]$$

Example 6 If $x^x y^y z^z$ = constant (k), show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$

Solution Given $x^x y^y z^z = k$ (1)

Here, z can be regarded as a function of x and y. Taking logarithm on both sides of (1), we get

$$x \log x + y \log y + z \log z = \log k \tag{2}$$

Since z is a function of x and y, differentiating (2) partially w.r.t. x taking 'y' as constant, we get

$$x \cdot \frac{1}{x} + \log x + \left[z \cdot \frac{1}{z} + \log z \right] \frac{\partial z}{\partial x} = 0$$

or $$(1 + \log x) + (1 + \log z) \frac{\partial z}{\partial x} = 0$$

$\therefore$ $$\frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \tag{3}$$

Similarly, differentiating (2) partially w.r.t. 'y', we get

$$(1 + \log y) + (1 + \log z) \frac{\partial z}{\partial y} = 0$$

or $$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{1 + \log z} \tag{4}$$

Differentiating (4) partially w.r.t. 'x', we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y} \right)$$

$$= \frac{\partial}{\partial x}\left[-\frac{1 + \log y}{1 + \log z} \right]$$

$$= -(1 + \log y) \cdot \frac{\partial}{\partial x}\left(\frac{1}{1 + \log z} \right)$$

$$= -(1 + \log y)\left[-(1 + \log z)^{-2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x} \right]$$

$$= \frac{(1 + \log y)}{z(1 + \log z)^2} \cdot \left[-\frac{1 + \log x}{1 + \log z} \right] \qquad \text{[By Eq. (3)]}$$

At $x = y = z$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x(\log e + \log x)} = -(x \log ex)^{-1}$$ **Hence, proved.**

Example 7 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Solution Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$ (1)

Differentiating (1) partially w.r.t. x, y and z, we get

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{(x^3 + y^3 + z^3 - 3xyz)} \tag{2}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \tag{3}$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \tag{4}$$

Adding (2), (3) and 4, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} u = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$$

[$\because x^3 + y^3 + z^3 - 3xyz = (x + y + z)\cdot\{x^2 + y^2 + z^2 - xy - yz - zx\}$]

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[\frac{-1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{-1}{(x+y+z)^2}\right]$$

$$= 3 \cdot \frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$ **Hence, proved.**

Example 8 If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Solution Given $r^2 = x^2 + y^2$ (1)

Differentiating (1) partially w.r.t. x and y, we have

$$2r\frac{\partial r}{\partial x} = 2x \text{ and } 2r\frac{\partial r}{\partial y} = 2y$$

or
$$\frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r}$$

But $u = f(r)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)$$

$$= \frac{1}{r} f'(r) + xf'(r)\left[-\frac{1}{r^2}\frac{\partial r}{\partial x}\right] - \frac{x}{r} f''(r)\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \quad (2)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r) \quad (3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{2}{r} f'(r) - \frac{x^2 + y^2}{r^3} f'(r) + \frac{x^2 + y^2}{r^2} f''(r)$$

$$= \frac{2}{r} f'(r) - \frac{1}{r} f'(r) + f''(r)$$

$$= f''(r) + \frac{1}{r} f'(r)$$

Hence, proved.

Example 9 If $z(x + y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Solution Here
$$z = \frac{x^2 + y^2}{x + y}$$

We have
$$\frac{\partial z}{\partial x} = \frac{(x + y)\cdot 2x - (x^2 + y^2)\cdot 1}{(x + y)^2} = \frac{x^2 - y^2 + 2xy}{(x + y)^2}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y^2 - x^2 + 2xy}{(x+y)^2}$$

Now, $$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2 - y^2)}{(x+y)^2} = \frac{2(x-y)}{(x+y)}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \tag{1}$$

Again, $$1 - \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 1 - \frac{4xy}{(x+y)^2}$$

$$= \frac{(x+y)^2 - 4xy}{(x+y)^2}$$

$$= \frac{x^2 + y^2 - 2xy}{(x+y)^2}$$

$$= \frac{(x-y)^2}{(x+y)^2}$$

$$\therefore \quad 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = \frac{4(x-y)^2}{(x+y)^2} \tag{2}$$

From (1) and (2), we have

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{dz}{\partial y}\right]$$ **Hence, proved.**

Example 10 If $u = \log(\tan x + \tan y + \tan z)$, prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2x \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Solution Here, $u = \log(\tan x + \tan y + \tan z)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \cdot \sec^2 x}{\Sigma \tan x}$$

or $$\sin 2x \frac{\partial u}{\partial x} = \frac{2 \sin x \ \cos x \cdot \dfrac{1}{\cos^2 x}}{\Sigma \tan x}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2 \tan x}{\Sigma \tan x} \tag{1}$$

Similarly,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\Sigma \tan y} \tag{2}$$

and

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\Sigma \tan z} \tag{3}$$

Adding (1), (2), and (3), we get

$$\begin{aligned}\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + 2z \frac{\partial u}{\partial z} &= \frac{2(\tan x + \tan y + \tan z)}{\Sigma \tan x} \\ &= \frac{2\, \Sigma \tan x}{\Sigma \tan x} \\ &= 2\end{aligned}$$

Hence, proved.

EXERCISE 3.1

1. Verify $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$ for the following functions:

   (i) $u = \tan^{-1}\left(\dfrac{x}{y}\right)$

   (ii) $u = \sin^{-1}\left(\dfrac{x}{y}\right)$

   (iii) $u = x \log y$

   (iv) $u = \log_e \dfrac{x^2 + y^2}{x + y}$

   (v) $u = ax^2 + by^2 + 2hxy$

   (vi) $u = \sin (x \cdot y^2)$

2. If $u = x^2 y + y^2 z + z^2 x$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2 .$$

3. If $u = \log (x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4. If $u = x^3 y - xy^3$, show that

$$\left[\frac{1}{u_x} + \frac{1}{u_y}\right]_{\substack{x=1\\y=2}} = -\frac{13}{22} .$$

5. If $u = \log (x^2 + y^3 - x^2y - xy^2)$, show that

$$u_{xx} + 2\, u_{xy} + u_{yy} = -4(x + y)^{-2}$$

6. If $u(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, show that

$$u_x + u_y + u_z = 0$$

7. If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial x} = \dfrac{\partial^3 u}{\partial x^2\, \partial y}$

8. If $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right]$$

9. If $z = y \cdot f(x^2 - y^2)$, show that

$$y \cdot z_x + x \cdot z_y = \frac{xz}{y}.$$

10. If $z = (x^2 - y^2) \cdot f(xy)$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (x^4 - y^4)\, f''(xy)$$

11. If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that $x\, u_x + y\, u_y + z\, u_z = 0$

12. If $u = \log \sqrt{x^2 + y^2 + z^2}$, show that

$$(x^2 + y^2 + z^2)\,(u_{xx} + u_{yy} + u_{zz}) = 1$$

13. If $u = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$

14. If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial z^2} = \frac{2}{r}\, f'(r) + f''(r)$$

3.5 HOMOGENEOUS FUNCTION

A function is called homogeneous if the degree of each term is same, i.e., an expression of the form

$$f(x, y) = a_0\, x^n + a_1\, x^{n-1} \cdot y + a^2\, x^{n-2} \cdot y^2 + \ldots + any^n.$$

in which each term of n^{th} degree is called a *homogeneous function of degree n*.

Now, the above expression can be written as

$$f(x, y) = x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y}{x}\right)^2 + \cdots + a_n \left(\frac{y}{x}\right)^n \right]$$

or $$= x^n \phi\left(\frac{y}{x}\right)$$

Thus, the function $f(x, y)$ which can be expressed as $x^n \phi(y/x)$ or $y^n \psi(x/y)$ is called a homogeneous function of degree n in the variables x and y.

Note: A function $f(x, y)$ is said to be a homogeneous function of degree 'n' in the variables x and y if

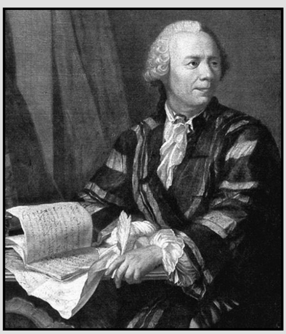
$$f(tx, ty) = t^n f(x, y).$$

Example: Function $f(x, y) = \cos^{-1}(x/y)$ is a homogeneous function of degree 0.

Example: Function $f(x, y) = x^5 + 2xy^4 + x^2y^3 + 3x^3y^2 + x^4y + y^5$ is a homogeneous function of degree 5.

Example: If $f(x, y)$ is a homogeneous function of degree n in the variables x and y then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are homogeneous functions of x and y of degree $(n - 1)$ each.

3.6 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS



Leonhard Euler (15 April 1707–7 September 1783) was born in Basel, Switzerland. A great mathematician and a brilliant physicist, Euler is best described as a genius. Euler's father was a friend of Johann Bernoulli, a renowned European mathematician who had a great influence on Euler. At thirteen, Euler entered the University of Basel and received his Master of Philosophy in 1723. Euler was a leading Swiss mathematician and physicist who made significant discoveries in varied fields such as graph theory and infinitesimal calculus. He also defined a substantial amount of contemporary mathematical terminology and notation, specifically for mathematical analysis, such as the notion of a mathematical function. He is also well renowned for his works in fluid dynamics, astronomy, mechanics, and optics. Euler worked on his subjects mostly in St. Petersburg (Russia) and in Berlin, during his adulthood. All his collections, if printed, would occupy 60-80 quarto volumes reflecting his outstanding mathematical abilities. Euler was a versatile mathematician who worked on a spectrum of topics such as number theory, algebra, geometry, and trigonometry and in different areas of physics such as continuum physics, lunar theory, etc. Contrary to Monadism and Wolffian science, Euler firmly believed in knowledge found on the basis of exact quantitative laws.

Theorem 1

If u is a homogeneous function in the variables x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof Since u is a homogeneous function of x and y of degree n then u can be written as

$$u = x^n \phi\left(\frac{y}{x}\right) \tag{3}$$

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial u}{\partial x} = x^n \cdot \phi'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) + nx^{n-1}\,\phi\left(\frac{y}{x}\right)$$

or
$$\frac{\partial u}{\partial x} = -y\,x^{n-2}\,\phi'\left(\frac{y}{x}\right) + nx^{n-1}\,\phi\left(\frac{y}{x}\right)$$

Multiplying both sides by 'x', we have

$$x\frac{\partial u}{\partial x} = -y\,x^{n-1}\,\phi'\left(\frac{y}{x}\right) + nx^{n}\,\phi\left(\frac{y}{x}\right) \quad (4)$$

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial u}{\partial y} = x^n\,\phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

or
$$\frac{\partial u}{\partial y} = x^{n-1}\,\phi'\left(\frac{y}{x}\right)$$

Multiplying both sides by 'y', we have

$$y\frac{\partial u}{\partial y} = y\,x^{n-1}\,\phi'\left(\frac{y}{x}\right) \quad (5)$$

Adding (4) and (5), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot x^n\,\phi\left(\frac{y}{x}\right)$$

or
$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u}$$

Hence, proved.

Generalized Form

Euler's theorem can be extended to a homogeneous function of any number of variables. If $u(x_1, x_2, x_3, \ldots, x_n)$ be a homogeneous function of degree n in the variables $x_1, x_2, x_3, \ldots, x_n$ then

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + x_3\frac{\partial u}{\partial x_3} + \cdots + x_n\frac{\partial u}{\partial x_n} = n\cdot u.$$

3.7 RELATIONS BETWEEN SECOND-ORDER DERIVATIVES OF HOMOGENEOUS FUNCTIONS

Theorem 2

If u is a homogeneous function of degree n then

(i) $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}$$

(ii) $$x \frac{\partial^2 u}{\partial x \, \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y}$$

(iii) $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)\, u$$

Proof Since u is a homogeneous function in the variables x and y of degree n then by Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \tag{6}$$

Differentiation (6) partially both sides w.r.t. 'x', we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x}$$

or $$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1) \frac{\partial u}{\partial x} \tag{7}$$

Again, differentiating (6) partially both sides w.r.t. 'y', we get

$$x \frac{\partial^2 u}{\partial y \, \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n \frac{\partial u}{\partial y}$$

or $$x \frac{\partial^2 u}{\partial x \, \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y} \tag{8}$$

Multiplying (7) by x and (8) by y and then adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y^2} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$= n(n-1) \cdot u \qquad \text{[Using (6)]}$$

Hence, proved.

3.8 DEDUCTION FROM EULER'S THEOREM

Theorem 3

If z is a Homogeneous Function in the variables x and y of degree n and $z = f(u)$ then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

Proof Since z is a homogeneous function in the variables x and y of degree n then by Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \tag{9}$$

Now, $z = f(u)$

$$\therefore \qquad \frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

Then (9) becomes

$$x\,f'(u)\frac{\partial u}{\partial x} + y\,f'(u)\frac{\partial u}{\partial y} = n\,f(u)$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} \qquad (10)$$ **Hence, proved.**

Theorem 4

If z is a homogeneous function in the variables x and y of degree n and $z = f(u)$ then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)\,[\phi'(u) - 1]$$

where $f(u) = n\dfrac{f(u)}{f'(u)}$.

Proof From Theorem 3, let $\phi(u) = \dfrac{n\,f(u)}{f'(u)}$; then Eq. (10), becomes

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \phi(u) \qquad (11)$$

Differentiating (11) partially w.r.t. 'x', we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 y}{\partial y^2} = \phi'(u)\cdot\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = [\phi'(u) - 1]\frac{\partial u}{\partial x} \qquad (12a)$$

Similarly, differentiating (11) partially w.r.t. 'y', we get

$$x\frac{\partial^2 u}{\partial y\,\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \phi'(u)\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial x\,\partial x} + y\frac{\partial^2 u}{\partial y^2} = [\phi'(u) - 1]\frac{\partial u}{\partial y} \qquad (12b)$$

Multiplying (12) by x and (13) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [\phi'(u) - 1]\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= [\phi'(u) - 1]\,.\,\phi(u) \quad \text{[From (11)]}$$

Example 11 If $u = \dfrac{x^3 + y^3}{x + y}$, apply Euler's theorem to find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

Solution

$$u(x, y) = \frac{x^3 + y^3}{x + y}$$

$$u(tx, ty) = \frac{t^3(x^3 + y^3)}{t(x + y)}$$

$$= t^2\left(\frac{x^3 + y^3}{x + y}\right) = t^2\, u(x, y)$$

$\therefore$ u is a homogeneous function in x and y of degree 2.
Hence, by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

Example 12 If $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$, prove that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$
$= 2\cos 3u \cdot \sin u$

Solution Here, the given function u is not a homogeneous function.

Let
$$\tan u = \frac{x^3 + y^3}{x - y} = f(x, y) \tag{1}$$

Now,
$$f(tx, ty) = \frac{t^3(x^3 + y^3)}{t(x - y)}$$
$$= t^2 f(x, y)$$

Hence, $f(x, y) = \tan u$ is a homogeneous function of degree 2.

(i) By Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

or
$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y} = (\tan u) = 2\tan u$$

or
$$x\sec^2 u\,\frac{\partial u}{\partial x} + y\sec^2 u\,\frac{\partial u}{\partial y} = 2\tan u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= \frac{2\sin u}{\cos u}\cdot\cos^2 u$$

$$= 2\sin u\cos u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u$$ **Hence, proved.**

(ii) Using Theorem 4,

let $\phi(u) = \sin 2u$

$\therefore$ $\phi'(u) = 2\cos 2u$

Hence,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)\,[\phi'(u)-1]$$

$$= \sin 2u\,(2\cos 2u-1)$$

$$= 2\cos 2u\sin 2u-\sin 2u$$

$$= \sin 4u-\sin 2u$$

$$= 2\cos 3u\,.\,\sin u$$ **Hence, proved.**

Example 13 If $u=\sin^{-1}\left(\dfrac{x+2y+3z}{\sqrt{x^8+y^8+x^8}}\right)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-3\tan u$$

Solution Here, u is not a homogeneous function.

Let $$f=\sin u=\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+x^8}}\right) \tag{1}$$

$\therefore$ $$f(tx,ty,tz)=\frac{t(x+2y+3z)}{t^4\sqrt{x^8+y^8+x^8}}=t^{-3}f(x,y,z)$$

Hence, $f(x, y, z) = \sin u$ is a homogeneous function in the variables x, y, z of degree –3.
By Euler's theorem, we have

$$x\frac{\partial}{\partial x}f+y\frac{\partial}{\partial y}f+z\frac{\partial}{\partial z}f=-3f$$

or $$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)+z\frac{\partial}{\partial z}(\sin u)=-3\sin u$$

or $$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} z \cos u \frac{\partial u}{\partial z} = -3 \sin u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} z \frac{\partial u}{\partial z} = -\frac{3 \sin u}{\cos u}$$
$$= -3 \tan u$$

Hence, proved.

Example 14 If $u = \log_e \left(\frac{x^4 + y^4}{x + y} \right)$, show that $x \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial y} = 3$

Solution Here u is not a homogeneous function

Let $$f(x, y) = e^u = \frac{x^4 + y^4}{x + y} \tag{1}$$

$\therefore$ $$f(tx, ty) = \frac{t^4 (x^4 + y^4)}{t (x + y)} = t^3 \cdot \left(\frac{x^4 + y^4}{x + y} \right) = t^3 \cdot f(x, y)$$

Hence, $f(x, y) = e^u$ is a homogeneous function in the variables x and y of degree 3.
By Euler's theorem, we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3 \cdot f$$

or $$x \frac{\partial}{\partial x} (e^u) + y \frac{\partial}{\partial y} (e^u) = 3 \cdot e^u$$

or $$x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 3 \cdot e^u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$$

Example 15 If $u = \sin^{-1} \left(\frac{x^3 + y^3 + z^3}{ax + by + cz} \right)$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial x} = 2 \tan u$$

Solution Do same as Example 12.

Example 16 If $u = log \left[\frac{x^2 + y^2}{\sqrt{x} + \sqrt{y}} \right]$, find the value of

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Solution Given function u is not a homogeneous function.

We have $$e^u = \frac{x^2 + y^2}{\sqrt{x} + \sqrt{y}} = f(x, y)$$

$$\therefore \quad f(tx, ty) = \frac{t^2 (x^2 + y^2)}{t^{1/2} (\sqrt{x} + \sqrt{y})} = t^{3/2} \left(\frac{x^2 + y^2}{\sqrt{x} + \sqrt{y}} \right) = t^{3/2} \cdot f(x, y)$$

Thus, $f(x, y)$ is a homogeneous function in x and y of degree $\frac{3}{2}$.

(i) By Euler's theorem,

$$x \frac{\partial f}{\partial u} + y \frac{\partial f}{\partial y} = \frac{3}{2} f$$

or $$x \frac{\partial}{\partial x} e^u + y \frac{\partial}{\partial y} e^u = \frac{3}{2} e^u$$

or $$x\, e^u \frac{\partial u}{\partial x} + y\, e^u \frac{\partial u}{\partial y} = \frac{3}{2} e^u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2}$$

(ii) Here, $f(u) = \frac{3}{2}$, $f'(u) = 0$

Using the deduction of Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = f(u)\,.\,[f'(u) - 1]$$

$$= \frac{3}{2} [0 - 1]$$

$$= \frac{-3}{2}$$

Example 17 If $u = \tan^{-1} \left(\frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}} \right)$, find the value of

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial xy} + y^2 \frac{\partial^2 u}{\partial y^2}$

Solution Do same as Example 19.

(i) $\frac{5}{4} \sin 2u$

(ii) $\dfrac{25}{16}\sin 4u - \dfrac{5}{4}\sin 2u$

Example 18 If $\log u = \left(\dfrac{x^3 + y^3}{3x + 4y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \cdot \log u$$

Solution Left as an exercise to the student.

EXERCISE 3.2

1. Verify Euler's theorem in each case:

   (i) $u = (x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})$

   (ii) $u = \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

   (iii) $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$

   (iv) $u = x^2 \log\left(\dfrac{y}{x}\right)$

   (v) $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$

   (vi) $u = ax^2 + by^2 + 2h\,xy$

   (vii) $u = 3x^2 yz + 5xy^2 z + 4z^4$

   (viii) $u = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$

2. If $u(x, y) = x^4 y^2 \sin^{-1}\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 6.u$

3. If $u = x^2 + y^3 + z^3 + 3xyz$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = z\dfrac{\partial u}{\partial z} = 3 \cdot u$

4. If $u = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$

5. If $u = \log\left(\dfrac{x^2 + y^2}{x + y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$

6. If $u = \log\left(\dfrac{x^4 + y^4}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$

7. If $u = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + 2u = 0$

8. If $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2\cdot u$

9. If $u = x \sin^{-1}\left(\dfrac{x}{y}\right) - y \sin^{-1}\left(\dfrac{y}{x}\right)$, show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$

10. If $u = \sec^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$, prove that

 (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2 \cot u$

 (ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -2 \cot u\,[\,2 \operatorname{cosec}^2 u + 1]$

11. If $u = \sin^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, prove that

 (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2} \tan u$

 (ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cdot \cos 2u}{4 \cos^3 u}$

12. If $u = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

13. If $u = \log\left(\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$

14. State and prove Euler's theorem for partial differentiation of a homogeneous function $u(x, y)$.

15. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(\tan^2 u - 11)$

3.9 COMPOSITE FUNCTION

Let $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$ then u is called a composite function of the variable 't'

$$\therefore \qquad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Also, if $w = f(x, y)$ where $x = \phi(u, v)$ and $y = \psi(u, v)$ then w is called a composite function of the variables u and v so that we can obtain $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

and
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Hence, $\frac{du}{dt}$ is called the total derivative of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Remark I: If $u = f(x, y, z)$ and x, y, z are functions of t then u is a composite function of t and is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

Remark II: If $u = f(x, y, z)$ and y, z are function of x then

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}$$

or
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

Remark III: If we are given an implicit function $f(x, y)$ = constant then $u = f(x, y)$ where u = constant.

We know that $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

Given u = constant $\Rightarrow \frac{du}{dx} = 0$

$\therefore$
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \text{ or } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

or
$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Remark IV: If $f(x, y)$ = constant then from Remark III, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \tag{13}$$

Differentiating (13) w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = -\left[\frac{f_y\cdot\frac{d}{dx}(f_x) - f_x\cdot\frac{d}{dx}(f_y)}{f_y^2}\right]$$

$$= -\frac{f_y \cdot \left[\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y} \cdot \frac{dy}{dx}\right] - f_x \left[\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y} \cdot \frac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y \cdot \left[f_{xx} - f_{yx} \cdot \frac{f_x}{f_y}\right] - f_x \left[f_{xy} - f_{yy} \cdot \frac{f_x}{f_y}\right]}{f_y^2}$$

$$\frac{d^2 y}{dx^2} = \left[\frac{f_{xx} f_y^2 - 2 f_x f_y\, f_{xy} + f_{yy} f_x^2}{f_y^3}\right]$$

Example 19 If $u = x \log (xy)$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

Solution Given $u = x \log (xy)$ (1)

and $x^3 + y^3 + 3xy = 1$ (2)

We know that $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$ (3)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial u}{\partial x} = \log xy + x \cdot \frac{1}{xy} \cdot y = 1 + \log xy \tag{4}$$

Again, differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} \cdot x = \frac{x}{y} \tag{5}$$

Now, differentiating (2) w.r.t. 'x', we get

$$3x^2 + 3y^2 \frac{dy}{dx} + 3\left[x \frac{dy}{dx} + y\right] = 0$$

or

$$\frac{dy}{dx} = -\left(\frac{x^2 + y}{x + y^2}\right) \tag{6}$$

Substituting the value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{dy}{dx}$ in (3), we obtain

$$\frac{du}{dx} = (1 + \log xy) + \frac{x}{y} \cdot \left[-\frac{x^2 + y}{x + y^2}\right]$$

$$= (1 + \log xy) - \frac{x}{y}\left(\frac{x^2 + y}{x + y^2}\right)$$

Example 20 If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0$.

Solution Given $u = f(X, Y, Z)$, where $X = y - z$, $Y = z - x$ and $Z = x - y$.

Since u is a composite function of x, y and z,

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}\cdot 0 + \frac{\partial u}{\partial Y}\cdot(-1) + \frac{\partial u}{\partial Z}\cdot(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$

$$= \frac{\partial u}{\partial X}\cdot(1) + \frac{\partial u}{\partial Y}\cdot(0) + \frac{\partial u}{\partial Z}\cdot(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \tag{2}$$

and
$$\frac{\partial u}{\partial Z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}\cdot(-1) + \frac{\partial u}{\partial Y}\cdot(1) + \frac{\partial u}{\partial z}\cdot(0)$$

$$\frac{\partial u}{\partial Z} = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \tag{3}$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Example 21 If $f(x, y) = 0$, $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$$

Solution We know that

$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} \tag{1}$$

Given
$$f(x, y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

and
$$f(y, z) = 0 \Rightarrow \frac{dz}{dy} = -\frac{\partial \phi/\partial y}{\partial \phi/\partial z}$$

$\therefore$ Eq. (1), becomes

$$\frac{dz}{dx} = \left(-\frac{\partial\phi/\partial y}{\partial\phi/\partial z}\right)\cdot\left(-\frac{\partial f/\partial x}{\partial f/\partial y}\right)$$

or $$\frac{\partial f}{\partial y}\cdot\frac{\partial\phi}{\partial z}\cdot\frac{dz}{dx} = \frac{d\phi}{\partial y}\cdot\frac{\partial f}{\partial x}$$ **Hence, proved.**

Example 22 If $u = x^2y$ and the variables x, y are connected by the relation $x^2 + y^2 + xy = 1$, find $\frac{du}{dx}$.

Solution Given $u = x^2y$ (1)

and $$x^2 + y^2 + xy = 1 \tag{2}$$

We know that $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \tag{3}$$

Differentiating (1) partially both sides w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = 2xy \text{ and } \frac{\partial u}{\partial y} = x^2$$

Again, differentiating (2) w.r.t. 'x', we get

$$2x + 2y\frac{dy}{dx} \quad x\frac{dy}{dx} + y = 0$$

or $$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

$\therefore$ Eq. (3), becomes

$$\frac{du}{dx} = 2xy + x^2\cdot\left(-\frac{2x + y}{x + 2y}\right)$$

$$\frac{du}{dx} = 2xy - \frac{x^2(2x + y)}{(x + 2y)}$$

Example 23 If $f(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z = -1$.

Solution Given $f(x, y, z) = 0$ (1)

$\left(\frac{\partial y}{\partial z}\right)_x$ means the partial derivative of y w.r.t. 'z', treating x as a constant.

$\therefore$ $$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} \tag{2}$$

Similarly, $$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial z} \tag{3}$$

and $$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial f/\partial y}{\partial f/\partial x} \qquad (4)$$

Multiplying (2), (3), and (4), we get

$$\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$$

Hence, proved.

EXERCISE 3.3

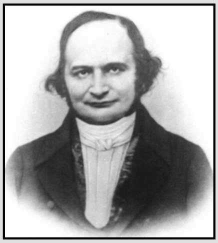
1. If $u = x^2 - y^2 + \sin yz$, where $y = e^x$ and $z = \log x$, find $\frac{du}{dx}$.
2. If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$
3. If $u = f(r, s, t)$, where $r = \frac{x}{y}$, $s = \frac{y}{z}$ and $t = \frac{z}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$
4. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.
5. If $z = f(x, y)$, where $x = e^u \cos v$ and $y = e^u \sin v$, show that $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.
6. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{(1-t^2)}}$.
7. If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$, find $\frac{du}{dx}$.
8. If $u = x^2 + y^2$, where $x = at^2$, $y = 2\,at$, find $\frac{du}{dt}$.
9. If x increases at the rate of 2 cm/s at the instant when $x = 3$ cm and $y = 1$ cm, at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?
10. If $x^y + y^x = a^b$, find $\frac{dy}{dx}$.

Answers

1. $(x - e^{2x}) + e^x \cos(e^x \log x)\left(\log x + \frac{1}{x}\right)$
7. $2\left(\frac{b^2 - a^2}{b^2}\right) x \cdot \cos(x^2 + y^2)$
8. $4a^2t\,(t^2 + 2)$
9. y is decreasing at the rate of $\frac{32}{21}$ cm/s
10. $-\left(\frac{y\,x^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$

3.10 JACOBIAN

A Jacobian is a functional determinant (whose elements are functions). It is very useful in the transformation of variables from Cartesian to polar, and cylindrical and spherical coordinates in multiple integral. The Jacobian is named after the German mathematician Carl Gustav Jacob-Jacobi.



Karl Gustav Jacob Jacobi (10 December 1804–18 February 1851) was born in Potsdam, Prussia (Germany). Karl Jacobi founded the theory of elliptic functions. His official biography says, "Carl Jacobi came from a Jewish family but he was given the French name Jacques Simon". He founded the theory of elliptic functions based on four theta functions. His *Fundamenta nova theoria functionum ellipticarum* in 1829 and its later supplements made basic contributions to the theory of elliptic functions. In 1834, Jacobi proved that if a single-valued function of one variable is doubly periodic then the ratio of the periods is imaginary. This result prompted much further work in this area, in particular by Liouville and Cauchy. Jacobi carried out important research in partial differential equations of the first order and applied them to the differential equations of dynamics. He also worked on determinants and studied the functional determinant now called the Jacobian. Jacobi was not the first to study the functional determinant which now bears his name; it appears first in a 1815 paper of Cauchy. However, Jacobi wrote a long memoir *De determinantibus functionalibus* in 1841 devoted to this determinant. He proves, among many other things, that if a set of *n* functions in *n* variables are functionally related then the Jacobian is identically zero, while if the functions are independent the Jacobian cannot be identically zero.

Let u and v be given functions of two independent variables x and y.

The Jacobian of u, v w.r.t. x, y denoted by $\dfrac{\partial(u, v)}{\partial(x, y)}$ of $J(u, v)$ is a second order functional determinant defined as

$$\frac{\partial(u, v)}{\partial(x, y)} = J(u, v) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Similarly, the Jacobian of three functions u, v, w of three independent variables x, y, z is defined as

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = J(u, v, w) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

In a similar manner, we can define the Jacobian of n functions $u_1, u_2, u_3, \ldots, u_n$ of n independent variables $x_1, x_2, x_3, \ldots, x_n$;

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = J(u_2, u_3, \ldots, u_n)$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

3.11 IMPORTANT PROPERTIES OF JACOBIANS

(i) If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

Proof Since u and v are composite functions of x, y, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} \tag{14}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} \tag{15}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial x} \tag{16}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial y} \tag{17}$$

From the definition of Jacobian,

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

By interchanging the rows and columns in the second determinant,

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix}$$

$$= \begin{vmatrix} u_r\, r_x + u_s \cdot s_x & u_r\, r_y + u_s \cdot s_y \\ v_r\, r_x + v_s s_x & v_r\, v_y + v_s\, s_y \end{vmatrix} \qquad \text{[Multiplying the determinant row-wise]}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \qquad \text{[Using (14), (15), (16), and (17)]}$$

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}$$

Note: It can be extended to n(any number of) variables.

(ii) If $J_1 = \frac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \frac{\partial(x, y)}{\partial(u, v)}$, then $J_1 J_2 = 1$

i.e., $$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1$$

Proof Let $u = f(x, y)$ and $v = g(x, y)$ (18)
be two given functions in terms of x and y, which are transformable from u, v to x, y. Then x and y can be written is the form of

$$x = \phi(u, v) \text{ and } y = \psi(u, v) \tag{19}$$

Differentiation partially w.r.t u and v, we get

$$1 = \frac{\partial u}{\partial u} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} = u_x x_u + u_y\ y_u \tag{20}$$

$$0 = \frac{\partial u}{\partial v} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} = u_x x_v + u_y \cdot y_v \tag{21}$$

$$0 = \frac{\partial v}{\partial u} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} = v_x x_u + v_y\ y_u \tag{22}$$

$$1 = \frac{\partial v}{\partial v} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial v} = v_x x_v + v_y\ y_v \tag{23}$$

Now,

$$J_1 \cdot J_2 = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & y_u \\ y_v & y_v \end{vmatrix}$$ [Interchanging rows and columns in the second determinants].

$$= \begin{vmatrix} u_x\ x_u & u_y\ y_u \\ v_x\ x_u & v_y y_v \end{vmatrix} \begin{vmatrix} u_x\ x_v & u_y\ y_v \\ v_x\ x_v & v_y y_v \end{vmatrix}$$ [$\because$ Multiplying the determinant row-wise]

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$ [Using Eqs (3), (4), (5), and (6)]

$= 1.$

Hence, $\boldsymbol{J_1 J_2 = 1}$. **Hence, proved.**

3.12 THEOREMS ON JACOBIAN (WITHOUT PROOF)

Theorem 5

If the functions $u_1, u_2, u_3, \ldots, u_n$ of the variables $x_1, x_2, x_3, \ldots, x_n$ be defined by the relation

$u_1 = f_1(x_1)$, $u_2 = f_2(x_1, x_2)$, $u_3 = f_3(x_1, x_2, x_3)$, …, $u_n = f_n(x_1, x_2, x_3, \ldots, x_n)$; then

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3}, \cdots, \frac{\partial u_n}{\partial x_n}$$

Theorem 6

If the functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.

i.e., $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = J(u, v, w) = 0.$$

3.13 JACOBIANS OF IMPLICIT FUNCTIONS

Theorem 7

If u_1, u_2 be connected by the implicit relations of the forms

$$F_1(u_1, u_2, x_1, x_2) = 0$$

and $$F_2(u_1, u_2, x_1, x_2) = 0$$

then $$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2 \left[\frac{\partial(F_1, F_2)/\partial(x_1, x_2)}{\partial(F_1, F_2)/\partial(u_1, u_2)} \right]$$

In general, if $u_1, u_2, u_3, \ldots, u_n$ are connected with n independent variables $x_1, x_2, x_3, \ldots, x_n$ by the relation of the form

$$F_1(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0$$
$$F_2(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0$$
$$\cdots$$
$$\cdots$$
$$F_n(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0$$

then,

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} = (-1)^n \left[\frac{\partial(F_1, F_2, \cdots, F_n)}{\partial(x_1, x_2, \cdots, x_n)} \Big/ \frac{\partial(F_1, F_2, \cdots, F_n)}{\partial(u_1, u_2, \cdots, u_n)} \right]$$

3.14 FUNCTIONAL DEPENDENCE

Let $u = f(x, y)$, $v = g(x, y)$ be two given differentiable functions of two independent variables x and y. Suppose u and v are connected by a relation $F(u, v) = 0$, where $F(u, v)$ is differentiable. Then u and v are said to be functionally dependent on each other if the partial derivatives u_x, u_y, v_x and v_y are not all zero simultaneously.

3.14.1 Necessary and Sufficient Condition

The necessary and sufficient condition for functional dependence can be expressed in terms of a determinant as follows:

Differentiating $F(u, v) = 0$ partially w.r.t. x and y, we obtain

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}=0$$

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}=0$$

A nontrivial solution $\frac{\partial F}{\partial u}\neq 0,\ \frac{\partial F}{\partial v}\neq 0$ to this system exists if the coefficient determinant is zero,

i.e. $$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=0$$

Two functions u and v are functionally dependent if their Jacobian $\boldsymbol{J(u, v)=\frac{\partial(y, v)}{\partial(x, y)}=0}$.

If the Jacobian is not equal to zero then u and v are said to be functionally independent,

We can extend this concept more than two functions suppose three given functions u, v, w of three independent variables x, y, z connected by the relation $F(u, v, w) = 0$ are functionally dependent if Jacobian of u, v, w w.r.t. x, y, z is zero, i.e., $\frac{\partial(u, v, w)}{\partial(x, y, z)}=0$. Otherwise, they are functionally independent.

Note: m functions of n variables are always functionally dependent if $m > n$.

Example 24 Find the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$, where $u = e^x \sin y$, $v = x + \log \sin y$.

Solution Given $u = e^x \sin y$; $\qquad v = x + \log \sin y$

$$\therefore\quad \frac{\partial u}{\partial x}=e^x\sin y \qquad \frac{\partial v}{\partial x}=1$$

$$\frac{\partial u}{\partial y}=e^x\cos y \qquad \frac{\partial v}{\partial y}=\frac{\cos y}{\sin y}$$

Now, $$\frac{\partial(u, v)}{\partial(x, y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}e^x\sin y & e^x\cos y\\ 1 & \frac{\cos y}{\sin y}\end{vmatrix}$$

$$=e^x\cos y-e^x\cos y=0$$

Example 25 Calculate $J_1=\frac{\partial(u, v)}{\partial(r, \theta)}$ and $J_2=\frac{\partial(r, \theta)}{\partial(u, v)}$. Also verify $J_1 J_1 = 1$, where $u = r\cos\theta$, $v = r\sin\theta$.

Solution Given $u = r\cos\theta, \quad v = r\sin\theta$

$$\therefore \quad \frac{\partial u}{\partial r} = \cos\theta \qquad \frac{\partial v}{\partial r} = \sin\theta$$

$$\frac{\partial u}{\partial \theta} = -r\sin\theta \qquad \frac{\partial v}{\partial \theta} = r\cos\theta$$

Now,

$$J_1 = \frac{\partial(u, v)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r.$$

From (1), $\quad r^2 = u^2 + v^2$ and $\theta = \tan^{-1}\left(\dfrac{v}{u}\right)$

$$\therefore \quad \frac{\partial r}{\partial u} = \frac{u}{r} \quad \text{and} \quad \frac{\partial \theta}{\partial u} = \frac{1}{1+\dfrac{v^2}{u^2}} \cdot \left(-\frac{v}{u^2}\right) = -\frac{v}{u^2+v^2}$$

$$\frac{\partial r}{\partial v} = \frac{v}{r} \qquad \frac{\partial \theta}{\partial v} = \frac{1}{1+\dfrac{v^2}{u^2}} \cdot \left(\frac{1}{u}\right) = -\frac{u}{u^2+v^2}$$

Now,

$$J_2 = \frac{\partial(r, \theta)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial r}{\partial u} & \dfrac{\partial r}{\partial v} \\ \dfrac{\partial \theta}{\partial u} & \dfrac{\partial \theta}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{u}{r} & \dfrac{v}{r} \\ -\dfrac{v}{u^2+v^2} & \dfrac{u}{u^2+v^2} \end{vmatrix} = \frac{u^2}{r(u^2+v^2)} + \frac{v^2}{r(u^2+v^2)}$$

$$= \frac{1}{r}\left[\frac{u^2+v^2}{u^2+v^2}\right] = \frac{1}{r}$$

$$J_1 J_2 = \frac{\partial(u, v)}{\partial(r, \theta)} \cdot \frac{\partial(r, \theta)}{\partial(u, v)}$$

$$= r \cdot \frac{1}{r}$$

$$= 1$$

Hence, proved.

Example 26 If $y_1 = \dfrac{x_2\ x_3}{x_1}$, $y_2 = \dfrac{x_3\ x_1}{x_2}$, $y_3 = \dfrac{x_1\ x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 w.r.t x_1, x_2, x_3 is 4.

Solution Given $y_1 = \dfrac{x_2\ x_3}{x_1}$, $y_2 = \dfrac{x_3\ x_1}{x_2}$, $y_3 = \dfrac{x_1\ x_2}{x_3}$

$$\therefore \quad \frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}, \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_1} = -\frac{x_3}{x_2}, \frac{\partial y_2}{\partial x_2} = -\frac{x_3\ x_1}{x_2^2}, \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}, \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}, \frac{\partial y_3}{\partial x_3} = \frac{-x_1\ x_2}{x_3^2}$$

Now,

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2\ x_2^2\ x_3^2} \begin{vmatrix} -x_2\ x_3 & x_1\ x_3 & x_1\ x_2 \\ x_2\ x_3 & -x_1\ x_3 & x_1\ x_2 \\ x_2\ x_3 & x_1\ x_3 & -x_1\ x_2 \end{vmatrix}$$

$$= \frac{x_1^2\ x_2^2\ x_3^2}{x_1^2\ x_2^2\ x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1(1 - 1) - 1(-1 - 1) + 1(1 + 1)$

$= 0 + 2 + 2$

$= 4$ **Hence, proved.**

Example 27 If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \cos x_3$ then show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3$$

Solution Given $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \cos x_3$

$$\therefore \quad \frac{\partial y_1}{\partial x_1} = -\sin x_1, \frac{\partial y_2}{\partial x_2} = -\sin x_1 \sin x_2, \text{ and } \frac{\partial y_3}{\partial x_3} = -\sin x_1 \cdot \sin x_2 \cdot \sin x_3$$

Now,

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3}$$

$$= (-\sin x_1)(-\sin x_1 \sin x_2) \cdot (-\sin x_1 \sin x_2 \sin x_3)$$
$$= -\sin^3 x_1 \sin^2 x_2 \sin x_3.$$

Hence, proved.

Example 28 If $u = xyz$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$, find the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

Solution Givne $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} yz & zx & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y(z-x) & x(z-y) & xy \\ x-z & y-z & z \\ 0 & 0 & 1 \end{vmatrix} \quad \text{Applying } \begin{matrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= 2(z-x)(y-z)\begin{vmatrix} y & -x & xy \\ -1 & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(z-x)(y-z) \cdot (y-x)$$
$$= -2(x-y)(y-z)(z-x)$$

We know that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(u, v, w)} = 1$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{\dfrac{\partial(u, v, w)}{\partial(x, y, z)}}$$

$$= \frac{-1}{2(x-y)(y-z)(z-x)}$$

Example 29 If $x + y + z = u$, $y + z = uv$, $z = uvw$ then find the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

Solution Given $x + y + z = u$, $y + z = uv$, $z = uvw$

$$\therefore \quad x = u - (y + z) = u - uv = u(1 - v)$$
$$y = uv - z = uv - uvw = uv(1 - w)$$
$$z = uvw$$

$$\therefore \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} (1-v) & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} \quad \text{Applying } R_1 \to R_1 + (R_2 + R_3)$$

or
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v(1-w) + uvw$$
$$= u^2 v$$

Example 30 If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$ then show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{y^2 - x^2}{2uv(u-v)}$.

Solution Since the variables u, v are implicit functions of x, y and are related by the following relations,

$$F_1(u, v, x, y) \equiv u^3 + v^3 - (x + y) = 0$$
$$F_2(u, v, x, y) \equiv u^2 + v^2 - (x^3 + y^3) = 0$$

$$\therefore \qquad \frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \left[\frac{\partial(F_1, F_2)}{\partial(x, y)} \middle/ \frac{\partial(F_1, F_2)}{\partial(u, v)} \right] \tag{1}$$

Now,
$$\frac{\partial(F_1, F_2)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} = 3((y^2 - x^2)$$

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 3v \end{vmatrix} = 6u^2 v - 6uv^2 = 6uv(u - v)$$

Equation (1) becomes

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{3(y^2 - x^2)}{6uv(u - v)}$$

$$= \frac{(y^2 - x^2)}{2uv(u - v)}$$

Hence, proved.

Example 31 If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1 u_2 = x_2 + x_3 + x_4$

$u_1 u_2 u_3 = x_3 + x_4$, $u_1\, u_2\, u_3\, u_4 = x_4$, then show that $\dfrac{\partial(u_1, u_2, u_3, u_4)}{\partial(x_1, x_2, x_3, x_4)} = u_1^3\, u_2^2\, u_3$.

Solution Since u_1, u_2, u_3 and u_4 are implicit functions of the variables x_1, x_2, x_3, x_4 and they are related by the relations,

$$F_1 \equiv x_1 + x_2 + x_3 + x_4 - u_1 = 0$$
$$F_2 \equiv x_2 + x_3 + x_4 - u_1 u_2 = 0$$
$$F_3 \equiv x_3 + x_4 - u_1 u_2 u_3 = 0$$
$$F_4 \equiv x_4 - u_1\, u_2\, u_3\, u_4 = 0$$

$$\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(u_1, u_2, u_3, u_4)} = \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial F_3}{\partial u_3} \cdot \frac{\partial F_4}{\partial u_4}$$
$$= (-1)\,(-u_1)(-u_1\, u_2) \cdot (-u_1\, u_2\, u_3)$$
$$= u_1^3\, u_2^2\, u_3$$

and $$\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(x_1, x_2, x_3, x_4)} = \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdot \frac{\partial F_3}{\partial x_3} \cdot \frac{\partial F_4}{\partial x_4} = (1)\,(1)\,(1)\,(1) = 1.$$

Now,

$$\frac{\partial(u_1, u_2, u_3, u_4)}{\partial(x_1, x_2, x_3, x_4)} = (-1)^4 \left[\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(u_1, u_2, u_3, u_4)} \Big/ \frac{\partial(F_1, F_2, F_3, F_4)}{\partial(x_1, x_2, x_3, x_4)}\right]$$
$$= \frac{u_1^3\, u_2^2\, u_3}{1}$$
$$= u_1^3\, u_2^2\, u_3$$

Hence, proved.

Example 32 If $u^3 = xyz$, $\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, $w^2 = x^2 + y^2 + z^2$ then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{v(x-y)\,(y-z)\,(z-x)(x+y+z)}{3u^2 w(xy + yz + zx)}$$

Solution Let

$$F_1 \equiv u^3 - xyz = 0$$
$$F_2 \equiv \frac{1}{v} - \frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 0$$
$$F_3 \equiv w^2 - x^2 - y^2 - z^2 = 0$$

Now,

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} -yz & -xz & -xy \\ \frac{1}{x^2} & \frac{1}{y^2} & \frac{1}{z^2} \\ -2x & -2y & -2z \end{vmatrix}$$
$$= \frac{+2}{x^2\, y^2\, z^2} \begin{vmatrix} x^2yz & xy^2z & xyz^2 \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \frac{2\,xyz}{x^2\,y^2\,z^2}\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix} = \frac{2(x-y)(y-z)(z-x)(x+y+z)}{x\,y\,z}$$

and $$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & -\dfrac{1}{v^2} & 0 \\ 0 & 0 & 2w \end{vmatrix} = -\frac{6u^2 w}{v^2}$$

$\therefore$ $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3\left[\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}\bigg/\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}\right]$$

$$= -\frac{2(x-y)(y-z)(z-x)(x+y+z)}{x\,y\,z}\cdot\left(-\frac{v^2}{6u^2 w}\right)$$

$$= \frac{2(x-y)(y-z)(z-x)(x+y+z)}{x\,y\,z\cdot 6u^2 w\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)}$$

$$= \frac{2v(x-y)(y-z)(z-x)(x+y+z)\cdot xyz}{6\,xyz\,u^2 w\,(xy+yz+zx)}$$

$$= \frac{v(x-y)(y-z)(z-x)(x+y+z)}{3\,u^2 w\,(xy+yz+zx)}$$ **Hence, proved.**

Example 33 If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. Are u and v functionally related? If so find the relationship.

Solution

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1-x^2} & \dfrac{1}{1-y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence, u and v are functionally related.

Now, $$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$v = \tan^{-1} u$$

or $\boldsymbol{u = \tan v}$ which is the required relation between u and v.

Example 34 If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$. Show that they are not independent. Find the relation between u, v, and w.

Solution Here, $u_x = 1, u_y = 2, u_z = 1$

$v_x = 1, v_y = -1, v_z = 3$

and $w_x = 2y - z, w_y = 2x + 4z, w_z = -x + 4y - 4z$

$$\therefore \quad J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -2 & -8 & 3 \\ x - 2y + 3z & 4x - 8y + 12z & -x + 4y - 4z \end{vmatrix} \quad \begin{matrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - 3C_3 \end{matrix}$$

$$= \begin{vmatrix} -2 & -8 \\ x - 2y + 3z & 4x - 8y + 12z \end{vmatrix}$$

$$= -4 \begin{vmatrix} 2 & 2 \\ x - 2y + 3z & x - 2y + 3z \end{vmatrix} = 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

Hence, u, v, w are not independent.

Now, $u + v = 2x + 4z$

$u - v = 4y - 2z$

$$\therefore \quad u^2 - v^2 = (u + v)(u - v) = (2x + 4z)(4y - 2z) = 4(x + 2z)(2y - z)$$

$$= 4(2xy - zx + 4yz - 2z^2) = 4w$$

$$\boldsymbol{u^2 - v^2 = 4w}$$

which is the required relation between u, v, and w.

Example 35 If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$, show that the Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ vanishes. Find the relation between u, v, and w.

Solution Here, $u_x = 2x, u_y = 2y, u_z = 2z$

$v_x = 1, v_y = 1, v_z = 1$

and $w_x = (y + z), w_y = (x + z), w_z = (y + x)$

$$\therefore \quad J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y + z & z + x & x + y \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y + z & z + x & x + y \end{vmatrix} \qquad (R_1 \leftrightarrow R_2)$$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ y+z & x-y & x-z \end{vmatrix} \qquad [C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & -(x-y) & -(x-z) \\ y+z & (x-y) & (x-z) \end{vmatrix} = 0$$

Hence, u, v, and w are not independent.

Now, $\quad v = x + y + z \Rightarrow v^2 = (x + y + z)^2$

$$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$\boldsymbol{v^2 = u + 2w}$$

which is the required relation between u, v, and w.

Example 36 If $u = y\sqrt{1-x^2} + x\sqrt{1-y^2}$ and $v = \sqrt{(1-x^2)(1-y^2)} - xy$ then prove that u and v are not independent and find the relationship between them.

Solution $\quad J(u, v) = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{-xy}{\sqrt{1-x^2}} + \sqrt{1+y^2} & \sqrt{1-x^2} - \dfrac{xy}{\sqrt{1-y^2}} \\ -\dfrac{x\sqrt{1-y^2}}{1-x^2} - y & -\dfrac{y\sqrt{1-x^2}}{\sqrt{1-y^2}} - x \end{vmatrix}$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \begin{vmatrix} \sqrt{(1-x^2)(1-y^2)} - xy & \sqrt{(1-x^2)(1-y^2)} - xy \\ -x\sqrt{1-y^2)} - y\sqrt{1-x^2)} & -x\sqrt{1-y^2)} - y\sqrt{1-x^2)} \end{vmatrix}$$

$$= 0$$

Hence, u, v are not independent.

Now, $\quad u^2 + v^2 = \left[y\sqrt{1-x^2} + x\sqrt{1-y^2}\right]^2 + \left[\sqrt{(1-x^2)(1-y^2)} - xy\right]^2$

$$= y^2(1-x^2) + x^2(1-y^2) + 2xy\sqrt{(1-x^2)(1-y^2)}$$

$$+ (1-x^2)(1-y^2) + x^2y^2 - 2xy\sqrt{(1-x^2)(1-y^2)}$$

$$= y^2(1 - x^2 + x^2) + (1 - y^2)(x^2 + 1 - x^2)$$

$$= y^2 + 1 - y^2 = 1$$

$\therefore \quad$ $\boldsymbol{u^2 + v^2 = 1}$, which is the required relation.

EXERCISE 3.4

1. Find the Jacobian of the following transformation $u = ax + by$, $v = cx + dy$.
2. If $u = a \cos hx \cos y$, $v = a \sinh x \,.\, \sin y$ then show that $J(u, v) = \dfrac{a^2}{2}(\cosh 2x - \cos 2y)$.
3. If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ then show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.
4. If $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ then show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$.
5. If $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $z = r\sin\theta$, show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = -r^2\cos\theta$.
6. If $u_1 = 1 - x_1$, $u_2 = x_1(1 - x_2)$, $u_3 = x_1 x_2 (1 - x_3)$, find the value of $\dfrac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$.
7. If $y_1 = x_1 + x_2 + x_3$, $y_1^{\;2}\, y_2 = x_2 + x_3$, $y_1^{\;3}\, y_3 = x_3$, find the value of the Jacobian $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.
8. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ then show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{y^2 - x^2}{2uv(u - v)}$.
9. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$. Show that $\dfrac{\partial(x, y, z)}{\partial(u, v, z)} = \dfrac{1}{2(x - y)(y - z)(z - x)}$.
10. Find the Jacobian of $y_1, y_2, y_3, \ldots, y_n$, when it is given that $y_1 = (1 - x_1)$, $y_2 = x_1(1 - x_2)$, $y_3 = x_1 x_2(1 - x_3)$, $\ldots$, $y_n = x_1 x_2 \ldots x_{n-1}(1 - x_n)$.
11. Prove that the functions $u = xy + zx$, $v = xy + yz$ and $w = xz - zy$ are not independent and find the relation between them.
12. If $u = x + y + z$, $v = xy + yz + zx$, $w = x^3 + y^3 + z^3 - 3xyz$, show that u, v, w are not independent and find the relation between them.
13. Use the Jacobian to prove that the functions $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ are not independent. Find the relation between them.
14. If $u = x^3 + x^2y + x^2z - z^2(x + y + z)$, $v = x + z$ and $w = x^2 - z^2 + xy - zy$, prove that u, v, w are connected by a functional relation.

Answers

1. $ad - bc$
6. $-x_1^{\;2}\, x_2$
7. $\dfrac{1}{y_1^5}$
10. $J(y_1, y_2, \ldots, y_n) = (-1)^n\, x_1^{\;n-1}\, x_2^{\;n-2}, \ldots, x_{n-1}$.
11. $u - v = w$
12. $u^3 - 3uv = w$
13. $w^2 = v + 2u$
14. $u = wv$

3.15 EXPANSION OF FUNCTIONS OF SEVERAL VARIABLES

3.15.1 Expansion of Functions of One Variable

Taylor's Theorem (or Taylor's Series)

Let $f(x+h)$ be a function of h(x being independent of h) which can be expanded in powers of h and this function be differentiable any number of times. Then the theorem states that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots \tag{24}$$

Brook Taylor (18 August 1685–29 December 1731), was born in Edmonton, England. His parents, John Taylor and Olivia Tempest, had a very stable financial condition. Taylor was home-tutored before starting his studies in St. John's College, Cambridge. He acquired the degrees of LLB in 1709 and LLD in 1714. He was interested in Art and Music but his first love was Mathematics. He portrayed exceptional abilities in mathematics by writing a very important paper even before his graduation. It gave the explanation of the oscillation of a body. Taylor provided the solution to the 'Kepler's Law' to Machin in 1912. Noticing Taylor's extraordinary expertise in the subject, he was elected as a member of the Royal Society by Machin and Keill. In 1714, Brook Taylor became the secretary of the Royal Society.

Other Forms of Taylor's Theorem

(i) Putting $x = a$ in (24), we obtain

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \cdots \tag{25}$$

(ii) Putting $h = x - a$ in (25), we obtain.

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots \tag{26}$$

3.15.2 Corollary

Putting $h = x$ and $a = 0$ in (25), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots \tag{27}$$

which is called ***Maclaurin's theorem***.

3.16 EXPANSION OF FUNCTIONS OF TWO VARIABLES

Taylor's Series/Theorem Expansion of a Function of Two Variables

Statement

Let $f(x, y)$ be a function of two independent variables x and y. It possesses finite and continuous partial derivatives up to n^{th} order in any neighborhood D of a point (a, b) and let $(a + h, b + k)$ be any point of D. Then $f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$

$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(a,b)+\cdots$

where $a \le x \le a+h, \quad b \le y \le b+k.$

Proof Since $f(x, y)$ is a function of two independent variables x and y, if taking y as a constant then $f(x+h, y+k)$ is a function of x. Expanding with the help of Taylor's theorem for one variable, we have

$$f(x+h, y+k) = f(x, y+k) + h f_x(x, y+k) + \frac{h^2}{2!} f_{xx}(x, y+k) + \frac{h^3}{3!} f_{xxx}(x, y+k) + \cdots \tag{28}$$

Now, expanding functions on the RHS of (28) as functions of y by Taylor's theorem for one variable, we have

$$f(x, y+k) = f(x, y) + k f_y(x, y) + \frac{k^2}{2!} f_{yy}(x, y) + \cdots$$

$$h(f_x(x, y+k) = h\left[f_x(x, y) + k\, f_{xy}(x, y) + \frac{k^2}{2!} f_{xy^2}(x, y) + \cdots\right]$$

$$\frac{h^2}{2!} f_{xx}(x, y+k) = \frac{h^2}{2!}\left[f_{xx}(x, y) + k\, f_{xxy}(x, y) + \frac{k^2}{2!} f_{xxyy}(x, y) + \cdots\right], \text{ and so on.}$$

Substituting these values in (28), we obtain

$$f(x+h, y+k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})$$

$$+\frac{1}{3!}(h^3 f_{xxx} + 3h^2kf_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}) + \cdots \tag{29}$$

or symbolically, we have

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x, y)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x, y)+\cdots \tag{30}$$

Putting $x = a$ and $y = b$ in (30), we get

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(a, b)+\cdots \tag{31}$$

Putting $h = x - a$ and $k = y - b$ in (31), we get

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)]$$

$$\frac{1}{3!}[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2 (y-b) f_{xxx}(a, b) + (x-a(y-b)^2 f_{xyy}(a, b + (y-b)^3 f_{yyy}(a, b)] + \cdots \quad (32)$$

Now, putting $a = 0 = b$ is (32), we get

$$f(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\ f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$
$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y\ f_{xxy}(0,0) + 3xy^2\ f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \cdots \quad (33)$$

Equations (30), (31), and (32) are known as **Taylor's theorem** and Eq. (33) is called **Maclaurin's theorem**.

3.17 TAYLOR'S AND MACLAURIN'S THEOREMS FOR THREE VARIABLES

(i) Taylor's Theorem

$$f(x+h, y+k; z+p) = f(x, y, z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + p\frac{\partial}{\partial z}\right) f(x, y, z) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + p\frac{\partial}{\partial z}\right)^2 f(x, y, z) + \cdots$$

(ii) Maclaurin's Theorem

$$f(x, y, z) = f(0,0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) f(0,0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)^2 f(0,0,0) + \cdots$$

Example 37 Expand $\sin xy$ in powers of $(x - 1)$ and $\left(y - \frac{\pi}{2}\right)$ up to second-degree terms.

Solution The Taylor's series expansion in powers of $(x - a)$ and $(y - b)$ is given by

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \cdots \quad (1)$$

We have $f(x, y) = \sin xy$

$$f_x(x, y) = y \cos xy, f_y(x, y) = x \cos xy$$

$$f_{xx} = -y^2 \sin xy\, f_{yy} = -x \sin xy, f_{xy} = -xy \sin y + \cos xy$$

$\therefore$ at the point $\left(1, \frac{\pi}{2}\right)$, we have

$$f\left(1, \frac{\pi}{2}\right) = 1, f_x\left(1, \frac{\pi}{2}\right) = 0, f_y\left(1, \frac{\pi}{2}\right) = 0$$

$$f_{xx}\left(1, \frac{\pi}{2}\right) = -\frac{\pi^2}{4}, f_{yy}\left(1, \frac{\pi}{2}\right) = -1, f_{xy}\left(1, \frac{\pi}{2}\right) = -\frac{\pi}{2}$$

Putting these values is (1), we obtain

$$\sin xy = f\left(1, \frac{\pi}{2}\right) + \left[(x-1)\cdot 0 + \left(y - \frac{\pi}{2}\right)\cdot 0\right] + \frac{1}{2!}\left[(x-1)^2 \cdot \left(-\frac{\pi}{4}\right)^2\right.$$

$$\left. + 2(x-1)\left(y - \frac{\pi}{2}\right)\cdot\left(-\frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 (-1)\right]$$

$$= 1 + \frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)\left(y - \frac{\pi}{2}\right) - \left(y - \frac{\pi}{2}\right)^2\right]$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right)^2$$

Example 38 Expand $e^x \cos y$ in powers of $(x - 1)$ and $\left(y - \frac{\pi}{4}\right)$ by Taylor's theorem.

Solution Taylor's theorem in powers of $(x - 1)$ and $\left(y - \frac{\pi}{4}\right)$ is given by

$$f(x, y) = f\left(1, \frac{\pi}{4}\right) + (x-1)\, f_x\left(1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) f_y\left(1, \frac{\pi}{4}\right)$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1, \frac{\pi}{4}\right) + 2(x-1)\left(y - \frac{\pi}{4}\right) f_{xy}\left(1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2 f_{yy}\left(1, \frac{\pi}{4}\right)\right] + \cdots \quad (1)$$

We have

$$f(x, y) = e^x \cos y$$

$$f_x = e^x \cos y, f_y = -e^x \sin y, f_{xx} = e^x \cos y, f_{xy} = -e^x \sin y$$

and $$f_{yy} = -e^x \cos y$$

$\therefore$ at the point $\left(1, \frac{\pi}{4}\right)$, we have

$$f_y\left(1, \frac{\pi}{4}\right) = e \cos\frac{\pi}{4} = \frac{e}{\sqrt{2}}, f_x\left(1 - \frac{\pi}{4}\right) = e \cos\frac{\pi}{4} = \frac{e}{\sqrt{2}}$$

$$f_y\left(1, \frac{\pi}{4}\right) = -e \sin\frac{\pi}{4} = -\frac{e}{\sqrt{2}},\ f_{xx}\left(1 - \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}},\ f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

and $f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$ and so on.

Putting these values in (1), we get

$$f(x, y) = \frac{e}{\sqrt{2}} + \left[(x-1)\cdot\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)\right]$$

$$+\frac{1}{2!} + \left[(x-1)^2 \cdot \frac{e}{\sqrt{2}} + 2(x-1)\left(y - \frac{\pi}{4}\right)\cdot\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2 \cdot \left(-\frac{e}{\sqrt{2}}\right)\right] + \cdots$$

$$e^x \cos y = +\frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{1}{2!}[(x-1)^2 - 2(x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2\right] + \cdots$$

Example 39 Expand $f(x, y) = e^{-(x^2+y^2)} \cdot \cos xy$ about the point (0, 0) up to three terms by Taylor's series.

Solution The Taylor's series expansion of $f(x, y)$ at (0, 0) is given by

$$f(x, y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2\, f_{yy}(0,0)\right] + \cdots \quad (1)$$

We have

$$f(x, y) = e^{-(x^2+y^2)} \cdot \cos xy$$

$$f_x = -e^{-(x^2+y^2)}(2x \cos xy + y \sin xy)$$

$$f_y = -e^{-(x^2+y^2)}(2y \cos xy + x \sin xy)$$

$$f_{xx} = e^{-(x^2+y^2)}[4x^2 \cos xy + 4xy \sin xy - 2\cos xy - y^2 \sin xy]$$

$$f_{xy} = e^{-(x^2+y^2)}[3xy \cos xy + 2x^2 \sin xy + 2y^2 \sin xy - \sin xy]$$

$$f_{yy} = e^{-(x^2+y^2)}[4y^2 \cos xy + 4xy \sin xy - 2\cos xy - x^2 \sin xy]$$

$\therefore$ at the point (0, 0), we have

$$f(0,0) = 1, f_x(0,0) = 0, f_y(0,0) = 0$$

$$f_{xx}(0,0) = -2, f_{xy}(0,0) = 0, f_{yy}(0,0) = -2, \text{ etc.}$$

Substituting these values in (1), we obtain

$$e^{-(x^2+y^2)} \cdot \cos xy = 1 + [x\cdot 0 + y \cdot 0] + \frac{1}{2}\left[x^2 \cdot (-2) + 2xy \cdot (0) + y^2(-2)\right] + \cdots$$

$$= 1 - x^2 - y^2 + \ldots$$

Example 40 Expand $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ in the neighborhood of (1, 1) up to and inclusive of second-degree terms. Hence, find $f(1\cdot1, 0\cdot9)$ approximately.

Solution Taylor's series expansion of $f(x, y)$ at (1, 1) is given by

$$f(x, y) = f(1,1) + \left[(x-1)\, f_x(1,1) + (y-1)\, f_y(1,1)\right]$$
$$+\frac{1}{2!}\left[(x-1^2\; f_{xx}(1,1) + 2(x-1)(y-1)\, f_{xy}(1,1) + (y-1)^2\; f_{yy}(1,1)\right]$$
$$+\frac{1}{3!}\,[(x-1)^3 f_{xxx}(1, 1) + 3(x-1)^2(y-1) f_{xxy}(1, 1)$$
$$+ 3(x-1)(y-1)^2 f_{xyy}(1, 1) + (y-1)^3 f_{yyy}(1, 1)] + \cdots \tag{1}$$

We have

$$f(x, y) = \tan^{-1}\frac{y}{x} \qquad f(1, 1) = \frac{\pi}{4}$$

$$f_x = -\frac{y}{x^2+y^2} \qquad f_x(1, 1) = -\frac{1}{2}$$

$$f_y = \frac{x}{x^2+y^2} \qquad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx} = \frac{2\,xy}{(x^2+y^2)^2} \qquad f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad f_{xy}(1, 1) = 0$$

$$f_{yy} = -\frac{2xy}{(x^2+y^2)^2} \qquad f_{yy}(1, 1) = -\frac{1}{2}$$

Similarly, we obtain

$$f_{xxx}(1, 1) = -\frac{1}{2}, \quad f_{xxy}(1,1) = -\frac{1}{2}, \quad f_{xyy}(1,1) = \frac{1}{2}, \quad f_{yyy}(1,1) = \frac{1}{2}$$

Putting these values in (1), we get

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\cdot\frac{1}{2}\right] + \frac{1}{2!}\left[(x-1)^2\cdot\frac{1}{2} + 2(x-1)(y-1)\cdot 0 + (y-1)^2\cdot\left(-\frac{1}{2}\right)\right]$$
$$+\frac{\pi}{3!} + \left[(x-1)^3\left(\frac{-1}{2}\right) + 3(x-1)^2\,(y-1)\cdot\left(-\frac{1}{2}\right) + (3-1(y-1)^2\cdot\frac{1}{2} + (y-1)^3\cdot\frac{1}{2}\right] + \cdots$$

$$= \frac{\pi}{4} - \frac{1}{2}[(x-1)-(y-1)] + \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right]$$
$$+\frac{1}{12}[-(x-1)^3 - 3(x-1)^2(y-1) + 3(x-1)(y-1)^2 + (y-1)^3] + \cdots$$

Now $f(1{\cdot}1, 0{\cdot}9) = \dfrac{\pi}{4} - \dfrac{1}{2}\big[0.1 - (-0.1)\big] + \dfrac{1}{4}[(0.1)^2 - (-0.1)^2]$

$$+\frac{1}{12}\Big[-(0.1)^3 - 3(0.1)^2(-0.1)\Big] + 3(0.1)(-0.1)^2 + (-0.1)^3] + \cdots$$

$$= \frac{\pi}{4} - \frac{0.2}{2} + \frac{1}{12}[-0.001 + 0.003 + 0.003 - 0.001] + \cdots$$

$= 0.6857$ (approximately).

Example 41 Find the Taylor's series expansion of $f(x, y) = x^y$ in the powers of $(x-1)$ and $(y-1)$ up to third-degree terms,

Solution Taylor's series expansion of $f(x, y)$ in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x, y) = f(1, 1) + [(x-1)f_x(1, 1) + (y-1)\,f_y(1, 1)]$$

$$+\frac{1}{2!}[(x-1)^2\ f_{xx}(1,1) + 2(x-1)(y-1)\ f_{xy}(1,1) + (y-1)^2\ f_{yy}(1,1)]$$

$$+\frac{1}{3!}[(x-1)^3\ f_{xxx}(1,1) + 3(x-1)^2(y-1)\ f_{xxy}(1,1)$$

$$+ 3(x-1)(y-1)^2 f_{xyy}(1, 1) + (y-1)^3 f_{yyy}(1, 1)] + \ldots \qquad (1)$$

We have

$f(x, y) = x^y;$	$f(1, 1) = 1$
$f_x = yx^{y-1};$	$f_x(1, 1) = 1$
$f_y = x^y \log x;$	$f_y(1, 1) = 0$
$f_{xx} = y(y-1)\,x^{y-2};$	$f_{xx}(1, 1) = 0$
$f_{yy} = y\,x^{y-1}\log x + x^{y-1};$	$f_{xy}(1, 1) = 1$
$f_{yy} = x^y(\log x)^2;$	$f_{yy}(1, 1) = 0$
$f_{xxx} = y(y-1(y-2)x^{y-3};$	$f_{xxx}(1, 1) = 0$
$f_{xxy} = (y-1)x^{y-2} + y(y-1)\,x^{y-2}\log x + yx^{y-2};$	$f_{xxy}(1, 1) = 1$
$f_{xyy} = y\,x^{y-1}\cdot(\log x)^2\ 2x^{y-1}\cdot\log x;$	$f_{xyy}(1, 1) = 0$
$f_{yyy} = x^y(\log x)^3;$	$f_{yyy}(1, 1) = 0$

Substituting these values is (1), we obtain

$$x^y = 1 + [(x-1)\cdot 1 + (y-1)\cdot 0] + \frac{1}{2!}[(x-1)^2\cdot 0 + 2(x-1)(y-1) + (y-1)^2\cdot 0]$$

$$+\frac{1}{3!}[(x-1)^3\cdot 0 + 3(x-1)^2\ (y-1)\cdot + 3(x-1)(y-1)^2\cdot 0 + (y-1)^3\cdot 0] + \cdots$$

or $$x^y = 1 + (x-1) + (x-1(y-1) + \frac{1}{3!}[3(x-1)^2\cdot(y-1)] + \cdots$$

Example 42 Expand $f(x, y) = x^2 + xy + y^2$ is powers of $(x-2)$ and $(y-3)$ up to second-degree terms.

Solution The Taylor's series expansion of $f(x, y)$ in powers of $(x-2)$ and $(y-3)$ is given by

$$f(x, y) = f(2, 3) + [(x-2)\,f_x(2, 3) + (y-3)\,f_y(2, 3)]$$

$$+\frac{1}{2!}\left[(x-2)^2 f_{xx}(2,3)+2(x-2)(y-3)\cdot f_{xy}\,(2,3)+(y-3)^2\ f_{yy}(2,3)\right]+\cdots \quad (1)$$

We have

$$\begin{aligned} f(x, y) &= x^2 + xy + y^2; & f(2, 3) &= 19 \\ f_x &= 2x + y; & f_x(2, 3) &= 7 \\ f_y &= x + 2y; & f_y(2, 3) &= 8 \\ f_{xx} &= 2; & f_{xx}(2, 3) &= 2 \\ f_{xy} &= 1; & f_{xy}(2, 3) &= 1 \\ f_{yy} &= 2; & f_{yy}(2, 3) &= 2 \end{aligned}$$

Substituting thee values in (1), we obtain

$$f(x, y) = 19 + [(7(x-2) + 8(y-3)] + \frac{1}{2}[2(x-2)^2 + 2(x-2)(y-3) + 2(y-3)^2] + \cdots$$

EXERCISE 3.5

1. Find the Taylor's series expansion of $f(x, y) = x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ up to second-degree terms.
2. Obtain Taylor's series expansion of the function $f(x, y) = e^{xy}$ about (1, 1) up to third-degree terms.
3. Find Taylor's series expansion $f(x, y) = \tan^{-1}(xy)$. Hence, compute an approximate value of $f(0.9, -1.2)$.
4. Expand $f(x, y) = y^x$ in powers of $(x - 1)$ and $(y - 1)$ up to second-degree terms
5. Expand $e^x \cdot \sin y$ in powers of x and y up to third-degree terms.
6. Expand $e^{ax} \sin by$ in powers of x and y as far as their terms of third degree.
7. Find the Taylor's series expansion of the function e^x. log $(1 + y)$ in the neighborhood of the point (0, 0).
8. Expand $(x^2y + \sin y + e^x)$ in powers of $(x - 1)$ and $(y - \pi)$.
9. Expand $\sin(x + y)$ about (0, 0) up to and including the third-degree terms.

Answers

1. $f(x, y) = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2 (y + 2)$
2. $$f(x, y) = e\left[1+(x-1)+(y-1)+\frac{1}{2}(x-1)^2+2(x-1)(y-1)+\frac{1}{2}(y-1)^2\right.$$
$$\left.+\frac{1}{6}(x-1)^3+\frac{3}{2}(x-1)^2(y-1)+\frac{3}{2}(x-1)(y-1)^2+\frac{1}{6}(y-1)^3\right]+\cdots$$
3. At $(1, -1)$, $f(x, y) = -\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y+1)+\frac{1}{4}(x-1)^2+\frac{1}{4}(y+1)^2+\cdots$

   $f(0.9, -1.2) = -0.8229$.
4. $y^x = 1 + (y - 1) + (x - 1)\,(y - 1) + \ldots$

5. $e^x \sin y = y + xy + \dfrac{x^2 y}{2} - \dfrac{y^3}{6} + \cdots$

6. $f(x, y) = by + ab\,xy + \dfrac{1}{3!}(3a^2\ bx^2 y - b^3 y^3) + \cdots$

7. $f(x, y) = y + xy - \dfrac{y^2}{2} + \dfrac{x^2 y}{2} - \dfrac{xy^2}{2} + \dfrac{y^3}{3} + \cdots$

8. $\pi + e + (x-1)(2\pi + e) + \dfrac{1}{2}(x-1)^2\ (2\pi + e) + 2(x-1)(y-\pi) + \cdots$

9. $\sin(x + y) = x + y - \dfrac{(x+y)^3}{3} + \cdots$

SUMMARY

1. Homogeneous Function

A function $f(x, y)$ is called a homogeneous function if the degree of each term is same.

2. Euler's Theorem

If u is a homogeneous function in the variables x and y of degree n then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$.

Generalized Form Euler's theorem can be extended to a homogeneous function of any number of variables. If $u(x_1, x_2, x_3, \ldots, x_n)$ be a homogeneous function of degree n in the variables $x_1, x_2, x_3, \ldots, x_n$ then

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + x_3\frac{\partial u}{\partial x_3} + \cdots + x_n\frac{\partial u}{\partial x_n} = n\cdot u$$

3. If z is a homogeneous function in the variables x and y of degree n and $z = f(u)$ then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

4. If z is a homogeneous function in the variables x and y of degree n and $z = f(u)$, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u) - 1], \text{ where } \quad \phi(u) = n\frac{f(u)}{f'(u)}.$$

5. Jacobian

Let u and v be given functions of two independent variables x and y. The Jacobian of u, v w.r.t. x, y denoted by $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J(u, v)$ is a second-order functional determinant defined as $\dfrac{\partial(u, v)}{\partial(x, y)} = J(u, v) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

In a similar manner, we can define the Jacobian of n functions $u_1, u_2, u_3, \ldots, u_n$ of n independent variables $x_1, x_2, x_3, \ldots, x_n$;

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = J(u_2, u_3, \ldots, u_n)$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

(a) Important Properties of Jacobian

(i) If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

(ii) If $J_1 = \dfrac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \dfrac{\partial(x, y)}{\partial(u, v)}$ then $J_1 J_2 = 1$, i.e., $\dfrac{\partial(u, v)}{\partial(x, y)} \cdot \dfrac{\partial(x, y)}{\partial(u, v)} = 1$

(b) Theorems on Jacobian

Theorem I: If the functions $u_1, u_2, u_3, \ldots, u_n$ of the variables $x_1, x_2, x_3, \ldots, x_n$ be defined by the relation $u_1 = f_1(x_1), u_2 = f_2(x_1, x_2), u_3 = f_3(x_1, x_2, x_3), \ldots, u_n = f_n(x_1, x_2, x_3, ..., x_n)$ then

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdot \cdots \cdot \frac{\partial u_n}{\partial x_n}$$

Theorem II: If the functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.

(c) Jacobian of Implicit Functions

Theorem: Let $u_1, u_2, u_3, ..., u_n$ are connected with n independent variables $x_1, x_2, x_3, ..., x_n$ by the relation of the form

$F_1(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0, F_2(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0 \ldots$

$\ldots F_n(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0.$

Then,

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} = (-1)^n \left[\frac{\partial(F_1, F_2, \cdots, F_n)}{\partial(x_1, x_2, \cdots, x_n)} \bigg/ \frac{\partial(F_1, F_2, \cdots, F_n)}{\partial(u_1, u_2, \cdots, u_n)} \right]$$

(d) Necessary and Sufficient Condition

The necessary and sufficient condition for functional dependence can be expressed in terms of a determinant as follows: Two functions u and v are functionally dependent if their Jacobian $J(u, v) = \dfrac{\partial(u, v)}{\partial(x, y)} = 0$. If the Jacobian is not equal to zero then u and v are said to be functionally independent,

6. Expansion of Functions of Several Variables

(a) Expansion of Functions of One Variable

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

(b) Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

7. Expansion of Functions of Two Variables

Let $f(x, y)$ be a function of two independent variables x and y. Then the Taylor's series in the powers of $(x-a)$ and $(y-b)$ is given by

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b)$$

$$+ 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] +$$

$$\frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a, b) + 3(x-a(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)\Big] + \cdots$$

where $a \le x \le a + h$, $b \le y \le b + k$.

Now, putting $a = 0 = b$ in the above expression, we get

$$f(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)]$$

$$+ \frac{1}{2!}\Big[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + 3xy^2\, f_{xyy}(0,0) + \mathrm{y}^3 f_{yyy}(0,0)\Big] + \cdots$$

$$+ \frac{1}{3!}\Big[x^3 f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2\, f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\Big] + \cdots$$

which is called Macluarin's theorem.

OBJECTIVE-TYPE QUESTIONS

1. If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$ then $x\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}$ is

(a) u (b) $2u$
(c) $\tan u$ (d) $\sin u$

2. If $u = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$ then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) 0
(b) $3[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$
(c) $2[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$
(d) $\sin^{-1}(x/y) + \tan^{-1}(y/x)]$

3. If $x = r\cos\theta$, $y = r\sin\theta$ then

(a) $\dfrac{\partial r}{\partial x} = 0$ (b) $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$
(c) $\dfrac{\partial x}{\partial r} = 0$ (d) $\dfrac{\partial x}{\partial r} = 2$

4. If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$ then $\dfrac{\partial f}{\partial u} - \dfrac{\partial f}{\partial v}$ is equal to

(a) $2\left(x\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y}\right)$ (b) $2\left(x\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\right)$
(c) $2\left(u\dfrac{\partial f}{\partial x} - v\dfrac{\partial f}{\partial y}\right)$ (d) $2\left(u\dfrac{\partial f}{\partial x} + v\dfrac{\partial f}{\partial y}\right)$

5. If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ then $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ is

(a) $\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2$

(b) $\left(\frac{\partial f}{\partial r}\right)^2 + r^2\left(\frac{\partial f}{\partial \theta}\right)^2$

(c) $\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$

(d) $\left(\frac{\partial f}{\partial r}\right)^2 + r\left(\frac{\partial f}{\partial \theta}\right)^2$

6. If $f(x, y) = 0$ and $\phi(y, z) = 0$ then

(a) $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial x}\frac{dx}{dz} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$

(b) $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$

(c) $\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}\frac{df}{dy} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$

(d) $\frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial z}$

7. If $f(x, y, z) = 0$ then $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y$ is

(a) 0 (b) 1
(c) 2 (d) –1

8. If $u = x^2 + y^2$ then $\frac{\partial^2 u}{\partial x \partial y}$ is equal to

(a) 2 (b) 0
(c) $x + y$ (d) $2(x + y)$

9. If $u = \log\frac{x^4 + y^4}{x + y}$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(a) $3u$ (b) u
(c) 3 (d) $\log u$

10. If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$ then $x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ is

(a) $2u$ (b) u
(c) $2\tan u$ (d) $\tan u$

11. If $\log u = x^2y^2/(x + y)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(a) $\tan u$ (b) $3u\log u$
(c) $3\log u$ (d) $u\log u$

12. If $u = x^y$ then $\partial u/\partial y$ is

(a) yx^{y-1} (b) 0
(c) $x^y\log x$ (d) y^x

13. If $u = x^y$ then $\frac{\partial u}{\partial x}$

(a) yx^{y-1} (b) $x^y\log x$
(c) 0 (d) $y^x\log x$

14. If $u = f(y/x)$ then

(a) $x\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$ (b) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(c) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$ (d) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

15. The Taylor's series expansion of $\sin x$ about $x = \pi/6$ is given by

(a) $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \cdots$

(b) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

(c) $\left(x - \frac{\pi}{6}\right) - \frac{(x - \pi/6)^3}{3!} + \frac{(x - \pi/6)^5}{5!} - \cdots$

(d) $\frac{1}{2}$

16. Taylor's series expansion of $e^x\cos y$ at (0, 0) is

(a) $1 + x + \frac{1}{2}(x^2 - y^2) + \cdots$

(b) $1 + x + \frac{1}{2}(x^2 + y^2) + \cdots$

(c) $1 - x + \frac{1}{2}(x^2 + y^2)$

(d) $-1 - x + \frac{1}{2}(x^2 - y^2) + \cdots$

17. The Taylor' series expansion of $e^x\sin y$ in powers of x and y is

(a) $y + xy + \frac{x^2y}{2} + \frac{y^3}{6} + \cdots$

(b) $x + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \cdots$

(c) $y - xy - \frac{x^2 y}{2} + \frac{y^3}{6} + \cdots$

(d) $y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \cdots$

18. If $x = e^u \cos v$, $y = e^u \sin v$ then $\frac{\partial(x, y)}{\partial(u, v)}$ is equal to

(a) e^u (b) e^{2u}
(c) e^{-2u} (d) e^{-u}

19. The Jacobian $\frac{\partial(u, v)}{\partial(u, v)}$ is equal to

(a) 0 (b) 2
(c) 1 (d) 1/2

20. If $u = e^x \sin y$, $v = e^x \cos y$ then $\frac{\partial(u, v)}{\partial(x, y)}$ is equal to

(a) $-e^x$ (b) 1
(c) 0 (d) $e^{2x} \sin y \cos y$

21. Two functions u and v are functionally dependent if their Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ is equal to

(a) 1 (b) xy
(c) uv (d) 0

22. If $u = \sin x + \sin y$, $v = \sin(x + y)$ then u and v are

(a) functionally dependent
(b) functionally independent
(c) $u^2 = v$
(d) constants

23. If $u = x \sin y$, $v = y \sin x$ then $\frac{\partial(u, v)}{\partial(x, y)}$ is equal to

(a) $\sin x \sin y$
(b) $\cos x \cos y$
(c) $xy \sin y \sin x$
(d) $\sin x \sin y - xy \cos x \cos y$

24. The Taylor's series expansion of $\sin x$ is

(a) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ (b) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(c) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ (d) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

25. If $f = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} xy^{n-1} + a_n y^n$, where $a_i (i = 0$ to $n)$ is constant, then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$ is

(a) f/n (b) n/f
(c) nf (d) $n\sqrt{f}$

[GATE (IE) 2005]

26. Let $f = y^x$ the value of $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2$, $y = 1$ is

(a) 0 (b) log 2
(c) 1 (d) 1/log 2

[GATE (ME) 2008]

27. The Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ for the function $u = e^x \sin y$, $v = (x + \log \sin y)$ is equal to

(a) 1
(b) $\sin x \sin y - xy \cos x \cos y$
(c) 0
(d) $\frac{e^x}{x}$ **[UPTU 2008]**

28. In the Taylor's series expansion of exp (x) + sin (x), about the point $x = \pi$, the coefficient of $(x - \pi)^2$ is

(a) exp (π) (b) 0.5 exp (π)
(c) exp $(\pi) + 1$ (d) exp $(\pi) - 1$

[GATE (ECE) 2008]

29. In the Taylor's series expansion of e^x about $x = 2$, the coefficient of $(x - 2)^4$ is

(a) 1/4! (b) $2^4/4!$
(c) $e^2/4!$ (d) $e^4/4!$

30. Total derivative of the function 'xy'

(a) $x\,dy + y\,dx$ (b) $x\,dx + y\,dy$
(c) $dx + dy$ (d) $dx\,dy$

[GATE (IPE) 2009]

31. Taylor's series expansion of $\frac{\sin x}{x - \pi}$ at $x = \pi$ is given by

(a) $1 + \frac{(x - \pi)^2}{3!} + \cdots$

(b) $-1 - \frac{(x - \pi)^2}{3!} - \cdots$

(c) $1 - \frac{(x - \pi)^2}{3!} + \cdots$

(d) $-1+\frac{(x-\pi)^2}{3!}+\cdots$

[GATE (ECE) 2009]

32. If $u=\log(e^x+e^y)$ then $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=$

(a) e^x+e^y (b) e^x-e^y

(c) $\frac{1}{e^x+e^y}$ (d) 1

[GATE (BT) 2013]

33. If $z=xyln(xy)$ then

(a) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$ (b) $y\frac{\partial z}{\partial x}=x\frac{\partial z}{\partial y}$

(c) $x\frac{\partial z}{\partial x}=y\frac{\partial z}{\partial y}$ (d) $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=0$

[GATE (CE) 2014]

34. The Taylor series expansion of 3 sin x + 2 cos x is

(a) $2+3x-x^2-\frac{x^3}{2}+\cdots$

(b) $2-3x+x^2-\frac{x^3}{2}+\cdots$

(c) $2+3x+x^2+\frac{x^3}{2}+\cdots$

(d) $2-3x-x^2+\frac{x^3}{2}+\cdots$

[GATE (EC) 2014]

ANSWERS

1. (c)	2. (a)	3. (b)	4. (a)	5. (b)	6. (b)	7.(d)	8. (b)	9. (c)	10. (d)
11. (b)	12. (c)	13. (a)	14. (b)	15. (a)	16. (a)	17.(d)	18. (d)	19. (d)	20. (b)
21. (c)	22. (d)	23. (c)	24. (c)	25. (b)	26. (b)	27.(c)	28. (a)	29. (d)	30. (a)
31. (b)	32. (d)	33. (c)	34. (a)						

4 Maxima and Minima

4.1 INTRODUCTION

An important application of differential calculus is to find the maximum or minimum values of a function of several variables and to obtain where they occur.

4.2 MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES

Let $f(x, y)$ be a function of two independent variables x and y, which is continuous for all values of x and y in the neighborhood of (a, b), i.e., $(a + h, b + k)$, be a point in its neighborhood which lies inside the region R. We define the following:

(i) The point (a, b) is called a point of *relative* (*or local*) *minimum*, if
$f(a, b) \le f(a + h, b + k)$ for all h, k.
Then $f(a, b)$ is called the *relative* (*or local*) *minimum value*.

(ii) The point (a, b) is called a *point of relative* (*or local*) *maximum*, if
$f(a, b) \ge f(a + h, b + k)$ for all h, k.
Then $f(a, b)$ is called the *relative* (*or local*) *maximum value*.

A function $f(x, y)$ may also attain its minimum or maximum values on the boundary of the region R. The smallest and the largest values attained by a function over the region including the boundaries are called the ***absolute* (*or global*) *minimum*** and ***absolute* (*or global*) *maximum values***.

The points at which maximum/minimum values of the function exist are called the ***stationary points*** or the *points extrema*.

The maximum and minimum values taken together are called the ***extreme values*** of the function.

4.3 NECESSARY CONDITIONS FOR THE EXISTENCE OF MAXIMA OR MINIMA OF $f(x, y)$ AT THE POINT (a, b)

Statement

The necessary conditions for the existence of a maxima or a minima of $f(x, y)$ at the point (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$, where $f_x(a, b)$ and $f_y(a, b)$ respectively denote the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = a$, $y = b$.

Proof Let $(a + h, b + k)$ be a point in the neighborhood of the point (a, b). Then the point (a, b) will be a maximum if

$$\Delta f = f(a + h, b + k) - f(a, b) \leq 0 \text{ for all } h, k.$$

and a point of minimum if

$$\Delta f = f(a + h, b + k) - f(a, b) \geq 0 \text{ for all } h, k.$$

Using the Taylor's series expansion about the point (a, b), we get

$$f(a + h, b + k) = f(a, b) + [(hf_x + kf_y)]_{(a, b)} + \frac{1}{2!}[h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}]_{(a,b)} + \cdots \quad (1)$$

Neglecting the second and higher order terms, we get

$$\Delta f \approx [hf_x + kf_y]_{(a, b)} = hf_x(a, b) + kf_y(a, b) \quad (2)$$

The sign of Δf in (2) depends on the sign of $hf_x(a, b) + kf_y(a, b)$ which is a function of h and k.

$\therefore$ the necessary condition that Δf has the same positive or negative sign is when $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (even though h and k can take both positive and negative values).

Hence, the necessary conditions for $f(x, y)$ to have a maximum or minimum value at a point (a, b) are that the partial derivatives f_x and f_y vanish for $x = a$ and $y = b$.

4.4 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA (LAGRANGE'S CONDITION FOR TWO INDEPENDENT VARIABLES)

Let $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Suppose $f_{xx}(a, b) = r, f_{xy}(a, b) = s$ and $f_{yy}(a, b) = t$.

Then the following are true:

(i) If $r > 0$ and $(rt - s^2) > 0, f(x, y)$ is minimum at (a, b).

(ii) If $r < 0$ and $(rt - s^2) > 0, f(x, y)$ is maximum at (a, b).

(iii) If $(rt - s^2) < 0, f(x, y)$ is neither maximum nor minimum at (a, b), i.e., (a, b) is a saddle point.

(iv) If $(rt - s^2) = 0$, it is a doubtful case (point of inflection).

Proof By Taylor's theorem for two variables, we have

$$\Delta f = f(a + h, b + k) - f(a, b) = (hf_x + kf_y)_{(a, b)} + \frac{1}{2!}\left[h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right]_{(a,b)} + \cdots$$

If the necessary conditions are $f_x(a, b) = 0 = f_y(a, b)$ then

$$\Delta f = f(a + h, b + k) - f(a, b) = \frac{1}{2}\left[rh^2 + 2shk + tk^2\right] + R_3 \quad (3)$$

where R_3 consists of terms of third and higher orders in h and k. For sufficiently small values of the h and k, the sign of RHS and, hence, of LHS of (3) is governed by the second degree terms.

Now, $$rh^2 + 2shk + tk^2 = \frac{1}{r}\left[r^2 h^2 + 2rshk + rtk^2\right]$$

$$= \frac{1}{r}\left[(r^2 h^2 + 2rshk + s^2k^2) + rtk^2 - s^2k^2\right]$$

$$= \frac{1}{r}\left[(rh + sh)^2 + (rt - s^2)\, k^2\right] \quad (4)$$

Now, the following cases arise:

(i) When $rt - s^2 > 0$ and $r > 0$, clearly RHS of (4) is positive.

$\therefore$ RHS of (3) is positive and the LHS of (3), i.e.,

$\Delta f > 0$, for all h and k.

Hence, $f(x, y)$ has a minima at the point (a, b).

(ii) When $(rt - s^2) > 0$ and $r < 0$, the RHS of (4) is negative and the RHS of (3) is negative.

$\therefore$ LHS of (3), i.e., $\Delta f < 0$.

Hence, $f(x, y)$ has a maxima at point (a, b).

(iii) When $(rt - s^2) < 0$, in this case, we can say nothing about the sign of the second degree terms in the RHS of (3), i.e., $(rh^2 + 2shk + tk^2)$, is not an invariable sign.

Hence, in this case, $f(x, y)$ have neither maxima nor minima at (a, b).

(iv) When $(rt - s^2) = 0$, i.e., $rt = s^2$ and $r \neq 0$.

In this case, Eq. (4) becomes

$$rh^2 + 2shk + tk^2 = \frac{1}{r}(rh + ks)^2$$

and so it has the same sign as r.

Hence, $f(x, y)$ is maximum if $r < 0$ and minimum if $r > 0$.

Now, if $rt = s^2$ and $\frac{h}{k} = -\frac{s}{r}$ (i.e., $rh + ks = 0$) then second-degree terms vanish and we have to consider terms of higher order in (3). This case is doubtful, if

$$r = 0 \text{ then } rt - s^2 = 0 \Rightarrow s = 0$$

$$\therefore \qquad rh^2 + 2shk + tk^2 = tk^2$$

which is clearly zero when $k = 0\ \forall\ h$. Hence, this case is again doubtful.

$\therefore$ if $rt - s^2 = 0$, the case is doubtful and further investigation is required to decise whether the function $f(x, y)$ is a maximum or a minimum at (a, b).

Example 1 Discuss the extreme values (maxima and minima) of the function $x^3 + y^3 - 3axy$.

Solution Let $f(x, y) = x^3 + y^3 - 3axy$ (1)

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \text{ and } \frac{\partial f}{\partial y} = 3y^2 - 3ax.$$

For maxima or minima of $f(x, y)$,

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial x}$$

$\Rightarrow \qquad 3x^2 - 3ay = 0$ or $x^2 = ay$ (2)

and $\qquad 3y^2 - 3ax = 0$ or $y^2 = ax$ (3)

$\Rightarrow \qquad x = y = a$ and $x = y = 0$ [For solving (2) and (3)]

$\therefore$ the stationary or critical points are (a, a) and $(0, 0)$.

Now, $$r = \frac{\partial^2 f}{\partial x^2} = 6x, \quad s = \frac{\partial^2 f}{\partial x\,\partial y} = -3a, \; t = \frac{\partial^2 f}{\partial y^2} = 6y$$

(i) At the point (a, a), we have

$$rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 = 27a^2 > 0.$$

Since $rt - s^2 > 0$ and r is +ve or –ve depending on whether a is +ve or –ve.

Thus, $f(x, y)$ have a maximum or a minimum, whether a is –ve or +ve at (a, a).

(ii) At the point (0, 0), we have

$$r = 0,\ s = -3a \text{ and } t = 0$$

$$rt - s^2 = 0 - (-3a)^2 = -9a^2 < 0$$

Since $(rt - s^2) < 0$, thus, $f(x, y)$ has neither maximum nor minimum at (0, 0).

Example 2 Find a maximum and minimum values of $f(x, y) = 4x^2 + 9y^2 - 8x - 12y + 4$ over the rectangle is the first quadrant bounded by the lines $x = 2$, $y = 3$ and the coordinates axes.

Solution Given $f(x, y) = 4x^2 + 9y^2 - 8x - 12y + 4$ (1)

The given function $f(x, y)$ can attain maximum or minimum values at the stationary (or critical) points or on the boundary of the rectangle *OABC*.

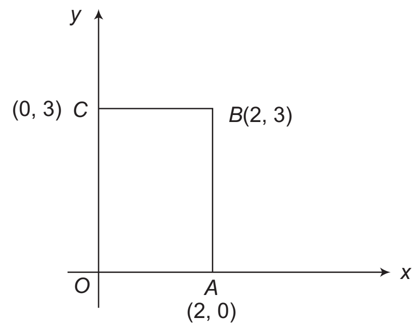


Fig. 4.1

$\therefore$ $$f_x = 8x - 8$$
$$f_y = 18y - 12$$

For stationary points, put

$$f_x = 0 \Rightarrow 8x - 8 = 0, \text{ or } x = 1$$

$$f_y = 0 \Rightarrow 18y - 12 = 0, \text{ or } y = \frac{2}{3}$$

$\therefore$ stationary points are $\left(1, \dfrac{2}{3}\right)$.

Now, $$r = f_{xx} = 9,\ s = f_{xy} = 0 \text{ and } t = f_{yy} = 18$$

$\therefore$ $$rt - s^2 = 8 \times 18 - 0 = 144$$

since $$rt - s^2 = 144 > 0 \text{ and } r = 8 > 0$$

Hence, $f(x, y)$ is minimum, i.e., the point $\left(1, \dfrac{2}{3}\right)$, is a point of relative minimum.

The minimum value is $f\left(1, \dfrac{2}{3}\right) = -4$ on the boundary of the rectangle.

***Along line OA*:** $y = 0$ and $f(x, y) = f(x, 0) = F(x) = 4x^2 - 8x + 4$
which is a function of one variable

Putting $\dfrac{dF}{dx} = 0 \Rightarrow 8x - 8 = 0 \Rightarrow x = 1$

Now, $\dfrac{d^2F}{dx^2} = 8 > 0$

$\therefore$ at $x = 1$, the function has a minimum and minimum value is $F(1) = 4 \times 1^2 - 8.1 + 4 = 0$

Also, at the corners (0, 0) and (2, 0),

we have $f(0, 0) = F(0) = 4$ and $f(2, 0) = F(2) = 4$.

Similarly, along the other boundary lines, we have the following results:

At $\quad x = 2,\ G(y) = 9y^2 - 12y + 4$

$$\frac{dG}{dy} = 18y - 12 = 0 \Rightarrow y = \frac{2}{3} \text{ and } \frac{d^2G}{dy^2} = 18 > 0$$

$\therefore\ y = \frac{2}{3}$ is a point of minimum and the minimum value is $f\left(2, \frac{2}{3}\right) = 0.$ At the corner (2, 3), we have $f(2, 3) = 49$.

At $y = 3$, $\quad H(x) = 4x^2 - 8x + 49$

$$\frac{dH}{dx} = 8x - 8 = 0 \Rightarrow x = 1 \text{ and } \frac{d^2H}{dx^2} = 8 > 0$$

$\therefore\ x = 1$ is a point of minimum and the minimum value is $f(1, 3) = 45$. At the corner point (0, 3), we have $f(0, 3) = 49$.

At $x = 0$, $I(y) = 9y^2 - 12\,y + 4$, which is the same case as for $x = 2$.

$\therefore$ the absolute minimum value is –4 which occurs at $\left(1, \frac{2}{3}\right)$ and the maximum value is 49 which occurs at the points (2, 3) and (0, 3).

Example 3 Find the maxima and minima of the function $\sin x + \sin y + \sin (x + y)$.

Solution $\quad f(x, y) = \sin x + \sin y + \sin (x + y) \qquad (1)$

$$\frac{\partial f}{\partial x} = \cos x + \cos (x + y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos (x + y)$$

For maximum or minimum of f, we have

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos (x + y) = 0 \qquad (2)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \cos y + \cos (x + y) = 0 \qquad (3)$$

Solving (2) and (3), we get $\cos x = \cos y$ or $x = y$.

Now, from (2), $\quad \cos x + \cos (x + x) = 0$

$$\cos x + \cos 2x = 0$$

or $\quad 2\cos^2 x + \cos x - 1 = 0$

or $\quad 2\cos^2 x + 2\cos x - \cos x - 1 = 0$

or $\quad 2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

or $$(\cos x + 1)(2\cos x - 1) = 0$$

When $2\cos x - 1 = 0 \Rightarrow \cos x = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

or $$x = 2n\pi \pm \frac{\pi}{3}$$

When $\cos x + 1 = 0 \Rightarrow \cos x = -1 = \cos\pi$

or $$x = 2n\pi \pm \pi$$

$\therefore$ stationary points are $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ and (π, π).

Now, $$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x + y)$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = -\sin(x + y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x + y)$$

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $$r = \sin\frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$s = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$t = -\sin\frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

Now, $$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4} = \frac{9}{4} > 0$$

Since $(rt - s^2) = \dfrac{9}{4} > 0$ and $r = -\sqrt{3} < 0$, the function has a maximum value at $x = y = \dfrac{\pi}{3}$

At (π, π), we have

$$r = -\sin\pi - \sin(\pi + \pi) = 0$$

$$s = -\sin(\pi + \pi) = 0 \text{ and } t = -\sin\pi - \sin(\pi + \pi) = 0$$

$\therefore$ $$rt = s^2 = 0 - 0 = 0$$

Hence, this case is doubtful and we shall leave it.

Example 4 Find the maximum and minimum values of the function $f(x, y) = 3x^2 + y^2 - x$. Over the region $2x^2 + y^2 \leq 1$.

Solution Given $f(x, y) = 3x^2 + y^2 - x$ (1)

and region $2x^2 + y^2 \leq 1$ (2)

For maximum/minimum of f, we have

$$f_x = \frac{\partial f}{\partial x} = 6x - 1 = 0 \Rightarrow x = \frac{1}{6}$$

$$f_y = \frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$$

$\therefore$ stationary (or critical) points are $\left(\frac{1}{6}, 0\right)$.

Now, $r = f_{xx} = 6,\ s = f_{xy} = 0,\ t = f_{yy} = 2$

At $\left(\frac{1}{6}, 0\right)$, $r = 6$, $s = 0$ and $t = 2$.

$\therefore$ $rt - s^2 = 6.2 - 0 = 12 > 0$

Hence, the point $\left(\frac{1}{6}, 0\right)$ is a minimum, because $rt - s^2 > 0$ and $r > 0$.

The minimum value at $\left(\frac{1}{6}, 0\right)$ is $f\left(\frac{1}{6}, 0\right) = -\frac{1}{12}$.

On the boundary, we have $y^2 = 1 - 2x^2$; $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.

Putting in $f(x, y)$, we get

$$f(x, y) = 3x^2 + 1 - 2x^2 - x$$
$$= x^2 - x + 1 \equiv F(x)$$ (which is a function of one variable 'x')

Putting $\frac{dF}{dx} = 0 \Rightarrow 2x - 1 = 0$

or $x = \frac{1}{2}$

Also, $\frac{d^2F}{dx^2} = 2 > 0$

For $x = \frac{1}{2}$, we get $y^2 = 1 - 2x^2 = 1 - \frac{1}{2} = \frac{1}{2}$

or $y = \pm\frac{1}{\sqrt{2}}$

$\therefore$ the points $\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ are points of minimum.

The minimum value is $f\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = 3 \cdot \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{3}{4}$.

At the corners (vertices), we have

$$f\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{3-\sqrt{2}}{2},\ f\left(-\frac{1}{\sqrt{2}}, 0\right) = \frac{3+\sqrt{2}}{2} \text{ and } f(0, \pm 1) = 1.$$

Thus, the given function has an absolute minimum value $-\frac{1}{12}$ at $\left(\frac{1}{6}, 0\right)$ and absolute maximum value.

$$\frac{(3+\sqrt{2})}{2} \text{ at } \left(-\frac{1}{\sqrt{2}}, 0\right)$$

Example 5 Find the maxima and minima of the function $f(x, y) = \sin x \cdot \sin y \cdot \sin(x + y)$.

Solution Given $f(x, y) = \sin x \cdot \sin y \cdot \sin(x + y)$ (1)

For maxima and minima of 'f', we have

$$\frac{\partial f}{\partial x} = \sin x \sin y \cos(x + y) + \cos x \sin y \sin(x + y) = \sin y(\sin(2x + y) = 0 \quad (2)$$

$$\frac{\partial f}{\partial y} = \sin x \sin y \cos(x + y) + \sin x \cos y \sin(x + y) = \sin x \sin(x + 2y) = 0 \quad (3)$$

Since the given function is periodic with the period π, both for x and y, it is sufficient to consider the values of x and y between 0 and π.

From (2), $\sin y = 0$ or $\sin(2x + y) = 0$

$\Rightarrow$ $y = 0$ or $2x + y = \pi$ or 2π

and from (3), $\sin x = 0$ or $\sin(x + 2y) = 0$ (4)

$\Rightarrow$ $x = 0$ or $x + 2y = \pi$ or 2π (5)

Solving equations (4) and (5), we obtain the critical points

$$\left(\frac{\pi}{3}, \frac{\pi}{3}\right) \text{ and } \left(\frac{2\pi}{3}, \frac{2\pi}{3}\right).$$

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = 2 \sin y \cos(2x + y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \cos x \cdot \sin(x + 2y) + \sin x \cdot \cos(x + 2y) = \sin(2x + 2y).$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2 \sin x \cos(x + 2y)$$

At the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$$r = 2 \cdot \frac{\sqrt{3}}{2} \cdot (-1) = -\sqrt{3}, \text{ and}$$

$$s = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$t = 2\cdot\left(\frac{\sqrt{3}}{2}\right)(-1) = -\sqrt{3}$$

$\therefore$ $$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0 \text{ and } r = -\sqrt{3} < 0$$

Hence, $f(x, y)$ is a maximum at $x = y = \dfrac{\pi}{3}$, i.e., $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

At $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$,

$$r = 2\left(\frac{\sqrt{3}}{2}\right)\cdot 1 = \sqrt{3},$$

$$s = \frac{\sqrt{3}}{2} \text{ and}$$

$$t = \sqrt{3}$$

$\therefore$ $$rt - s^2 = (\sqrt{3})(\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0 \text{ and } r = \sqrt{3} > 0$$

Hence, $f(x, y)$ is a minimum at $x = y = \dfrac{2\pi}{3}$, i.e., $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$.

Now, $$f_{\max} = \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\cdot\frac{2\pi}{3} = \frac{3\sqrt{3}}{8}.$$

$$f_{\min} = \sin\frac{2\pi}{3}\cdot\sin\frac{2\pi}{3}\cdot\sin\frac{4\pi}{3} = \frac{3\sqrt{3}}{8}.$$

Example 6 If x, y, z are angles of a triangle then find the minimum value of $\sin x \sin y \sin z$.

Solution Let $f(x, y, z) = \sin x \sin y \sin z$, where $x + y + z = \pi$

$$x + y = \pi - z$$

$\therefore$ $$\sin(x + y) = \sin(\pi - z)$$
$$= \sin z$$

$\therefore$ $$f(x, y) = \sin x \sin y \sin(x + y).$$

Now, do as Example 5.

Example 7 Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution Let x, y, z be the length, breadth, and height of the rectangular solid.

If V is the volume of the solid then

$$V = xyz \tag{1}$$

Since each diagonal of the solid passes through the centre of the sphere, therefore, each diagonal is equal to the diameter of sphere = a (say).

i.e., $x^2 + y^2 + z^2 = a^2$ or $z = \sqrt{a^2 - x^2 - y^2}$

Then (1), becomes

$$V = xy\sqrt{a^2 - x^2 - y^2}$$

or $$V^2 = x^2y^2(a^2 - x^2 - y^2) = a^2x^2y^2 - x^4y^2 - x^2y^4 = f(x, y) \tag{2}$$

For maximum/minimum of f, we have

$$f_x = 2xy^2(a^2 - 2x^2 - y^2) = 0$$

and $$f_y = 2x^2y(a^2 - x^2 - 2y^2) = 0$$

$$\Rightarrow \quad a^2 - 2x^2 - y^2 = 0 \qquad (\because x \neq 0, y \neq 0) \tag{3}$$

$$a^2 - x^2 - 2y^2 = 0 \tag{4}$$

Solving (3) and (4), we obtain

$$-x^2 + y^2 = 0 \text{ or } y = x.$$

$\therefore$ from (1), $$x = y = \frac{a}{\sqrt{3}}, \quad z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Now, $$r = f_{xx} = 2a^2y^2 - 12x^2y^2 - 2y^4$$
$$s = f_{xy} = 4a^2xy - 8x^3y - 8xy^3$$
$$t = f_{yy} = 2a^2x^2 - 2x^4 - 12x^2y^2$$

At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$, $r = -\frac{8}{9}a^4 < 0$

$$s = -\frac{4}{9}a^4 \text{ and}$$

$$t = -\frac{8}{9}a^4$$

$$\therefore \quad rt - s^2 = \left(-\frac{8}{9}a^4\right)\left(-\frac{8}{9}a^4\right) - \left(-\frac{4}{9}a^4\right)^2 = \frac{16a^8}{27} > 0 \text{ and } r < 0$$

$\therefore$ V is maximum when $x = y = z$, i.e., when the rectangular solid is a cube.

4.5 MAXIMUM AND MINIMUM VALUES FOR A FUNCTION $f(x, y, z)$

Let $u = f(x, y, z)$

For maximum/minimum of u is

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}$$

Now, find u_{xx}, u_{yy}, u_{zz}, u_{yz}, u_{zx}, u_{xy}. They are denoted by A, B, C, F, G, H respectively.

Again, find $D_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix}$ and $D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

Thus, the given function will have a minimum, if $A > 0$, $D_1 > 0$ and $D_2 > 0$.
and will have a maximum, if $A > 0$, $D_1 > 0$ and $D_2 < 0$.
If these above conditions are not satisfied then the function has neither maximum nor minimum.

Example 8 Find the maximum and minimum values for the function $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

Solution We have

$$f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z \tag{1}$$

For maximum/minimum of f, we have

$$f_x = 2x - y + 1 = 0 \tag{2}$$
$$f_y = 2y - x = 0 \tag{3}$$
$$f_z = 2z - 2 = 0 \tag{4}$$

Solving (2), (3), and (4), we get

$$x = -\frac{2}{3},\ y = -\frac{1}{3},\ z = 1$$

$\therefore$ the critical (or stationary) point is $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

Now, $A = f_{xx} = 2 (> 0)$ $\quad F = f_{yz} = 0$

$B = f_{yy} = 2$ $\quad G = f_{zx} = 0$

$C = f_{zz} = 2$ $\quad H = f_{xy} = -1$

Now, $$D_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

$$D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 > 0$$

$\therefore$ $A = 2 > 0$, $D_1 = 3 > 0$ and $D_2 = 6 > 0$.

Thus, the given function has a minimum at $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

$\therefore$ Minimum value $= f\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$

$$= \left(-\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + 1 - \frac{2}{9} - \frac{2}{3} - 2$$

$$= \frac{4}{9} + \frac{1}{9} + 1 - \frac{2}{9} - \frac{8}{3}$$

$$f_{\min} = -\frac{4}{3}$$

4.6 LAGRANGE'S METHOD OF MULTIPLIERS

In many practical and theoretical problems, it is required to find the extremum of the function $f(x_1, x_2, x_3, \ldots, x_n)$ under the conditions

$$\phi_i(x_1, x_2, x_3, \ldots, x_n) = 0;\ i = 1, 2, 3, \ldots, r \tag{5}$$

Now, we construct an auxiliary function of the form

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_r) = f(x_1, x_2, x_3, \ldots, x_n) + \sum_{i=1}^{r} \lambda_i\ \phi_i\ (x_1, x_2, \cdots, x_n) \tag{6}$$

where λ_i's are undetermined parameters and are known as ***Lagrange's multipliers***.

Now, we find the critical (or stationary) points of F; we have the necessary conditions

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \cdots = \frac{\partial F}{\partial x_n}$$

which implies that

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{r} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0;\ \ j = 1, 2, \cdots, n \tag{7}$$

From equations (5) and (7), we obtain $(n + r)$ equations in $(n + r)$ unknowns $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_r$. Solving these equations, we get the required stationary points $(x_1, x_2, \ldots, x_n)$ at which the function 'f' has an extremum. Further investigation is needed to find the exact nature of these points.

Example 9 Find the extreme values of the function $f(x, y, z) = 2x + 3y + z$ under the conditions $x^2 + y^2 = 5$ and $x + z = 1$.

Solution We have $f(x, y, z), = 2x + 3y + z$

Subject to conditions $\phi_1 \equiv x^2 + y^2 - 5 = 0$ (1)

and $\phi_2 \equiv x + z - 1 = 0$ (2)

Now, construct an auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \sum_{i=1}^{2} \lambda_i\ \phi_i\ (x, y, z)$$
$$= (2x + 3y + z) + \lambda_1\ \phi_1\ (x, y, z) + \lambda_2\ \phi_2(x, y, z)$$
$$= (2x + 3y + z) + \lambda_1(x^2 + y^2 - 5) + \lambda_2(x + z - 1)$$

For the extremum, we have

$$\frac{\partial F}{\partial x} = 2 + 2\ \lambda_1\ x + \lambda_2 = 0$$

$$\frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0$$

$$\frac{\partial F}{\partial z} = 1 + \lambda_2 = 0$$

For the above equations, we get $\lambda_2 = -1$, $3 + 2\lambda_1 y = 0$ and $1 + 2\lambda_1 x = 0$

or $$x = -\frac{1}{2\lambda_1}, y = -\frac{3}{2\lambda_1}$$

Substituting the values of x and y in $x^2 + y^2 = 5$, we obtain

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5 \text{ or } \lambda_1^2 = \frac{1}{2} \text{ or } \lambda_1 = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ for $\lambda_1 = \dfrac{1}{\sqrt{2}}, x = -\dfrac{\sqrt{2}}{2}, y = -\dfrac{3\sqrt{2}}{2}, z = 1 - x = \dfrac{(2+\sqrt{2})}{2}$

$$\therefore \quad f(x, y, z) = f\left(-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right)$$

$$= -\sqrt{2} - \frac{9\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2} = \frac{2-10\sqrt{2}}{2} = (1 - 5\sqrt{2})$$

For $$\lambda_1 = -\frac{1}{\sqrt{2}}, \quad x = \frac{\sqrt{2}}{2}, y = \frac{3\sqrt{2}}{2} \text{ and } z = 1 - x = \frac{2-\sqrt{2}}{2}$$

$$\therefore \quad f(x, y, z) = \sqrt{2} + \frac{9\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2} = \frac{2+10\sqrt{2}}{2} = 1 + 5\sqrt{2}$$

Example 10 Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

Solution Let $P(x, y, z)$ be any point on the sphere $x^2 + y^2 + z^2 = 1$.

The distance of the point $A(3, 4, 12)$ from the sphere is

$$AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x, y, z) \text{ say} \tag{1}$$

where $$\phi(x, y, z) \equiv x^2 + y^2 + z^2 - 1 = 0 \tag{2}$$

Now, construct the Lagrange's function or auxiliary function.

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

$$= [(x-3)^2 + (y-4)^2 + (z-12)^2] + \lambda(x^2 + y^2 + z^2 - 1) \tag{3}$$

For extremum, we have

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x = 0, \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y = 0 \text{ and}$$

$$\frac{\partial F}{\partial z} = 2(z - 12) + 2\lambda z = 0$$

Solving the above equations, we obtain

$$x = \frac{3}{\lambda + 1}, y = \frac{4}{\lambda + 1} \text{ and } z = \frac{12}{\lambda + 1}.$$

Putting the values of x, y, z in Eq. (2), we get

$$\left(\frac{3}{\lambda+1}\right)^2 + \left(\frac{4}{\lambda+1}\right)^2 + \left(\frac{12}{\lambda+1}\right)^2 = 1$$

or $$\frac{9+16+144}{(\lambda+1)^2} = 1 \text{ or } (\lambda + 1)^2 = 169$$

$$\lambda + 1 = \pm 13 \text{ or } \lambda = 12, -14$$

When $\lambda = 12$, $x = \frac{3}{13}, y = \frac{4}{13}, z = \frac{12}{13}$

When $\lambda = -14$, $x = -\frac{3}{13}, y = -\frac{4}{13}, z = -\frac{12}{13}$

Hence, we find the two points P and Q, i.e.,

$$P = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right) \text{ and } Q = \left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right).$$

$\therefore$ the distance from the point $A(3, 4, 12)$ to P and Q are

$$AP = \sqrt{\left(\frac{3}{13} - 3\right)^2 + \left(\frac{4}{13}, -4\right)^2 + \left(\frac{12}{13} - 12\right)^2} = 12$$

$$AQ = \sqrt{\left(-\frac{3}{13} - 3\right)^2 + \left(-\frac{4}{13}, -4\right)^2 + \left(-\frac{12}{13} - 12\right)^2} = 14$$

Thus, the minimum distance is 12 and the maximum distance is 14.

Example 11 Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution Let $2x$, $2y$, $2z$ be the length breadth and height of the rectangular solid and let r be the radius of the sphere.

Then, the volume $V = 8xyz \equiv f(x, y, z)$ (1)

Subject to constraint $\phi(x, y, z) \equiv x^2 + y^2 + z^2 - r^2 = 0$ (2)

Construct Lagrange's function,

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\, \phi(x, y, z)$$

$$= 8xyz + \lambda(x^2 + y^2 + z^2 - r^2) \quad (3)$$

For extremum of F, we have

$$\frac{\partial F}{\partial x} = 8\,yz + 2\lambda x = 0 \tag{4}$$

$$\frac{\partial F}{\partial y} = 8\,xz + 2\lambda y = 0 \tag{5}$$

$$\frac{\partial F}{\partial z} = 8\,xy + 2\lambda z = 0 \tag{6}$$

From (4), $2\lambda x^2 = -8\,xyz$

From (5), $2\lambda y^2 = -8\,xyz$

From (6), $2\lambda z^2 = -8\,xyz$

$\therefore$ $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$ or $x^2 = y^2 = z^2$

or $x = y = z$.

Hence, the rectangular solid is a cube. **Hence, proved.**

Example 12 Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution Let $2x$, $2y$, $2z$ be the length, breadth, and height of the rectangular parallelepiped then its volume

$$V = 8\,xyz \equiv f(x, y, z) \text{ say} \tag{1}$$

subject to constraints

$$\phi(x, y, z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{2}$$

Construct the Lagrange's function.

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\,\phi(x, y, z)$$

$$= 8\,xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) \tag{3}$$

For stationary points, we have

$$\frac{dF}{dx} = 0 \Rightarrow \left(8\,yz + \frac{2\lambda x}{a^2}\right) = 0 \tag{4}$$

$$\frac{dF}{dy} = 0 \Rightarrow 8\,xz + \frac{2\lambda y}{b^2} = 0 \tag{5}$$

$$\frac{dF}{dz} = 0 \Rightarrow 8\,xy + \frac{2\lambda z}{c^2} = 0 \tag{6}$$

Multiplying (4), (5), and (6) by x, y, and z respectively, and adding we get

$$24\,xyz + 2\lambda = 0 \Rightarrow \lambda = -12\,xyz$$

From (4), $\quad 8\,yz - 12\,xyz \cdot \dfrac{2x}{a^2} = 0 \quad \text{or} \quad x = \dfrac{a}{\sqrt{3}}$

From (5), $\quad 8\,xz - 12\,xyz \cdot \dfrac{2y}{b^2} = 0 \quad \text{or} \quad y = \dfrac{b}{\sqrt{3}}$

From (6), $\quad 8\,xy - 12\,xyz \cdot \dfrac{2z}{c^2} = 0 \quad \text{or} \quad z = \dfrac{c}{\sqrt{3}}$

At the point $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right)$,

volume of the largest rectangular parallelepiped = $8\,xyz$

$$= 8 \cdot \frac{abc}{3\sqrt{3}}$$

$$V = \frac{8\,abc}{3\sqrt{3}}$$

EXERCISE 4.1

1. Examine the following functions for extreme values
   (i) $x^3 + y^3 - 3xy$
   (ii) $x^2y^2 - 5x^2 - 8xy - 5y^2$
   (iii) $2(x - y)^2 - x^4 - y^4$
2. Discuss the extreme values of the function
   $f(x, y) = x^3 - 4xy + 2y^2$
3. Examine for extreme values the function
   $f(x, y) = xy(a - x - y)$
4. A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring least material for its construction.
5. The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.
6. Find the extreme values of the function
   $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$
7. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when
   (i) box is open at the top
   (ii) box is closed
8. Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.
9. The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.
10. Using the method of Lagrange's multiplier, find the largest product of the number x, y, and z, when $x + y + z^2 = 16$.

11. Using the method of Lagrange's multiplier, find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.
12. Find the extreme value of $x^2 + y^2 + z^2$ subject to the constraints $xy + yz + zx = p$.
13. Use Lagrange's method to find the minimum distance from the origin to the plane $3x + 2y + z = 12$.
14. Find the maxima and minima of $f(x, y, z) = x^2 + y^2 + z^2$ where $ax^2 + by^2 + cz^2 = 1$.
15. The temperature T at any point (x, y, z) in space is $T = 400\, xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.

Answers

1. (i) Minimum value = –1 at (1, 1), i.e., $f_{\min}(1, 1) = -1$.
   (ii) $f_{\max}(0, 0) = 0$.
   (iii) $f_{\max} = 8$ at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.
2. $f_{\min}\left(\frac{4}{3}, \frac{4}{3}\right) = -\frac{32}{27}$.
3. $f(x, y)$ has a maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$ and has no extreme value at (0, 0), (0, a), and (a, 0).
4. Function is minimum at $x = y = (2\,v)^{1/3}$
6. $f(x, y)$ is minimum at (0, 1) and (0, –1), the minimum value is –1 at (0, 1) and (0, –1). Also, the points (0, 0), (±1, ±1) are neither the points of maximum nor minimum
7. (i) When the box is open at the top, $n = 1$.

   $x = \sqrt{\frac{s}{3}}$, dimensions are $x = y = \sqrt{\frac{s}{3}}, z = \frac{1}{2}\sqrt{\frac{s}{3}}$

   (ii) When the box is closed, $n = 2$

   $x = \sqrt{\frac{s}{6}}$, dimensions are $x = y = z = \sqrt{\frac{s}{6}}$

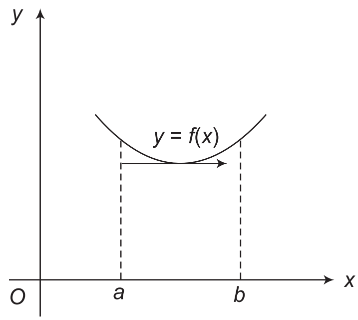
   where s is the surface area of the box.
8. 3.
10. $f_{\max}\left(\frac{32}{5}, \frac{32}{5}, \frac{4}{\sqrt{5}}\right) = \frac{4096}{25\sqrt{5}}$.
11. Minimum is 27.
12. p
13. 3. 2071
14. Minimum and maximum of f are the roots of $\left(f - \frac{1}{a}\right)\left(f - \frac{1}{b}\right)\left(f - \frac{1}{c}\right) = 0$
15. $T = 50$

4.7 CONVEXITY, CONCAVITY, AND POINT OF INFLECTION

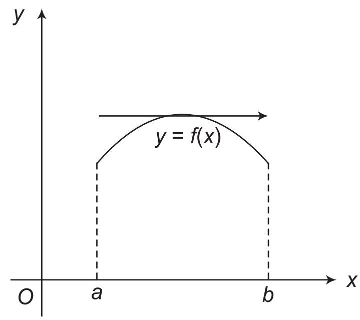
Let $y = f(x)$ be a curve, this curve has continuous first- and second-order derivatives in an open interval (a, b). Then we define the convexity, concavity, and point of inflection of the curve as follows:

(i) Concave Upward

A curve $y = f(x)$ is said to be concave upward if and only if the derivative $f'(x)$ is an increasing function on (a, b) and all points on the curve in the interval (a, b) lie above the tangent to the curve at any point in this interval. In terms of the second-order derivative, we define that a curve is concave upward if $f''(x) > 0$ on (a, b). Such a curve is also known as a ***convex curve***.



(a) Concave upward curve



(b) Concave downward curve

Fig. 4.2

(ii) Concave Downward

A curve $y = f(x)$ is said to be concave downward if and only if the derivative $f'(x)$ is a decreasing function on (a, b) and all the points on the curve in the interval (a, b) lie below the tangent to the curve at any point in (a, b).

In terms of the second-order derivative, we define that a curve is concave downward if $f''(x) < 0$ in (a, b). Such a curve is known as a ***concave curve***.

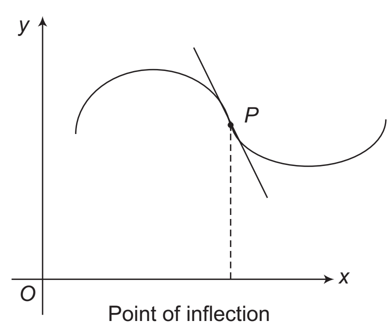
(iii) Point of Inflection

A point P on the curve $y = f(x)$ is called a point of inflection if the curve changes its concavity from concave upward to concave downward or from concave downward to concave upward.

In other words, the point that separates the convex part of a continuous curve from the concave part is called the **point of inflection**.

If $y = f(x)$ be a continuous curve and if $f''(p) = 0$ or $f''(p)$ does not exist, and if the derivative $f''(x)$ changes sign when passing through $x = p$ then the point of the curve with abscissa $x = p$ is the point of inflection.

Thus, a point of inflection P, $f'(x)$ is +ve on one side of P and –ve on the other side. Hence, at a point of inflection, $f''(x) = 0$ and $f'''(x) \neq 0$.



Point of inflection

Fig. 4.3

4.8 ASYMPTOTES TO A CURVE

A point $P(x, y)$ on a curve $y = f(x)$, if the distance of the point $P(x, y)$ from a straight line (L) tends to zero as x or y or both x, y tends to infinity then the line (L) is called an *asymptote* of the curve.

In other words, a straight line, at a finite distance from the origin to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.

Asymptotes are usually classified as the following:

(i) Vertical asymptotes
(ii) Horizontal asymptotes
(iii) Inclined, or oblique asymptotes

(i) Vertical Asymptotes or Asymptotes Parallel to the *y*-axis

A line $x = a$ is called a vertical asymptote to the curve $y = f(x)$, or is called an asymptotes parallel to the y-axis to the curve $y = f(x)$.

The asymptotes parallel to the y-axis are obtained by equating to zero the coefficient of the highest power of y.

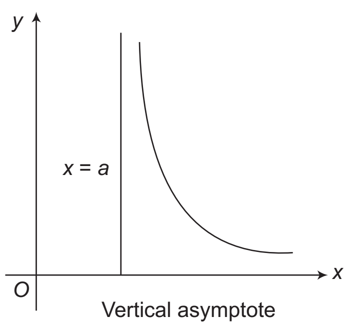


Fig. 4.4

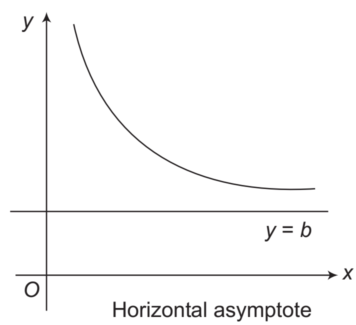


Fig. 4.5

(ii) Horizontal Asymptotes or Asymptotes Parallel to the *x*-axis

A line $y = b$ is called a horizontal or parallel to x-axis asymptote.

Asymptotes parallel to the x-axis are obtained by equating to zero the coefficient of the highest power of x.

(iii) Inclined (or Oblique) Asymptotes

Oblique asymptote (not parallel to the x-axis and y-axis) are given by $y = mx + c$, where $m = \lim_{x\to\infty}\left(\frac{y}{x}\right)$ and $c = \lim_{x\to\infty}(y - mx)$.

Oblique asymptotes are obtained when the curve is represented by an implicit function of the form $f(x, y) = 0$. (8)

Let $y = mx + c$ be the asymptote of (8), where m and c are constants whose values are determined as follows:

(i) Put $x = 1$ and $y = m$ in (8) (i.e., terms of highest power 'n' in the equation of the curve) and put the expression thus obtained equal to $\phi_n(m)$.

(ii) Put $x = 1$ and $y = m$ in the $(n - 1)^{\text{th}}$ degree terms in the equation of the curve and put the expression thus obtained equal to $\phi_{n-1}(m)$.

(iii) Put $x = 1$ and $y = m$ in the $(n - 2)^{\text{th}}$ degree terms and we obtained $\phi_{n-2}(m)$.

Repeat the above process for the terms of lower degrees.

To find the value of *m*, we put

$$\phi_n(m) = 0 \tag{9}$$

Solve (9) and find all values of m.

To find the value of *c*, the value of c is given by the following formula

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} \tag{10}$$

For the different values of m, we obtain the values of c and putting the values of m and c in $y = mx + c$, the asymptotes are obtained.

(iv) Two Mutually Parallel Asymptotes

From Eq. (9),

$$\phi_n(m) = 0$$

If two values of m are equal then there will be two mutually parallel asymptotes.

In this case, the two different values of 'c' cannot be determined from (10).

Hence, in this case, the values of 'c' are given by the formula

$$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0 \tag{11}$$

If three values of m are equal then the three different values of c are determined by the given formula.

$$\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + \frac{c}{1!}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0 \tag{12}$$

Example 13 Find the asymptote parallel to the axis of 'x' of the curve $y^3 + x^2y + 2xy^2 - y + 2 = 0$.

Solution Let $f(x, y) \equiv y^3 + x^2y + 2xy^2 - y + 2 = 0$ (1)

Asymptote parallel to x-axis = coefficient of highest power of $x = 0$

i.e., $y = 0$

Example 14 Find the asymptotes parallel to the y-axis of the curve $x^4 + x^2y^2 - a^2(a^2 + y^2) = 0$.

Solution Let $f(x, y) \equiv x^4 + x^2y^2 - a^2(a^2 + y^2) = 0$

Asymptote parallel to y-axis is coefficient of highest power of $y = 0$.

i.e., $x^2 - a^2 = 0$

$$x = \pm a$$

Example 15 Find the asymptotes of the curve

$$4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 - 7 = 0.$$

Solution Given $4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 - 7 = 0$ (1)

Here, the coefficient of the highest power of x is 4 which is a constant and the coefficient of the highest power of y is 1, which is a constant.

Therefore, there are no asymptotes parallel to x-axis and y-axis.

Thus, the curve has only oblique asymptotes.

Let $y = mx + c$ be any asymptote to the curve (1).

Putting $x = 1$ and $y = m$ in the highest power terms (i.e., in third-degree terms), we have

$$\phi_3(m) = 4(1)^3 - (1)^2 m - 4(1)(m)^2 + m^3$$
$$= m^3 - 4m^2 - m + 4$$

Now, putting $x = 1$ and $y = m$ in second-degree terms, we have

$$\phi_2(m) = 3(1)^2 + 2(1)\,m - m^2 = 3 + 2\,m - m^2$$

Now, putting $\phi_3(m) = 0$

i.e., $m^3 - 4m^2 - m + 4 = 0$

or $(m-1)(m^2 - 3m - 4) = 0$

$$m = 1, -1, 4.$$

To find the value of 'c', we have

i.e., $$c = \frac{\phi_{n-1}(m)}{\phi_n'(m)} = -\frac{\phi_2(m)}{\phi_3'(m)}$$

$$c = -\frac{(3+2m-m^2)}{(3m^2-8m-1)}$$

When $m = 1$, $c = -\left(\dfrac{3+2-1}{3-8-1}\right) = \dfrac{2}{3}$

When $m = -1$, $c = -\left(\dfrac{3-2-1}{3+8-1}\right) = 0$

When $m = 4$, $c = -\left(\dfrac{3+8-16}{3\times16-8\times4-1}\right) = \dfrac{1}{3}$

Putting the values of m and c in $y = mx + c$, the three asymptotes of the given curve are

$$y = 1\cdot x + \frac{2}{3}, \text{ i.e., } y = x + \frac{2}{3}$$

$$y = (-1)\,x + 0, \text{ i.e., } y = -x$$

and $$y = 4x + \frac{1}{3}, \text{ i.e., } y = 4x + \frac{1}{3}$$

4.9 CURVE TRACING

Curve tracing is an analytical method to sketch an approximate shape of the curve involving a study of some important points such as symmetry, multiple points, asymptotes, tangents, region, and sign of the first- and second-order derivatives.

4.9.1 Curve Tracing in Cartesian Coordinates

We shall study the following properties of the curve to trace/sketch it.

(i) Symmetry

(a) If the equation of the curve involves even powers of y or if y is replaced by $-y$ and the equation of the curve remains unaltered then the curve is symmetrical about the x-axis.
Example: $y^2 = 4ax$ is symmetrical about the x-axis.

(b) If the equation of the curve involves even powers of x or if x is replaced by $-x$ and the equation of the curve remains unaltered then the curve is symmetrical about the y-axis.
Example: $x^2 = 4ay$ is symmetrical about the y-axis.

(c) If the equation of a curve remains unaltered when x and y are interchanged then the curve is symmetrical about the line $y = x$.
Example: $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

(d) If x is replaced by $-x$ and y is replaced by $-y$ then the equation of the curve remains unaltered, i.e., $f(-x, -y) = f(x, y)$ then the curve is symmetrical in the opposite quadrants.
Example: $xy = c^2$ and $x^2 + y^2 = a^2$ are symmetrical in opposite quadrants.

(e) If the equation of the curve remains unaltered when x and y are replaced by $-x$ and $-y$, i.e., $f(-x, -y) = f(x, y)$ then the curve is symmetrical about the origin.
Example: $x^5 + y^5 = 5a^2 x^2y$ is symmetrical about the origin.

(f) If the equation of the curve remains unaltered when x is replaced by $-y$ and y is replaced by $-x$, i.e., $f(-y, -x) = f(x, y)$ then the curve is symmetrical about the line $y = -x$.
Example: $x^3 - y^3 = 3axy$

(ii) Origin

If the equation of the curve does not contain any constant term then the curve passes through the origin, i.e., (0, 0).
Example: $x^3 + y^3 = 3axy$

(iii) Tangents to the Curve at the Origin

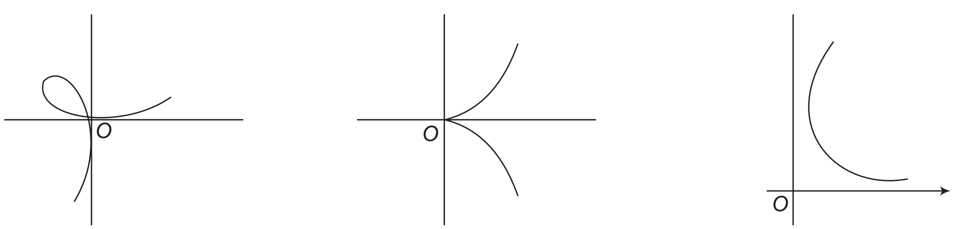
If the equation of the curve passes through the origin then the tangents to the curve at the origin are obtain by equating to zero the lowest degree terms in the equation of the curve.

Example: $x^2 = 4ay$, lowest degree term $4ay = 0$, i.e., $y = 0$, is the tangent to the curve at the origin.

Now, some important points arise:

(a) If there are two tangents at the origin then the origin is a double point.

- When the two tangents are real and distinct then the origin is a node.
- If the two tangents real and coincident then the origin is a cusp.
- If the two tangents are imaginary then the origin is a conjugate point or an isolated point.



(a) Node (b) Cusp (c) Conjugate

Fig. 4.6

(iv) Asymptotes

To obtain the asymptotes of the curve parallel to the axes and the oblique asymptotes.

(v) Region or Extent

To find the region by solving y in terms of x or vice versa.

(a) Real Region It is defined by values of x for which y is defined. The real vertical region is defined by values of y for which x is defined.

(b) Imaginary Region It is defined by values of x for which y becomes imaginary (i.e., undefined) or vice versa.

(vi) Point of Inflection

A point P on a curve is said to be a point of inflection if the curve is concave on one side and convex on the other side of the point P with respect to any line, not passing through the point P. Then, there will be a point of inflection at a point P on the curve if $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$.

(vii) Intersection with the Coordinate Axes

To obtain the points where the curve cuts the coordinate axes, we put $y = 0$ in the equation of the curve to find where the curve cuts the x-axis. Similarly, put $x = 0$ in the equation of the curve to find where the curve cuts the y-axis.

(viii) Sign of First Derivative *dy/dx*

(a) If $\frac{dy}{dx} > 0$ in $[a, b]$ then the curve is increasing in $[a, b]$.

(b) If $\frac{dy}{dx} < 0$ in $[a, b]$ then the curve is decreasing in $[a, b]$.

(c) If $\frac{dy}{dx} = 0$ at $x = a$ then the point (a, b) is a stationary point where maxima and minima can occur.

(ix) Sign of the Second Derivative d^2y/dx^2

(a) If in $\frac{d^2y}{dx^2} > 0$ in $[a, b]$ then the curve is convex or concave upward.

(b) If $\frac{d^2y}{dx^2} < 0$ in $[a, b]$ then the curve is concave downward.

(c) If $\frac{d^2y}{dx^2} = 0$ at a point P, it is called an inflection point where the curve changes the direction of concavity from downward to upward or vice versa.

Example 16 Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution Given $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (1)

Rewrite (1) as

$$(x^{\frac{1}{3}})^2 + (y^{\frac{1}{3}})^2 = (a^{\frac{1}{3}})^2 \tag{2}$$

(i) *Symmetry*: Equation of the curve contains even powers of x as well as y, so the curve is symmetrical about both the axes.

(ii) *Origin*: The curve does not pass through the origin because the curve contains a constant term.

(iii) *Asymptotes*: The curve has no asymptotes.

(iv) *Region*: From (1), we obtain $y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}}$.

If $x > a$ and $x < -a$, y becomes imaginary; hence, the curve does not pass the left side of the line $x = -a$ and the right side of the line $x = a$.
Similarly, if $y > a$ and $y < -a$, x becomes imaginary; hence, the curve does not pass above the line $y = a$ and below the line $y = -a$.

(v) *Points of intersection*: Put $x = 0$ in (1). We get $y = \pm a$; thus, the curve (1) meets the y-axis at the points $(0, a)$ and $(0, -a)$.
Also, putting $y = 0$ in (1), we obtain $x = \pm a$.
Thus, the curve (1) meets the x-axis at the points $(a, 0)$ and $(-a, 0)$.

(vi) *Special points*: From (1), we have

$$y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

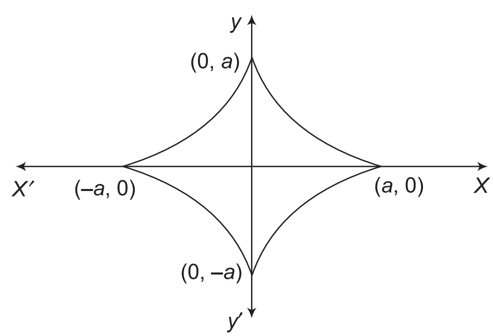


Fig. 4.7

$\therefore$ $\frac{dy}{dx} = 0$, when $y = 0$

Hence, when $y = 0$, $x = \pm a$, the tangents to the curve are parallel to the x-axis at the points $(\pm a, 0)$,

Again, $\frac{dy}{dx} = \infty$, when $x = 0$

From (1), $y = \pm a$, when $x = 0$

Hence, the curve (1) has a tangent parallel to the y-axis at $(0, \pm a)$.

The shape of the curve is as shown in the Fig. 4.7.

Example 17 Trace the curve $x^3 + y^3 = 3axy$.

Solution Given $x^3 + y^3 = 3axy$ (1)

(i) *Symmetry*: If x and y are interchanged, then the curve (1) remains unaltered. Thus, the curve is symmetrical about the line $y = x$.

(ii) *Origin*: Curve (1) passes through (0, 0).

(iii) *Tangent at origin*: Equating the lowest degree term of (1) to zero, i.e., $3axy = 0$

$\Rightarrow \quad x = 0$ and $y = 0$

$\therefore$ tangents at the origin are $x = 0$ and $y = 0$

Thus, the origin is a node (since tangents are real and distinct)

(iv) *Intercept*: There are no x-intercepts and no y-intercepts except the origin (0, 0), because if we put $x = 0$ in (1), we obtain $y = 0$ and if we put $y = 0$ in (1), we obtain $x = 0$.

(v) *Asymptotes*: The coefficients of highest powers of x and y are constant; there are no asymptotes parallel to the coordinate axes.

Now, for oblique asymptotes, putting $x = 1$ and $y = m$, we get

$\phi_3(m) = 1 + m^3 = 0 \quad \Rightarrow m = -1$ (left imaginary root).

$\phi_2(m) = -3am$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(-3am)}{3m^2} = +\frac{a}{m}$$

At $m = -1$, $c = \dfrac{+a}{-1} = -a$

$\therefore$ the required asymptote is $y = mx + c = -x - a$ or $y + x + a = 0$

(vi) *Region*:

(a) If x and y are –ve then the equation of the curve (1) is not satisfied. Thus, no part of the curve exists in the third quadrant.

(b) If x is –ve and y is +ve, or vice versa then the equation of the curve is satisfied. Thus, the curve lies in second and fourth quadrants. Also, if both x and y are +ve then the equation of the curve is satisfied. Hence, the curve lies in the first quadrant.

(vii) *Intersection of the curve with the line $y = x$*

Putting $y = x$ in the curve (1), we obtain.

$$x^3 + x^3 = 3ax^2 \text{ or } x(2x^2 - 3ax) = 0$$

$$\text{or } x = 0 \text{ or } x = \frac{3a}{2} = y$$

Thus, the line $y = x$ meets the curve in two points (0, 0) and $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$.

$$\therefore \qquad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

At $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$, $\qquad \dfrac{dy}{dx} = -1$

$\therefore$ equation of the tangent to the curve at the point $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$

is $\left(y - \dfrac{3a}{2}\right) = (-1)\left(x - \dfrac{3a}{2}\right)$

or $\qquad x + y - 3a = 0$

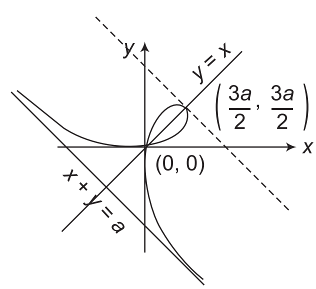


Fig. 4.8

Thus, this tangent is parallel to the asymptote $x + y + a = 0$
Hence, the shape of the curve is as shown in the Fig. 4.8.

Example 18 Trace the curve $y^2(a + x) = x^2(b - x)$.

Solution Given $y^2(a + x) = x^2(b - x)$ (1)

(i) *Symmetry*: The given curve (1) contains an even power of y. Thus, the curve is symmetrical about the x-axis.

(ii) *Origin*: Curve (1) passes through the origin.

(iii) *Tangent at origin*: Equating the lowest degree term to zero,
i.e., $bx^2 - ay^2 = 0 \Rightarrow ay^2 = bx^2$

or $$y = \pm\sqrt{\frac{b}{a}}\,x$$

Thus, the origin is a node.

(iv) *Intercept*: For x-intercept, putting $y = 0$ in (1), we get

$$x = 0 \text{ or } x = b$$

The curve meets the x-axis at (0, 0) and (b, 0).
For y-intercept, putting $x = 0$ in (1), we get $y = 0$
The curve meets the y-axis at (0, 0).

(v) *Asymptotes*: The coefficient of the highest order term of x is constant; thus, no asymptotes exist parallel to the X-axis.
Now, $x = -a$ is the asymptote parallel to the y-axis, obtained by equating the coefficient of y^2, i.e., $(x + a)$, to zero.

(vi) *Region*: From (1), we have

$$y = \pm x\left(\frac{b-x}{a+x}\right)^{\frac{1}{2}}$$

when $x > b$ and $x < -a$, y becomes imaginary; thus, the curve exists only between $x = -a$ and $x = b$.

(vii) *Derivative:* From (1), we have

$$\frac{dy}{dx} = \frac{(-2x^2 - 3ax + 2ab + bx)}{2(a+x)^{\frac{3}{2}} \cdot (b-x)^{\frac{1}{2}}}$$

$\frac{dy}{dx} = \infty$ at $x = b$, so the tangent to the curve at (b, 0) is parallel to the y-axis.

$\therefore$ the curve cuts the x-axis at right angles at (b, 0).
Hence, the shape of the curve is as shown in the Fig. 4.9.

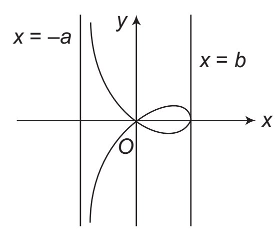


Fig. 4.9

Example 19 Trace the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

Solution Given $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ (1)

(i) *Symmetry*: The curve is symmetrical about the line $y = x$.

(ii) *Origin*: The curve does not pass through the origin (0, 0).

(iii) *Asymptotes*: There are no asymptotes parallel to the coordinate axes.

(iv) *Point of intersection*: Putting $y = 0$ in (1), we get $x = a$; thus, the curve meets the x-axis at $(a, 0)$.

Now, putting $x = 0$ in (1), we get $y = a$; thus, the curve meets the y-axis at $(0, a)$.

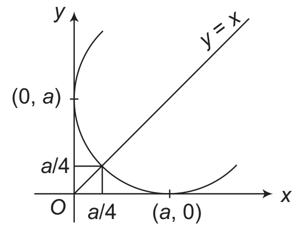


Fig. 4.10

(v) *Region*: The curve exists only between $x > 0$ and $y > 0$.

(vi) *Special points/Derivative*: From (1), we have

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{2}}$$

At $(a, 0)$, $\frac{dy}{dx} = 0$, thus, tangent at $(a, 0)$ is parallel to the x-axis (or y-axis).

At $(0, a)$, $\frac{dy}{dx} = \infty$; thus the tangent at $(0, a)$ is parallel to the y-axis.

Hence, the shape of the curve is as shown in the Fig. 4.10.

Example 20 Trace the curve $x = (y - 1)(y - 2)(y - 3)$.

Solution Given $x = (y - 1)(y - 2)(y - 3)$ (1)

(i) *Symmetry*: Here, the curve has odd powers of x and y, so it is not symmetrical about the axes and opposite quadrant.

(ii) *Origin*: The curve does not passes through the origin.

(iii) *Asymptotes*: The curve has no linear asymptotes, because $y \to \pm\infty$ as $x \to \pm\infty$.

(iv) *Point of intersection*: Put $x = 0$ in (1), we get $y = 1, 2, 3$. Therefore, the curve cuts the y-axis at (0, 1), (0, 2), and (0, 3). Similarly, put $y = 0$, we get $x = -6$, i.e., the curve meets on x-axis at $(-6, 0)$.

(v) *Region*: When $0 < y < 1$ then all the factors are –ve and so x is –ve. When $1 < y < 2$, x is +ve, similarly, when $2 < y < 3$ then x is –ve. At $y = 3$, $x = 0$, when $y > 3$, x is +ve, when $y < 0$, x is –ve. Hence, the shape of the curve is as shown in the Fig. 4.11.

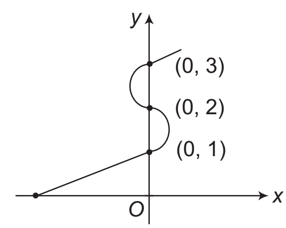


Fig. 4.11

4.9.2 Curve Tracing in Polar Coordinates

The general equation of the curve in polar coordinates (r, θ) in the explicit form is $r = f(\theta)$ or $\theta = f(r)$ and in the implicit form is $f(r, \theta) = 0$. Now, to trace a curve in a polar form of equation, **we adopt the following procedure:**

(i) Symmetry

(a) If the equation of the curve does not change when θ is replaced by $-\theta$, the curve is symmetrical about the initial line.

(b) If r is replaced by $-r$ and the equation of the curve remains unaltered then the curve is symmetrical about the pole and the pole is the center of the curve.

(c) If θ is changed to $-\theta$ and r is changed to $-r$, and the equation of the curve does not change, i.e., $f(-r, -\theta) = f(r, \theta)$,

then the curve is symmetrical about the line $\theta = \dfrac{\pi}{2}$

(ii) Pole

If $r = f(\theta) = 0$ at $\theta = \alpha$ (constant), then the curve passes through the pole and the tangent at the pole is $\theta = \alpha$.

(iii) Asymptotes

If α is a root of the equation $f(\theta) = 0$ then $r \sin(\theta - \alpha) = \dfrac{1}{f'(\alpha)}$ is an asymptote of the polar curve $\dfrac{1}{r} = f(\theta)$.

Region or Extent Find the region where the given curve does not exist. If r is imaginary in $\alpha < \theta < \beta$ then the curve does not exist between the lines $\theta = \alpha$ and $\theta = \beta$.

(iv) Points of Intersection

Points of intersection of the curve with the initial line and the line $\theta = \dfrac{\pi}{2}$ are obtained by putting $\theta = 0$ and $\theta = \dfrac{\pi}{2}$ respectively in the polar equation.

(vi) Direction of Tangent

The tangent at a point (r, θ) on the curve is obtained by $\tan\phi = r\dfrac{d\theta}{dr}$ where ϕ is the angle between radius vector and the tangent.

(vii) Derivative

If $\dfrac{dr}{d\theta} > 0$ then r increases and if $\dfrac{dr}{d\theta} < 0$ then r is decreases.

Example 21 Trace the curve $r = a(1 - \cos\theta)$ (cardioid)

Solution Given $r = a(1 - \cos\theta)$ (1)

(i) *Symmetry:* If θ is replaced by $-\theta$ then the equation of the curve (1) remains unchanged. Thus, the curve is symmetrical about the initial line.

(ii) *Pole:* Putting $r = 0$ in (1), we get $1 - \cos\theta = 0$ or $\theta = 0$. Hence, the curve passes through the pole and the line $\theta = 0$ is tangent to the curve at the pole.

(iii) *Asymptotes:* No asymptote to the curve, because for any finite value of θ, r does not tend to infinity.

(iv) *Points of intersection:* The curve cuts the line $\theta = \pi$ at $(2a, \pi)$

(v) *Tangent:* From (1), we get $\dfrac{dr}{d\theta} = a\sin\theta$

$$\therefore \qquad \tan\phi = r\frac{d\theta}{dr} = \frac{r}{a\sin\theta} = \frac{a(1-\cos\theta)}{a\sin\theta} = \tan\frac{\theta}{2}$$

If $\qquad \dfrac{\theta}{2} = \dfrac{\pi}{2} \Rightarrow \phi = 90°$

Thus, at the point $\theta = \pi$, the tangent to the curve is perpendicular to the radius vector.

(vi) *Region*: The values of θ and r are

θ:	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$2\frac{\pi}{3}$	π
r:	0	$\frac{a}{3}$	a	$3\frac{a}{3}$	$2a$

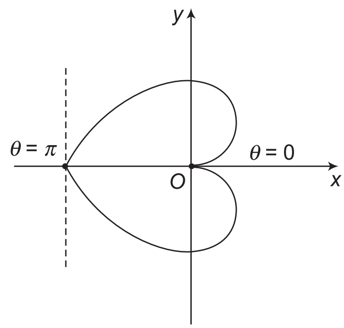


Fig. 4.12

We observed that as θ increases from 0 to π, r increases from 0 to $2a$ and r is never greater than $2a$. Thus, no portion of the curve lies to the left of the tangent at $(2a, 0)$. Since $|r| \le 2a$ then the curve lies entirely within the circle $r = 2a$.

The shape of the curve is shown in Fig. 4.12.

Example 22 Trace the curve $r = a\cos 3\theta$ (three-leaved rose).

Solution Given $r = a\cos 3\theta$ (1)

(i) *Symmetry*: If θ is replaced by $-\theta$ and the equation of the curve remains unchanged; hence, the curve is symmetric about the initial line.

(ii) *Pole*: Putting $r = 0$ in (1), we get $a\cos 3\theta = 0$ or $\cos 3\theta = 0$

or $\qquad 3\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}, \dfrac{11\pi}{2}.$

or $\qquad \theta = \dfrac{\pi}{6}, \dfrac{3\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{9\pi}{6}, \dfrac{11\pi}{6}.$

Hence, the curve passes through the pole.

Thus, the tangents to the curve at pole 0 are given by the six lines, i.e.,

$$\theta = \frac{\pi}{6}, \theta = \frac{\pi}{2}, \theta = \frac{5\pi}{6}, \theta = \frac{7\pi}{6}, \theta = \frac{3\pi}{2}, \text{ and } \theta = \frac{11\pi}{6}.$$

Hence, the pole is a node.

(iii) *Asymptotes:* No asymptote, because r is finite for any value of θ.

(iv) *Points:* Curve intersects the initial line $\theta = 0$ at the point $(a, 0)$ only.

(v) *Region*: Curve lies within a circle $r = a$, since the maximum value of $\cos 3\theta$ is one.

(vi) *Tangent*: $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{\cos 3\theta}{-3\sin 3\theta}.$

At $\theta = 0$, $\tan\phi = \infty$

Thus, the tangent at the point $(a, 0)$ is perpendicular to the initial line.

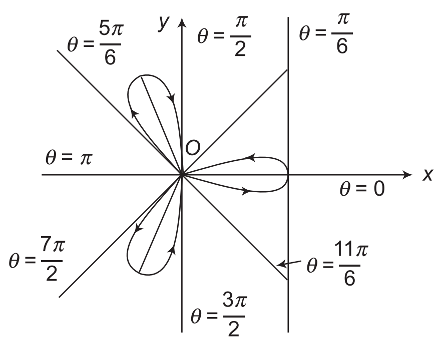


Fig. 4.13

(vii) For $n = 3$, the curve has 3 loops.

Example 23 Trace the curve $r = a \cos 2\theta$, $a > 0$. [Four-leaved rose]

Solution Given $r = a \cos 2\theta$ (1)

Here, $n = 2$, the curve consists of $2n = 2.2 = 4$ equal loops.

(i) *Symmetry*:

(a) The curve is symmetrical about the initial line, because θ is changed by $-\theta$ and the equation of the curve remains unchanged.

(b) If $r(\pi - \theta) = a \cos 2(\pi - \theta) = a \cos 2\pi$

$\cos 2\theta = a \cos 2\theta = r(\theta)$.

$\therefore$ The curve is symmetric about the line $\theta = \dfrac{\pi}{2}$

(ii) *Pole*: The curve passes through the pole, when

$$r = a \cos 2\theta = 0 \text{ or } \cos 2\theta = 0 \text{ so that}$$

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

or $$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

Hence, the tangents to the curve at the pole are the lines

$$\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}, \theta = \frac{5\pi}{4} \text{ and } \theta = \frac{7\pi}{4}.$$

(iii) *Asymptotes*: No asymptote, since r is finite for any value of θ.

(iv) *Variation of r*: The following table gives values of $r = a \cos 2\theta$ for different values of θ. $\left(\dfrac{1}{\sqrt{2}} = 0.70\right)$

$\theta = 0$	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π	$\frac{9\pi}{8}$	$\frac{5\pi}{4}$	$\frac{11\pi}{8}$	$\frac{3\pi}{2}$	$\frac{13\pi}{8}$	$\frac{7\pi}{4}$	$\frac{15\pi}{8}$	2π
$2\theta = 0$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{8}$	3π	$\frac{13\pi}{4}$	$\frac{7\pi}{2}$	$\frac{15\pi}{4}$	4π
$r = a$	$0.7a$	0	$-0.7a$	$-a$	$-0.7a$	0	$0.7a$	a	$0.7a$	0	$0.7a$	$-a$	$-0.7a$	0	$0.7a$	a

As θ varies from 0 to $\dfrac{\pi}{4}$, r varies from a to 0.

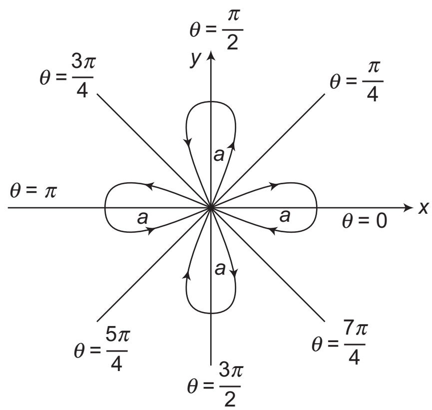


Fig. 4.14

Example 24 Trace the curve $r^2 \cos 2\theta = a^2$.

Solution Given $r^2 \cos 2\theta = a^2$ (1)

or $$r^2(\cos^2 \theta - \sin^2 \theta) = a^2$$

or $$x^2 - y^2 = a^2, \text{ (2) where } x = r \cos \theta, y = r \sin \theta$$

Equation (2) represents a rectangular hyperbola.

(i) *Symmetry*: The curve is symmetrical about both the axes.

(ii) *Origin*: The curve does not passes through the origin (0, 0).

(iii) *Asymptotes*: No asymptotes parallel to coordinate axes and the oblique asymptotes are $y = x$ and $y = -x$.

(iv) *Region*: The equation of the curve can be written as $y^2 = x^2 - a^2$. when $0 < x < a$, y^2 is –ve, i.e., y is imaginary.

$\therefore$ The curve does not lies in the region $0 < x < a$. But when $x > a$, y^2 is +ve and y is a real so the curve exists in the region $x > a$. Further, when $x \to \infty$, $y^2 \to \infty$.

(v) *Tangents*: Shifting the origin to $(a, 0)$, we obtain

$(x + \text{a})^2 - y^2 = a^2$ or $x^2 - y^2 + 2ax = 0$.

$\therefore$ The tangents at $(a, 0)$ is given by $2ax = 0$ or $x = 0$ the tangent at $(a, 0)$ is the line parallel to the y-axis.

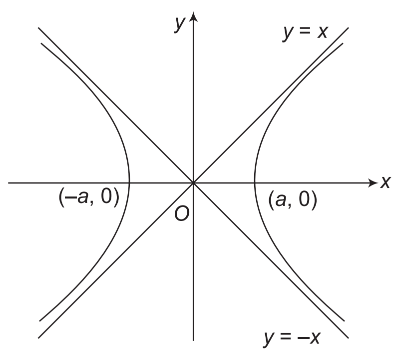


Fig. 4.15

4.9.3 Curve Tracing in Parametric Form

If the equation of the curve are $x = f(t)$ and $y = g(t)$, then eliminate the parameter and obtain a Cartesian equation of the curve. Then trace the curve as dealt with in case of the Cartesian equation. When the parameter (t) cannot be eliminated easily then a series of values are given to 't' and the corresponding values of x, y, and $\frac{dy}{dx}$ are found.

We explain the process with the help of an example.

Example 25 Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution Given $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (1)

(i) The parametric equations of the curve are $x = a\cos^3 t$, $y = a\sin^3 t$.

$\therefore$ $|x| \le a$ and $|y| \le a$.

$\Rightarrow$ The curve lies between the lines $x = \pm a$ and $y = \pm a$.

(ii) The equation of the curve (1) can be rewritten as

$$\left(\frac{x^2}{a^2}\right)^{\frac{1}{3}} + \left(\frac{y^2}{a^2}\right)^{\frac{1}{3}} = 1 \tag{2}$$

This equation shows the curve is symmetric about the line $y = x$.

(iii) *Asymptotes*: No asymptotes

(iv) *Points of intersection*: The curve cuts the x-axis at $(a, 0)$ and $(-a, 0)$, and meets the y-axis at $(0, a)$ and $(0, -a)$.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{t=0} = \left[\frac{dy}{dt}\Big/\frac{dx}{dt}\right]_{t=0} = (-\tan t)_{t=0} = 0$$

Thus, at the point $(a, 0)$, the x-axis is the tangent to the curve.
Similarly, at the point $(0, a)$, the y-axis is the tangent to the curve.
The shape of the curve is as shown in Fig. 4.16.

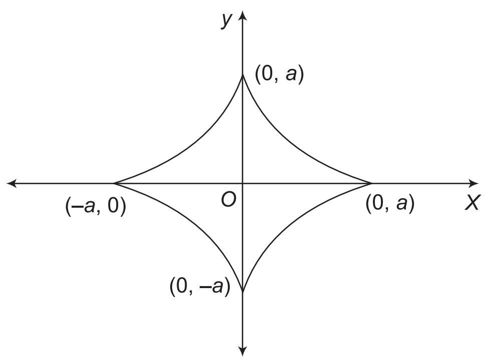


Fig. 4.16

EXERCISE 4.2

1. Trace the curve $a^2y^2 = x^2(a^2 - x^2)$.
2. Trace the curve $y^2(2a - x) = x^3$ (cissoid).
3. Trace the curve $xy = a^2(a - x)$.

4. Trace the curve $y = x^3 - 12x - 16$.
5. Trace the curve $9ay^2 = x(x - 3a)^2$.
6. Trace the curve $r = a \cos 2\theta$
7. Trace the curve $r^2 = a^2 \sin 2\theta$ (lemniscate).
8. Trace the curve $r = \dfrac{a \sin^2 \theta}{\cos \theta}$ (cissoid).
9. Trace the curve $r = a(1 + \sin \theta)$.
10. Trace the curve $x = a(t + \sin t)$, $y = a(1 + \cos t)$.
11. Trace the curve $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$.

Answers

1.

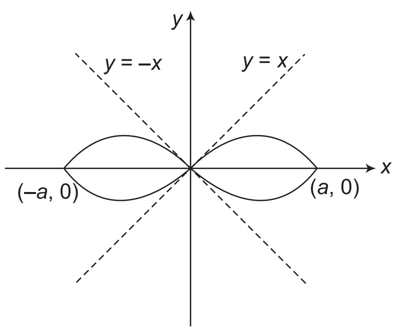


Fig. 4.17

2.

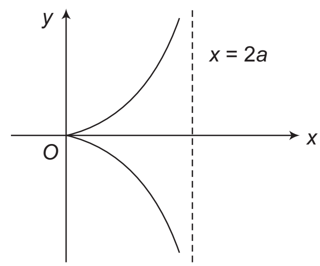


Fig. 4.18

3.

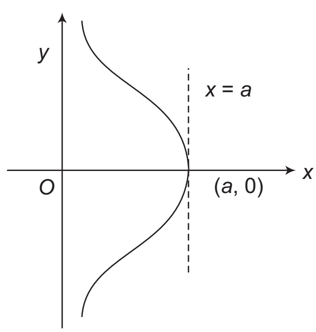


Fig. 4.19

4.

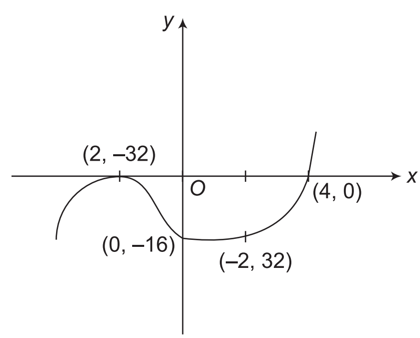


Fig. 4.20

5.

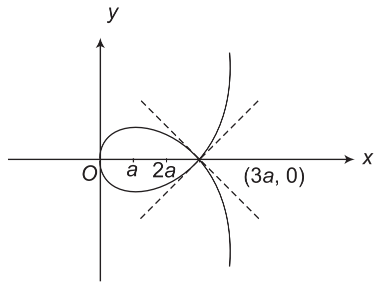


Fig. 4.21

6.

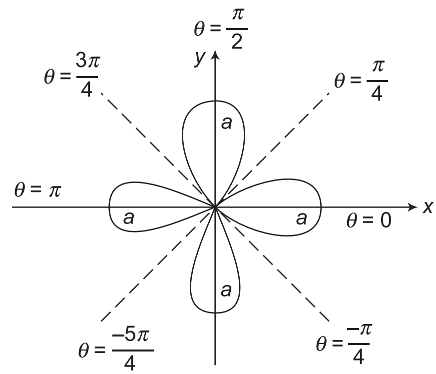


Fig. 4.22

7.

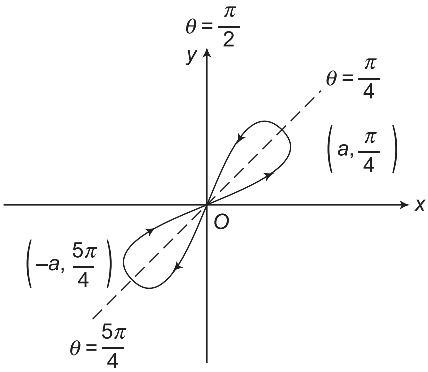


Fig. 4.23

8.

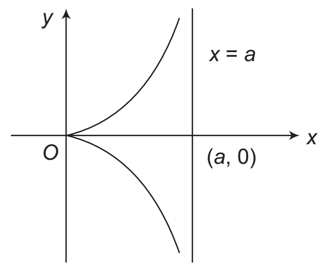


Fig. 4.24

9.

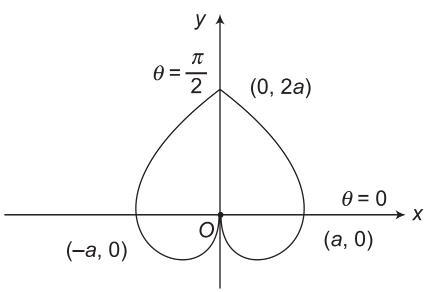


Fig. 4.25

10.

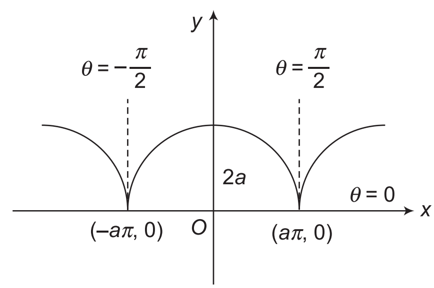


Fig. 4.26

11.

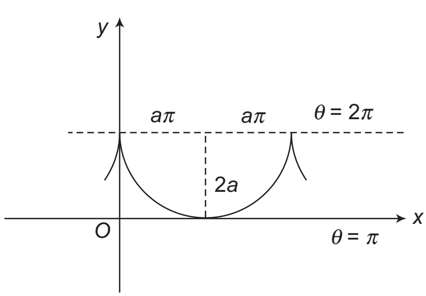


Fig. 4.27

SUMMARY

1. Maxima and Minima of Functions of two Independent Variables

Let $f(x, y)$ be a function of two independent variables x and y, which is continuous for all values of x and y in the neighborhood of (a, b), i.e., $(a + h, b + k)$ be a point in its neighborhood and lies inside the region R. We define the following:

(i) The point (a, b) is called a point of relative (or local) minimum, if

$$f(a, b) \le f(a + h, b + k) \text{ for all } h, k.$$

(ii) The point (a, b) is called a point of relative (or local) maximum, if

$$f(a, b) \ge f(a + h, b + k) \text{ for all } h, k.$$

2. Necessary Conditions for the Existence of Maxima or Minima of $f(x, y)$ at the Point (a, b)

Statement: The necessary conditions for the existence of a maxima or a minima of $f(x, y)$ at the point (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$, where $f_x(a, b)$ and $f_y(a, b)$ respectively denote the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = a$, $y = b$.

3. Sufficient Conditions for Maxima and Minima (Lagrange's Condition for Two Independent Variables)

Let $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Suppose $f_{xx}(a, b) = r$, $f_{xy}(a, b) = s$ and $f_{yy}(a, b) = t$.

Then

(i) If $r > 0$ and $(rt - s^2) > 0$, $f(x, y)$ is minimum at (a, b).

(ii) If $r < 0$ and $(rt - s^2) > 0$, $f(x, y)$ is maximum at (a, b).

(iii) If $(rt - s^2) < 0$, $f(x, y)$ is neither maximum nor minimum at (a, b), i.e., (a, b) is a saddle point.

(iv) If $(rt - st) = 0$, it is a doubtful case.

4. Maximum and Minimum Values for a Function $f(x, y, z)$

Let $u = f(x, y, z)$. For the maximum/minimum of u is $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}$

Now, find u_{xx}, u_{yy}, u_{zz}, u_{yz}, u_{zx}, u_{xy}. They are denoted by A, B, C, F, G, H respectively.

Again, find $D_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix}$ and $D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

Thus, the given function will have a minimum if $A > 0$, $D_1 > 0$ and $D_2 > 0$, and will have a maximum if $A > 0$, $D_1 > 0$ and $D_2 < 0$.

If these above conditions are not satisfied, we have neither maximum nor minimum.

5. Curve Tracing

Curve tracing is an analytical method to sketch an approximate shape of the curve involving a status of some important points such as symmetry, multiple points, asymptotes, tangents, region, and sign of the first- and second-order derivatives.

OBJECTIVE-TYPE QUESTIONS

1. With usual notations, a function $f(x, y)$ has a maximum at (a, b), if
(a) $r > 0, rt - s^2 > 0$ (b) $r > 0, rt - s^2 < 0$
(c) $r < 0, rt - s^2 < 0$ (d) $r < 0, rt - s^2 > 0$

2. With usual notations, a function $f(x, y)$ has a saddle point (a, b), if
(a) $rt - s^2 = 0$ (b) $rt - s^2 < 0$
(c) $rt - s^2 > 0$ (d) $rt = s$

3. The stationary points of $f(x, y)$ are given by
(a) $f_x = 0, f_y = 0$ (b) $f_{xy} = 0$
(c) $f_{xx} = 0$ and $f_{yy} = 0$ (d) $f_{xx}^2 = 0 + f_{yy}^2 = 0$

4. The curve given by the equation $x^3 + y^3 = 3axy$ is
(a) symmetrical about x-axis
(b) symmetrical about y-axis
(c) symmetrical about $y = x$ line
(d) tangent to $x = y = a/3$
[GATE (CE) 1997]

5. The maxima and minima of the function $f(x) = 2x^3 + 15x^2 + 36x + 10$ occur, respectively if
(a) $x = 3$ and $x = 2$ (b) $x = 1$ and $x = 3$
(c) $x = 2$ and $x = 3$ (d) $x = 3$ and $x = 4$
[GATE (CE) 2000]

6. The function $f(x, y) = 2x^2 + 2xy - y^3$ has
(a) only one stationary point at (0, 0)
(b) two stationary points (0, 0) and $\left(\frac{1}{6}, -\frac{1}{3}\right)$
(c) two stationary points at (0, 0) and $\left(\frac{1}{6}, \frac{1}{3}\right)$
(d) two stationary points at (0, 0) and $\left(-\frac{1}{3}, \frac{-1}{3}\right)$
[GATE (CE) 2002]

7. The function $f(x) = 2x^3 - 3x^2 - 36x + 2$ has its maxima at
(a) $x = -2$ only (b) $x = 0$ only
(c) $x = 3$ only
(d) both $x = -2$ and $x = -3$
[GATE (Civil) 2004]

8. Consider function $f(x) = (x^2 - 4)^2$, where x is real number then the function has
(a) only one minima (b) only two minima
(c) three minima (d) three maxima
[GATE (EE) 2008]

9. The continuous function $f(x, y)$ is said to have saddle point at (a, b), if
(a) $f_x(a, b) = f_y(a, b) = 0, f_{xx}f_{yy} - (f_{xy})^2 < 0$ at (a, b)
(b) $f_x(a, b) = 0,\ f_y(a, b) = 0, (f_{xy})^2 - f_{xx}f_{yy} > 0$ at (a, b)
(c) $f_x(a, b) = 0,\ f_y(a, b) = 0, f_{xy}$ and $f_{yy} < 0$ at (a, b)
(d) $f_x(a, b) = 0,\ f_y(a, b) = 0, (f_{xy})^2 - f_{xx}f_{yy} = 0$ at (a, b)
[GATE (CE) 2008]

10. For real values of x, the minimum value of the function $f(x) = \exp(x) + \exp(-x)$ is
(a) 2 (b) 1
(c) 0.5 (d) 0
[GATE (ECE) 2008]

11. Maxima and minima occur
(a) simultaneously (b) once
(c) alternately (d) rarely

12. The maximum value of $\frac{1}{\sqrt{2}}(\sin x - \cos x)$ is
(a) 1 (b) $\sqrt{2}$
(c) $1/\sqrt{2}$ (d) 3

13. If $x + y = k$, $x > 0$, $y > 0$ then xy is the maximum, when
(a) $x = ky$ (b) $kx = y$
(c) $x = y$ (d) none of these

14. Curve is symmetrical about origin means
(a) curve has the same shape in the first and fourth quadrant
(b) curve has the same shape in the first and second quadrant
(c) curve has the same shape in the first and third quadrant
(d) curve has the same shape in all the four quadrants

15. The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is
(a) 21 (b) 25
(c) 41 (d) 46
[GATE (EE) 2014]

16. The maximum value of $f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is _____.
[GATE (EC) 2014]

17. Let $f(x) = xe^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is

(a) e^{-1} (b) e

(c) $1 - e^{-1}$ (d) $1 + e^{-1}$

[GATE (EE) 2014]

18. Maximum of the real value function of $f(x) = (x-1)^{2/3}$ occurs at x equal to

(a) $-\infty$ (b) 0

(c) 1 (d) ∞

[GATE (EE) 2014]

19. The maximum value of the function $f(x) = \ln(1 + x) - x$ (where $x > -1$) occurs at x = _____.

[GATE (EC) 2014]

20. The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval $[-3, 3]$ is

(a) 20 (b) 28

(c) 16 (d) 32

[GATE (EE) 2014]

ANSWERS

1. (d)	2. (b)	3. (a)	4. (c)	5. (c)	6. (c)	7.(d)	8. (c)	9. (a)	10. (a)
11. (c)	12. (d)	13. (b)	14. (c)	15. (c)	16. (6)	17.(a)	18. (c)	19. (0)	20. (b)

5 Integral Calculus

5.1 INTRODUCTION

In the present chapter, we examine two processes and their relation to one another. In the first process, we determine functions from their derivatives and in the second process, we arrive at exact formulas for such things as area and volume by successive approximation. Both processes are called *integration*.

There are two types of the integration:

(i) **Indefinite integral**

(ii) **Definite integral**

5.2 INDEFINITE INTEGRAL

A function $f(x)$ is an antiderivative or primitive of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of $f(x)$.

The collection of all antiderivatives of f is called the *indefinite integral* of $f(x)$ and is denoted by $\int f(x)\, dx$.

Thus,

$$\frac{d}{dx}[F(x) + C] = f(x) \Leftrightarrow \int f(x)dx = F(x) + C \qquad (1)$$

where C is an arbitrary constant known as the *constant of integration*.

Here, $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration, and dx is the element of integration or differential of x.

5.3 SOME STANDARD RESULTS ON INTEGRATION

(i) $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$

(ii) $\int k \cdot f(x)dx = \int kf(x)\, dx$, where k is a constant

(iii) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\, dx$

(iv) $\int -f(x)dx = -\int f(x)dx$

5.4 DEFINITE INTEGRAL

Let $F(x)$ be the antiderivative of a function $f(x)$ defined on $[a, b]$, i.e., $\dfrac{d}{dx} F(x) = f(x)$.

Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\, dx$ and is defined as $[F(b) - F(a)]$.

i.e.,
$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{2}$$

The numbers a and b are called the *limits of integration*, 'a' is called the *lower limit*, and 'b' is called the *upper limit*. The interval $[a, b]$ is called the *interval of integration*.

5.5 GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

If $f(x) > 0$ for all $x \in [a, b]$ then $\int_a^b f(x)\, dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x-axis, and the straight lines $x = a$ to $x = b$ [see Fig. 5.1(a)], i.e., $\int_a^b f(x)\, dx$ = Area $ABCD$.

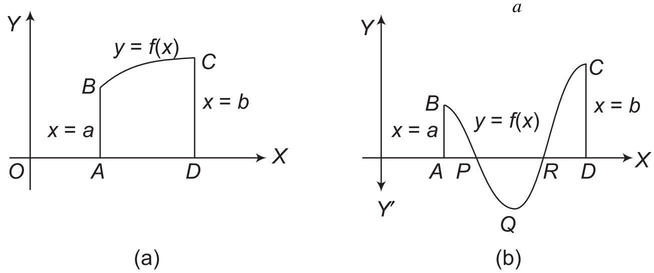


Fig. 5.1

In general, $\int_a^b f(x)\, dx$ represents an algebraic sum of the areas of the figures bounded by the curve $y = f(x)$, the X-axis, and the straight lines $x = a$ and $x = b$.

The areas above the x-axis are taken with a plus sign and the areas below the x-axis are taken with a minus sign (see Fig. 5.1(b)], i.e.,

$$\int_a^b f(x)\, dx = \text{Area } ABP - \text{Area } PQR + \text{Area } RCD$$

5.6 LEIBNITZ'S RULE OF DIFFERENTIATION UNDER THE SIGN OF INTEGRATION

Let $F(x, t)$ and $\dfrac{\partial F}{\partial x}$ be continuous functions of both x and t, and let the first derivative of $G(x)$ and $H(x)$ be continuous.

Then, $$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\,dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\,dt + F[x, H(x)]\cdot\frac{dH}{dx} - F[x, G(x)]\cdot\frac{dG}{dx} \quad (3)$$

Particular Case

If F and G are absolute constant then Eq. (3) reduces to

$$\frac{d}{dx}\int_{G}^{H} F(x,t)\,dt = \int_{G}^{H} \frac{\partial F}{\partial x}\,dt \quad (4)$$

5.7 REDUCTION FORMULA FOR THE INTEGRALS

A formula which connects an integral with another in which the integrand is of the same type, but is of lower degree or order, or is otherwise easier to integrate, is called a *reduction formula*.

5.7.1 Reduction Formula for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$

(i) $$\int \sin^n x\,dx = \frac{-\sin^{n-1} x\cos x}{n} + \frac{(n-1)}{n}\int \sin^{n-2} x\,dx$$

(ii) $$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1} x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} x\,dx$$

Deduction

We have

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \dfrac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}; & \text{if } n \text{ is odd} \\[2ex] \dfrac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}\cdot\dfrac{\pi}{2}; & \text{if } n \text{ is even} \end{cases}$$

Proof Using reduction formula for $\int \sin^n x\,dx$, we have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx$$

$$= -\left[\frac{\sin^{n-1} x\cos x}{n}\right]_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx^2$$

$$= \frac{(n-1)}{n} I_{n-2}$$

If n is odd, we have

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$I_5 = \frac{4}{5} I_3, \quad I_3 = \frac{2}{3} I_1 = \frac{2}{3}\int_0^{\pi/2} \sin x \, dx = \frac{2}{3}$$

Hence,

$$I_n = \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}$$

If n is even, we have

$$I_{n-2} = \frac{n-3}{n-2} \cdot I_{n-4}, I_{n-4} = \frac{n-5}{n-4} I_{n-6}, I_4 = \frac{3}{4} I_2$$

and $$I_2 = \frac{1}{4} I_0 = \frac{1}{2}\int_0^{\pi/2} \sin^0 x dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

Hence,

$$I_n = \frac{(n-1)(n-3)(n-5)\cdots 3}{n(n-2)(n-4)\cdots 4 \cdot 2} \cdot \frac{\pi}{2}$$

Example 1 Evaluate $\int_0^{\pi/2} \sin^6 x \, dx$.

Solution Here, $n = 6$(even), so we have

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 3}{n(n-2)(n-4)\cdots 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$\therefore \quad \int_0^{\pi/2} \sin^6 x \, dx = \frac{(6-1)(6-3)(6-5)\cdots 3 \cdot 1}{6 \cdot (6-2)(6-4)\cdots 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{5 \cdot 3 \cdot 1 \cdots}{6 \cdot 4 \cdot 2 \cdots} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi}{32}$$

Example 2 Evaluate $\int_0^{\pi/2} \cos^7 x \, dx$.

Solution Here, $n = 7$ (odd power)

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-1)\cdots 3}$$

$$\therefore \quad \int_0^{\pi/2} \cos^7 x \, dx = \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 6} = \frac{8}{35}$$

5.7.2 Reduction Formula for $\int \sin^m x \cos^n x dx$

$$\int \sin^m x \cdot \cos^n x\, dx = \int \sin^m x \cdot \cos x \cdot \cos^{n-1} x\, dx$$

$$\frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \left(\frac{n-1}{m+1}\right) \int \sin^{m+1} x \cos^{n-2} x\, dx \qquad \text{[Integration by parts]}$$

$$= \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+1} + \left(\frac{n-1}{m+1}\right) \int \sin^m x \cos^{n-2} x\,(1-\cos^2 x)dx$$

$$= \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+1} + \left(\frac{n-1}{m+1}\right) \int \sin^m x \cos^{n-2} x\, dx - \left(\frac{n-1}{m+1}\right) \int \sin^m x \cos^n dx$$

Hence,

$$\int \sin^m x \cdot \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+1} \int \sin^m x \cdot \cos^{n-2} x\, dx$$

Deduction

(i) If m and n are +ve integers then

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

(ii) If n be an even +ve integer and m be an odd +ve integer then

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot \Gamma(n+1)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

(iii) If n is an odd and m is an even +ve integers then the integral can be transformed into the one considered in (ii) by writing $x + \frac{\pi}{2}$ for x.

Hence,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

Example 3 Evaluate $\int_0^{\pi/2} \sin^6 \theta \cdot \cos^2 \theta \, d\theta$

Solution Here, $m = 6$, $n = 2$

$$\int_0^{\pi/2} \sin^6 \theta \cdot \cos^2 \theta \, d\theta = \frac{\Gamma\left(\frac{6+1}{2}\right) \cdot \Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{6+2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{7}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{2\,\Gamma(5)} \qquad \left[\because \Gamma(n+1) = n\Gamma(n)\right]$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{5 \cdot \cancel{3} \cdot \pi}{16 \times 8 \times \cancel{3} \cdot 2}$$

$$= \frac{5\pi}{256}$$

Example 4 Evaluate $\int_0^a x^2 (a^2 - x^2)^{3/2} \, dx$.

Solution

$$\int_0^a x^2 (a^2 - x^2)^{3/2} \, dx = \int_0^{\pi/2} a^6 \sin^2 \theta \cdot \cos^4 \theta \, d\theta$$

Put $x = a \sin \theta$

$dx = a \cos \theta \, d\theta$

$$= a^6 \cdot \frac{\Gamma\left(\frac{2+1}{2}\right) \cdot \Gamma\left(\frac{4+1}{2}\right)}{2 \cdot \Gamma\left(\frac{2+4+2}{2}\right)}$$

$$= a^6 \cdot \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{5}{2}\right)}{2 \cdot \Gamma(4)}$$

$$= \frac{\pi a^6}{32}$$

Example 5 Evaluate $\int_0^{2a} x^{5/2} \cdot \sqrt{2a - x} \cdot dx$.

Solution Put $x = 2a \sin^2\theta \Rightarrow dx = 4a \sin \theta \cos \theta \, d\theta$.

$$\therefore \qquad \int_0^{2a} x^{5/2} \cdot \sqrt{2a-x} \cdot dx = \int_0^{\pi/2} \sqrt{2a} \cdot (2a \sin^2 \theta)^{5/2} \quad (4a \sin\theta \cos\theta) \cdot \cos\theta \, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \sin^6 \theta \cdot \cos^2 \theta d\theta$$

$$= 32\, a^4 \cdot \frac{\Gamma\left(\frac{6+1}{2}\right) \cdot \Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{6+2+2}{2}\right)}$$

$$= 32\, a^4 \cdot \frac{\Gamma\left(\frac{7}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{2 \cdot \Gamma(5)}$$

$$= 32\, a^4 \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{\cancel{32}\, a^4 \cdot \pi \times 5 \cdot \cancel{3}}{\cancel{16} \times 2 \times 4 \times \cancel{3} \times \cancel{2}}$$

$$= \frac{5\, a^4\, \pi}{8}$$

5.7.3 Reduction Formula for $\sec^n x$ and $\operatorname{cosec}^n x$

(i) $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \cdot \tan x}{n-1} + \left(\frac{n-2}{n-1}\right) \int \sec^{n-2} x \, dx$$

(ii) $$\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cdot \cot x}{n-1} + \left(\frac{n-2}{n-1}\right) \int \operatorname{cosec}^{n-2} x \, dx$$

Example 6 Evaluate $\int \sec^4 x \, dx$.

Solution

$$\int \sec^4 x \, dx = \frac{\sec^2 x \cdot \tan x}{3} + \frac{2}{3} \int \sec^2 x \, dx$$

$$= \frac{\sec^2 x \cdot \tan x}{3} + \frac{2}{3} \tan x$$

$$= \frac{1}{3} [\tan x \sec^2 x + 2 \tan x]$$

Example 7 Evaluate $\int \operatorname{cosec}^4 x\, dx$.

Solution

$$\begin{aligned}\int \operatorname{cosec}^4 x\, dx &= -\frac{\operatorname{cosec}^2 x \cdot \cot x}{3} + \frac{2}{3}\int \operatorname{cosec}^2 x\, dx \\ &= -\frac{\operatorname{cosec}^2 x \cdot \cot x}{3} + \frac{2}{3}(-\cot x) \\ &= -\frac{1}{3}[\cot x \cdot \operatorname{cosec}^2 x + 2\cot x]\end{aligned}$$

5.7.4 Integration of $\tan^n x$ and $\cot^n x$

(i) $\int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

(ii) $\int \cot^n x\, dx = \dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$

Example 8 Evaluate $\int \tan^3 x\, dx$.

Solution

$$\begin{aligned}\int \tan^3 x\, dx &= \frac{\tan^2 x}{2} - \int \tan x\, dx \\ &= \frac{\tan^2 x}{2} - \log \sec x \\ &= \frac{1}{2}\tan^2 x + \log \cos x\end{aligned}$$

Example 9 Evaluate $\int \cot^4 x\, dx$.

Solution

$$\begin{aligned}\int \cot^4 x\, dx &= -\frac{\cot^3 x}{3} - \int \cot^2 x\, dx \\ &= -\frac{1}{3}\cot^3 x - \int (\operatorname{cosec}^2 x - 1)\, dx \\ &= -\frac{1}{3}\cot^3 x - \cot x + x\end{aligned}$$

EXERCISE 5.1

1. Evaluate

(i) $\int_0^{\pi/2} \sin^4 \theta\, d\theta$ (ii) $\int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \cdot d\theta$ (iii) $\int_0^{\pi/2} \sin^5 x \cos^8 x \cdot dx$

2. Show that $\int_0^a x^4 \cdot \sqrt{(a^2 - x^2}\, dx = \frac{\pi a^6}{32}$.

3. Prove that $\int_0^1 x^{3/2} \cdot (1-x)^{3/2}\, dx = \frac{3\pi}{128}$.

4. Show that $\int_0^1 x^2 (1-x^2)^{3/2}\, dx = \frac{\pi}{32}$.

5. Evaluate $\int_0^{2a} (x)^{9/2} (2a-x)^{-1/2}\, dx$.

Answers

1. (i) $\frac{3\pi - 8}{32}$ (ii) $\frac{\pi}{32}$ (iii) $\frac{8}{1827}$

5. $\frac{63}{8} \pi a^5$

5.8 AREAS OF CURVES

5.8.1 Areas of Curves given by Cartesian Equations

Area bounded by the curve $y = f(x)$, the axis of x, and the ordinates $x = a$ and $x = b$ where $f(x)$ is single-valued, finite, and a continuous function of x in (a, b) is given by

$$\int_a^b f(x)\, dx. \text{ or } \int_a^b y\, dx$$

Proof Let DC be the curve $y = f(x)$, and AD and BC be the two ordinates at $x = a$ and $x = b$.

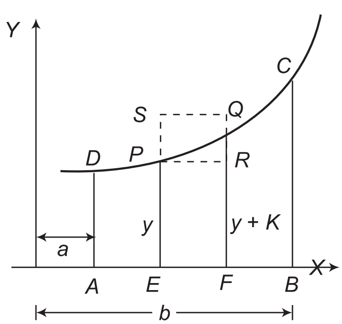


Fig. 5.2

Let P be any point (x, y) on the curve and let PE be its ordinate.

Then the area *AEPD* is some function of x, say $\phi(x)$.

Let Q be any other point $(x + h, y + k)$ on the curve, and let QF be its ordinate. Let PR and QS be the perpendiculars from P and Q to FQ and EP produced respectively.

Then the area $AFQD = \phi(x + h)$.

Hence,
$$\frac{\phi(x+h)-\phi(x)}{h} = \frac{\text{Area } AFQD - \text{Area } AEPD}{h}$$
$$= \frac{\text{Area } EFQP}{h} \tag{5}$$

Now, the area of the rectangle $EFRP = y \cdot h$ and that of the rectangle $EFQS = (y + k) \cdot h$.

Assuming the area *EFQP* lies in magnitude between the areas of the rectangles *EFRP* and *EFQS*, it follows from (5) that

$\dfrac{\phi(x+h)-\phi(x)}{h}$ lies between $y + k$ and y

$$\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h} = y$$

i.e.,
$$\frac{d\phi(x)}{dx} = f(x) \tag{6}$$

Consequently, if $F(x)$ is any known integral of $f(x)$, $\phi(x) = F(x) + C$ (7)
where C is any constant.

For the constant 'C', put $x = a$ in (7).

$$\phi(a) = 0 \Rightarrow F(a) + C = 0$$

$\Rightarrow$ $$C = -F(a)$$

$\therefore$ $$\phi(x) = F(x) - F(a)$$

Hence, the area $ABCD = \phi(b) = F(b) - F(a)$

$$= \int_a^b f(x)\,dx$$ **Proved.**

Note 1 Area bounded by the curve $x = \phi(y)$, the axis of y, and the two abscissas at points $y = a$ and $y = b$ is given by

$$\int_a^b x\,dy. \text{ or } \int_a^b \phi(y)\,dy$$

Note 2 Area bounded by the two curves $y = f_1(x)$ and $y = f_2(x)$, and the ordinates at $x = a$ and $x = b$ is

$$\int_a^b [f_1(x) - f_2(x)]\,dx$$

Example 10 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

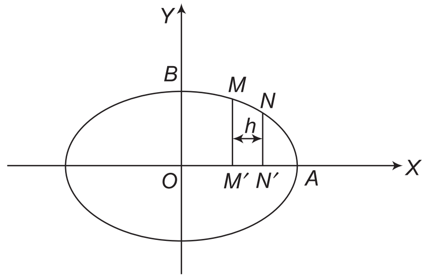


Fig. 5.3

Solution The curve is symmetrical about both the axis; hence, the whole area

$$= 4 \times \text{Area in the first quadrant}$$

$$= 4 \times \text{Area } (AOB)$$

Now, area of $AOB = \int_{x=0}^{a} y \, dx$

$$= \int_{x=0}^{a} \left(\frac{b}{a}\right) \sqrt{(a^2 - x^2)} \, dx$$

$$= \frac{b}{a} \left[\frac{x}{2} \cdot \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{2a} [0 + a^2 \sin^{1} (1)]$$

$$= \frac{b}{2a} \left[a^2 \frac{\pi}{2}\right]$$

$$= \frac{\pi ab}{4}$$

Hence, the required area = $4 \times \frac{\pi ab}{4} = \pi ab$

Example 11 Find the area included between the cure $xy^2 = 4a^2(2a - x)$ and its asymptote.

Solution The curve is symmetrical about the x-axis and is as shown in Fig. 5.4. It cuts the x-axis at $x = 2a$. The asymptote is the y-axis.

Hence, the required area

$$= 2\int_0^{2a} y\,dx$$

$$= 4a\int_0^{2a} \frac{\sqrt{2a-x}}{\sqrt{x}}\,dx \qquad \text{Putting } x = 2a\sin^2\theta,\ dx = 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 16a^2 \frac{\left|\frac{3}{2}\cdot\right|\frac{1}{2}}{2\left|2\right.}$$

$$= 16a^2 \frac{\frac{1}{2}\pi}{2}$$

$$= 4\pi a^2$$

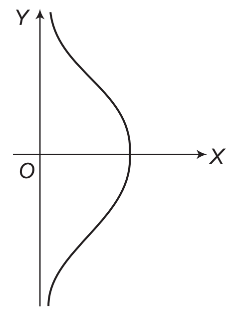


Fig. 5.4

Example 12 Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.

Solution Let S be the point of intersection in the first quadrant of the circle and the ellipse shown in Fig. 5.5.

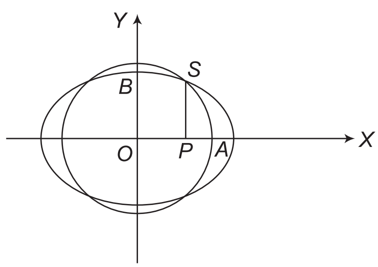


Fig. 5.5

The required area

$$= 4 \times \text{Area } OASB$$

$$= 4 \times (\text{Area } OPSB + \text{Area } PAS)$$

$$= 4\int_0^{\sqrt{7/3}} \frac{1}{2}\sqrt{(9-x^2)}\,dx + 4\int_{\sqrt{7/3}}^{2} \sqrt{(4-x^2)}\,dx.$$

Solving $x^2 + y^2 = 4$ and $x^2 + 4y^2 = 9$, $x = \sqrt{7/3}$

$$= 2\left[\frac{x}{2}\sqrt{(9-x^2)} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^{\sqrt{7/3}} + 4\left[\frac{x}{2}\sqrt{(4-x^2)} + 2\sin^{-1}\frac{x}{2}\right]_{\sqrt{7/3}}^{2}$$

$$= 2\left[\frac{\sqrt{7/3}}{2}\sqrt{\left(9-\frac{7}{3}\right)}+\frac{9}{2}\sin^{-1}\left(\frac{\sqrt{7/3}}{x}\right)-0\right]+4\left[\pi-\left(\frac{\sqrt{7/3}}{2}\sqrt{4-\frac{7}{3}}+2\sin^{-1}\frac{\sqrt{7/3}}{2}\right)\right]$$

$$= 2\left[\frac{\sqrt{7/3}}{2}\cdot\sqrt{\frac{20}{3}}+\frac{9}{2}\sin^{-1}\left(\frac{\sqrt{7/3}}{x}\right)\right]+4\left[\pi-\frac{\sqrt{7/3}}{2}\cdot\sqrt{\frac{5}{3}}-2\sin^{-1}\frac{\sqrt{7/3}}{x}\right]$$

Example 13 Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Solution The required Area = 2 × Area OAC. The curve is symmetrical about the x-axis.

∴ the area between parabola and latus rectum

$$= 2\int_0^a y\,dx$$

$$= 2\int_0^a \sqrt{4\,ax}\,dx$$

$$= 4\int_0^a \sqrt{a}\cdot x^{1/2}\,dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}\cdot x^{3/2}\right]_0^a$$

$$= \frac{2\times 4}{3}\sqrt{a}\left[a^{3/2}-0\right]$$

$$= \frac{8}{3}a^{\frac{1}{2}+\frac{3}{2}}$$

$$= \frac{8}{3}a^2$$

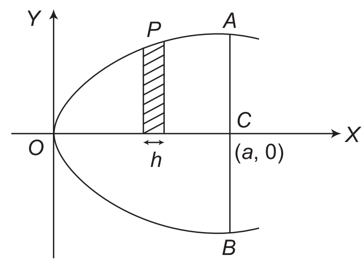


Fig. 5.6

5.8.2 Area of Curves in Polar Coordinates

Let $f(\theta)$ be continuous for every value of θ in the domain (α, β). Then the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ is equal to

$$\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

Proof Let O be the pole and OX, the initial line. AB is the portion of the arc included between the radii vectors $\theta = \alpha$ and $\theta = \beta$.

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + d\theta)$ be the two neighboring points on the curve $r = f(\theta)$.

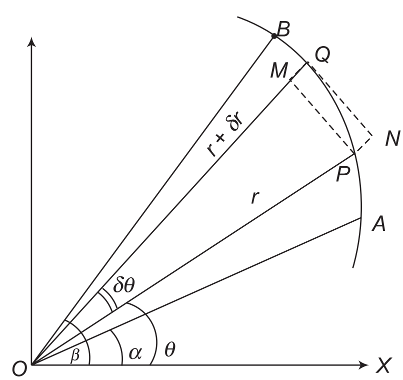


Fig. 5.7

Let the sectorial areas OAP and OAQ be denoted by A and $A + \delta A$. Now, taking O as the centre, draw circles OPM an OQN of radius r and $r + \delta r$ respectively.

The sectorial area $OPQ = A + \delta A - A = \delta A$ (8a)

Sectorial area $OPM = \frac{1}{2} r^2 \ \delta\theta$ (8b)

Sectorial area $OLQ = \frac{1}{2}(r + \delta r)^2 \ \delta\theta$ (8c)

Evidently, the sectorial area OPQ lies in between sectorial areas $\frac{1}{2} r^2 \ \delta\theta$ and $\frac{1}{2}(r + \delta r)^2 \ \delta\theta$,

i.e., $$\frac{1}{2} r^2 \delta\theta < \delta A < \frac{1}{2}(r + \delta r)^2 \ \delta\theta$$

or $$\frac{r^2}{2} < \frac{\delta A}{\delta\theta} < \frac{1}{2}(r + \delta r)^2 \tag{9}$$

Let $Q \to P$ so that $\delta\theta$ and δr both tend to zero.

$\therefore$ $$\frac{dA}{d\theta} = \frac{r^2}{2} \Rightarrow dA = \frac{r^2}{2} d\theta$$

Integrating both sides, we get

$$\int_{\theta=\alpha}^{\beta} dA = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$$

$$(A)_{\theta=\alpha}^{\beta} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$$

or $$(A)_{\theta=\beta} - (A)_{\theta=\alpha} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$$

Hence, the sectorial area $OAB = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$

Example 14 Find the area of the cardioid $r = a(1 + \cos\theta)$.

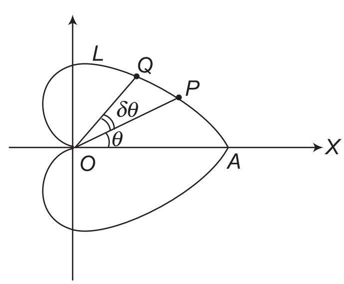


Fig. 5.8

Solution

Area of cardioid = 2 × Area ($OALO$)

$$= 2 \times \frac{1}{2} \int_{\theta=0}^{\pi} r^2 \, d\theta$$

$$= \int_{\theta=0}^{\pi} a^2 \, (1 + \cos\theta)^2 \, d\theta = \int_{0}^{\pi} 4a^2 \cos^4 \frac{\theta}{2} \, d\theta$$

$$= 4a^2 \int_{0}^{\pi} \cos^4 \left(\frac{\theta}{2}\right) d\theta$$

$$= 8a^2 \int_{0}^{\pi/2} \cos^4 \phi \, d\phi \qquad \left[\text{Putting } \phi = \frac{\theta}{2}\right]$$

$$= 8a^2 \cdot \frac{\Gamma\left(\frac{4+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{2 \cdot \Gamma\left(\frac{4+2}{2}\right)}$$

$$= 8a^2 \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \pi}{2 \cdot 2}$$

$$= {}^{2}\cancel{8}\, a^2 \cdot \frac{3\pi}{\cancel{4} \times \cancel{4}_2}$$

$$= \frac{3a^2 \pi}{2}$$

Example 15 Find the area enclosed by the curve $r = 3 + 2\cos\theta$.

Solution The given curve $r = 3 + 2\cos\theta$.

The area of the given curve is

$$= 2 \times \frac{1}{2} \int_{\theta=0}^{\pi} r^2 \, d\theta$$

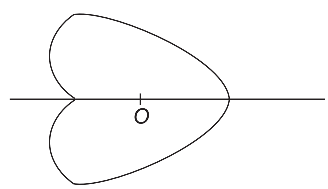


Fig. 5.9

$$= \int_{\theta=0}^{\pi} (3 + 2\cos)^2 \, d\theta$$

$$= \int_{\theta=0}^{\pi} [9 + 4\cos^2\theta + 12\cos\theta] \, d\theta$$

$$= 9\int_{0}^{\pi} \mathrm{d}\theta + 2\int_{0}^{\pi} (1 + \cos 2\theta) \, d\theta + 12\int_{0}^{\pi} \cos\theta \, d\theta \qquad [\because 2\cos^2\theta = 1 + \cos 2\theta]$$

$$= 11\int_0^{\pi} d\theta + 2\int_0^{\pi} \cos 2\,\theta\, \mathrm{d}\theta + 12\int_0^{\pi} \cos \theta d\,\theta$$

$$= 11\,(\theta)_0^{\pi} + 2\left(\frac{\sin 2\theta}{2}\right)_0^{\pi} + 12\,(\sin\theta)_0^{\pi}$$

$$= 11\,(\pi - 0) + 2\cdot\frac{(0-0}{2} + 12(0-0)$$

$$= 11\pi$$

Example 16 Find the area of a loop of the curve $r = a \sin 3\theta$.

Solution The given curve $r = a \sin 3\theta$

One loop is obtained by the values of θ from $\theta = 0$ to $\theta = \dfrac{\pi}{3}$

$\therefore$ the area of a loop

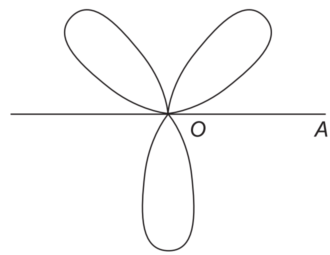


Fig. 5.10

$$= \frac{a^2}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta$$

$$= \frac{2\cdot a^2}{6}\int_0^{\pi/2} \sin^2 \phi\, d\phi;$$

[Putting $3\theta = \phi \Rightarrow 3d\theta = d\phi$]

$$= \frac{2\cdot a^2}{6}\cdot\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(2)}$$

$$= \frac{a^2}{3}\cdot\frac{\pi}{2\cdot 2}$$

$$= \frac{\pi a^2}{12}$$

EXERCISE 5.2

1. Find the area bounded by the curve $y = x^2$, the y-axis, and the lines $y = 1$ and $y = 8$.
2. Find the area of the curve $a^2y = x^2\,(x + a)$.
3. Find the area common to the two curves $y^2 = ax$ and $x^2 + y^2 = 4ax$.
4. Find the area of a loop of the curve $x = a \sin 2t$, $y = a \sin t$.
5. Find the area of a loop of the curve $ay^2 = x^2\,(a - x)$.
6. Find the common area between curve $y^2 = 4ax$ and $x^2 = 4ay$.
7. Find the area included between of the curve $x = a\,(\theta - \sin\,\theta)$, $y = a(1 - \cos\,\theta)$ and its base.
8. Find the area of one loop of $r = a \cos 4\theta$.
9. Find the area of the curve $r = 8a \cos\,\theta$.
10. Find the area of the curve $r = a \sin 3\theta$.

Answers

1. $\dfrac{45}{4}$

2. $\dfrac{a^2}{12}$

3. $\left(\dfrac{4}{3}\pi\, a^2 + 3\sqrt{3}\, a^2\right)$

4. $\dfrac{4a^2}{3}$

5. $\dfrac{8a^2}{15}$

6. $\dfrac{16a^2}{3}$

7. $3\,\pi\, a^2$

8. $\dfrac{\pi\, a^2}{16}$

9. πa^2

10. $\dfrac{\pi\, a^2}{12}$

5.9 AREA OF CLOSED CURVES

Let a closed curve represented by $x = f(t)$ and $y = g(t)$, t being the parameter and assume that the curve does not interact itself. Then the area bounded by such a curve is

$$\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

If the curve intersect itself once as shown in Fig. 5.12, then to compute area. Then the area bounded by such curves is

$$\frac{1}{2}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

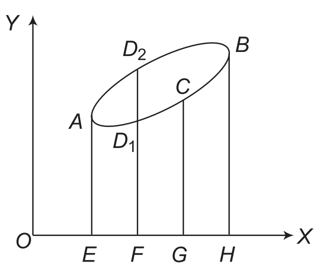


Fig. 5.11

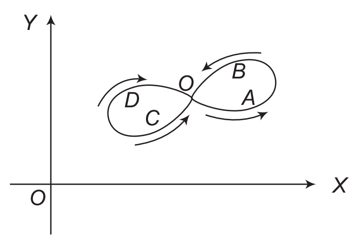


Fig. 5.12

Example 17 Find the area of the ellipse $x = a \cos t$ and $y = b \sin t$.

Solution The ellipse is a closed curve and $0 \le t \le 2\pi$,

Therefore, the area of the ellipse is

$$= \frac{1}{2}\int_{0}^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi} [a\cos t \cdot b\cos t - b\sin t\,(-a\sin t)]\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} [a\,b\,(\cos^2 t + \sin^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (a\,b)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt$$

$$= \frac{ab}{2}(t)_0^{2\pi}$$

$$= \frac{ab}{2}(2\pi - 0)$$

$$= \pi\,ab$$

Example 18 Find the area of the loop of the curve

$x = a(1 - t^2)$, $y = at(1 - t^2)$; $-1 \le t \le 1$.

Solution Area of the loop of the curve

$$= \frac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_{-1}^{1} [a\,(1 - t^2)\,(a - 3at^2) - a\,t(1 - t^2)\cdot(-2a\,t)]\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} [a^2(1 - 4t^2 + 3t^4) + 2a^2(t^2 - t^4)]\,dt$$

$$= \frac{a^2}{2}\int_{-1}^{1} [1 - 4t^2 + 3t^4 + 2t^2 - 2t^4]\,dt$$

$$= \frac{a^2}{2}\int_{-1}^{1} [1 - 2t^2 + t^4]\,dt$$

$$= \frac{a^2}{2}\left[t - \frac{2}{3}t^3 + \frac{1}{3}t^5\right]_{-1}^{1}$$

$$= \frac{a^2}{2}\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right]$$

$$= \frac{a^2}{2}\left[\frac{8}{15} - \left(-\frac{8}{15}\right)\right] = \frac{a^2}{2}\cdot\frac{16}{15} = \frac{8a^2}{15}$$

EXERCISE 5.3

1. Find the area of a loop of the curve $x = a \sin 2t$, $y = a \sin t$.
2. Prove that the area of the curve $x = a \sin 2\theta\,(1 + \cos 2\theta)$, $y = a \cos 2\theta\,(1 - \cos 2\theta)$ is $\dfrac{\pi a^2}{2}$.
3. Prove that the area of the curve
$$x = a \cos \theta + b \sin \theta + c,\ y = a' \cos \theta + b' \sin \theta + c' \text{ is equal to } \pi(ab' - a'b).$$
4. Find the area of the loop of the curve
$$x = \frac{a \sin 3\theta}{\sin\theta},\ y = \frac{a \sin 3\theta}{\cos\theta}$$
5. Prove that the area of the curve $x = a(3 \sin \theta - \sin^3\theta)$, $y = a^3\theta$ is $\dfrac{15\,a^2\pi}{8}$.

Answers

1. $\dfrac{4\,a^3}{3}$ 4. $\dfrac{3\sqrt{3}\,a^2}{2}$

5.10 RECTIFICATION

The process to obtain the length of an arc of a curve between two given points on it is called the *length of curve* or *rectification*.

Let AB be the curve $y = f(x)$, where A and B are two points with abscissae a and b.

Let s denote the length of the arc from the fixed point A to B.

By differential calculus, we know that

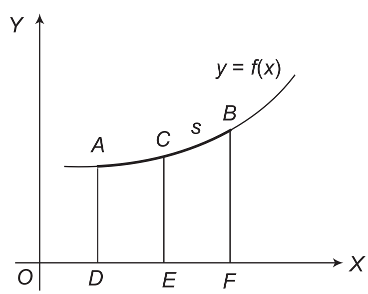


Fig. 5.13

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

or $$ds = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\,dx$$

Integrating both sides, we get

$$\int_a^b ds = \int_a^b \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right] dx$$

or $$Arc\ (AB) = \int_a^b \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right] dx$$

Note The length of the arc of the curve $x = f(y)$ between the points $y = c$ to d is

$$\int_c^d \left[\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right] dy$$

If x and y are expressed in terms of a parameter t, and to the points A and B correspond the values t_1 and t_2 of t then evidently the length of the arc AB is equal to

$$\int_{t_1}^{t_2} \left[\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\right] dt$$

5.10.1 The length of the arc of the curve $r = f(\theta)$ between the points where $\theta = \alpha$, $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\beta} \left[\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right] d\theta$$

Note The length of the arc of the curve $\theta = f(r)$ between the points $r = a$ to $r = b$ is

$$\int_a^b \sqrt{\left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right]}\, dr$$

Example 19 Find the length of the curve $y = \log \sec x$ between the points $x = 0$ and $x = \dfrac{\pi}{3}$.

Solution The equation of the curve $y = \log \sec x$.

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sec x}\cdot \sec x \cdot \tan x = \tan x$$

The required length of the arc

$$= \int_0^{\pi/3} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^{\pi/3} \sqrt{(1+\tan^2 x)}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx = [\log(\sec x + \tan x)]_0^{\pi/3}$$

$$= \log\left(\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right) - \log(1+0)$$

$$= \log(2+\sqrt{3}) - 0$$

$$\text{Arc} = \log(2+\sqrt{3})$$

Example 20 Find the length of the arc of the parabola $x^2 = 4\ ay$ from the vertex to an extremity of the latus rectum.

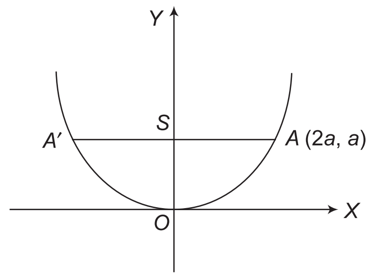


Fig. 5.14

Solution Let O be the node and A be the extremity of the latus rectum of the parabola $x^2 = 4ay$.

Given

$$x^2 = 4ay \tag{1}$$

or
$$y = \frac{x^2}{4a}$$

$\therefore$
$$\frac{dy}{dx} = \frac{x}{2a}$$

Now, the required length of the arc OA

$$= \int_0^{2a} \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^{2a} \sqrt{\left[1 + \frac{x^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + x^2}\, dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{4a^2 + x^2}}{2} + \frac{4a^2}{2}\sinh^{-1}\left(\frac{x}{2a}\right)\right]_0^{2a}$$

$$= \frac{1}{4a}\left[2a \cdot 2\sqrt{2}a + 4a^2 \sinh^{-1}(1)\right]$$

$$= a\left[\sqrt{2} + \sinh^{-1}(1)\right]$$

$$= a\left[\sqrt{2} + \log(1 + \sqrt{2})\right] \qquad \left[\because\ \sinh_x^{-1} = \log(x + \sqrt{1 + x^2})\right]$$

Example 21 Find the length of the arc of the semicubical parabola $ay^2 = x^3$ from the vertex to the point (0, a).

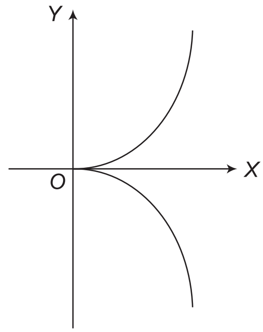


Fig. 5.15

Solution Given $$ay^2 = x^3 \text{ or } y^2 = \frac{x^3}{a} \tag{1}$$

$$2yy' = \frac{3x^2}{a}$$

or $$4y^2(y')^2 = \frac{9x^4}{a^2} \text{ (squaring both sides)}$$

$$(y') = \frac{9x^4}{4a^2\, y^2} = \frac{9x^4}{4a \cdot x^3} = \frac{9x}{4a}$$

The required arc

$$= \int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx = \int_0^a \sqrt{\left[1 + \frac{9x}{4a}\right]}\, dx$$

$$= \frac{1}{2\sqrt{a}} \int_0^a \sqrt{(4a + 9x)}\, dx = \frac{1}{2\sqrt{a}} \left[\frac{(4a + 9x)^{3/2}}{9 \cdot 3/2}\right]_0^a$$

$$= \frac{1}{27\sqrt{a}} \left[(13a)^{3/2} - (4a)^{3/2}\right]$$

$$= \frac{1}{27}[13\sqrt{13} - 8]\, a$$

Example 22 Show that in the catenary $y = c \cosh \dfrac{x}{c}$, the length of the arc from the vertex to any point is given by

(i) $s = c \sinh \dfrac{x}{c}$

(ii) $s^2 = y^2 - c^2$.

Solution

(i) The equation of the curve is

$$y = c\cosh\left(\frac{x}{c}\right) \tag{1}$$

Differentiating (1) both sides, we get

$$\frac{dy}{dx} = c\cdot\frac{1}{c}\sinh\left(\frac{x}{c}\right) = \sinh\left(\frac{x}{c}\right)$$

The length of the arc from $x = 0$ to $x = x$ is

$$s = \int_0^x \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$= \int_0^x \sqrt{\left[1+\sinh^2\frac{x}{c}\right]}dx$$

$$= \int_0^x \cosh\left(\frac{x}{c}\right)dx = c\left[\sinh\frac{x}{c}\right]_0^x = c\sin\left(\frac{x}{c}\right) \tag{2}$$

(ii) Squaring (1) and (2) and subtracting, we get

$$y^2 - s^2 = c^2\cosh^2\left(\frac{x}{c}\right) - c^2\sinh^2\left(\frac{x}{c}\right)$$

$$= c^2\left[\cosh^2\left(\frac{x}{c}\right) - \sinh^2\left(\frac{x}{c^2}\right)\right]$$

$$y^2 - s^2 = c^2$$

Example 23 Find the length of the loop of the curve

$$x = t^2,\ y = t - t^3/3.$$

Solution The equations of the curve are

$$x = t^2\ ,\ \ y = t - t^3/3$$

$$\therefore \qquad \frac{dx}{dt} = 2t, \frac{dy}{dt} = 1 - t^2$$

For the half-loop, put $y = 0$; then $t - \dfrac{t^3}{3} = 0$, i.e., t varies from 0 to $\sqrt{3}$.

The required length of the loop is

$$= 2\int_{t=0}^{\sqrt{3}} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\,dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{\left[(2t)^2 + (1-t^2)^2\right]}\,dt = 2\int_0^{\sqrt{3}} \sqrt{(1+t^2)^2}\,dt$$

$$= 2\int_0^{\sqrt{3}} (1+t^2)\,dt = 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}}$$

$$= 2\left[\sqrt{3} + \frac{1}{3}\cdot 3\sqrt{3} - 0\right]$$

$$= 4\sqrt{3}$$

Example 24 Show that $8a$ is the length of an arc of the cycloid whose equations are $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution The equation of cycloid are

$$x = a(t - \sin t) \text{ and } y = a(1 - \cos t)$$

$$\therefore \quad \frac{dx}{dt} = a(1 - \cos t) \text{ and } \frac{dy}{dt} = a \sin t$$

For the loop of curve, put $y = 0$, i.e., $a(1 - \cos t) = 0$, t varies 0 to 2π.

Hence, the length of arc of the curve

$$= \int_0^{2\pi} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\,dt$$

$$= \int_0^{2\pi} \sqrt{[a(1-\cos t)^2 + (a \sin t)^2]}\,dt$$

$$= a\int_0^{2\pi} \sqrt{1 + \cos^2 t - 2\cos t + \sin^2 t]}\,dt$$

$$= a\int_0^{2\pi} \sqrt{[2(1-\cos t)]}\,dt = a\int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\,dt$$

$$= 2a\int_0^{2\pi} \sin\frac{t}{2}\,dt = 4a\left[-\cos\frac{t}{2}\right]_9^{2\pi}$$

$$= 8a$$ **Hence, proved.**

Example 25 Find the entire length of the curve given by the parametric equations $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Solution The equations of the curve are

$$x = a\cos^3\theta,\ y = a\sin^3\theta \tag{1}$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta} = 3a\sin^2\theta\cdot\cos\theta$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2 \sin^2\theta \cos^2\theta. \qquad (2)$$

The given curve (1) is symmetrical about both the axis. $|x| < a$, $|y| < a$, i.e., $x = \pm a$, $y = \pm a$.
The corresponding values of x, y for some values of θ are

$\theta = 0$	$\pi/2$	π	2π
$x = a$	0	$-a$	0
$y = 0$	a	0	0

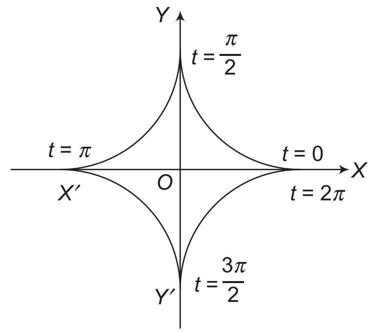


Fig. 5.16

In the first quadrant, θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ the required length of the curve is

$$= 4\int_0^{\pi/2} \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]}\, d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{9a^2 \sin^2\theta \cos^2\theta}\, d\theta$$

$$= 12a \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta$$

$$= 6a \int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= 6a\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi/2} = -3a(\cos\pi - \cos 0)$$

$$= -3a(-1 - 1) = 6a$$

Example 26 Find the perimeter of the cardioid $r = a(1 - \cos\theta)$.

Solution The equation of the curve is

$$r = a(1 - \cos\theta) \qquad (1)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

The curve is symmetrical about the initial line.
When $r = 0$, $\theta = 0$ (tangent to the curve)
When $r = 2a$, $\theta = \pi$
The perimeter of the curve

$$= 2 \times \text{Area above } OA$$

$$= 2\int_{\theta=0}^{\pi} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]}\, d\theta$$

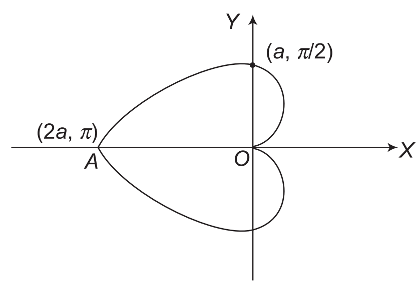


Fig. 5.17

$$= 2a\int_0^{\pi}\sqrt{\left[1+\cos^2\theta - 2\cos\theta + \sin^2\theta\right]}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{2(1-\cos\theta)}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{4\sin^2\frac{\theta}{2}}\,d\theta$$

$$= 4a\int_0^{\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 4a\left[\frac{-\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= 8a$$

Example 27 Find the entire length of the cardiod $r = a(1 + \cos\theta)$.

Solution Do the same as Example 26.

EXERCISE 5.4

1. Find the length of an arc of the parabola $y = x^2$, measured from the vertex. Calculate the length of the arc to the point (1, 1), given $\log_e(2+\sqrt{5}) = 1.45$.
2. Find the length of the curve $y = \log\left(\frac{e^x - 1}{e^x + 1}\right)$ from $x = 1$ to $x = 2$.
3. Find the length of one quadrant of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ and, hence, find its whole length.
4. Find the length of an arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$.
5. Find the length of the arc of the curve $x = e^{\theta}\cdot\sin\theta$, $y = e^{\theta}\cdot\cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.
6. Find the length of arc of the equiangular spiral $r = a\,e^{\theta\cot\theta}$ between the points for which the radius vectors r_1 and r_2.
7. Find the length of the arc of the spiral $r = a\theta$ between the points whose radii vectors are r_1 and r_2.
8. Find the length of an arc of the cissoid $r = \frac{a\sin^2\theta}{\cos\theta}$.
9. Find the perimeter of the following curves.

   (i) $r^{1/3} = 8\cos\frac{\theta}{3}$ (ii) $r^{1/2} = a^{1/3}\cos\frac{\theta}{3}$

Answers

1. $x\sqrt{\left(x^2+\frac{1}{4}\right)}+\frac{1}{4}\log\left[x+\sqrt{x^2+\frac{1}{4}}\right]+\frac{1}{4}\log e^2; 1.48$

2. $\log\left(e+\frac{1}{e}\right)$

3. $\frac{3a}{2}, 6a$

4. $8a$

5. $\sqrt{2}\,(e^{\pi/2}-1)$

6. $\sec\alpha\,(r_2-r_1)$

7. $f\left(\frac{r_2}{a}\right)-f\left(\frac{r_1}{a}\right)$, where $f(\theta)=\frac{a}{2}\left[\theta\sqrt{1+\theta^2}+\log\{\theta+\sqrt{1+\theta^2}\}\right]$

8. $f(\theta_2)-f(\theta_1)$, where $f(\theta)=a\left[\sqrt{(3+\sec^2\theta)}-\sqrt{3}\log\left[\sqrt{(1+3\cos^2\theta)}+\sqrt{3}\cos\theta\right]\right]$

9. (i) 76π (ii) $\frac{3a\pi}{2}$

5.11 INTRINSIC EQUATIONS

The intrinsic equation of a curve is a relation between s and ψ, where s is the length of the arc of a curve measure from a fixed point A on it to any point P, and ψ is the angle which the tangent at P makes with the tangent at A to the curve. s and ψ are called the intrinsic co-ordinates.

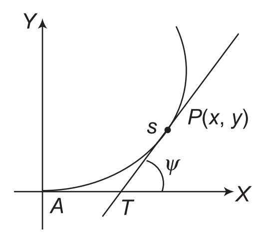


Fig. 5.18

5.11.1 To Find the Intrinsic Equation from the Cartesian Equation

Let $y=f(x)$ be the Cartesian equation of a given curve. Let the x-axis be the tangent to the curve at the origin O from which the arc is measured. Let $P(x, y)$ be any point on the curve and PT be the tangent at the point P.

Now, $$\tan\psi=\frac{dy}{dx}=f'(x) \tag{10}$$

$\therefore$ $$s=\int_0^x\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$=\int_0^x\sqrt{\left[1+\{f'(x)\}^2\right]}\,dx \tag{11}$$

Eliminating x between (10) and (11), we get a **relation between s and ψ, which is the required intrinsic equation of the curves**.

5.11.2 To Find the Intrinsic Equation from the Polar Equation

Let $r = f(\theta)$ be the polar equation of a given curve. Let O be the pole and the initial line be the tangent to the curve at the pole from which the arc is measured.

Let arc $OP = s$ and $\angle xTP = \psi$

$$\therefore \quad \frac{dr}{d\theta} = f'(\theta) \tag{12}$$

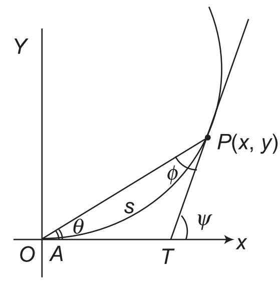


Fig. 5.19

If ϕ be the angle between the radius vector and the tangent at P,

$$\tan\phi = r\frac{d\theta}{dr} = r\cdot\frac{1}{\dfrac{dr}{d\theta}} = \frac{f(\theta)}{f'(\theta)} \tag{13}$$

Also, $$\psi = \theta + \phi \tag{14}$$

$$\text{The arc } s = \int_0^\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta = \int_0^\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{15}$$

Eliminating θ and ϕ, from (13), (14), and (15), we get a **relation between s and ψ, which is the intrinsic equation of the curve**.

5.11.3 Intrinsic Equation in Parametric Equation

Let $x = f(t)$, $y = g(t)$ be the parametric equation of a given curve. Let the x-axis be the tangent to the curve, at the point $P(x, y)$. If PT is the tangent then.

$$\tan\psi = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{g'(t)}{f'(t)} \tag{16}$$

$$\therefore \quad s = \int_0^t \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\, dt$$

$$s = \int_0^t \sqrt{\left[\{f'(t)\}^2 + \{g'(t)\}^2\right]}\, dt \tag{17}$$

Eliminating t between (16) and (17), we get a **relation between s and ψ which is the required intrinsic equation of the curve**.

Example 28 Find the intrinsic equation of the catenary $y = a \cosh (x/a)$.

Solution Let s be measured from the point whose abscissa is O. Then.

$$s = \int_0^x \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^x \sqrt{\left[1 + \left(-\sinh \frac{x}{a}\right)^2\right]}\, dx = \int_0^x \sqrt{\left[1 + \sinh^2\left(\frac{x}{a}\right)\right]}\, dx$$

$$s = \int_0^x \cosh\left(\frac{x}{a}\right) dx = a \sinh\left(\frac{x}{a}\right) \tag{1}$$

Also, $$\tan \psi = \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right) \tag{2}$$

From (1) and (2), we get

$$s = a \tan \psi$$

Example 29 Find the intrinsic equation of the cardioid $r = a(1 - \cos \theta)$.

Solution The given curve is

$$r = a(1 - \cos \theta)$$

$$\therefore \quad \frac{dr}{d\theta} = a \sin \theta$$

Then, the arc $$(s) = \int_0^\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta$$

$$= \int_0^\theta \sqrt{[a^2(1 - \cos \theta)^2 + (a \sin \theta)^2}\, d\theta$$

$$= a\int_0^\theta \sqrt{[1 + \cos^2 \theta - 2 \cos \theta + \sin^2 \theta]}\, d\theta$$

$$= 2a\int_0^\theta \sin \frac{\theta}{2}\, d\theta = 4a\left[-\cos\frac{\theta}{2}\right]_0^\theta$$

$$= 4a\left(1 - \cos\frac{\theta}{2}\right)$$

$$s = 8a \sin^2\left(\frac{\theta}{4}\right) \tag{1}$$

Also, $$\tan \phi = r \cdot \frac{d\theta}{dr} = \frac{(1 - \cos\theta)}{\sin \theta} = \tan \frac{\theta}{2}$$

$$\therefore \quad \phi = \frac{\theta}{2} \text{ so that } y = \theta + \phi = \theta + \frac{\theta}{2} = \frac{3\theta}{2} \tag{2}$$

From (1) and (2), we get

$$s = 8a \sin^2\left(\frac{\psi}{8}\right)$$

Example 30 Find the intrinsic equation of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ and prove that $s^2 + \rho^2 = 16a^2$.

Solution The parametric equations of the cycloid are

$$x = a(t + \sin t),\ y = a(1 - \cos t)$$

$$\frac{dx}{dt} = a(1 + \cos t),\quad \frac{dy}{dt} = a \sin t$$

$$\therefore \qquad \tan \psi = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)}$$

$$= \frac{\sin t}{1 + \cos t} = \frac{2 \sin t/2 \cdot \cos t/2}{2 \cos^2 \dfrac{t}{2}}$$

$$\tan \psi = \tan \frac{t}{2} \Rightarrow \psi = \frac{t}{2} \tag{1}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2 (1 + \cos t)^2 + a^2 \sin^2 t$$

$$= 2a^2(1 + \cos t) = 4a^2 \cos^2 \left(\frac{t}{2}\right)$$

$$\therefore \qquad s = \int_0^t \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\, dt = \int_0^t \sqrt{4a^2 \cos^2 \left(\frac{t}{2}\right)}\, dt$$

$$= \int_0^t 2a \cos \frac{t}{2}\, dt = 2a \cdot 2 \sin \frac{t}{2}$$

$$s = 4a \sin \frac{1}{2} \tag{2}$$

From (1) and (2), we get

$$s = 4a \sin \psi \tag{3}$$

$$\rho = \frac{ds}{d\psi} = 4a \cos \psi \tag{4}$$

where ρ is the radius of curvature of the curve at any point.

Squaring (3) and (4) and adding, we get

$$s^2 + \rho^2 = 16a^2 \sin^2 \psi + 16a^2 \cos^2 \psi = 16a^2.$$

$$s^2 + \rho^2 = 16a^2$$

EXERCISE 5.5

1. Find the intrinsic equation of $y = a \log \sec (x/a)$.
2. Find the intrinsic equation of $y^3 = ax^2$.

3. Show that the intrinsic equation of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, taking $(a, 0)$ as the fixed point is $s = \frac{3a}{2} \sin^2 \psi$.
4. Show that the intrinsic equation of the semicubical parabola $ay^2 = x^3$ taking its cusp as the fixed point is $27s = 8a (\sec^3 \psi - 1)$.
5. Show that the intrinsic equation of the cycloid
$$x = a(t + \sin t),\ y = a(1 - \cos t) \text{ is } s = 4a \sin \psi.$$
6. Find the intrinsic equation of the spiral $r = a\theta$, the arc being measured from the pole.
7. Find the intrinsic equation of the parabola $x = at^2$, $y = 2at$, the arc being measured from the vertex.
8. Show that the intrinsic equation of the semicubical parabola $3ay^2 = 2x^3$ is
$$9s = 4a(\sec^3 \psi - 1).$$

Answers

1. $s = \log (\sec \psi \mp \tan \psi)$
2. $27s = 8a(\text{cosec}^2 \psi - 1)$
6. $s = \frac{a}{2}\left[\theta\sqrt{1+\theta^2} + \sinh^{-1} \theta\right], \psi = \theta + \tan^{-1} \theta$
7. $s = a[\cot \psi \,\text{cosec}\, \psi + \log (\cos \psi + \text{cosec}\, \psi)]$

5.12 VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION

5.12.1 Volume of the Solid of Revolution (Cartesian Equations)

(i) Revolution about *x*-axis

The volume of the solid generated by the revolution about the x-axis of the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, $x = b$ is

$$\int_a^b \pi y^2 \, dx$$

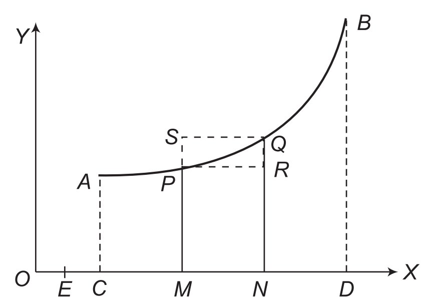


Fig. 5.20

(ii) Revolution about *y*-axis

The volume of the solid generated by the revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis, and the abscissae $y = c$, $y = d$ is

$$\int_c^d \pi x^2 \, dy$$

(iii) Revolution about a Line

If the generated curve AB revolves about any line (other than x-axis and y-axis), and AC, BD be the perpendiculars on the axis CD from A and B then the volume of the solid generated by the revolution of the arc AB is

$$\int_{EC}^{ED} \pi\, PM^2\, d(EM)$$

where PM is the perpendicular from any point P of the curve on the axis and E is a fixed point on the axis.

Example 31 Find the volume of the solid obtained by the revolution of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its minor axis.

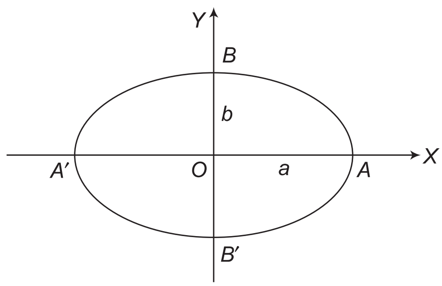


Fig. 5.21

Solution The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or
$$x^2=\frac{a^2}{b^2}(b^2-y^2) \qquad (1)$$

Volume of the ellipse = twice the volume generated by the revolution of the arc BA about the minor axis, i.e., y-axis. For the arc BA, y varies from $y = 0$ to $y = b$.

$\therefore$ the required volume $= 2\int_{y=0}^{b} \pi\, x^2\, dy$

$$= 2\pi\int_0^b \frac{a^2}{b^2}(b^2-y^2)\,dy = \frac{2\pi a^2}{b^2}\int_0^b (b^2-y^2)\,dy$$

$$= \frac{2\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_0^b = \frac{2\pi a^2}{b^2}\left(b^3-\frac{b^3}{3}\right)$$

$$= \frac{4\pi a^2 b}{3}$$

Example 32 Prove that the volume of the solid generated by the revolution of the curve $(a-x)y^2 = a^2x$ about its asymptote is $\dfrac{\pi^2 a^3}{2}$.

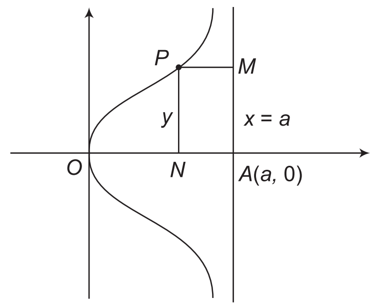


Fig. 5.22

Solution The given equation of the curve

$$(a-x)\, y^2 = a^2 x$$

or $$x = \frac{ay^2}{a^2+y^2} \qquad (1)$$

The curve is symmetrical about the x-axis and the equation of the asymptote is $x = a$. The revolution of the axis is $x = a$.

Let $P(x, y)$ be any point on the curve and PM, the perpendicular on the asymptote.

Let PN be the perpendicular on the x-axis. Then

$$PM = NA = OA - ON = (a-x)$$

Therefore, the required volume

$$= \int \pi\, \text{PM}^2\, d(\text{AM})$$

$$= 2\int_0^\infty \pi\,(a-x)^2\, dy = 2\pi \int_0^\infty \left[a - \frac{ay^2}{a^2+y^2}\right]^2 dy$$

$$= 2\pi a^6 \int_0^\infty \frac{dy}{(a^2+y^2)^2}$$

$$= 2\pi a^6 \int_0^{\pi/2} \frac{a \sec^2\theta}{(a^2+a^2 y \tan^2\theta)^2}\, d\theta$$

[Putting $y = a\tan\theta$, $dy = a\sec^2\theta\, d\theta$, and $0 \le \theta \le \pi/2$]

$$= \frac{2\pi a^6 \cdot a}{a^4} \int_0^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2\theta\, d\theta = \pi a^3 \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \pi a^3 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi a^2 \left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right]$$

$$= \frac{\pi^2 a^3}{2}$$

Example 33 The curve $y^2(a+x) = x^2(3a-x)$ revolves about the x-axis. Find the volume generated by the loop.

Solution The equation of the curve is

$$y^2(a + x) = x^2(3a - x) \text{ or } y^2 = \frac{x^2(3a - x)}{(a + x)} \quad (1)$$

The curve is symmetrical about the x-axis and $0 \le x \le 3a$. Then the required volume

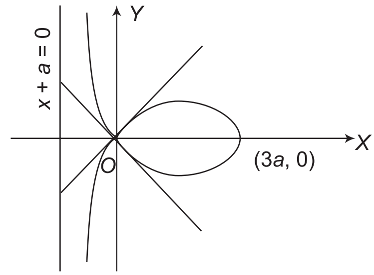


Fig. 5.23

$$= \int_0^{3a} \pi y^2 \, dx$$

$$= \pi \int_0^{3a} \frac{x^2 (3a - x)}{(a + x)} dx \quad [\because \text{ by (1)}]$$

$$= \pi \int_0^{3a} \left[-x^2 - 4a^2 + 4ax + \frac{4a^3}{a + x} \right] dx$$

$$= \pi \left[-\frac{x^3}{3} - 4a^2 x + 4a\frac{x^2}{2} + 4a^3 \log(a + x) \right]_0^{3a}$$

$$= \pi[-9a^3 - 12a^3 + 18a^3 + 4a^3 (\log 4a - \log a)]$$
$$= \pi [-3a^3 + 4a^3 \log 4]$$
$$= \pi a^3(8 \log 2 - 3)$$

5.12.2 Volume of Revolution for Parametric Equations

Let $x = f(t)$, $y = g(t)$ be the equations of the parametric form, where 't' is a parameter. There are two cases arise.

Case I

The volume generated by the revolution of the area bounded by the given equations of curve, the x-axis, and the ordinates $x = a$, $x = b$ about the x-axis is

$$\int_a^b \pi y^2 \, dx = \pi \int_{t_1}^{t_2} [g(t)]^2 \frac{dx}{dt} dt$$

where $t = t_1$ when $x = a$, and $t = t_2$ when $x = b$.

Case II

The volume generated by the revolution of the area bounded by the given equations of the curve, the y-axis, and the abscissae $y = c$, $y = d$ about the y-axis is

$$\int_c^d \pi x^2 \, dy = \pi \int_{t_3}^{t_4} [f(t)]^2 \frac{dy}{dt} dt$$

where $t = t_3$ when $y = c$, and $t = t_4$ when $y = d$.

Example 34 Find the volume of the solid generated by the revolution of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ about the x-axis.

Solution The parametric equations of the astroid are $x = a\cos^3 t$ and $y = a\sin^3 t$ (1)

The curve is symmetrical about both the axes. It cuts the axis at $t = 0$ and $t = \dfrac{\pi}{2}$. For the portion of the curve in the first quadrant, t varies from 0 to $\dfrac{\pi}{2}$.

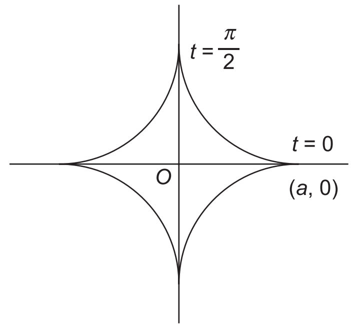


Fig. 5.24

The required volume

$$= 2\int_0^{\pi/2} \pi\, y^2 \frac{dx}{dt}\, dt$$

$$= 2\pi \int_0^{\pi/2} (a\sin^3 t)^2 \cdot (-3a\cos^2 t \sin t)\, dt$$

$$= -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cdot \cos^2 t\, dt$$

$$= -6\pi a^3 \cdot \frac{\Gamma(4)\cdot\Gamma(3/2)}{2\,\Gamma(11/2)}$$

$$= -6\pi a^3 \cdot \frac{3\cdot 2\cdot 1\cdot \frac{1}{2}\sqrt{\pi}}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= -\frac{32\,\pi\, a^3}{105}$$

$$\text{Volume} = \frac{32\,\pi\, a^2}{105} \text{ (in magnitude)}$$

Example 35 Find the volume of the solid generated by the revolution of the cissoid

$$x = 2a\sin^2 t,\ y = 2a\sin^3 t/\cos t$$

about its asymptote.

Solution The parametric equation of the cissoid

$$x = 2a\sin^2 t,\ y = 2a\sin^3 t/\cos t \quad (1)$$

The curve is symmetrical about the x-axis, and the asymptote is $x = 2a$.

Hence, the required volume

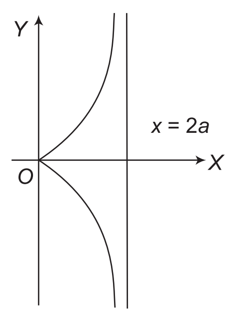


Fig. 5.25

$$= 2\pi \int_{y=0}^{\infty} (2a - x)^2\, dy$$

$$= 2\pi \int_{t=0}^{\pi/2} (2a - x)^2 \cdot \frac{dy}{dt}\cdot dt$$

$$= 2\pi \int_{t=0}^{\pi/2} 4a^2 \cos^4 t \cdot 2a \cdot \frac{3\sin^2 t \cos^2 t + \sin^4 t}{\cos^2 t}\, dt$$

$$= 16\pi a^3 \int_0^{\pi/2} \cos^2 t \cdot \sin^2 t\,(1 + 2\cos^2 t)\, dt$$

$$= 16\pi a^3 \left[\int_0^{\pi/2} \cos^2 t \cdot \sin^2 t\, dt + 2\int_0^{\pi/2} \sin^2 t \cdot \cos^4 t\, dt \right]$$

$$= 16\pi a^3 \left[\frac{\Gamma(3/2)\cdot\Gamma(3/2)}{2\Gamma(3)} + 2\cdot\frac{\Gamma(5/2)\cdot\Gamma(3/2)}{2\Gamma(4)} \right]$$

$$= 2\pi^2 a^3$$

5.12.3 Volume of the Solid of Revolution for Polar Coordinates

(i) The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ about the initial line $\theta = 0$ is given by

$$\frac{2\pi}{3}\int_\alpha^\beta r^3 \sin\theta\, d\theta$$

(ii) The volume of the solid generated by the revolution of the area about the line $\theta = \pi/2$ of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is given by

$$\frac{2\pi}{3}\int_\alpha^\beta r^3 \cos\theta\, d\theta$$

(iii) If the curve $r = f(\theta)$ revolves about the initial line (i.e., x-axis) then the volume generated is

$$\int_a^b \pi y^2\, dx = \int_\alpha^\beta \pi y^2 \frac{dx}{d\theta}\, d\theta$$

where $\theta = \alpha$ when $x = a$, and $\theta = \beta$, when $x = b$.

But $x = r\cos\theta$, $y = r\sin\theta$, $\dfrac{dx}{d\theta} = \dfrac{d}{d\theta}(r\cos\theta)$

$$\therefore \quad \text{Volume} = \int_\alpha^\beta \pi (r\sin\theta)^2 \frac{d}{d\theta}\cdot(r\cos\theta)\, d\theta$$

Similarly, if $r = f(\theta)$ revolves about the y-axis then

$$\text{Volume} = \int_{\alpha'}^{\beta'} \pi (r\cos\theta)^2 \frac{d}{d\theta}(r\sin\theta)\, d\theta$$

where $\theta = \alpha'$ when $y = c$, and $\theta = \beta'$ when $y = d$.

Example 36 If the cardioid $r = a(1 + \cos\theta)$ revolves about the initial line, find the volume generated.

Solution The equation of the curve $r = a(1 + \cos\theta)$ (1)

The curve (1) is symmetrical about the initial line and for the upper half, θ varies from 0 to π.

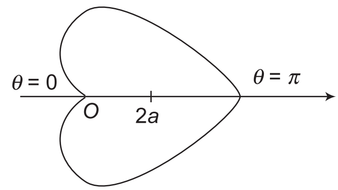


Fig. 5.26

$$\therefore \text{ the required volume} = \frac{2\pi}{3}\int_0^{\pi} r^3 \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \cdot \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3 \cdot \sin\theta\, d\theta$$

$$= -\frac{2\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi} = -\frac{1}{6}\pi a^3\,(0-16)$$

$$= \frac{8}{3}\pi a^3$$

Example 37 Find the volume of the solid generated by revolving one loop of the curve $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$.

Solution The equation of the curve is

$$r^2 = a^2\cos 2\theta \qquad (1)$$

The curve (1) is symmetrical about the initial line.

Putting $r = 0$ in (1), we get

$$\cos 2\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{4}$$

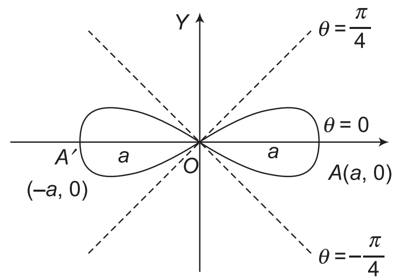


Fig. 5.27

For the upper half of the first loop, θ varies from $\theta = 0$ to $\frac{\pi}{4}$.

$\therefore$ the required volume

$$= 2\cdot\frac{2\pi}{3}\int_0^{\pi/4} r^3\cos\theta\, d\theta$$

$$= \frac{4\pi}{3}a^3\int_0^{\pi/4}\cos^{3/2} 2\theta\cdot\cos\theta\, d\theta$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4}(1-2\sin\theta)^{3/2}\cos\theta\cdot d\theta$$

[Put $\sqrt{2}\sin\theta = \sin\phi$

$\because \sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\phi$.

Also, when $\theta = \frac{\pi}{4}, \phi = \frac{\pi}{2}$

$\theta = 0, \phi = 0$.]

$$\therefore \qquad \text{Volume} = \frac{4\pi a^3}{3}\frac{1}{\sqrt{2}}\int_{\phi=0}^{\pi/2}\cos^4\phi\, d\phi$$

$$= \frac{4\pi\, a^3}{3\sqrt{2}} \cdot \frac{\overline{|1/2} \cdot \overline{|5/2}}{2\sqrt{3}} = \frac{4\pi\, a^3}{3\sqrt{2}} \cdot \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{2 \cdot 2 \cdot 1}$$

$$= \frac{\pi^2\, a^3}{4\sqrt{2}}$$

EXERCISE 5.6

1. Show that the volume of the solid formed by the revolution of the loop of the curve $y^2(a + x) = x^2(a - x)$ about the x-axis is $2\pi\left(\log 2 - \frac{2}{3}\right)a^3$.
2. Find the volume of the solid generated by the revolution of the catenary $y = c \cosh\left(\frac{x}{c}\right)$ about the x-axis.
3. Find the volume of the solid generated by the revolution of the plane area bounded by $y^2 = 9x$ and $y = 3x$ about the x-axis.
4. Find the volume of the solid generated by revolving the curve $xy^2 = 4(2 - x)$ about the y-axis.
5. Find the volume when the loop of the curve $y^2 = x(2x - 1)^2$ revolves about the x-axis.
6. Find the volume of the solid generated by the revolution of the cycloid $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$ about the y-axis.
7. Find the volume generated by the revolution of the loop of the curve $x = t^2$, $y = \left(t - \frac{t^3}{3}\right)$ about the x-axis.
8. Find the volume generated by the revolution of the curve $x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}$, $y = a\sin t$ about its asymptote.
9. Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.
10. Find the volume of the solid formed by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line.
11. Find the volume generated by the revolution of $r = 2a\cos\theta$ about the initial line.
12. The area lying within the cardioid $r = 2a(1 + \cos\theta)$ and within the parabola $r(1 + \cos\theta) = 2a$ revolves about the initial line. Show that the volume generated is $18\pi a^3$.

Answers

2. Volume $= \frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh\frac{2x}{c}\right]$
3. $\frac{3\pi}{2}$
4. $4\pi^2$
5. $\pi/48$
6. $6\pi 3a^3$
7. $\frac{3\pi}{4}$
8. $\frac{2\pi\, a^3}{3}$
9. $\frac{8\pi\, a^3}{3}$

10. $\dfrac{8\pi a^3}{3}$ 11. $4\pi a^3$

5.13 SURFACES OF SOLIDS OF REVOLUTION

5.13.1 Surface of the Solid of Revolution in Cartesian Coordinates

The curved surface of a solid generated by revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, $x = b$ is

$$\int_{x=a}^{b} 2\pi\, y\, dS$$

where S is the length of the arc of the curve measured from a fixed point on it to any point (x, y).

Note 1 We know that $\dfrac{dS}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Therefore, the above formula may be rewritten as the required curved surface $= 2\pi \displaystyle\int_{x=a}^{b} y\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$.

Note 2 Similarly, the curved surface of the solid generated by the revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis, and the abscissae $y = c$ and $y = d$ is

$$\int_{y=c}^{d} 2\pi\, x\, dS = \int_{y=c}^{d} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

or

$$= 2\pi \int_{y=c}^{d} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 38 The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the area of the surface of the reclithus generated.

Solution The given equation of the parabola is

$$y^2 = 4\,ax \qquad (1)$$

The required curved surface is generated by the revolution of the arc BOB', cut off by the latus rectum about the axis of y. Here, BAB' is the latus rectum. For half the arc, x varies from 0 to a.

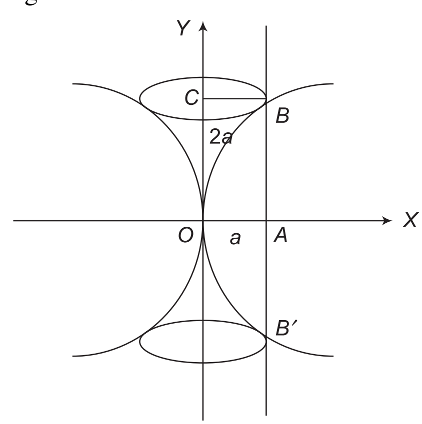


Fig. 5.28

From (1), $y = 2\sqrt{ax}$

$$\frac{dy}{dx} = \sqrt{a/x}$$

$$\therefore \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{a}{x}} = \sqrt{\frac{a + x}{x}}$$

The required surface = 2 × Surface generated by the revolution of the arc OB

$$= 2\int_0^a 2\pi\, x\, dS$$

$$= 4\pi \int_0^a x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_0^a x\sqrt{\left(\frac{a+x}{x}\right)}\, dx = 4\pi \int_0^a \sqrt{(x^2+ax)}\, dx$$

$$= 4\pi \int_0^a \sqrt{\left[\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right]}\, dx$$

$$= 4\pi \left[\frac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{x^2+ax} - \frac{1}{2}\left(\frac{a}{2}\right)^2 \log\left\{\left(x+\frac{a}{2}\right)+\sqrt{x^2+ax}\right\}\right]_0^a$$

$$= \pi a^2 \left[3\sqrt{2} - \log(\sqrt{2}+1)\right]$$

5.13.2 Surface of the Solid of Revolution in Parametric Equations

The curved surface of the solid generated by the revolution about the x-axis of the area bounded by the curve $x = \phi(t)$, $y = \psi(t)$, the x-axis, and the ordinates $t = t_1$, $t = t_2$ is

$$\int 2\pi\, y\, dS = \int_{t=t_1}^{t_2} 2\pi\, y \frac{dS}{dt}\, dt$$

where $$\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Similarly, the revolution about the y-axis gives

$$\int 2\pi\, x\, dS = \int_{t=t_1}^{t_2} 2\pi\, y \frac{dS}{dt}\, dt$$

where $$\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example 39 Find the surface of the solid generated by the revolution of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ about the x-axis.

Solution The parametric equations are

$$x = a\cos^3 t,\ y = a\sin^3 t \tag{1}$$

$$\therefore \qquad \frac{dx}{dt} = -3a\cos^2 t \cdot \sin t, \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\frac{dS}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{(-3a\cos^2 t\sin)^2 + (3a\sin^2 t\cos t)^2}$$

$$= 3a\sqrt{\cos^4 t \cdot \sin^2 t + \sin^4 t\cos^2 t}$$

$$\frac{dS}{dt} = 3a \cdot \sin t\cos t$$

The required surface = 2 × Area of the surface generated by revolution about the x-axis

$$= 2\int_{t=0}^{\pi/2} 2\pi\, y\, dS = 2\int_{t=0}^{\pi/2} 2\pi\, y\frac{dS}{dt}\cdot dt$$

$$= 4\pi\int_{t=0}^{\pi/2} a\sin^3 t\,(3a\sin t\cos t)\, dt$$

$$= 4\pi\, a^2\int_{t=0}^{\pi/2} 3\sin^4 t\cos t\, dt = 12\,\pi\, a^2\int_0^{\pi/2}\sin^4 t\cdot\cos t\, dt$$

$$= 12\pi\, a^2\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = 12\,\pi\, a^2\left[\frac{1}{5} - 0\right]$$

$$= \frac{12}{5}\pi a^2$$

5.13.3 Surface of the Solid of Revolution in Polar Form

The curved surface of the solid generated by the revolution about the x-axis of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\beta} 2\pi\, y\, dS = 2\pi\int_{\alpha}^{\beta} y\frac{dS}{d\theta}\, d\theta$$

where $\dfrac{dS}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$ and $y = r\sin\theta$

Similarly, revolution about the y-axis of the area bounded by the curve $\theta = f(r)$ is

$$\int_{\theta=\alpha}^{\beta} 2\,\pi\, x\, dS = 2\pi\int_{\alpha}^{\beta} x\frac{dS}{d\theta}\, d\theta$$

where $\dfrac{dS}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$ and $x = r\cos\theta$.

Example 40 Find the surface area generated by the revolution of the loops of the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.

Solution The given curve is symmetrical about the initial line and the line $\theta = \pi/2$. The curve consist of two loops and in the first quadrant, for half the loop, θ varies from 0 to $\pi/4$.

$$\therefore \qquad r^2 = a^2 \cos 2\theta \text{ or } r = a\sqrt{\cos 2\theta} \qquad (1)$$

$$2r\frac{dr}{d\theta} = a^2(-2\sin 2\theta)$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\frac{dS}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{\left[a^2\cos^2 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}\right]}$$

$$= \sqrt{\left[a^2\cos^2 2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}\right]} = \sqrt{\frac{a^2}{\cos 2\theta}}$$

$$= \frac{a}{\sqrt{\cos 2\theta}}$$

The required surface

$$= 2\int_0^{\pi/4} 2\pi\, y \frac{dS}{d\theta} d\theta = 4\pi \int_0^{\pi/4} r \sin\theta \frac{dS}{d\theta} d\theta$$

$$= 4\pi \int_0^{\pi/4} a\sqrt{\cos 2\theta}\cdot \sin\theta \cdot \frac{a}{\sqrt{\cos 2\theta}} d\theta$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta = 4\pi a^2 [-\cos\theta]_0^{\pi/4}$$

$$= 4\pi a^2\left(1 - \frac{1}{\sqrt{2}}\right)$$

EXERCISE 5.7

1. Find the area of the surface formed by the revolution of $x^2 + 4y^2 = 16$ about its major axis.
2. Find the surface of the solid formed by the revolution about the x-axis of the loop of the curve $x = t^2,\ y = t - \dfrac{t^3}{3}$.
3. Show that the surface area of the solid generated by revolving one complete arc of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ about the line $y = 2a$ is $\dfrac{32}{3}\pi a^2$.

4. Find the surface generated by the revolution of the curve $x = a\cos t + \frac{1}{2}a\log\left(\tan^2\frac{t}{2}\right)$, $y = a\sin t$ about its asymptote.
5. Find the surface of the solid generated by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.
6. Find the surface of the solid formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.
7. Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.
8. The lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface of the solid generated is $4\pi a^2$.

Answers

1. $8\pi + \frac{32\sqrt{3}}{9}\pi^2$
2. 3π
4. $4\pi a^2$
5. $\frac{64}{3}\pi a^2$
6. $\frac{32}{5}\pi a^2$
7. $4\pi a^2$

5.14 APPLICATIONS OF INTEGRAL CALCULUS

5.14.1 Centre of Gravity

A point fixed in the body through the weight of the body irrespective of whatever position the body may be placed is said to be the centre of gravity of the body.

(i) Centre of Gravity of a System of Particles

Let a system of particles of mass $m_1, m_2, m_3, \ldots, m_n$ situated at points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, respectively. Then the centre of gravity (CG) of this system is the point $(\bar{x}, \bar{y})$, where

$$\bar{\boldsymbol{x}} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i},\ \bar{\boldsymbol{y}} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \tag{18}$$

The centre of gravity is also known as *centre of mass* or *centre of inertia*.

(ii) Centre of Gravity of a Plane Area

Let $(\bar{x}, \bar{y})$ be the CG of a plane area A.

Then $$\bar{x} = \frac{\int x\, dm}{\int dm}, \bar{y} = \frac{\int y\, dm}{\int dm}$$

where dm is an element of the mass of A and (x, y) are coordinates of the CG of dm.

Let dA be an element of area A and let (x, y) denote the CG of dA.

Let ρ be mass per unit area. Then $dm = \rho\, dA$ and (x, y) is CG of $\rho\, dA$ or dm.

Therefore,

$$\bar{x} = \frac{\int x\, \rho\, dA}{\int \rho\, dA}, \bar{y} = \frac{\int y\, \rho\, dA}{\int \rho\, dA}$$

or $$\bar{x} = \frac{\int x\, dA}{\int dA}, \bar{y} = \frac{\int y\, dA}{\int dA}, \text{ if } \rho \text{ is a constant.}$$

Special Cases

(i) When the area is bounded by the curve $y = f(x)$, x-axis, and the ordinates $x = a$, $x = b$.
Then the area of the elementary strip of width δx is $\delta A = y\delta x$. The centre of gravity of the elementary strip is $(x, y/2)$.

$\therefore$ CG of the area is given by

$$\bar{x} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}, \bar{y} = \frac{\int_a^b \frac{y}{2} \cdot y dx}{\int_a^b y\, dx}$$

or $$\bar{x} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}, \bar{y} = \frac{\frac{1}{2}\int_a^b y^2 dx}{\int_a^b y\, dx}$$

(ii) Similarly, the CG of a plane area bounded by the curve $x = f(y)$, y-axis, and the abscissae $y = a$ and $y = b$, is given by

$$\bar{x} = \frac{\frac{1}{2}\int_a^b x^2\, dy}{\int_a^b x\, dy}, \bar{y} = \frac{\int_a^b yx\, dy}{\int_a^b x\, dy}$$

(iii) Consider the CG of the sectorial area bounded by the curve $r = f(\theta)$ and the two radii $\theta = \alpha$ and $\theta = \beta$. Then the area of the elementary sectorial area $\delta A = \frac{1}{2} r^2\, \delta\theta$ and the elementary strip $\left(\frac{2}{3} r \cos\theta, \frac{2}{3} r \sin\theta\right)$ is given by

$$\bar{x} = \frac{\frac{2}{3}\int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta}{\int_{\alpha}^{\beta} r^2 \, d\theta} = \frac{2}{3}\frac{\int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta}{\int_{\alpha}^{\beta} r^2 \, d\theta}$$

$$\bar{y} = \frac{\frac{2}{3}\int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta}{\int_{\alpha}^{\beta} r^2 \, d\theta} = \frac{2}{3}\frac{\int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta}{\int_{\alpha}^{\beta} r^2 \, d\theta}$$

(iv) ***CG of a surface of revolution:*** The CG of a surface of revolution about the x-axis of the curve $y = f(x)$ from $x = a$ to $x = b$, is $(\bar{x}, 0)$ is given by

$$\bar{x} = \frac{\int_a^b x \cdot 2\pi \, y \, dS}{\int_a^b 2\pi \, y \, dS} = \frac{\int_a^b xy \, dS}{\int_a^b y \, dS} \text{ and}$$

$$\bar{y} = 0$$

Similarly, the surface of revolution about y-axis of the curve $x = f(y)$ from $y = a$ to $y = b$ for $(0, \bar{y})$ is given by

$$\bar{x} = 0 \text{ and } \bar{y} = \frac{\int_a^b xy \, dS}{\int_a^b x \, dS}$$

(v) ***Centre of gravity of an arc:*** Suppose δs be a small portion of the curve between the points (x, y) and $(x + \delta x, y + \delta y)$. Let ρ be the density at point (x, y) of the curve $y = f(x)$. Then the mass of the small portion is $\delta m = \rho \, \delta s$.

$$\therefore \qquad \bar{x} = \frac{\int x \rho \, dS}{\int \rho \, dS} \text{ and } \bar{y} = \frac{\int y \rho \, dS}{\int \rho \, dS}$$

where dS is given by

(a) Cartesian equation: $dS = \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} \, dx$ or $dS = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} dy$

(b) Parametric equation: $dS = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} \, dt$

(c) Polar form:
$$dS = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta \text{ or } dS = \sqrt{\left[\left(r\frac{d\theta}{dr}\right)^2 + 1\right]}\, dr$$

Example 41 Find the CG of the area enclosed by the parabola $y^2 = 4ax$ and the double ordinates $x = h$.

Solution Let $(\bar{x}, \bar{y})$ be the CG, $\bar{y} = 0$ by the symmetry. The abscissa of the CG will be the same even if we consider the area on one side only. Hence.

$$\bar{x} = \frac{\int_0^h xy\,dx}{\int_0^h y\,dx} = \frac{\int_0^h x^{3/2}\,dx}{\int_0^h x^{1/2}\,dx} \qquad [\text{By } y^2 = 4\,ax]$$

$$= \frac{\frac{2}{5}h^{5/2}}{\frac{2}{3}h^{3/2}} = \frac{3}{5}h$$

Example 42 Find the CG of a uniform lamina bounded by the coordinate axis and arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

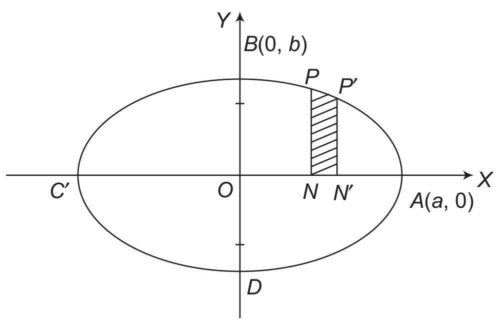


Fig. 5.29

Solution The equation of the ellipse can be written as

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad (1)$$

Let the CG of the ellipse be $(\bar{x}, \bar{y})$

$\therefore$ The CG of the ellipse in the first quadrant is given by

$$\bar{x} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx} = \frac{\int_0^a x\cdot\frac{b}{a}\sqrt{(a^2 - x^2)}\,dx}{\int_0^a \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx}$$

$$\bar{x} = \frac{\int_0^a x\sqrt{(a^2 - x^2)}\,dx}{\int_0^a \sqrt{(a^2 - x^2)}\,dx} = \frac{4a}{3\pi} \qquad \text{[By putting } x = a\cos\theta]$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx} = \frac{\frac{b^2}{2a^2}\int_0^a (a^2 - x^2)\,dx}{\frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx} = \frac{4b}{3\pi} \qquad \text{[Put } x = a\cos\theta]$$

Hence, $$(\bar{x}, \bar{y}) = \left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$$

Example 43 Find the centre of gravity of a uniform hemispherical shell.

Solution A hemispherical shell is obtained by revolving the circle $x^2 + y^2 = a^2$ about the x-axis.

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y} \text{ and } \frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(-\frac{x}{y}\right)^2}$$

$$= \sqrt{\frac{x^2 + y^2}{y^2}} = \frac{a}{y} \qquad [\because x^2 + y^2 = a^2]$$

Let the CG of the hemispherical shell be $(\bar{x}, \bar{y})$.
Hence, the CG $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\int_0^a x \cdot 2\pi y\,dS}{\int_0^a 2\pi y\,dS} = \frac{\int_0^a xy\,\frac{dS}{dx}\,dx}{\int_0^a y\,\frac{dS}{dx}\,dx}$$

$$= \frac{\int_0^a ax\,dx}{\int_0^a a\,dx} = \frac{\int_0^a x\,dx}{\int_0^a dx} = \frac{a}{2}$$

and $$\bar{y} = 0$$

Hence, $\left(\frac{a}{2}, 0\right)$ is the required CG of the given curve.

Example 44 Find the position of the centroid of the area of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ lying in the positive quadrant.

Solution The parametric forms of the given curve are

$$x = a\cos^3\theta,\ y = b\sin^3\theta \tag{1}$$

$$\therefore \quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

Let the CG of the given curve for $(\bar{x}, \bar{y})$ and is given by

$$\bar{x} = \frac{\int_0^{\pi/2} xy\,dx}{\int_0^{\pi/2} y\,dx} = \frac{\int_0^{\pi/2} xy\,\frac{dx}{d\theta}\cdot d\theta}{\int_0^{\pi/2} y\,\frac{dx}{d\theta}\,d\theta}$$

$$= \frac{\int_0^{\pi/2} ab\cos^3\theta\cdot\sin^3\theta\,(-3a\cos^2\theta\cdot\sin\theta)\,d\theta}{\int_0^{\pi/2} b\sin^3\theta\cdot(-3a\cos^2\theta\cdot\sin\theta)\,d\theta}$$

$$= \frac{\int_0^{\pi/2} \sin^4\theta\cdot\cos^5\theta\,d\theta}{\int_0^{\pi/2} \sin^4\theta\cdot\cos^2\theta\,d\theta} = \frac{\Gamma\frac{5}{2}\cdot\Gamma 3\Big/2\cdot\Gamma\frac{11}{2}}{\Gamma\frac{5}{2}\cdot\Gamma\frac{3}{2}\Big/2\cdot\Gamma 4}$$

$$= \frac{\Gamma 3\cdot\Gamma 4}{\Gamma\frac{11}{2}\cdot\Gamma\frac{3}{2}} = \frac{256\,a}{315\,\pi}$$

Similarly, $\quad \bar{y} = \dfrac{256\,b}{315\,\pi}$

Hence, the required CG is $\left(\dfrac{256\,a}{315\,\pi}, \dfrac{256\,b}{315\,\pi}\right)$

Example 45 Find the CG of the area of a loop of the lemniscate $r^2 = a^2\cos 2\theta$.

Solution The given equation of the curve is

$$r^2 = a^2\cos 2\theta \tag{1}$$

Let $(\bar{x}, \bar{y})$ be the required CG of the loop of the curve. Since the curve is symmetrical about the x-axis, $y = 0$, also for this loop θ varies from $= -\pi/4$ to $\pi/4$.

$$\therefore \quad \bar{x} = \frac{\frac{2}{3}\int_{-\pi/4}^{\pi/4} \rho\,r^3\cos\theta\,d\theta}{\int_{-\pi/4}^{\pi/4} \rho\,r^2\,d\theta} = \frac{2}{3}\,\frac{\int_{-\pi/4}^{\pi/4} a^3(\cos 2\theta)^{3/2}\cos\theta\,d\theta}{\int_{-\pi/4}^{\pi/4} a^2\cos 2\theta\,d\theta}$$

$$\bar{x} = \frac{2a^2}{3}\frac{\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta \; d\theta}{2\int_0^{\pi/4} \cos 2\theta \, d\theta}$$

$$= \frac{2a}{3}\frac{\int_0^{\pi/4} (1-2\sin^2 \theta)^{3/2} \cos\theta \; d\theta}{\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}}$$

$$= \frac{2a}{3}\frac{\int_0^{\pi/2} (1-\sin^2 t)^{3/2} \frac{\cos t \, dt}{\sqrt{2}}}{\left(\frac{1}{2}-0\right)_0^{\pi/4}}$$

[Put $\sqrt{2} \sin\theta = \sin t$

$\sqrt{2} \cos\theta \, d\theta = \cos t \, dt$]

$$= \frac{4a}{3\sqrt{2}} \int_0^{\pi/2} \cos^4 t \, dt$$

$$= \frac{4a}{3\sqrt{2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \frac{\pi}{2} = \frac{\pi a\sqrt{2}}{8}$$

and $\bar{y} = 0$

Hence, the required CG is $\left(\frac{\pi a\sqrt{2}}{8}, 0\right)$

Example 46 Find the CG of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ in the positive quadrant.

Solution Let $(\bar{x}, \bar{y})$ be the required CG of the curve in the first quadrant.

Since the curve is symmetrical about the line $y = x$,

we have $\bar{y} = \bar{x}$ (1)

The equation of the curve is

$$x = a\cos^3 t, \; y = a\sin^3 t \quad (2)$$

$$\frac{dx}{dt} = 3a\cos^2 t \cdot (-\sin t), \; \frac{dy}{dt} = 3a\sin^2 t \cdot \cos t$$

$\therefore$

$$dS = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} \, dt$$

$$= \sqrt{[(3a\cos^2 t \cdot (-\sin t))^2 + (3a\sin^2 t \cdot \cos t)^2} \; dt$$

$$= \sqrt{9a^2 \; [(\cos^4 t \cdot \sin^2 t + \sin^4 t \cdot \cos^2 t]} \, dt$$

$$= 3a \sin t \cdot \cos t \sqrt{\sin^2 t + \cos^2 t}\ dt$$

$$dS = 3a \sin t \cos t\ dt$$

For the arc of the curve in the first quadrant, t varies from $t = 0$ to $t = \pi/2$.

$$\therefore \qquad \bar{x} = \frac{\int x \rho\, dS}{\int \rho\, dS} = \frac{\int_0^{\pi/2} a \cos^3 t \cdot 3a \sin t \cdot \cos t\ dt}{\int_0^{\pi/2} 3 \sin t \cdot \cos t\ dt}$$

$$= \frac{\int_0^{\pi/2} \cos^4 t\,(-\sin t)\ dt}{\int_0^{\pi/2} \cos t\,(-\sin t)\, dt}$$

$$= a \frac{\left(\frac{\cos^5 t}{5}\right)_0^{\pi/2}}{\left(\frac{\cos^2 t}{2}\right)_0^{\pi/2}} = \frac{2a}{5}$$

From (1), $\bar{y} = \bar{x} = \frac{2a}{5}$

Hence, the CG is $\left(\frac{2a}{5}, \frac{2a}{5}\right)$.

EXERCISE 5.8

1. Find the CG of the area bounded by the parabola $y^2 = 4ax$, the x-axis, and the latus rectum.
2. Find the CG of the area included between the curve $y^2(2a - x) = x^3$ and its asymptote.
3. Find the CG of a uniform circular wire of radius a in the form of a quadrant of a circle.
4. Find the CG of a uniform wire of radius a in the form of a semicircle.
5. Find the CG of the arc of the cardioid $r = a(1 + \cos\theta)$ lying above the initial line.
6. Find the CG of a solid formed by revolving a quadrant of an ellipse about its minor axis.
7. Find the centroid of the surface formed by the revolution of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.

Answers

1. $\left(\frac{2a \sin\alpha}{3\alpha}, 0\right)$
2. $\left(\frac{5a}{3}, 0\right)$
3. $\left(\frac{4\sqrt{2}a}{\pi}, 0\right)$
4. $\left(\frac{2a}{\pi}, 0\right)$

5. $\left(\dfrac{4a}{5}, \dfrac{4a}{5}\right)$

6. $\left(0, \dfrac{3b}{8}\right)$

7. $\left(\dfrac{a}{6}(2+\sqrt{2}), 0\right)$

5.14.2 Moment of Inertia

Let a particle of mass m lie at a distance r from a given line. Then mr^2 is said to be the *Moment of Inertia (MI)* of the particle about the given line.

If particles of masses $m_1, m_2, m_3, \ldots, m_n$ be placed at distances $r_1, r_2, r_3, \ldots, r_n$ then

$$m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2 \tag{19}$$

If the MI of a body of mass M about any axis AB be MK^2 then K is called the *radius of gyration* of the body about AB.

Similarly, the MI of a plane lamina about any straight line OZ perpendicular to it is equal to the sum of the MI about any two perpendicular straight lines OX and OY in the lamina which pass through the point of intersection O of the lamina and OZ, i.e.,

$$I_z = I_x + I_y$$

5.14.3 Some Standard Results of Moment of Inertia

(i) The MI of a rod of length $2a$ and mass M about a line through

(a) one of its extremities perpendicular to its length is $\dfrac{4a^2}{3} \cdot M$

(b) its centre perpendicular to its length is $\dfrac{a^2}{3} \cdot M$

(ii) The MI of a rectangular parallelepiped of edges $2a$, $2b$, $2c$, and mass m about a line through the centre parallel to the edge

(a) $2a$ is $\dfrac{m(b^2+c^2)}{3}$

(b) $2b$ is $\dfrac{m(a^2+c^2)}{3}$

(c) $2c$ is $\dfrac{m(a^2+b^2)}{3}$

(iii) The MI of a rectangular lamina of sides $2a$, $2b$, and mass m about a line

(a) through the center parallel to the side $2a$ is $\dfrac{mb^2}{3}$

(b) through the center parallel to the side $2b$ is $\dfrac{ma^2}{3}$

(c) perpendicular to the plate through the centre is $\dfrac{m(a^2+b^2)}{3}$

(iv) The MI of a circular ring of radius a and mass m about a

(a) diameter is $\dfrac{ma^2}{2}$

(b) line through its center and perpendicular to its plane is ma^2

(v) The MI of a circular disc (plate) of radius a and mass m about a

(a) diameter is $\dfrac{ma^2}{4}$

(b) line through its centre and perpendicular to its plane is $\dfrac{ma^2}{2}$

(vi) The MI of the elliptic disc $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ of mass m about the

(a) major axis is $\dfrac{mb^2}{4}$

(b) minor axis is $\dfrac{ma^2}{4}$

(vii) The MI of a solid sphere of radius a and mass m about a diameter is $\dfrac{2ma^2}{5}$.

(viii) The MI of a hollow sphere of radius a and mass m about a diameter is $\dfrac{2ma^2}{3}$

Example 47 Find the MI of a right circular cylinder of radius a and mass m about its axis.

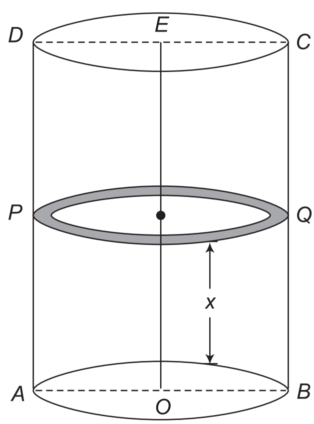


Fig. 5.30

Solution Let a be the height of the cylinder, and let O be the center of base AB and OE be its axis. Suppose a circular disc of thickness δx at a distance x from O.

Then the volume of the elementary disc

$$= \pi a^2 \delta x$$

Mass per unit volume of the cylinder

$$= \frac{m}{\pi a^2 h}$$

$\therefore$ the mass of the elementary disc

$$= \pi a^2 \ \delta x \cdot \frac{m}{\pi a^2 h}$$

$$= \frac{m \cdot \delta x}{h}$$

The MI of the elementary disc about OE

$$= \frac{1}{2} \frac{m \ \delta x}{h} \cdot a^2$$

Hence, MI of the circular cylinder about its axis

$$= \int_0^h \frac{1}{2} \frac{m}{h} a^2 \ dx = \frac{ma^2}{2h} [x]_0^h$$

$$= \frac{1}{2} ma^2$$

Example 48 Find the moment of inertia of a solid right circular cone about its axis.

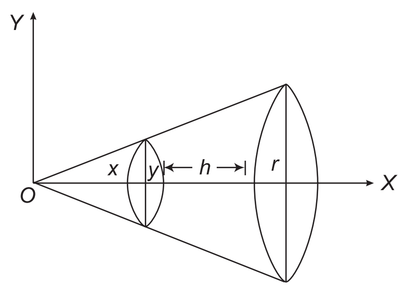


Fig. 5.31

Solution Let the right circular cone be with vertex O, height h, radius r, and its mass is

$$M = \frac{1}{3} \pi r^2 h \cdot \rho \tag{1}$$

Consider y is the radius of an elementary disc perpendicular to the x-axis at a distance x from the vertex O and width δx.

Then $\dfrac{y}{r} = \dfrac{x}{h} \Rightarrow y = \dfrac{rx}{h}$ and the mass of this strip is $\pi y^2 \, x\rho$.

The MI about the x-axis is $\pi y^2 x \cdot \rho \cdot \dfrac{y^2}{2}$

$\therefore$ MI of the solid right circular cone is

$$\int_0^h \pi y^2 x \cdot \rho \frac{y^2}{2} dx = \frac{\rho\pi}{2} \int_0^h y^4 \, dx$$

$$= \frac{\rho\pi}{2} \int_0^h \left(\frac{rx}{h}\right)^4 dx = \frac{\rho \pi r^4}{2h^4} \cdot \int_0^h x^4 \, dx$$

$$= \frac{\rho\pi r^4}{2h^4} \frac{h^5}{5} = \frac{\rho \pi h r^4}{10} = \frac{3Mr^2}{10} \qquad \text{[By (1)]}$$

EXERCISE 5.9

1. Find the MI of a hollow right circular cone of base radius r and mass M.
2. Find the MI of a solid right circular cone about the diameter of its base.
3. Find the MI of a uniform solid sphere of radius a, mass M about any tangent line.
4. Show that the MI of a cone of mass M standing on a circular base of radius a is $\dfrac{3}{10} Ma^2$ about its axis.
5. Find the MI of a circular disc about an axis through its centre perpendicular to its plane, assuming that the density at any point varies as the square of its distance from the center.

Answers

1. $\dfrac{Mr^2}{2}$
2. $\dfrac{M}{20}(3r^2 + 2h^2)$
3. $\dfrac{7}{5} Ma^2$
5. $\dfrac{2}{3} Ma^2$

5.15 IMPROPER INTEGRALS

The integral

$$\int_a^b f(x) \, dx \tag{20}$$

when the range of integration is finite and the integrand function $f(x)$ is bounded for all x in range $[a, b]$ is said to be a **proper (definite) integral**. But if either a or b (or both) are infinite or when a and b are finite but $f(x)$ becomes infinite at $x = a$ or $x = b$ or at one or more points within $[a, b]$ then the integral (20) is called an ***improper*** or ***infinite integral***.

5.15.1 Kinds of Improper Integrals

Improper integrals be divided into the following two kinds:

(i) Improper Integrals of the First Kind

Those integrals in which the integrand function is bounded and the range of integration is infinite are called improper integrals of the first kind.

(ii) Improper Integrals of the Second Kind

Those integrals in which the integrand function is unbounded, i.e., infinite, for some values in the range of integration are called improper integrals of the second kind.

5.15.2 Convergence of Improper Integrals

(i) Improper Integrals of the First Kind

In this section, we shall discuss to evaluate improper integrals of the following forms:

(a) $\int_a^{\infty} f(x)dx$ (b) $\int_{-\infty}^{b} f(x)dx$ (c) $\int_{-\infty}^{\infty} f(x)dx$

(a) Integral $\int_a^{\infty} f(x)dx$

If $f(x)$ is bounded and integrable in the interval (a, b), where $b > a$ then the integral

$$\int_a^{\infty} f(x)dx = \lim_{x\to\infty}\int_a^{x} f(x)dx \text{ provided this limit exists.} \tag{21}$$

The improper integral (21) is said to be convergent if the limits on the right of (21) are unique and finite, otherwise the improper integral is divergent.

Example 49 Test the convergence of $\int_0^{\infty} e^{-x}dx$.

Solution

$$\begin{aligned}\int_0^{\infty} e^{-x}dx &= \lim_{x\to\infty}\int_0^{x} e^{-x}dx\\ &= \lim_{x\to\infty}\left[-e^{-x}\right]_0^x\\ &= \lim_{x\to\infty}\left[-e^{-x}\right]\\ &= \lim_{x\to\infty}(1-e^{-x})\\ &= 1. \text{ (finite and unique)}\end{aligned}$$

Hence, the given integral is convergent.

Example 50 Test the convergence of $\int_0^{\infty} e^x \, dx$.

Solution

$$\int_0^{\infty} e^x \, dx = \lim_{x\to\infty} \int_0^{x} e^x \, dx$$

$$= \lim_{x\to\infty} (e^x)_0^x = \lim_{x\to\infty} (e^x - 1)$$

$$= \infty$$

Hence, the integral diverges to $+\infty$

Note If the integral neither converges nor diverges to a definite limit, it is said to be oscillating. e.g. $\int_0^{\infty} x \sin x \, dx$ oscillates infinitely.

(b) Integral $\int_{-\infty}^{b} f(x)dx$

If $f(x)$ is bounded and integrable in the interval (a, b) where b is fixed and $b < a$ then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{n\to-\infty} \int_x^{b} f(x)dx \text{ provided the limit exists.}$$

If the limit is finite and exists then the improper integral converges, otherwise it is diverges.

Example 51 Evaluate $\int_{-\infty}^{0} e^x \, dx$.

Solution

$$\int_{-\infty}^{0} e^x \, dx = \lim_{x\to-\infty} \int_x^{0} e^x \, dx$$

$$= \lim_{x\to-\infty} (e^x)_x^0$$

$$= \lim_{x\to-\infty} (1 - e^x) = 1$$

Hence, the integral converges to 1.

Example 52 Evaluate $\int_{-\infty}^{0} e^{-x} \, dx$.

Solution

$$\int_{-\infty}^{0} e^{-x} \, dx = \lim_{x\to-\infty} \int_x^{0} e^{-x} dx = \lim_{x\to-\infty} \int_x^{0} (e^{-x} - 1) = \infty$$

Hence, the integral diverges to $+\infty$.

(c) Integral $\int_{-\infty}^{\infty} f(x)dx$

If $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{x\to-\infty}\int_{x}^{c} f(x)\,dx + \lim_{y\to\infty}\int_{c}^{y} f(x)\,dx \tag{22}$$

where c is any finite constant including zero.

If both the limits on the RHS of (22) exist separately and are finite then the improper integral converges. If one or both the limits do not exist or are infinite then the improper integral diverges.

Example 53 Test the convergence of $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \\
&= \lim_{x\to-\infty}\int_{x}^{0} \frac{dx}{1+x^2} + \lim_{y\to\infty}\int_{0}^{y} \frac{dx}{1+x^2} \\
&= \lim_{x\to-\infty}\left[-\tan^{-1} x\right] + \lim_{y\to\infty}\left[\tan^{-1} y\right] \\
&= \frac{\pi}{2} + \frac{\pi}{2} \\
&= \pi \text{ (finite)}
\end{aligned}$$

Hence, the integral converges to the value π.

Necessary and Sufficient Condition for the Convergence of $\int_{0}^{\infty} f(x)dx$

The integral $\int_{0}^{\infty} f(x)\,dx$ converges to the value I, when corresponding to any positive number $\in$ having chosen, there exists a positive number m such that

$$I = \left|\int_{0}^{\infty} f(x)\,dx\right| < \in; \text{ provide } x \geq m.$$

Integral $\int_{a}^{\infty} f(x)\,dx$ when integrand $f(x)$ is positive.

The integral $\int_{a}^{\infty} f(x)\,dx$ converges to the positive number l, such that $\int_{a}^{\infty} f(x)\,dx < l$, when $x > a$ and in this case, $\int_{a}^{\infty} f(x)\,dx \leq l$.

Otherwise, it diverges.

Example 54 Test the convergence of $\int_a^{\infty} \frac{\sin^2 x}{x^2} dx$.

Solution We know that $0 < \sin x < 1$, when $x > 0$.

Now, $$\frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \text{ when } x \ge a > 0$$

Hence, $$\int_a^{\infty} \frac{\sin^2 x}{x^2} dx \le \int_a^{\infty} \frac{1}{x^2} dx$$

$$\le \left(-\frac{1}{x}\right)_a^{\infty}$$

$$\le \frac{1}{a} \text{ (which is positive)}$$

Hence, the integral is convergent.

Example 55 Test the convergence of $\int_2^{\infty} \frac{dx}{\sqrt{(x^2-1)}}$.

Solution We know that $\frac{1}{\sqrt{(x^2-1)}} > \frac{1}{x}$ when $x \ge 2$,

$$\int_2^{\infty} \frac{1}{\sqrt{(x^2-1)}} dx > \int_2^{\infty} \frac{1}{x} dx$$

Hence, the integral is divergent.

We do not always study the convergence or divergence of an improper integral by evaluating, it as was done in previous problems. For example, $\int_2^{\infty} \frac{1}{\log x} dx$ which cannot be evaluated directly. **Now, we discuss the following tests for evaluating the convergence or divergence of an improper integral.**

(i) Comparison Test Let $f(x)$ and $g(x)$ be two positive functions, bounded and integrable in the interval (a, b) in which a is fixed and $b > a$. Also, let $f(x) \le g(x)$ when $x \ge a$. Then

$$\int_0^{\infty} f(x)\,dx \text{ is convergent if } \int_0^{\infty} g(x)\,dx \text{ is convergent, and } \int_0^{\infty} f(x)\,dx \le \int_0^{\infty} g(x)\,dx.$$

If $f(x) \ge g(x)$ and $\int_0^{\infty} g(x)\,dx$ diverges, then $\int_0^{\infty} f(x)\,dx$ also diverges.

(ii) μ-test for the Convergence of $\int_0^\infty f(x)dx$

Let $f(x)$ be bounded and integrable in the interval (a, b) where $a > 0$. If a positive number $\mu > 1$, such that $\lim_{x\to\infty} x^\mu \cdot f(x)$ exists finitely (***the limit being neither zero nor infinite***) then the integral $\int_0^\infty f(x)\,dx$ converges.

If $\mu \le 1$, the integral $\int_0^\infty f(x)\,dx$ is divergent.

(iii) Abel's Test for Convergence of Integral of a Product

If $\int_a^\infty f(x)\,dx$ is convergent and $g(x)$ is bounded and monotonic in (a, ∞) then the integral $\int_a^\infty f(x)\cdot g(x)\,dx$ is convergent.

(iv) Dirichlet's Test for Convergence of the Integral of a Product

Let $f(x)$ be bounded and monotonic and $\lim_{x\to 0} f(x) = 0$, also consider $\left|\int_a^x \phi(x)\,dx\right|$ be bounded when $x \ge a$; then $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

Example 56 Prove that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

Solution

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx \tag{1}$$

In the RHS of (1), the first integral $\int_0^{-1} e^{-x^2}\,dx$ is proper (definite) so it is convergent.

And
$$\int_1^\infty e^{-x^2}\,dx < \int_0^\infty xe^{-x^2}\,dx \quad \text{as} \quad x > 1$$
$$< \left(-\frac{1}{2}e^{-x^2}\right)_0^\infty$$
$$< \frac{1}{2} \text{ (a finite quantity)}$$

$\therefore$
$$\int_1^\infty e^{-x^2}\,dx \text{ is convergent}$$

Hence, $\int_0^\infty e^{-x^2}$ is convergent. **Hence, proved**

Example 57 Test the convergence of $\int_0^\infty \frac{x}{(1+x)^3}dx$.

Solution Here, $f(x) = \frac{x}{(1+x)^3}$

μ = Highest power of x in the denominator – Highest power of 'x' in the numerator.

$\therefore \quad \mu = 3 - 1 = 2$

Using μ-test,

$$\therefore \quad \lim_{x\to\infty}\left[x^\mu \cdot f(x)\right] = \lim_{x\to\infty} x^2 \cdot \frac{x}{(1+x)^3}$$

$$= \lim_{x\to\infty} \cdot \frac{x^3}{x^3\left(1+\frac{1}{x}\right)^3}$$

$$= \lim_{x\to\infty} \cdot \frac{1}{\left(1+\frac{1}{x}\right)^3} = 1 \qquad \text{[a finite quantity]}$$

Since $\mu = 2 > 1$; hence, the integral is convergent.

Example 58 Test the convergence of $\int_0^\infty \frac{x^3}{(a^2+x^2)^2}dx$.

Solution Here, $f(x) = \frac{x^3}{(a^2+x^2)^2}$ and $\mu = 4 - 3 = 1$.

$$\therefore \lim_{x\to\infty} x^\mu \cdot f(x) = \lim_{x\to\infty} x \cdot \frac{x^3}{(a^2+x^2)^2} = \lim_{x\to\infty} \frac{x^4}{x^4\left(1+\frac{a^2}{x^2}\right)^2} = 1 \text{ (finite quantity)}$$

Since $\mu = 1$, i.e., not > 1.

Hence, the integral is divergent.

Example 59 Show that $\int_b^\infty \frac{x^{3/2}}{\sqrt{(x^4-a^4)}}dx$ is divergent, $b > a$.

Solution Here, $f(x) = \frac{x^{3/2}}{\sqrt{(x^4-a^4)}}$ is bounded in (b, ∞) and $\mu = 2 - \frac{3}{2} = \frac{1}{2}$

$$\therefore \qquad \lim_{x\to\infty}\left[x^{\mu}\cdot f(x)\right] = \lim_{x\to\infty}\left[x^{1/2}\cdot\frac{x^{3/2}}{\sqrt{(x^4-a^4)}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x^2}{\sqrt{(x^4-a^4)}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x^2}{x^2\sqrt{1-\frac{a^4}{x^4}}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{1}{\sqrt{1-\frac{a^4}{x^4}}}\right]=1$$

Since $\mu=\frac{1}{2}<1$, Hence, the integral is divergent.

Example 60 Test the convergence of $\int_a^{\infty} e^{-x}\frac{\sin x}{x^2}dx$.

Solution Let $f(x)=\frac{\sin x}{x^2}$ and $g(x)=e^{-x}$

But $\int_a^{\infty}\frac{\sin x}{x^2}dx<\int_a^{\infty}\frac{1}{x^2}dx = \left(-\frac{1}{x}\right)_a^{\infty}<\frac{1}{a}$

Hence, $\int_a^{\infty}\frac{\sin x}{x^2}dx$ is convergent.

Also, e^{-x} is monotonic increasing and bounded in (a, ∞).

Hence, by Abel's test, $\int_a^{\infty}e^{-x}\frac{\sin x}{x^2}dx$ is convergent.

Example 61 Show that $\int_a^{\infty}\frac{(\cos ax-\cos bx)}{x}dx$ is convergent.

Solution Here, $\frac{1}{x}$ is monotonic and $\lim_{x\to\infty}\frac{1}{x}=0$.

Also, $\left|\int_a^{\infty}(\cos ax-\cos bx)\,dx\right|$ is bounded. Hence, the given integral is convergent.

Example 62 Show that $\int_1^{\infty} \frac{x}{1+x^2} \sin x\, dx$ is convergent.

Solution Here, $\frac{x}{1+x^2}$ is monotonic and $\lim_{x\to\infty} \frac{x}{1+x^2} = 0$

Also, $\left|\int_1^{\infty} \sin x\, dx\right| = |-\cos x + \cos 1| \le 2$

Hence, by Dirichlet's test, the integral $\int_1^{\infty} \frac{x}{1+x^2} \sin x\, dx$ is convergent.

5.15.3 Absolute Convergence

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent when $f(x)$ is bounded and integrable in (a, b) and $\int_a^b |f(x)|\, dx$ is convergent.

A convergent integral which is not absolutely convergent is said to be a *conditionally convergent integral*.

Example 63 Test the convergence of $\int_0^1 \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}}\, dx$.

Solution Here, $f(x) = \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}}$

There is no neighborhood of the point 0 in which $f(x)$ constantly keeps the same sign.

Now, $\forall\, x \in [0, 1]$, we have

$$|f(x)| = \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| = \frac{\left|\sin\frac{1}{x}\right|}{\left|\sqrt{x}\right|} \le \frac{1}{\sqrt{x}}$$

Also, $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ is convergent.

$\therefore \quad \int_0^1 \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| dx$ is convergent.

Hence, $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\, dx$ is absolutely convergent.

EXERCISE 5.10

1. Examine the convergence of $\int_0^\infty \frac{x^2}{(4+x^2)^2}\,dx$.

2. Test the convergence of $\int_0^\infty \frac{x^{3/2}}{b^2 x^2 + c^2}\,dx$.

3. Show that $\int_0^\infty \frac{\sin x}{x}\,dx$, is convergent.

4. Show that $\int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx,\ a > 0$ is convergent.

5. Show that $\int_0^{\pi/2} \log \sin x\,dx$, is convergent.

6. Show that $\int_0^1 \frac{\sin 1/x}{(\overline{x})^n}\,dx,\ x > 0$, converges absolutely, if $n < 1$.

Answers

1. Convergent 2. Divergent

5.16 MULTIPLE INTEGRALS

In this section, we shall study double and triple integrals along with their applications.

5.16.1 Double Integrals

A double integral is the counterpart, in two dimensions, of the definite integral of a function of a single variable.

Let R be a closed and bounded region (domain) of the xy-plane and let $f(x, y)$ be a function of the independent variables x, y defined at every point in the region R. Divide the region R in to n-parts, of areas δR_1, δR_2, δR_3, …, δR_n. Let (x_r, y_r) be any point inside the r^{th} elementary area δR_r.

Consider the sum.

$$S_n = f(x_1, y_1)\,\delta R_1 + f(x_2, y_2)\,\delta R_2 + \cdots + f(x_r, y_r)\,\delta R_r + \cdots + f(x_r, y_n)\,\delta R_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta R_r \tag{23}$$

When $n \to \infty$, the number of subdivisions increases indefinitely such that the largest of the areas $\delta R_r \to 0$.

The $\lim\limits_{\substack{n\to\infty \\ \delta R_r \to 0}} S_n$, if it exists, called the ***double integral*** of the function $f(x, y)$ over the region R, and is defined as

$$\iint_R f(x, y)\,dR \tag{24}$$

Thus, $$\iint_R f(x, y)\, dR = \lim_{\substack{n\to\infty \\ \delta R_r \to 0}} S_n$$

or $$\iint_R f(x, y)\, dR = \lim_{\substack{n\to\infty \\ \delta R_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta R_r \tag{25}$$

If the region R is divided into a rectangular form of a network of lines parallel to the coordinate axis dx (length) and dy (breadth) of a rectangular then $dx\,dy$ is the elementary area such that

$$\iint_R \boldsymbol{f(x, y)\, dR} = \iint_R \boldsymbol{f(x, y)\, dx\, dy} \tag{26}$$

5.16.2 Properties of Double Integrals

(i) $\iint_R [f(x, y) + g(x, y)]dR = \iint_R f(x, y)\, dR + \iint_R g(x, y)\, dR$

(ii) If the region R is divided into two regions R_1 and R_2 then

$$\iint_R f(x, y)\, dR = \iint_{R_1} f(x, y) dR + \iint_{R_2} f(x, y)\, dR$$

(iii) Let $\alpha \neq 0$ be any real number then

$$\iint_R \alpha \cdot f(x, y)\, dR = \alpha \iint_R f(x, y)\, dR$$

5.16.3 Evaluation of a Double Integral in Cartesian Coordinates

The double integral can be evaluated by successive single integrations.

(i) Let R be a Region Bounded by the Curves

$$y = f_1(x),\ y = f_2(x),\ x = a \text{ and } x = b.$$

Then $$\iint_R \boldsymbol{f(x, y)\, dR} = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} \boldsymbol{f(x, y)\, dy\, dx}$$

where the integration with respect to y is performed first treating x as a constant.

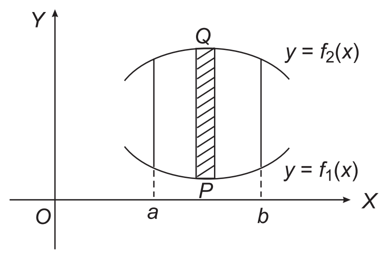


Fig. 5.32

(ii) If R is a Region Bounded by the Curves

$$x = f_1(y),\ x = f_2(y),\ y = a \text{ and } y = b \text{ then}$$

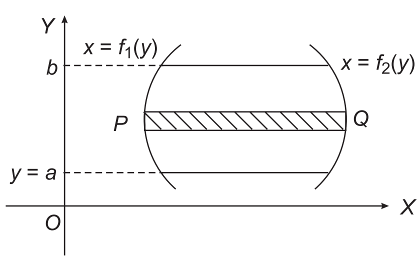


Fig. 5.33

$$\iint_R f(x, y)\, dR = \int_{y=a}^{b} \int_{x=f_1(y)}^{f_2(y)} f(x, y)\, dx\, dy$$

where the integration w.r.t. x is performed first treating y as a constant.

(iii) If R is a Region Bounded by the Lines

$x = a$, $x = b$, $y = c$ and $y = d$ then

$$\iint_R f(x, y)\, dR = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\, dy\, dx$$

OR

$$= \int_{y=c}^{d} \int_{x=a}^{b} f(x, y)\, dx\, dy$$

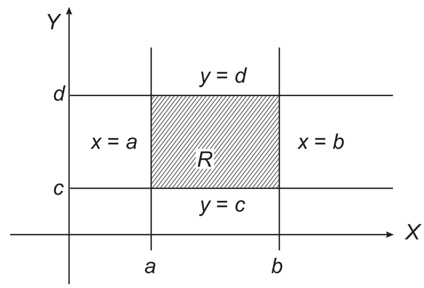


Fig. 5.34

Example 64 Evaluate $\iint_R xy\, dx\, dy$ over the region R in the positive quadrant for which $x + y \le 1$.

Solution Since the region R is bounded by $x = 0$, $y = 0$ and $x + y = 1$, and the order of given integration is $dxdy$, so we consider the strip parallel to the x-axis.

We can consider it as the area bounded by the lines

$$x = 0, x = 1 - y, y = 0 \text{ and } y = 1$$

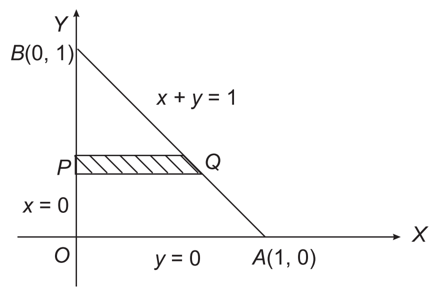


Fig. 5.35

$$\therefore \qquad \iint_R xy\, dx\, dy = \int_{y=0}^{1} \int_{x=0}^{1-y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y \left(\frac{x^2}{2} \right)_0^{1-y} dy$$

$$= \int_{y=0}^{1} \frac{y(1-y)^2}{2}\, dy$$

$$= \frac{1}{2} \int_{y=0}^{1} y(1 + y^2 - 2y)\, dy = \frac{1}{2} \int_{y=0}^{1} [y + y^3 - 2y^2]\, dy$$

$$= \frac{1}{2} \left[\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3} \right]_0^1 = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} - 0 \right)$$

$$= \frac{1}{24}$$

Example 65 Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$.

Solution

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dx\, dy = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{\left[\sqrt{1+x^2}\right]^2 + y^2}\, dy dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} 1 - \tan^{-1} 0\right] dx = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4} \log\left[x + \sqrt{1+x^2}\right]_0^1 = \frac{\pi}{4} \log(1+\sqrt{2})$$

Example 66 Evaluate $\iint_R (x^2+y^2)\, dx\, dy$, where R is bounded by $y = x$ and $y^2 = 4x$.

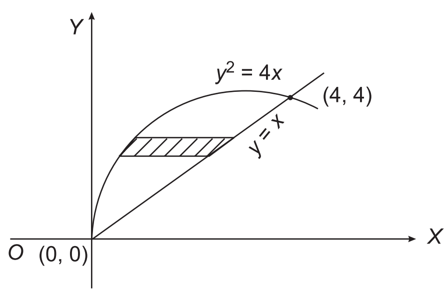


Fig. 5.36

Solution

$$\iint_R (x^2+y^2)\, dx\, dy = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{y} (x^2+y^2)\, dx\, dy$$

$$= \int_{y=0}^{4} \left[\frac{x^3}{3} + xy^2\right]_{\frac{y^2}{4}}^{y} dy = \int_0^4 \left[\left(\frac{y^3}{3} + y^3\right) - \left(\frac{y^6}{3\times 64} + \frac{y^4}{4}\right)\right] dy$$

$$= \int_0^4 \left[\frac{4y^3}{3} - \frac{y^4}{4} - \frac{y^6}{192}\right] dy$$

$$= \left[\frac{y^4}{3} - \frac{y^5}{20} - \frac{y^7}{192 \times 7} \right]_0^4 = \frac{768}{35}$$

Example 67 Evaluate $\iint_R y\, dx\, dy$; where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

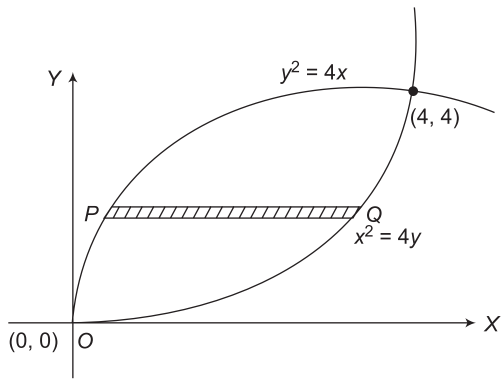


Fig. 5.37

Solution

$$\iint_R y\, dx\, dy = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{2\sqrt{y}} y\, dx\, dy$$

$$= \int_0^4 y \cdot (x)_{\frac{y^2}{4}}^{2\sqrt{y}}\, dy$$

$$= \int_0^4 y\left(2\sqrt{y} - \frac{y^2}{4}\right) dy$$

$$= \int_0^4 \left[2y^{3/2} - \frac{y^3}{4}\right] dy$$

$$= \left[\frac{4}{5} y^{5/2} - \frac{1}{16} y^4\right]_0^4$$

$$= \frac{4}{5} \cdot (4)^{5/2} - \frac{1}{16}(4)^4 = \frac{4}{5} \cdot 32 - \frac{16 \times 16}{16}$$

$$= \frac{128}{5} - 16 = -\frac{128 - 80}{5} = \frac{48}{5}$$

Example 68 Evaluate $\iint (x+y)^2 \, dxdy$ over the area bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

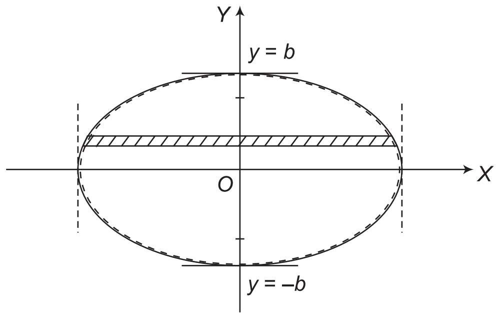


Fig. 5.38

Solution For the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, so the region of integration can be considered bounded by the lines

$$x=-\frac{a}{b}\sqrt{b^2-y^2} \text{ to } x=\frac{a}{b}\sqrt{b^2-y^2} \text{ and } y=-b \text{ to } y=b.$$

$$\therefore \quad \iint (x+y)^2 \, dx\,dy = \int_{y=-b}^{b} \int_{x=-\frac{a}{b}\sqrt{b^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} (x^2+2xy+y^2)\, dx\,dy$$

$$= 2\int_{-b}^{b} \int_{0}^{\frac{a}{b}\sqrt{b^2-y^2}} (x^2+y^2)\, dx\,dy$$

$$= 4\int_{0}^{b} \int_{0}^{\frac{a}{b}\sqrt{b^2-y^2}} (x^2+y^2)\, dx\,dy$$

$$= 4\int_{0}^{b} \left[\frac{x^3}{3}+xy^2\right]_{0}^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= 4\int_{0}^{b} \left[\frac{a^3}{3b^3}(b^2-y^2)^{3/2}+\frac{ay^2}{b}(b^2-y^2)^{1/2}\right] dy$$

[Putting $y = b\sin\theta$; $dy = b\cos\theta\, d\theta$]

$$= 4\int_{0}^{\pi/2} \left[\frac{a^3}{3b^3}\cdot(b^2-b^2\sin^2\theta)^{3/2}+\frac{ab^2}{b}\sin^2\theta(b^2-b^2\sin^2\theta)^{1/2}\right]\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{a^3}{3}\cos^3\theta + ab^2\sin^2\theta\cdot\cos\theta\right]b\cos\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2}\left[\frac{a^2}{3}\cos^4\theta + b^2\sin^2\theta\cdot\cos^2\theta\right]d\theta$$

$$= 4ab\left[\frac{a^2}{3}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} + b^2\cdot\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{2\Gamma(3)}\right]$$

$$= \frac{4ab}{16}\pi\ (a^2 + b^2)$$

$$= \frac{\pi\, ab}{4}(a^2 + b^2)$$

5.16.4 Evaluation of Double Integrals in Polar Coordinates

Consider a function $f(r, \theta)$ of the polar coordinates r, θ which is to be integrated over a certain region R whose boundary is also given in polar coordinates, and it is defined as

$$\iint_R f(r,\theta)\, dR = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta) r\, dr\, d\theta$$

where the region R is bounded by the lines

$$\theta = \theta_1 \text{ to } \theta = \theta_2 \text{ and } r = f_1(\theta) \text{ to } r = f_2(\theta)$$

The first integration is performed with respect to r, taking θ as a constant. After substituting the limits, the second integration with respect to θ is performed.

Example 69 Evaluate $\iint_R r\sin\theta\, dR$; R is the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line.

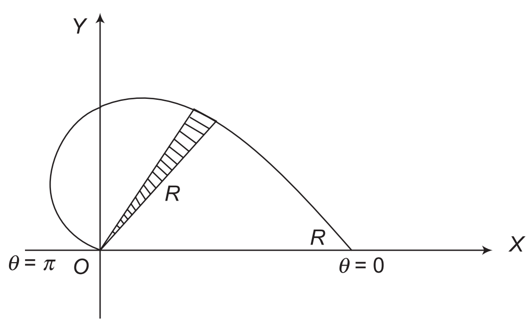


Fig. 5.39

Solution Here, the region of integration R can be covered by radial strips whose ends are at $r = 0$ and $r = a(1 + \cos\theta)$ and end at $\theta = \pi$.

$\therefore$ the required integral

$$\iint_R r\sin\theta\, dR = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta\; dr\,d\theta$$

$$= \int_0^{\pi}\left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)}\sin\theta\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}\sin\theta\,(1+\cos\theta)^3\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot\left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/2}\sin\phi\cos^7\phi\,2d\phi \qquad [\text{Putting } \theta = 2\phi;\ d\theta = 2d\phi]$$

$$= \frac{32a^3}{3}\left[-\frac{\cos^8\phi}{8}\right]_0^{\pi/2}$$

$$= \frac{4a^3}{3}$$

Example 70 Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2a\cos\theta$, $r = 2b\cos\theta$; $b < a$.

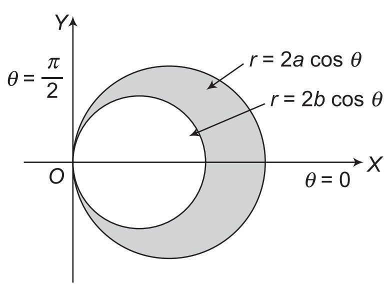


Fig. 5.40

Solution In the given region of integration, r varies from $2b\cos\theta$ to $2a\cos\theta$, and θ varies from

$$\theta = -\frac{\pi}{2} \text{ to } \theta = \frac{\pi}{2}$$

$$\therefore \qquad \iint r^3\, dr\, d\theta = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \left[\int_{2b\cos\theta}^{2a\cos\theta} r^3\, dr\right] d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\right)_{2b\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\pi/2} [16a^4 \cos^4\theta - 16b^4 \cos^4\theta]\, d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\pi/2} (a^4 - b^4)\cos^4\theta\, d\theta$$

$$= 8(a^4 - b^4)\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 8(a^4 - b^4)\cdot \frac{\left|\frac{1}{2}\cdot\right|\frac{5}{2}}{2\cdot\left|2\right.} = \frac{3\pi}{2}(a^4 - b^4)$$

Change of Variables in a Double Integral

Change from Cartesian to Polar Coordinates The evaluation of some double integrals by changing from Cartesian to polar coordinates can be easily effected by noting that the element of area in the latter system is $r\, dr d\theta$.

Thus,

$$\iint_R f(x, y)\, dx\, dy = \iint_R f(x, y)\, dR$$

$$= \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r, \theta)\, r\, dr\, d\theta$$

where $F(r, \theta) = f(r\cos\theta, r\sin\theta) = f(x, y)$ and R' is the corresponding domain in polar coordinates.

Example 71 Transform the integral $\int_0^a \int_0^{\sqrt{a^2 - x^2}} y^2 \sqrt{(x^2 + y^2)}\, dx\, dy$ by changing to polar coordinates, and, hence, evaluate it.

Solution The given limits of integration show that the region of integration lies between the curves

$$y = 0 \text{ to } y = \sqrt{a^2 - x^2}$$

$$x = 0 \text{ to } x = a.$$

Thus, the region of integration is the part of the circle $x^2 + y^2 = a^2$ in the first quadrant.

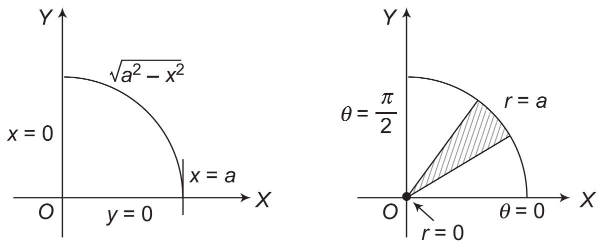


Fig. 5.41

In polar coordinates, the equation of the circle is

$$r^2 \cos^2\theta + r^2 \sin^2 \theta = a^2 \text{ or } r^2 = a^2$$

$$\text{or } r = a$$

Thus, in polar co-ordinates, the region of integration is bounded by the curves

$$r = 0,\ r = a \quad \text{and} \quad \theta = 0,\ \theta = \frac{\pi}{2}$$

$$\therefore \quad \int_0^a \int_0^{\sqrt{a^2 - x^2}} y^2 \sqrt{x^2 + y^2}\, dx\, dy$$

$$= \int_0^{\pi/2} \int_0^a r^2 \sin^2 \theta \cdot r\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^a r^4 \sin^2 \theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \sin^2 \theta \cdot \left(\frac{r^5}{5}\right)_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \sin^2 \theta \cdot d\theta$$

$$= \frac{a^5}{5} \cdot \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{2 \cdot \Gamma(2)} \qquad \text{[By Gamma function]}$$

$$= \frac{\pi a^5}{20}$$

Example 72 Evaluate $\displaystyle\int_0^{\pi/2} \int_0^{a\cos\theta} r \sin \theta\, dr\, d\theta$.

Solution

$$\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\sin\theta\cdot\left(\frac{r^2}{2}\right)_0^{a\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\sin\theta\cdot\frac{a^2\cos^2\theta}{2}\,d\theta = \frac{a^2}{2}\int_0^{\pi/2}\sin'\theta\cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\cdot\frac{\Gamma\left(\frac{2}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{2\cdot\Gamma(2)} = \frac{a^2}{2}\cdot\frac{1\cdot\Gamma\left(\frac{3}{2}\right)}{2\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)} = \frac{a^2}{6}$$

Example 73 Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.

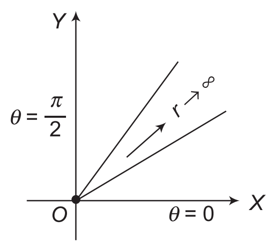


Fig. 5.42

Solution Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$.

Since the first quadrant lies in the xy-plane, x varies from 0 to ∞ and y varies from 0 to ∞, and is covered where r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^\infty e^{-r^2}(\theta)_0^{\pi/2}\,r\,dr\,d\theta$$

$$= \frac{\pi}{2\times 2}\int_0^\infty e^{-r^2}\cdot 2r\;dr\,d\theta$$

$$= \frac{\pi}{4}\int_0^\infty e^{-r^2}\cdot 2r\;dr\,d\theta$$

$$= \frac{\pi}{4}\left[-e^{-r^2}\right]_0^{\infty}$$

$$= \frac{\pi}{4}$$

EXERCISE 5.11

1. Evaluate $\displaystyle\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy$.

2. Evaluate $\displaystyle\iint_R x^2\,dx\,dy$; where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$, and $x = 8$.

3. Evaluate $\displaystyle\iint_R (4xy - y^2)\,dx\,dy$, where R is the rectangle bounded by $x = 1$, $x = 2$, $y = 0$, and $y = 3$.

4. Evaluate $\displaystyle\int_0^3\int_1^2 xy(1 + x + y)\,dx\,dy$.

5. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} (1 + x^2 + y^2)^{-1}\,dx\,dy$.

6. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{x}} (x^2 + y^2)\,dx\,dy$.

7. Evaluate $\displaystyle\iint_R x^2\,y^2\,dx\,dy$; R is the region bounded by $x = 0$, $y = 0$, and $x^2 + y^2 = 1$.

8. Evaluate $\displaystyle\iint xy\,(x + y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.

9. Evaluate $\displaystyle\int_0^{\pi}\int_a^{(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta$.

10. Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\,dr\,d\theta$.

11. Evaluate $\displaystyle\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$.

12. Using polar coordinates, evaluate $\displaystyle\iint xy\,(x^2 + y^2)^{3/2}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

13. Evaluate $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr\, d\theta$ by changing it into Cartesian coordinates.

Hint: $\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy\, dx\, dy = 0$

14. Using polar coordinates, evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} dy\, dx$.

15. Evaluate $\iint \sqrt{(a^2 - x^2 - y^2)}\, dx\, dy$ over the semicircle $x^2 + y^2 = ax$ in the positive quadrant.

Hint: $\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2 - r^2} \cdot r\, dr\, d\theta$

Answers

1. $8 \log 8 - 16 + e$
2. 448
3. 18
4. $30\frac{3}{4}$ or $\frac{123}{4}$
5. $\frac{\pi}{4}\log(1+\sqrt{2})$
6. $\frac{3}{35}$
7. $\frac{\pi}{96}$
8. $\frac{3}{56}$
9. $\frac{5\pi a^3}{8}$
10. $(3\pi - 4)\frac{a^3}{18}$
11. $\frac{45\pi}{2}$
12. $\frac{1}{14}$
13. 0
14. $\frac{\pi a^2}{2}$
15. $\frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$

5.16.5 Change of Order of Integration

We have seen that in evaluating a double integral of successive integrations, if the limits of the both variables are constant then we can change the order of integration without affecting the result. But if the limits of integration are variable, i.e., if the limits of x are constant and limits of y are function of x and order of integration is $dy\, dx$ or if the limits of x are function of y and y are constants and order of integration is $dx\, dy$ then we take a strip parallel to the y-axis, or parallel to the x-axis.

When it is required to change the order of integration in an integral for which the limits are given, we first of all ascertain from the given limits the region R of integration. Knowing the region of integration, we can then put in the limits for integration in the reverse order.

Example 74 Change the order of integration in $\int_0^a \int_0^y f(x, y)dx\,dy$.

Solution From the limits of integration, it is clear that the region of integration is bounded by $x = 0$, $x = y$, $y = 0$, and $y = a$.

According to limits of the given integration, strips PQ parallel to the x-axis, i.e., horizontal strips.

For changing the order of integration, we consider the strips parallel to y-axis, i.e., vertical strips.

The new limits of integration become: y varies from x to a and x varies from 0 to a.

Thus,

$$\int_0^a \int_0^y f(x, y)dx\,dy = \int_0^a \int_x^a f(x, y)dy\,dx$$

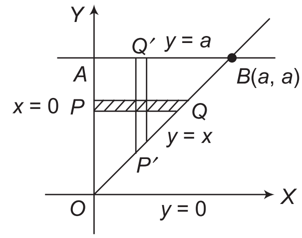


Fig. 5.43

Example 75 Change the order of integration in $\int_0^1 \int_{e^x}^e \frac{dy\,dx}{\log y}$ and, hence, evaluate it.

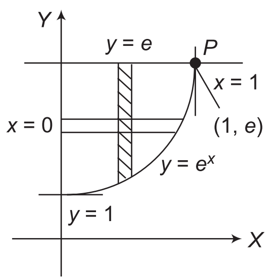


Fig. 5.44

Solution From the limits, the given integration is bounded by the curves $y = e^x$, $y = e$, $x = 0$, and $x = 1$.

According to limits of the given integration, the strips are parallel to y-axis, i.e., vertical strips. For changing the order of integration, we consider strips parallel to the x-axis i.e., horizontal strips. The new limits of integration become: x varies from 0 to $\log y$ and y varies from 1 to e.

$$\text{Thus,} \quad \int_0^1 \int_{e^x}^e \frac{dy\,dx}{\log y} = \int_1^e \int_0^{\log y} \frac{dy\,dx}{\log y}$$

$$= \int_1^e \left(\frac{x}{\log y}\right)_0^{\log y} dy$$

$$= \int_1^e 1 \cdot dy$$
$$= (y)_1^e$$
$$= (e - 1)$$

Example 76 Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and, hence, evaluate the same

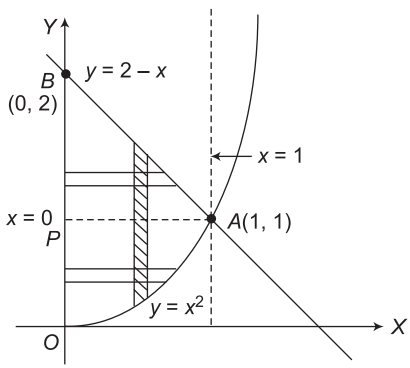


Fig. 5.45

Solution From the limits of the given integral, the region of integration is bounded by $y = x^2$, $y = 2 - x$, $x = 0$, and $x = 1$.

According to the given limits, the strip is parallel to the y-axis, i.e., it is a vertical strip. For changing the order of integration, we divide the region of integration into horizontal strips. The region of integration is divided into two parts, OAP and PAB.

For the region OAP, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1, and for the region PAB, x-varies 0 to $2 - y$ and y varies from 1 to 2.

Hence,

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 y \cdot \left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\, dy + \frac{1}{2}\int_1^2 y \cdot \left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\, dy + \frac{1}{2}\int_1^2 y \cdot (2-y)^2\, dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{3}{8}$$

Example 77 By changing the order of integration, evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$.

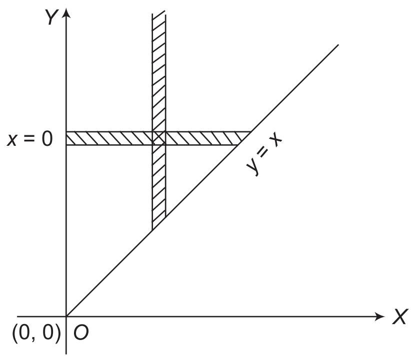


Fig. 5.46

Solution The given limits show that the area of integration lies between $y = x$, $y = \infty$, $x = 0$ and $x = \infty$.

New limits of the region are x varies from 0 to y and y varies from 0 to ∞.

$$\therefore \qquad \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot (x)_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy = \int_0^\infty e^{-y}\, dy$$

$$= (-e^{-y})_0^\infty$$

$$= 1$$

Example 78 Evaluate $\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax - y^2}}\, dx\, dy$ by changing the order of integration.

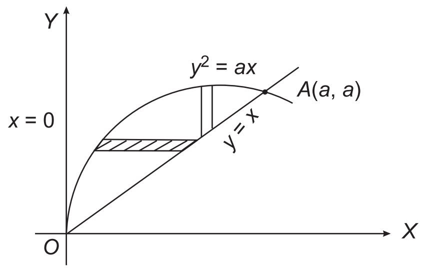


Fig. 5.47

Solution From the limits of integration, it is clear that the region of integration is bounded by $x = \dfrac{y^2}{a}$ $x = y$, $y = 0$ and $y = a$.

For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become: y varies from x to $\sqrt{ax}$ and x varies from 0 to a.

Thus,

$$\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy = \int_0^a \int_x^{\sqrt{ax}} \frac{y\, dy\, dx}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^a \left[-\frac{(ax-y^2)^{1/2}}{(a-x)} \right]_x^{\sqrt{ax}} dx$$

$$= \int_0^a \frac{(ax-x^2)^{1/2}}{(a-x)}\, dx = \int_0^a \sqrt{\left(\frac{x}{a-x}\right)}\, dx$$

$$= \int_0^{\pi/2} \sqrt{\left(\frac{a\sin^2\theta}{a\cos^2\theta}\right)}\, 2a\sin\theta\cos\theta\, d\theta$$

[Putting $x = a\sin^2\theta$; $dx = 2a\sin\theta \cdot \cos\theta\, d\theta$]

$$= \int_0^{\pi/2} 2a\sin^2\theta d\theta = 2a \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a}{2}$$

Example 79 By changing the order of integration for $\int_0^\infty \int_0^\infty e^{-xy}\sin px\, dx\, dy$, show that $\int_0^\infty \frac{\sin px}{x}\, dx = \frac{\pi}{2}$.

Solution The given limits show that the area of integration lies between $y = 0$ to ∞ and $x = 0$ to ∞.

For changing the order of integration, the new limits become:
x varies 0 to ∞ and y varies 0 to ∞.

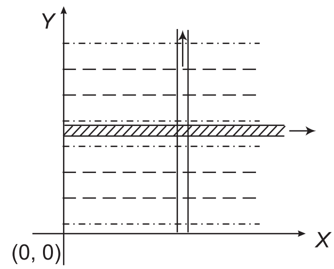


Fig. 5.48

$$\therefore \quad \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dy \, dx$$

$$= \int_0^\infty \left(\frac{e^{-xy}}{-x} \right)_0^\infty \sin px \, dx$$

$$= \int_0^\infty \frac{\sin px}{x} \, dx \tag{1}$$

$$\text{Again,} \quad \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \left[\int_0^\infty e^{-xy} \sin px \, dx \right] dy$$

$$= \int_0^\infty \left[\frac{e^{-xy}}{p^2 + y^2} (p \cos px + y \sin px) \right]_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2 + y^2} dy$$

$$= \left[\tan^{-1} \left(\frac{y}{p} \right) \right]_0^\infty = \frac{\pi}{2} \tag{2}$$

From (1) and (2), we get

$$\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$$

Hence, proved.

EXERCISE 5.12

Change the order of integration and then evaluate the following:

1. $\displaystyle\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

2. $\displaystyle\int_{-2}^1 \int_{x^2+4x}^{3x+2} dy \, dx$

3. $\displaystyle\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) \, dy \, dx$

4. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx$

5. $\displaystyle\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \, dx$

6. $\displaystyle\int_0^4 \int_y^4 \frac{x}{x^2 + y^2} dx \, dy$

7. $\int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy$

8. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dx\, dy$

9. $\int_0^4 \int_{2y}^2 e^{x^2}\, dx\, dy$

10. $\int_0^2 \int_1^{e^x} dy\, dx$

11. $\int_0^{2a} \int_{x^4/4a}^{3a-x} (x^2 + y^2)\, dy\, dx$

12. $\int_0^a \int_0^x \frac{f'(y)\, dy\, dx}{\sqrt{(a-x)(x-y)}}$

Answers

1. $\frac{1}{24}$

2. $\frac{9}{2}$

3. $\frac{a^3}{28} + \frac{a}{20}$

4. $\frac{\pi a^3}{6}$

5. $\frac{16a^2}{3}$

6. π

7. $\frac{(e^{16} - 1)}{8}$

8. $\frac{\pi a^2}{6}$

10. $(e^2 - 3)$

11. $\frac{314\, a^4}{35}$

12. $\pi[f(a) - f(0)]$

5.16.6 Applications of Double Integrals

(i) Area of a Plane Curve

(a) In Cartesian Coordinates The area of a plane region R: $\{(x, y);\, a \le x \le b, f_1(x) \le y \le f_2(x)\}$ is given by the double integral

$$A = \iint_R dS = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} dy\, dx$$

$$= \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} dx\, dy \qquad R: \{(x, y);\, c \le y \le d,\, g_1(y) \le x \le g_2(y)\}$$

(b) In Polar Coordinates The area A of the region R:$\{r, \theta);\, \alpha \le \theta \le \beta; f_1(\theta) \le r \le f_2(\theta)\}$ is given by the double integral

$$A = \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta$$

$$= \int_{r=r_1}^{r_2} \int_{\theta=g_1(r)}^{g_2(r)} d\theta\, r\, dr \qquad R: \{(r, \theta):\, r_1 \le r \le r_2 \text{ and } g_1(r) \le \theta \le g_2(r)\}.$$

(ii) Mass Contained in a Plane Curve

Suppose $f(x, y)$ be the surface density (mass/unit area) of a given region R. Then the quantity of mass M contained in the plane region R is given by

$$M = \iint_R f(x, y)\, dx\, dy$$

(iii) Center of Gravity (CG) of a Plane Region *R*

Let (x_{CG}, y_{CG}) be the CG of a plane region R with surface density $f(x, y)$ and containing mass M. Then

$$x_{CG} = \frac{\iint_R x \cdot f(x, y)\, dx\, dy}{M}$$

$$y_{CG} = \frac{\iint_R y \cdot f(x, y)\, dx\, dy}{M}$$

(iv) Moment of Inertia (MI) of a Plane Curve

The MI of a plane region R with surface density $f(x, y)$ relative (about) to x-axis, y-axis and the origin 0 are given by

$$I_{xx} = \iint_R y^2 \cdot f(x, y)\, dx\, dy$$

$$I_{yy} = \iint_R x^2 \cdot f(x, y)\, dx\, dy$$

$$I_O = I_{xx} + I_{yy}$$

or $$I_O = \iint_R (x^2 + y^2)\, dx\, dy;\ \text{where } I_O \text{ is the polar MI}$$

5.16.7 Volume Under Surface Area as a Double Integral

Suppose a surface $z = f(x, y)$. Consider the region S be the orthogonal projection of the region S' of $z = f(x, y)$ on the XY-plane.

Then the volume of the prism between the surface S and S' is $z\delta x\delta y$.

$\therefore$ the volume is given by

$$V = \iint z\, dx\, dy$$

$$= \iint_S f(x, y)\, dx\, dy$$

In the polar coordinates, the region S is divided into elements of area $r\, \delta r\delta\theta$.

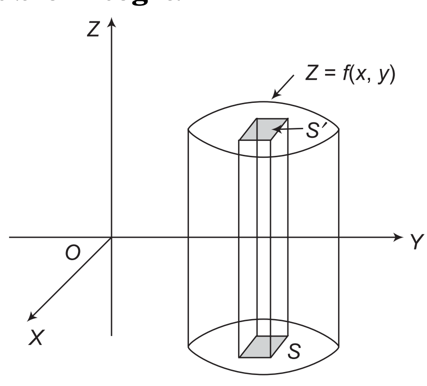


Fig. 5.49

$\therefore$ the volume is given by

$$V = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example 80 Find the area of the region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$.

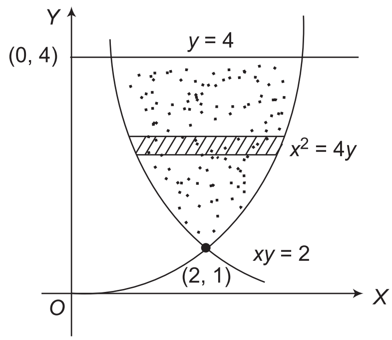


Fig. 5.50

Solution The required area of the shaded region is

$$= \int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\, dy$$

$$= \int_1^4 \left(2\sqrt{y} - \frac{2}{y}\right) dy = 2\left(\frac{2}{3}\, y^{3/2} - \log y\right)_1^4$$

$$= 2\left[\left(\frac{16}{3} - 2\log 2\right) - \frac{2}{3}\right] = \left(\frac{28}{3} - 4\log 2\right)$$

Example 81 Find the area bounded by the curves $y^2 = x^3$ and $x^2 = y^3$.

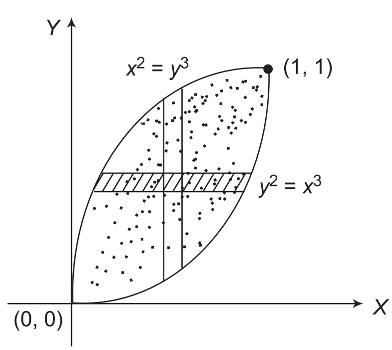


Fig. 5.51

Solution The required area of the shaded region is

$$A = \iint_R dx\, dy$$

$$= \iint_R dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^{3/2}}^{x^{2/3}} dy\, dx = \int_0^1 (x^{2/3} - x^{3/2})\, dx$$

$$= \left(\frac{3}{5} x^{5/3} - \frac{2}{5} x^{5/2}\right)_0^1$$

$$= \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$$

OR

$$A = \iint_R dx\, dy = \int_{y=0}^{1} \int_{x=y^{3/2}}^{y^{2/3}} dx\, dy = \int_0^1 (y^{2/3} - y^{3/2})\, dy$$

$$= \left(\frac{3}{5} y^{5/3} - \frac{2}{5} y^{5/2}\right)_0^1$$

$$= \frac{1}{5}$$

Example 82 Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

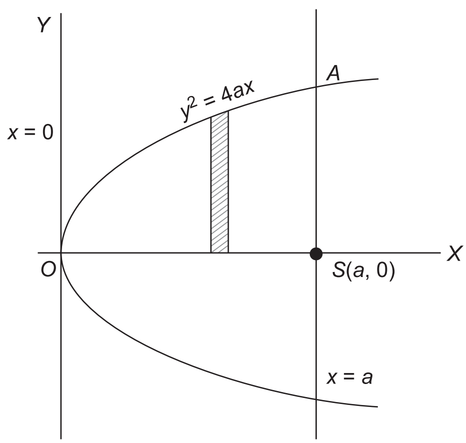


Fig. 5.52

Solution The required area of the bounded region is

$$= 2(\text{area of } OSA)$$

$$= 2\int_{x=0}^{a}\int_{0}^{2\sqrt{ax}} dy\, dx$$

$$= 2\int_{0}^{a} 2\sqrt{ax}\, dx = 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^a$$

$$= \frac{8a^2}{3}$$

Example 83 Find the mass, coordinates of the center of gravity and moments of inertia relative to the x-axis, y-axis, and origin of a rectangle $0 \le x \le 4$, $0 \le y \le 2$ having the mass density function $f(x, y) = xy$.

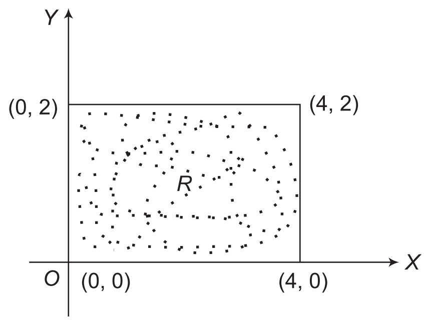


Fig. 5.53

Solution The mass of the region R is given by

$$M = \iint_R f(x, y)\, dx\, dy$$

$$= \int_{y=0}^{2}\int_{x=0}^{4} xy\, dx\, dy$$

or $$M = \int_{y=0}^{2} y\cdot\left(\frac{x^2}{2}\right)_0^4 dy$$

$$= \int_0^2 y\cdot\frac{16}{2}\, dy = 8\int_0^3 y\, dy = 8\cdot\left(\frac{y^2}{2}\right)_0^2$$

$$= 8\cdot 2 = 16$$

Let x_{CG} and y_{CG} be the coordinates of the CG of the region R; Then

$$x_{CG} = \frac{1}{M}\iint_R x\cdot f(x, y)\, dx\, dy = \frac{1}{16}\int_0^4\int_0^2 x\cdot(xy)\, dy\, dx$$

$$= \frac{1}{16} \int_0^4 x^2 \cdot \left(\frac{y^2}{2}\right)_0^2 dx = \frac{1}{8} \int_0^4 x^2 \, dx = \frac{8}{3}$$

$$y_{CG} = \frac{1}{M} \iint_R y \cdot f(x, y) \, dx \, dy = \frac{1}{16} \int_0^4 \int_0^2 y \cdot (xy) \, dy \, dx$$

$$= \frac{1}{16} \int_0^4 x^2 \cdot \left(\frac{y^3}{3}\right)_0^2 dx = \frac{1}{6} \int_0^4 x \, dx = \frac{4}{3}$$

The MI relative to x-axis

$$I_{xx} = \iint_R y^2 \, f(x, y) \, dx \, dy = \int_0^4 \int_0^2 y^2 \cdot (xy) \, dy \, dx = 32$$

Similarly, relative to y-axis

$$I_{yy} = \iint_R x^2 \, f(x, y) \, dx \, dy = \int_0^4 \int_0^2 x^2 \cdot xy \, dy \, dx = 128$$

Now,

$$I_O = \iint_R (x^2 + y^2) \cdot f(x, y) \, dx \, dy$$

$$= \int_0^4 \int_0^2 (x^2 + y^2) \cdot xy \, dy \, dx = \int_0^4 \left(\frac{x^3 y^2}{2} + \frac{xy^4}{4}\right)_0^2 dx$$

$$= \int_0^4 (2x^3 + 4x) \, dx = \left(\frac{2x^4}{4} + \frac{4x^2}{2}\right)_0^4$$

$$I_O = 128 + 32 = 160$$

Example 84 Find the volume of the space below the paraboloid $x^2 + y^2 + z = 4$ and above the square in the xy-plane with vertices at (0, 0), (0, 1), (1, 0), and (1, 1).

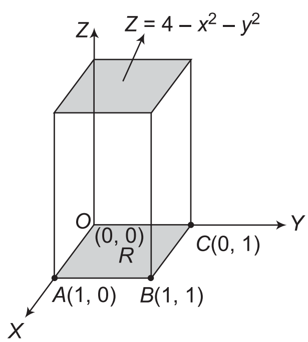


Fig. 5.54

Solution The top surface of the given figure is

$$z = 4 - x^2 - y^2 = f(x, y).$$

The projection R of the surface on the xy-plane into the square itself with coordinates $O(0, 0)$, $A(1, 0)$, $B(1, 1)$, and $C(0, 1)$

$$V = \iint_R f(x, y)\, dx\, dy = \int_0^1 \int_0^1 (4 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^1 \left(4y - x^2 y - \frac{y^3}{3}\right)_0^1 dx = \int_0^1 \left(\frac{11}{3} - x^2\right) dx = \frac{10}{3}$$

5.16.8 Triple Integral

The triple integral is a generalized form of a double integral. Let V be a region of three-dimensional space and let $f(x, y, z)$ be a function of the independents variables x, y, z defined at every point in V. Divide the region V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$ and let (x_r, y_r, z_r) be any point inside the r^{th} subdivision δV_r.

$$\therefore \qquad \sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r \qquad (27)$$

Then the limit sum of (27), as $n \to \infty$ and the dimensions of each subdivision tend to zero, is called the triple integral of $f(x, y, z)$ over the region V and is denoted by

$$\iiint_V f(x, y, z)\, dV$$

If the region is bounded by $z = f_1(x, y)$, $z = f_2(x, y)$; $y = g_1(x)$, $y = g_2(x)$ and $x = a$, $x = b$

$$\therefore \qquad V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

If the limits of integration are constant then, the triple integral is

$$\iiint_V f(x, y, z)\, dV = \int_a^b \int_c^d \int_e^f f(x, y, z)\, dx\, dy\, dz$$

The order of integration may be changed with suitable change in the limits of integration.

Similarly, volume integral/triple integral in **cylindrical coordinates** is

$$V = \iiint_V r\, dr\, d\phi\, dz$$

and in **spherical polar coordinates** is

$$V = \iiint_V r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Note: We integrate w.r.t. x first, then y and finally z here, but in fact there is no region to the integration in this order. There are 6 different possible orders to do the integral in and which order you do the integral in will depend upon the function and the order that you feel will be the easiest. We will find the same result regardless of the order however.

Example 85 Evaluate the triple integral of the function $f(x, y, z) = x^2$ over the region V enclosed by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = a$.

Solution A vertical column is bounded by the planes $z = 0$, $z = a - x - y$. The latter plane cuts the xy-plane in the line $a - x - y = 0$ so the area A above which the volume of the region in XY-plane bounded by the lines $y = 0$, $y = a - x$, $x = 0$, $x = a$

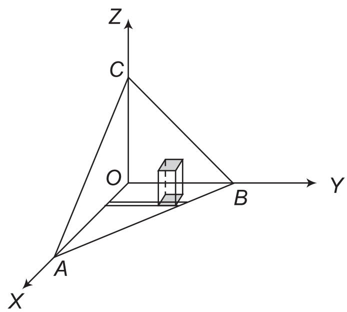


Fig. 5.55

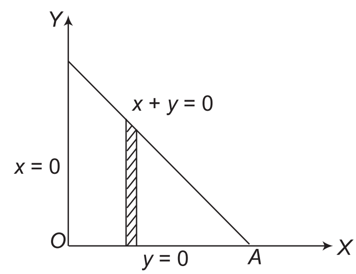


Fig. 5.56

$\therefore$ the triple integral

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} x^2 \, dz \, dy \, dx$$

Or
$$= \int_0^a \int_0^{a-x} x^2 (z)_0^{a-x-y} dy \, dx = \int_0^a \int_0^{a-x} x^2 (a - x - y) \, dy \, dx$$

$$= \int_0^a \left[x^2 (a - x) \, y - x^2 \cdot \frac{y^2}{2} \right]_0^{a-x} dx$$

$$= \int_0^a \left[x^2 (a - x)^2 - \frac{x^2}{2} (a - x)^2 \right] dx$$

$$= \frac{1}{2} \int_0^a x^2 (a - x)^2 \, dx = \frac{1}{2} \left[\frac{a^2 x^3}{3} - \frac{2}{4} a x^4 + \frac{x^5}{5} \right]_0^a$$

$$= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) a^5$$

$$= \frac{a^5}{60}$$

Example 86 Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2y$.

Solution The required volume is

$$V = \iiint_A dV \tag{1}$$

where A is the area above which this volume stands.

Eliminating z between the equations $z = x^2 + y^2$ and $z = 2y$, we obtain

$$x^2 + y^2 = 2y \tag{2}$$

This is the equation of the cylinder through the curve of intersection of the plane and the paraboloid, whose generators are parallel to the z-axis.

The section of this cylinder by the xy-plane gives the area A. Taking a strip parallel to the x-axis, the area A can be considered enclosed by the curves.

$$x = -\sqrt{2y - y^2} \text{ to } x = \sqrt{2y - y^2}$$
$$y = 0 \text{ to } y = 2.$$

$\therefore$ the integral (1) becomes

$$V = \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\,dx\,dy$$

$$= 2\int_0^2 \int_0^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\,dx\,dy$$

$$= 2\int_0^2 \int_0^{\sqrt{2y-y^2}} (2y - x^2 - y^2)\,dx\,dy$$

$$= 2\int_0^{\pi/2} \int_0^{2\sin\theta} (2r\sin\theta - r^2)\,r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \int_0^{2\sin\theta} (2r^2\sin\theta - r^3)\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{2}{3}r^3\sin\theta - \frac{r^4}{4}\right]_0^{2\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{16}{3}\sin^4\theta - 4\sin^4\theta\right] d\theta = \frac{8}{3}\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= \frac{8}{3}\cdot\frac{\left|\!\overline{\tfrac{5}{3}}\cdot\left|\!\overline{\tfrac{1}{2}}\right.\right.}{2\left|\!\overline{3}\right.} = \frac{8}{3}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot 2\cdot 1} = \frac{\pi}{2}$$

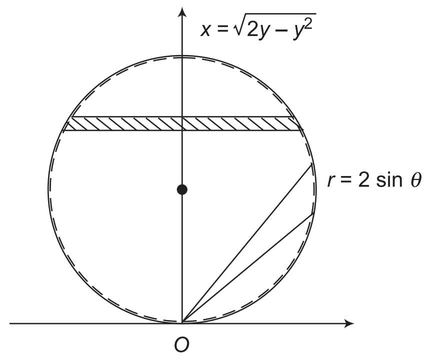


Fig. 5.57

Putting $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

and $r^2 = 2r\sin\theta$

or, $r = 2\sin\theta$

$0 \le \theta \le \dfrac{\pi}{2}$

Example 87 A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the planes $z = 0$ and the surface $z = c + xy$.

Solution The required volume of the prism is given as

$$V = \int_{x=0}^{a} \int_{y=0}^{\frac{bx}{a}} \int_{z=0}^{c+xy} dz\, dy\, dx$$

$$= \int_0^a \int_0^{bx/a} (c + xy)\, dy\, dx = \int_0^a \left[cy + \frac{xy^2}{2} \right]_0^{bx/a} dx$$

$$= \int_0^a \left[\frac{cbx}{a} + \frac{b^2 x^3}{2a^2} \right] dx = \frac{bc}{a} \left(\frac{x^2}{2} \right)_0^a + \frac{b^2}{2a^2} \left(\frac{x^4}{4} \right)_0^a$$

$$= \frac{abc}{2} + \frac{b^2 a^2}{8} = \frac{ab}{8}(4c + ab)$$

Example 88 Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution The required volume $= \displaystyle\int_{x=-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, dx$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2}\, dy\, dx = 4 \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= 8 \int_0^a (a^2 - x^2)\, dx = 8 \left(a^2 x - \frac{x^3}{3} \right)_0^a$$

$$= \frac{16a^3}{3}$$

Example 89 Evaluate $\displaystyle\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$.

Solution The function $f(x, y, z) = x^2 + y^2 + z^2$ is symmetrical in x, y, and z.

$\therefore$ the limits of integration can be assigned as per our preference; we get

$$I = \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= 2 \int_{-c}^{c} \int_{-b}^{b} \left[\int_0^a (x^2 + y^2 + z^2)\, dx \right] dy\, dz$$

$$= 2\int_{-c}^{c}\int_{-b}^{b}\left[\frac{x^3}{3} + xy^2 + xz^2\right]_0^a dy\,dz$$

$$= 2\int_{-c}^{c}\int_{-b}^{b}\left[\frac{a^3}{3} + ay^2 + az^2\right]dy\,dz$$

$$= 4\int_{-c}^{c}\left[\int_{-b}^{b}\left\{\frac{a^3}{3} + ay^2 + az^2\right\}dy\right]dz$$

$$= 4\int_{-c}^{c}\left[\frac{a^3 y}{3} + \frac{ay^3}{3} + az^2 y\right]_0^b dz$$

$$= 4\int_{-c}^{c}\left[\frac{a^3 b}{3} + \frac{ab^3}{3} + abz^2\right]dz$$

$$= 8\int_{0}^{c}\left[\frac{a^3 b}{3} + \frac{ab^3}{3} + abz^2\right]dz$$

$$= 8\left[\frac{a^3 bz}{3} + \frac{ab^3 z}{3} + \frac{abz^3}{3}\right]_0^c$$

$$= 8\left[\frac{a^3 bc}{3} + \frac{ab^3 c}{3} + \frac{abc^3}{3}\right] = \frac{8\,abc}{3}(a^2 + b^2 + c^2)$$

EXERCISE 5.13

1. Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area bounded by the circle $x^2 + y^2 = 9$ and the line $x + y = 3$.
3. Find the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.
4. Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.
5. Find by double integration, the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.
6. Find the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.
7. Calculate the volume of the solid bounded by the surfaces $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
8. Find the volume of the region bounded by the surfaces $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = 3$.
9. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y-axis.

10. Find the mass contained in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with surface density $(x + y)^2$.
11. Find the mass, centroid, and moments of inertia relative to the x-axis, y-axis, and origin of the plane region R having mass density $(x + y)$ and bounded by the parabola $x = y - y^2$ and the straight line $x + y = 0$.
12. Find the centroid of the plane region bounded by $y^2 + x = 0$ and $y = x + 2$.
13. A thin plate of uniform thickness and constant density ρ covers the region of the xy-plane and is bounded by $y = x^2$ and $y = x + 2$. Find the mass M, moment of inertia I_{yy} about the y-axis.
14. Evaluate $\int_0^1 \int_0^x \int_0^{x+y} (x + y + z)\, dz\, dy\, dx$.
15. Find the total mass of the region in the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with density at any point given by xyz.

    Hint: Mass = $\int_0^1 \int_0^1 \int_0^1 x\,y\,z \; dx\, dy\, dz$
16. Find the mass of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
17. Evaluate $\int_1^2 \int_1^z \int_0^{yz} (x\,y\,z) \; dx\, dy\, dz$.
18. Evaluate $\int_1^2 \int_y^{y^2} \int_0^{\log x} e^z \; dz\, dx\, dy$.
19. Evaluate $\int_0^a \int_0^x \int_0^{y+x} e^{(x+y+z)} \; dz\, dy\, dx$.
20. Evaluate $\iiint x^2 yz\, dx\, dy\, dz$; taken over the volume bounded by the surface $x^2 + y^2 = 9$, $z = 0$ and $z = 2$.
21. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.

Answers

1. $\frac{9}{2}$
2. $\frac{9}{4}(\pi - 2)$
3. $\frac{32}{3}$
4. $\frac{3}{2}(\pi - 2)$

5. $\frac{a^2}{4}(\pi+8)$

6. 3π

7. $\frac{1}{6}$

8. **Hint:** $V = \int_0^1 \int_{x^2}^{\sqrt{x}} 3dy\,dx = 1$

9. $\frac{4}{3}\pi a^2 b$

10. $M = \frac{\pi ab}{4}(a^2 + b^2)$

11. $M = \frac{8}{15}, (x_{CG}, y_{CG}) = \left(-\frac{8}{35}, \frac{11}{8}\right), I_{xx} = \frac{64}{105}, I_{yy} = \frac{256}{105}$ and $I_o = \frac{64}{21}$

12. $\left(-\frac{8}{5}, -\frac{1}{2}\right)$

13. $M = \frac{9\rho}{2}, I_{yy} = \frac{63}{20}\rho$

14. $\frac{7}{8}$

15. $\frac{1}{8}$

16. $\frac{\rho\, abc}{6}$

17. $\frac{7}{2}$

18. $\frac{47}{24}$

19. $\left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}\right)$

20. $\frac{648}{5}$

21. $\pi a^3\left(\frac{4\sqrt{2}}{3} - \frac{7}{6}\right)$

5.16.9 Dirichlet's Integral

This integral is useful to evaluate the double and triple integrals by expressing in terms of Beta and Gamma functions. The following theorems are useful in evaluating the multiple integrals.

Theorem 1

Let R be the region in the xy-plane bounded by $x \geq 0$, $y \geq 0$ and $x + y \leq k$, then

$$\iint_R x^{p-1}\, y^{q-1} = \frac{\Gamma p\, \Gamma q}{\Gamma(p+q+1)} \cdot k^{p+q}$$

Theorem 2

Let V be the region, where $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x + y + z \leq 1$; then

$$\iiint_V x^{p-1}\, y^{q-1}\, z^{r-1}\, dx\, dy\, dz = \frac{\Gamma p\, \Gamma q\, \Gamma r}{\Gamma(p+q+r+1)}$$

Remark 1: Dirichlet's theorem can be extended to a finite number of variables.

Remark 2: If $x + y + z \leq h$ then by putting $\frac{x}{h} = X, \frac{y}{h} = Y$ and $\frac{z}{h} = Z$ we have $X + Y + Z \leq \frac{h}{h} = 1$ and the Dirichlet's theorem is

$$\iiint_V x^{p-1}\, y^{q-1} z^{r-1}\, dx\, dy\, dz \frac{\Gamma p\, \Gamma q\, \Gamma r}{\Gamma(p+q+r+1)} \cdot h^{p+q+r}$$

5.16.10 Liouville's Extension of Dirichlet's Theorem

If the variables x, y, z are all positive such that $h_1 < (x + y + z) < h_2$ then

$$\iiint_V f(x+y+z)\, x^{p-1}\, y^{q-1}\, z^{r-1}\, dx\, dy\, dz = \frac{\Gamma p\, \Gamma q\, \Gamma r}{\Gamma(p+q+r)} \int_{h_1}^{h_2} f(u) \cdot u^{(p+q+r-1)}\, du$$

Example 90 Apply Dirichlet's theorem to evaluate $\iiint_V x\,yz\, dx\, dy\, dz$ taken throughout the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.

Solution Put $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z,$ we get $x = a\sqrt{X}, y = b\sqrt{Y}$ and $z = c\sqrt{Z}$

$$\therefore \qquad xdx = \frac{a^2}{2}\, dX,\; ydy = \frac{b^2}{2} dY \text{ and } zdz = \frac{c^2}{2}\, dZ$$

The condition, under this substitution, becomes $X + Y + Z \le 1$.

Now,

$$\iiint xyz\, dx\, dy\, dz = \iiint (x\, dx)\,(y\, dx)\,(z\, dz)$$

$$= \iiint \left(\frac{a^2}{2}\, dX\right)\left(\frac{b^2}{2}\, dY\right)\left(\frac{c^2}{2}\, dZ\right)$$

$$= \frac{a^2b^2c^2}{8} \iiint X^{1-1}\, Y^{1-1}\, Z^{1-1}\, dX\, dY\, dZ$$

$$= \frac{a^2b^2c^2}{8} \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma(1+1+1+1)}$$

$$= \frac{a^2b^2c^2}{8} \frac{1}{\Gamma 4}$$

$$= \frac{a^2b^2c^2}{48}$$

$\therefore$ The value for the whole ellipsoid is

$$8\left(\frac{a^2b^2c^2}{48}\right) = \frac{a^2b^2c^2}{6}$$

Example 91 Apply Dirichlet's integral to find the mass of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point being $\rho = Kxyz$.

Solution Put $\frac{x^2}{a^2}=X, \frac{y^2}{b^2}=Y$, and $\frac{z^2}{c^2}=Z$, then $X \ge 0, Y \ge 0, Z \ge 0$, and $X+Y+Z=1$,

Also, $$dx=\frac{a}{2\sqrt{X}}\,dX,\ dy=\frac{b}{2\sqrt{Y}}\,dY \text{ and } dz=\frac{c}{2\sqrt{Z}}\,dZ$$

$\therefore$ The required mass of an octant of the ellipsoid is

$$= \iiint K\ xyz\ dx\,dy\,dz$$

$$= \iiint K\cdot(a\sqrt{X})\,(b\sqrt{Y})\,(x\sqrt{Z}).\frac{abc}{8\sqrt{X}\sqrt{Y}\sqrt{Z}}\,dX\ dY\ dZ$$

$$= K\,\frac{a^2b^2c^2}{8}\iiint dX\ dY\ dZ$$

$$= \frac{K\,a^2b^2c^2}{8}\iiint X^{1-1}\,Y^{1-1}\,Z^{1-1}\,dX\ dY\ dZ$$

$$= \frac{K\,a^2b^2c^2}{8}\,\frac{\Gamma 1\ \Gamma 1\ \Gamma 1}{\Gamma(1+1+1+1)}$$

$$= \frac{K\,a^2b^2c^2}{8\cdot\Gamma 4}$$

$$= \frac{K\,a^2b^2c^2}{48}$$

Example 92 The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axis in *A*, *B*, and *C*. Apply Dirichlet's theorem to find the volume of the tetrahedron *OABC*. Also, find its mass if the density at any point is *Kxyz*.

Solution Put $\frac{x}{a}=X, \frac{y}{b}=Y, \frac{z}{c}=Z$; then $X \ge 0$, $Y \ge 0$, and $Z \ge 0$

$X+Y+Z=1$

Also, $$x=aX,\ y=bY,\ z=cZ$$
$$dx=adX,\ dy=bdY\ dz=cdZ.$$

$\therefore$ The volume of tetrahedron *OABC* is

$$V=\iiint_V dx\ dy\ dz=\iiint_V a\,b\,c\ \ dX\ dY\ dZ$$

$$= abc \iiint_V X^{1-1}\, Y^{1-1}\, Z^{1-1}\, dX\, dY\, dZ$$

$$= \frac{abc}{3!} = \frac{abc}{6}$$

Now, $$\text{Mass} = \iiint_V \rho\, dx\, dy\, dz$$

$$= \iiint_V K\, xyz\, dx\, dy\, dz$$

$$= K \iiint_V (abc)\, XYZ\, (a dX)\, (b dY)\, (c dZ)$$

$$= K\, a^2 b^2 c^2 \iiint_V X^{2-1}\, Y^{2-1}\, Z^{2-1}\, dX\, dY\, dZ$$

$$= K\, a^2 b^2 c^2 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)}$$

$$= K\, a^2 b^2 c^2 \frac{1!\; 1!\; 1!}{6!}$$

$$= \frac{K\, a^2 b^2 c^2}{720}$$

Example 93 Apply Liouville's theorem to evaluate $\iiint \log(x+y+z)\, dx\, dy\, dz$, the integral extending over all positive and zero values of x, y, z subject to $x + y + z < 1$.

Solution The given condition $0 \le (x + y + z) < 1$.

$$\therefore \quad \iiint \log(x+y+z)\, dx\, dy\, dz = \iiint x^{1-1}\, y^{1-1}\, z^{1-1} \log(x+y+z)\, dx\, dy\, dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 u^{1+1+1-1} \log u\, du$$

$$= \frac{1}{2}\int_0^1 u^2 \cdot \log u\, du$$

$$= \frac{1}{2}\left[\left(\frac{u^3}{3}\log u\right)_0^1 - \int_0^1 \frac{u^3}{3}\cdot\frac{1}{u}\, du\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3}\left(\frac{u^3}{3}\right)_0^1\right]$$

$$= -\frac{1}{18}$$

Example 94 Evaluate $\iiint (x+y+z+1)^{-3}\, dx\, dy\, dz$, the integral being taken throughout the volume bounded by planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution For the given condition $0 \le x + y + z \le 1$

$$\therefore \quad \iiint (x+y+z+1)^{-3}\, dx\, dy\, dz = \iiint \frac{x^{1-1}\, y^{1-1}\, z^{1-1}}{(x+y+z+1)^3}\, dx\, dy\, dz$$

$$= \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3}\, u^{1+1+1-1}\, du$$

$$= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^3}\, du$$

Put $\qquad u + 1 = t \Rightarrow du = dt$

$$\therefore \quad \frac{1}{2} \int_1^2 \frac{(t-1)^2}{t^3}\, dt = \frac{1}{2} \int_1^2 \frac{(t^2 - 2t + 1)}{t^3}\, dt$$

$$= \frac{1}{2} \int_1^2 \left[\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3}\right] dt = \frac{1}{2}\left[\log t + \frac{2}{t} - \frac{1}{2t^2}\right]_1^2$$

$$= \frac{1}{2}\left[\log 2 - \frac{5}{8}\right] = \left(\frac{1}{2}\log 2 - \frac{5}{16}\right)$$

Example 95 Show that the integral $\iiint x^{l-1}\, y^{m-1}\, z^{m-1}\, dx\, dy\, dz$ integrated over the region in the first octant below the surface $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$ is $\frac{a^l\, b^m\, c^n}{p\, q\, r} \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\left[\Gamma(l/p) + (m/q) + (n/r) + 1\right]}$.

Solution The required integral $\iiint x^{l-1}\, y^{m-1}\, z^{m-1}\, dx\, dy\, dz$, when the integral is extended to all positive values of the variables x, y, and z subject to the condition

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$$

Let us put $\left(\frac{x}{a}\right)^p = u$, i.e. $x = au^{1/p}$ so that $dx = \left(\frac{x}{a}\right)^p u^{(1-1)} du, \left(\frac{y}{b}\right)^q v$, i.e., $y = bv^{1/q}$ so that $dy = \left(\frac{b}{q}\right) v^{(1/q)}\, dv$ and $\left(\frac{z}{c}\right)^r = w$ i.e, $z = (w)yr$ so that $dz = \left(\frac{c}{r}\right) w^{(1/r)-1}\, dw$

$$\Rightarrow \quad \text{Required integral} = \iiint (a^{l-1}\, u^{(l-1)/p})\, (b^{m-1}\, v^{(m-1)/q}\, (c^{n-1}\, w^{(n-1)/r})$$

$$\times \frac{a}{p} u^{(1/p)-1}\, \frac{b}{q} v^{(1/q)-1} \cdot \frac{c}{r} w^{(1/r)-1}\, du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{(1/p)-1}\ v^{(m/q)-1}\ w^{(n/r)-1}\ du\ dv\ dw$$

$$\frac{a^l b^m c^n}{pqr} \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p)+(m/q)+(n/r)+1} \qquad \text{[By Dirichlet's integral]}$$

Example 96 Prove that $\iint\limits_D x^{l-1}\ y^{m-1}\ dx\ dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}\ h^{l+m}$

where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq h$.

Solution Put $x = Xh$ and $y = Yh$ so that $dx\ dy = h^2\ dXdY$.

The new domain D' is given by $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$.

$$\therefore \qquad \iint\limits_D x^{l-1}\ y^{m-1}\ dx\ dy = \iint\limits_{D'} (Xh)^{l-1}\ (Yh)^{m-1}\ dX\ dY$$

$$= h^{l+m} \int_0^1 \int_0^{1-X} X^{l-1} Y^{m-1}\ dX\ dY$$

$$= h^{l+m} \int_0^1 X^{l-1} \int_0^{1-X} Y^{m-1}\ dX\ dY$$

$$= h^{l+m} \int_0^1 X^{l-1} \left[\frac{Y^m}{m}\right]_0^{1-X} dX \;=\; \frac{h^{l+m}}{m} \int_0^1 X^{l-1}\ (1-X)^m\ dX$$

$$= \frac{h^{l+m}}{m}\ \beta(l, m+1) = \frac{h^{l+m}}{m} \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= \frac{h^{l+m}}{m} \frac{m\,\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} = h^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}$$

Example 97 Evaluate $\iiint \sqrt{\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}}\ dx\ dy\ dz$, integral being taken over all positive value of x, y, z such that $x^2 + y^2 + z^2 \leq 1$.

Solution Let $x^2 = u$, $y^2 = v$, $z^2 = w$ so that $u + v + w \leq 1$

Also,
$$x = \sqrt{u} \;\Rightarrow\; dx = \frac{1}{2\sqrt{-u}}\ du$$

$$y = \sqrt{v} \;\Rightarrow\; dy = \frac{1}{2\sqrt{v}}\ dv$$

$$z = \sqrt{w} \;\Rightarrow\; dz = \frac{1}{2\sqrt{w}}\ dw$$

Therefore, the given integral

$$= \iiint \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\,\frac{du\,dv\,dw}{8\sqrt{u\,v\,w}}$$

$$= \frac{1}{8}\iiint u^{\frac{1}{2}-1}\,v^{\frac{1}{2}-1}\,w^{\frac{1}{2}-1}\sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\,du\,dv\,dw$$

$$= \frac{1}{8}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}\int_0^1 \sqrt{\frac{1-u}{1+u}}\;u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1} \qquad \text{[By Liouville's extension]}$$

$$= \frac{1}{8}\,\frac{\left(\Gamma\frac{1}{2}\right)^3}{\frac{1}{2}\Gamma\frac{1}{2}}\int_0^1 \sqrt{\frac{1-u}{1-u^2}}\,u^{\frac{1}{2}}\,du$$

$$= \frac{\pi}{4}\int_0^1 \frac{1-\sqrt{t}}{\sqrt{(1-t)}}+t^{\frac{1}{4}}\,\frac{dt}{2\sqrt{t}} \qquad (\text{where } u^2 = t)$$

$$= \frac{\pi}{8}\int_0^1 \frac{(1-\sqrt{t})t^{-1/4}}{\sqrt{1-t}}\,dt = \frac{8}{\pi}\left[\int_0^1 t^{\frac{3}{4}-1}(1-t)^{\frac{1}{2}-1}\,dt - \int_0^1 t^{\frac{5}{4}-1}(1-t)^{\frac{1}{2}-1}\,dt\right]$$

$$= \frac{\pi}{8}\left[\beta\left(\frac{3}{4},\frac{1}{2}\right)-\beta\left(\frac{5}{4},\frac{1}{2}\right)\right]$$

Example 98 Evaluate the integral $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ the integral being extended to all positive values of the variables for which the expression is real.

Solution Given expression is real when $x^2 + y^2 + z^2 < a^2$.

Therefore, the required integral is to be extended to all positive values of x, y, and z such that $0 < x^2 + y^2 + z^2 < a^2$,

i.e., $0 < \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} < 1\cdot$

Putting $\left(\frac{x^2}{a^2}\right) = u_{1,}\ \left(\frac{y^2}{a^2}\right) = u_2$ and $\left(\frac{z^2}{a^2}\right) = u_3$

i.e. $x = au_1^{1/2},\ y = au_2^{1/2}$ and $z = au_3^{1/2}$

so that $dx = \frac{1}{2}au_1^{-1/2}\,du_1;\quad du = \frac{1}{2}au_2^{-1/2}\,du_2\,;\quad dz = \frac{1}{2}au_3^{-1/2}\,du_3$

With these substitutions, the given condition reduces to $0 < u_1 + u_2 + u_2 < 1$ and the required integral becomes

$$= \iiint \frac{\left(\frac{1}{2}\right)^3 \cdot a^3\, u_1^{-1/2}\, u_2^{-1/2}\, u_3^{-1/2}\, du_1\, du_2\, dy_3}{a\sqrt{\{1-(u_1+u_2+u_3\}}}$$

$$= \frac{a^2}{8}\cdot \iiint \frac{u_1^{1/2-1}\, u_2^{1/2-1}\, u_3^{1/2-1}\, du_1\, du_2\, dy_3}{\sqrt{\{1-(u_1+u_2+u_3\}}}$$

$$= \frac{a^2}{8}\cdot \frac{\left[\Gamma(1/2)\right]^3}{\Gamma(1/3)} \int_0^1 u^{3/2-1} \frac{1}{\sqrt{(1-u)}}\, du$$

[By Liouville's extension of Dirichlet's theorem]

$$= \frac{a^2}{8} \frac{\left[\sqrt{\pi}\right]^3}{\frac{1}{2}\sqrt{\pi}} \int_1^{\pi/2} \frac{\sin\theta \cdot 2\sin\theta\cos\theta}{\sqrt{(1-\sin^2\theta}}\, d\theta$$

[Putting $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta\, d\theta$]

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{\pi a^2}{2}\cdot\frac{1}{2}\frac{\pi}{2} = \frac{\pi^2 a^2}{8}$$

EXERCISE 5.14

1. Evaluate $\iint x^{2l-1}\, y^{2m-1}\, dx\, dy$ for all positive values of x and y such that $x^2 + y^2 \le c^2$.

2. Evaluate $\iiint dx\, dy\, dz$, where $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.

3. (i) Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is $\frac{a^m b^n}{2n}\beta\left(\frac{m}{2}, \frac{n}{2}+1\right)$

   (ii) Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$

4. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinate axes in the points A, B, C. Use Dirichlet's integral to evaluate the mass of the tetrahedron $OABC$, the density at any point (x, y, z) being $kxyz$.

5. Find the volume enclosed by the surface

$$\left(\frac{x}{a}\right)^{2n} + \left(\frac{y}{b}\right)^{2n} + \left(\frac{z}{c}\right)^{2n} = 1$$

6. Find the mass of the region in the xy-plane bounded by $x = 0$, $y = 0$, $x + y = 1$ with density $\rho = K\sqrt{xy}$.

7. Find the volume of the solid bounded by the coordinate planes and the surface $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} + \sqrt{\dfrac{z}{c}} = 1$.

8. Apply Dirichlet's integral to find the moment of inertia about the z-axis of an octanet of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. $\left[\textit{Hint}: \text{MI} = \iiint\limits_V (x^2 + y^2)\, \rho\, dx\, dy\, dz)\right]$

9. Evaluate $\iiint e^{x+y+z}\, dx\, dy\, dz,$ taken over the positive octant such that $x + y + z \le 1$.

10. Show that $\iiint \dfrac{dx\, dy\, dz}{(x+y+z+1)^2} = \dfrac{3}{4} - \log 2$, the integral being taken throughout the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

11. Find the volume and the mass contained in the solid region in the first octant of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ if the density at any point $\rho(x, y, z) = Kxyz$. Also find the coordinates of the centroid.

12. Show that the entire volume of the solid $\left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{b}\right)^{2/3} + \left(\dfrac{z}{c}\right)^{2/3} = 1$ is $\dfrac{4\pi\, abc}{35}$.

Answers

1. $\dfrac{1}{4} c^{2l+2m} \cdot \dfrac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}$

2. $\dfrac{4}{3}\pi\, abc$

3. (ii) $\dfrac{4}{3}\pi\, a^3$

4. $\dfrac{k\, a^2\, b^2\, c^2}{720}$

5. $\dfrac{2}{3} \cdot \dfrac{abc}{n^2} \dfrac{\left[\Gamma\left(\frac{1}{2}n\right)\right]^3}{\Gamma\left(\frac{3}{2}n\right)}$

6. $\dfrac{K\pi}{24}$

7. $\dfrac{abc}{90}$

8. $\dfrac{\rho\, abc}{30}(a^2 + b^2)\pi$

9. $\left(\dfrac{e}{2} - 1\right)$

10. $V = \dfrac{\pi\, abc}{6}, M = \dfrac{K\, a^2b^2c^2}{48}$ and $(x_{CG}, y_{CG}, z_{CG}) = \left(\dfrac{16a}{35}, \dfrac{16b}{35}, \dfrac{16c}{35}\right)$

SUMMARY

1. The Definite Integral

Let $F(x)$ be the antiderivative of a function $f(x)$ defined on $[a, b]$, i.e., $\dfrac{d}{dx}F(x) = f(x)$. Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is defined as $\int_a^b f(x)\,dx = F(b) - F(a)$.

The numbers a and b are called the *limits of integration*: 'a' is called the *lower limit* and 'b' is called the *upper limit*. The interval $[a, b]$ is called the *interval of integration.*

2. Leibnitz's Rule of Differentiation under the Sign of Integration

Let $F(x, t)$ and $\dfrac{\partial F}{\partial x}$ be continuous functions of both x and t and let the first derivative of $G(x)$ and $H(x)$ be continuous. Then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\,dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\,dt + F[x, H(x)]\cdot\frac{dH}{dx}\cdot - F[x, G(x)]\cdot\frac{\partial G}{\partial x}$$

3. Improper Integrals

The integral $\int_a^b f(x)\,dx$ (1)

when the range of integration is finite and the integrand function $f(x)$ is bounded for all x in the range $[a, b]$, it is said to be proper (definite) integral. But if either a or b (or both) are infinite or when a and b are finite but $f(x)$ becomes infinite at $x = a$ or $x = b$ or at one or more points within $[a, b]$ then the integral (1) is called *improper or infinite integral*.

Kinds of Improper Integrals

The improper integrals can be divided into the following two kinds:

(i) **Improper Integrals of First Kind:** Those integrals in which the integrand function is bounded and the range of integration is infinite are called improper integrals of first kind.

(ii) **Improper Integrals of Second Kind:** Those integrals in which the integrand function is unbounded, i.e., infinite for some values in the range of integration.

4. Multiple Integrals

(a) Double Integrals

A double integral is the counterpart, in two dimensions of the definite integral of a function of a single variable. Then

$$\iint_R f(x, y)\,dR = \iint_R f(x, y)\,dx\,dy$$

(b) Change of Order of Integration

We have seen that in evaluating a double integral of successive integrations, if the limits of both the variables are constant then we can change the order of integration without affecting the result. But if the limits of integration are variable, i.e., if the limits of x are constant and limits of y are functions of x and order of integration is $dy\,dx$ or if the limits of x are functions of y and y are constants and order of integration is $dx\,dy$, then we take a strip parallel to the y-axis, or parallel to the x-axis. When it is required to change the order of integration in an integral for which the limits are given, first of all we ascertain from the given limits the region R of integration. Knowing the region of integration, we can then put in the limits for integration in the reverse order.

5. Triple Integral

The triple integral is a generalized form of a double integral. Let V be a region of three-dimensional space and let $f(x, y, z)$ be a function of the independent variables x, y, z defined at every point in V. The triple integral of $f(x, y, z)$ over the region V is denoted by $\iiint_V f(x, y, z)\, dV$.

If the region is bounded by $z = f_1(x, y)$, $z = f_2(x, y)$; $y = g_1(x)$, $y = g_2(x)$ and $x = a$, $x = b$

$$\therefore \quad V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

The order of integration may be changed with a suitable change in the limits of integration.

6. Dirichlet's Integral

Theorem I: Let R be the region in the xy-plane bounded by $x \ge 0$, $y \ge 0$ and $x + y \le k$, then

$$\iint_R x^{p-1} y^{q-1} dxdy = \frac{\Gamma p\, \Gamma q}{\Gamma(p+q+1)} \cdot k^{p+q}$$

Theorem II: Let V be the region, where $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x + y + z \le 1$; then

$$\iiint_V x^{p-1} y^{q-1} z^{r-1}\, dx\, dy\, dz = \frac{\Gamma p\, \Gamma q\, \Gamma r}{\Gamma(p+q+r+1)}$$

Remark I: The Dirichlet's theorem can be extended to a finite number of variables.

Remark II: If V be the closed region by coordinate planes and $x + y + z \le h$ then the Dirichlet's theorem is

$$\iiint_V x^{p-1} y^{q-1} z^{r-1}\, dx\, dy\, dz = \frac{\Gamma p\, \Gamma q\, \Gamma r}{\Gamma(p+q+r+1)} h^{(p+q+r)}$$

7. Liouville's Extension of Dirichlet's Theorem

If the variables x, y, z are all positive such that $h_1 < (x + y + z) < h_2$ then

$$\iiint_V f(x+y+z)\, x^{p-1} y^{q-1} z^{r-1}\, dx\, dy\, dz = \frac{\Gamma p\, \Gamma q\, \Gamma r}{\Gamma(p+q+r)} \int_{h_1}^{h_2} f(u) \cdot u^{(p+q+r+1)}\, du$$

OBJECTIVE-TYPE QUESTIONS

1. $\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dx\,dy$ is

(a) 0 (b) π
(c) $\pi/2$ (d) 2

[GATE (ME) 2000]

2. Value of the integral $\int_0^{\pi/4}\cos^2 x\,dx$ is

(a) $\frac{\pi}{8}+\frac{1}{4}$ (b) $\frac{\pi}{8}-\frac{1}{4}$
(c) $-\frac{\pi}{8}-\frac{1}{4}$ (d) $-\frac{\pi}{8}+\frac{1}{4}$

[GATE (CE) 2001]

3. The value of the following integral

$$\int_{-\pi/2}^{\pi/2}\frac{\sin 2x}{1+\cos x}dx$$

(a) –2log 2 (b) 2
(c) 0 (d) $(\log 2)^2$

[GATE (CE) 2002]

4. The area enclosed between the parabola $y = x^2$ and the straight line $y = x$ is

(a) 1/8 (b) 1/6
(c) 1/3 (d) 1/2

[GATE (ME) 2003]

5. The volume of an object expressed in spherical coordinates is given by

$$V=\int_0^{2/\pi}\int_0^{\pi/3}\int_0^1 r^2\sin\phi\,dr\,d\phi\,d\theta$$

The value of integral is

(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $\pi/4$

[GATE (ME) 2004]

6. $\int_{-a}^{a}(\sin^6 x+\sin^7 x)dx$ is equal to

(a) $2\int_0^a \sin^6 x\,dx$ (b) $2\int_0^a \sin^7 x\,dx$

(c) $2\int_0^a(\sin^6 x+\cos^6 x)dx$

(d) 0 **[GATE (ME) 2005]**

7. The value of the integral $\int_{-1}^{1}\frac{dx}{x^2}$ is

(a) 2 (b) –2
(c) does not exist (d) ∞

[GATE (AIE) 2005]

8. Changing the order of the integration in the double integral $I=\int_0^8\int_{x/4}^2 f(x,y)\,dy\,dx$ leads to $I=\int_r^s\int_p^q f(x,y)\,dx\,dy$. What is q?

(a) $4y$ (b) $16y^2$
(c) x (d) 8

[GATE (ME) 2005]

9. By a change of variables $x(u, v) = uv$, $y(u, v) = v/u$ in a double integral, the integral $f(x, y)$ changes to $f\left(u, v, \frac{v}{u}\right)=\phi(u,v)$. Then $\phi(u, v)$ is

(a) $2v/u$ (b) $2\,uv$
(c) v^2 (d) 1

[GATE (ME) 2005]

10. The value of the integral $\int_0^{\pi}\sin^3\theta\,d\theta$ is

(a) 1/2 (b) 2/3
(c) 4/3 (d) 8/3

[GATE (EC) 2006]

11. Value of $\int_0^3\int_0^x(6-x-y)\,dx\,dy$ is

(a) 13.5 (b) 27.0
(c) 40.5 (d) 54.0

[GATE (CE) 2008]

12. The value of the integral $\int_0^{\infty}\int_0^{\infty}e^{-x^2}e^{-y^2}dx\,dy$ is

(a) $\sqrt{\pi/2}$ (b) $\sqrt{\pi}$
(c) π (d) $\pi/4$

[GATE (AIE) 2007]

13. The following point shows a function y which varies linearly with x. The value of the integral $I=\int_1^2 y\,dx$

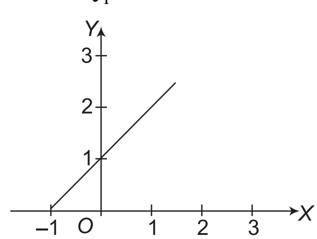


Fig. 5.58

(*a*) 1.0 (b) 2.5
(c) 4.0 (d) 5.0

[GATE (ECE) 2007]

14. Consider points P and Q in the xy plane with $P = (1, 0)$ and $Q = (0, 1)$. The line integral $2\int_P^Q (x\,dx + y\,dy)$ along the semicircle with the line segment PQ as its diameter is

(a) –1
(b) 0
(c) 1
(d) depends on the direction (clockwise or anticlockwise) of the semicircle

[GATE (ECE) 2008]

15. The volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$ is given as

(a) $\dfrac{\pi}{2a}$ (b) $\dfrac{\pi a^3}{2}$
(c) $\dfrac{4}{3}\pi a^3$ (d) none of the above

[UPTU 2008]

16. Considering the shaded triangular region P shown in the figure, what is $\iint_P xy\,dx\,dy$?

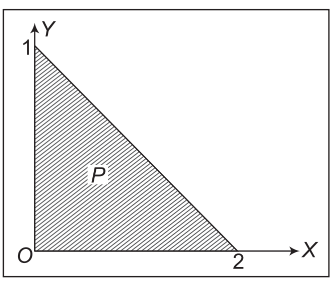


Fig. 5.59

(a) 1/6 (b) 2
(c) 1/16 (d) 1

[GATE (ME) 2008]

17. The length of the curve $y = \dfrac{2}{3}x^{3/2}$ between $x = 0$ and $x = 1$ is

(a) 2.27 (b) 0.67
(c) 0.27 (d) 1.22

[GATE (ME) 2008]

18. A path AB in the form of one quarter of a circle of unit radius is shown in Fig. 5.60. Integration of $(x + y)^2$ on path AB traversed in a counter-clockwise sense is

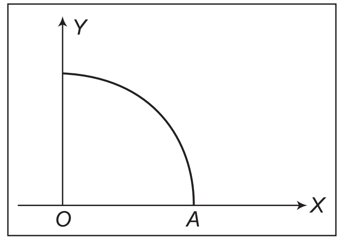


Fig. 5.60

(a) $\dfrac{\pi}{2} - 1$ (b) $\dfrac{\pi}{2} + 1$
(c) $\dfrac{\pi}{2}$ (d) 1

[GATE (ME) 2009]

19. The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is

(a) $\dfrac{16}{3}$ (b) 8
(c) $\dfrac{32}{3}$ (d) 16

[GATE (ME) 2009]

20. The area enclosed between the straight line $y = x$ and parabola $y = x^2$ in the xy plane is

(a) 1/6 (b) 1/4
(c) 1/3 (d) 1/2

[GATE (ME) 2013]

21. The value of the definite integral $\int_1^e \sqrt{x}\,\ln(x)\,dx$ is

(a) $\dfrac{4}{9}\sqrt{e^3} + \dfrac{2}{9}$ (b) $\dfrac{2}{9}\sqrt{e^3} - \dfrac{4}{9}$
(c) $\dfrac{2}{9}\sqrt{e^3} + \dfrac{4}{9}$ (d) $\dfrac{4}{9}\sqrt{e^3} - \dfrac{2}{9}$

[GATE (ME) 2013]

22. The value of $\displaystyle\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta\, d\theta$ is

(a) 0 (b) 1/15
(c) 1 (d) 8/3

[GATE (CE) 2013]

23. The value of integral $\int_0^2 \int_0^x e^{x+y} dy\,dx$ is

(a) $\frac{1}{2}(e-1)$ (b) $\frac{1}{2}(e^2-1)^2$

(c) $\frac{1}{2}(e^2-e)$ (d) $\frac{1}{2}\left(e-\frac{1}{e}\right)^2$

[GATE (ME) 2014]

24. The integral $\oint_c (ydx - xdy)$ is evaluated along the circle $x^2+y^2=\frac{1}{4}$ traversed in counter-clockwise direction. The integral is equal to

(a) 0 (b) $-\frac{\pi}{4}$

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

[GATE (ME) 2014]

25. The value of the integral $\int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^2+\cos(x-1)} dx$ is

(a) 3 (b) 0

(c) –1 (d) –2

[GATE (ME) 2014]

26. If $\int_0^{2\pi} |x \sin x|\, dx = k\pi$ then the value of k is equal to ______. **[GATE (CS) 2014]**

27. The volume under the surface $z(x, y) = x + y$ and above the triangle in the x-y plane defined by $\{0 \le y \le x \text{ and } 0 \le x \le 12\}$ is ______.

[GATE (EC) 2014]

28. To evaluate the double integral $\int_0^8 \left(\int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2} \right) dx \right) dy$, we make the substitution $u = \left(\frac{2x-y}{2} \right)$ and $v = \frac{y}{2}$. The integral will reduce to

(a) $\int_0^4 \left(\int_0^2 2u\,du \right) dv$ (b) $\int_0^4 \left(\int_0^1 2u\,du \right) dv$

(c) $\int_0^4 \left(\int_0^1 u\,du \right) dv$ (d) $\int_0^4 \left(\int_0^2 u\,du \right) dv$

[GATE (EE) 2014]

ANSWERS

1. (d)	2. (a)	3. (d)	4. (b)	5. (a)	6. (a)	7. (c)	8. (a)	9. (b)	10. (c)
11. (a)	12. (d)	13. (b)	14. (b)	15. (b)	16. (a)	17. (c)	18. (b)	19. (a)	20. (a)
21. (c)	22. (b)	23. (b)	24. (c)	25. (b)	26. (4)	27. (864)	28. (b)		

6 Special Functions

6.1 INTRODUCTION

The students are familiar with elementary functions like algebraic, trigonometric, exponential, and logarithmic. Special functions are those functions other than the elementary functions, such as Bessel's function, Legendre's polynomials, and Beta and Gamma functions.

Many integrals which cannot be evaluated in terms of elementary functions can be evaluated with the help of **Beta and Gamma functions**. The second-order linear differential equations are those whose solutions give rise to special functions (**Bessel's, Legendre's**).

6.2 BESSEL'S EQUATION

The differential equation of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

or $$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

is called *Bessel's equation* of order n, where n is a non-negative constant.

6.3 SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION

Bessel's equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \tag{1}$$

Let the series solution of (1) be

$$y = \sum_{r=0}^{\infty} a_r\, x^{k+r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r\,(k+r)\, x^{k+r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r\,(k+r)(k+r-1)\,x^{k+r-2}$$

Putting these values in Eq. (1), we have

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)\,x^{k+r-2} + \frac{1}{x}(k+r)\,x^{k+r-1} + \left(1-\frac{n^2}{x^2}\right)x^{k+r}\right] = 0$$

or $$\sum_{r=0}^{\infty} a_r\left[\{(k+r)^2 - n^2\}x^{k+r-2} + x^{k+r}\right] = 0 \tag{2}$$

Since the relation (2) is an identity, the coefficients of various powers of x must be zero.

Equating to zero the coefficient of the lowest power of x, i.e., of x^{k-2} in (2), we have

$$a_0(k^2 - n^2) = 0$$

Now, $a_0 \neq 0$ as it is the coefficient of the first term

$$k^2 - n^2 = 0 \text{ or } k = \pm n \tag{3}$$

Now, equating to zero the coefficient of x^{k-1} in (2), we get

$$a_1\,\{(k+1)^2 - n^2\} = 0$$

but $$(k+1)^2 = n^2 \neq 0 \text{ for } k = \pm n \text{ given by (3)}$$

$\therefore$ $$a_1 = 0$$

Again equating to zero the coefficient of the general term, i.e., of x^{k+r} in (2), we get

$$a_{r+2}\,\{(k+r+2)^2 - n^2\} + a_r = 0$$

or $$a_{r+2}\,(k+r+n+2)(k+r-n+2) = -a_r$$

$$a_{r+2} = -\frac{a_r}{(k+r+n+2)(k+r-n+2)} \tag{4}$$

Putting $r = 1$ in (4), we have

$$a_3 = -\frac{a_1}{(k+n+3)(k-n+3)} = 0\text{, since } a_1 = 0$$

Similarly, putting $r = 3, 5, 7$, etc., in (4), we have

$$a_1 = a_3 = a_5 = \ldots = 0$$

Now, two cases arise.

Case I

When $k = n$, from (4), we have

$$a_{r+2} = -\frac{a_r}{(2n+r+2)(r+2)}$$

Putting $r = 0, 2, 4$, etc., we get

$$a_2 = -\frac{a_0}{(2n+2)(2)} = -\frac{a_0}{2^2\cdot(n+1)}$$

$$a_4 = -\frac{a_2}{(2n+4)(4)} = -\frac{a_2}{2^2\cdot 2(n+2)} = \frac{a_0}{2^4\cdot \underline{|2}\,(n+1)(n+2)}\text{, etc.}$$

$\therefore$ $$y = a_0\left[x^n - \frac{x^{n+2}}{2^2\underline{|1}\,(n+1)} + -\frac{x^{n+4}}{2^4\,\underline{|2}\,(n+1)(n+2)}\cdots\right]$$

$$= a_0 x^n \left[1 + (-1)\frac{x^2}{2^2\,\underline{|1}\,(n+1)} + (-1)^2\frac{x^4}{2^4\,\underline{|2}\,(n+1)(n+2)} + \cdots\right]$$

By taking the arbitrary constant $a_0 = \dfrac{1}{2^n\,\overline{|(n+1)}}$, this solution is called $J_n(x)$, known as Bessel function of the first kind of order n.

$$\therefore \qquad J_n(x) = \frac{x^n}{2^n\,\overline{|(n+1)}}\left[1 + (-1)^2\frac{x^2}{2^2\,\underline{|1}\,(n+1)} + (-1)^2\frac{x^4}{2^4\,\underline{|2}\,(n+1)(n+2)} + \cdots\right]$$

$$= \frac{x^n}{2^n\,\overline{|(n+1)}}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^{2r}\,\underline{|r}\,(n+1)(n+2)\cdots(n+r)}$$

$$\boxed{J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{2^{2r}}\frac{x^{2r}}{\underline{|r}\,\overline{|n+r+1}}} = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\underline{|r}\cdot\overline{|(n+r+1)}}$$

Case II

When $k = -n$

The series solution is obtained by replacing n by $-n$ in the value of J_n.

$$\therefore \qquad J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{r}{2}\right)^{-n+2r}\frac{1}{\underline{|r}\,\overline{|(-n+r+1)}}$$

When n is not an integer $J_{-n}(x)$ is distinct from $J_n(x)$.

The general solution of Bessel's equation (1) when n is not an integer is

$$y = AJ_n(x) + B\,J_{-n}(x)$$

where A, B are two arbitrary constants.

6.4 RECURRENCE FORMULAE/RELATIONS OF BESSEL'S EQUATION

(I) $\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

Proof By the definition of $J_n(x)$,

$$\frac{d}{dx}\{x^n\,J_n(x)\} = \frac{d}{dx}\left[x^n\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\underline{|r}\,\overline{|(n+r+1)}}\right]$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{\underline{|r}\,\overline{|(n+r+1)}}\cdot\frac{1}{2^{n+2r}}\cdot\frac{d}{dx}x^{2n+2r}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\,(2n+2r)\,x^{2n+2r-1}}{\underline{|r}\,\overline{|(n+r+1)}\cdot 2^{n+2r}} = \sum_{r=0}^{\infty}\frac{(-1)^r\,2(n+r)}{\underline{|r}\,(n+r)\overline{|n+r}}\cdot\frac{x^n\cdot x^{n+2r-1}}{2^{n+2r}}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \,\overline{|n-1+r+1}} \left(\frac{x}{2}\right)^{n-1+2r} \qquad \left[\because \ \overline{|n+1} = n\overline{|n}\right]$$

$$= x^n J_{n-1}(x) \qquad \text{by the definition of } J_{n-1}(x)$$

(II) $\boldsymbol{\dfrac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)}$

Proof By the definition of $J_n(x)$,

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = \frac{d}{dx}\left\{x^{-n} \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\lfloor r \,\overline{|n+r+1}}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \,\overline{|n+r+1}} \cdot \frac{1}{2^{n+2r}} \cdot \frac{d}{dx} x^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r \, 2rx^{2r-1}}{r\lfloor(r-1) \,\overline{|(n+r+1)} \cdot 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r-1}}{\lfloor(r-1) \,\overline{|n+r+1}} \cdot \frac{1}{2^{n-1+2r}} = \sum_{r=1}^{\infty} \frac{(-1)^r x^{2r-1}}{\lfloor(r-1) \,\overline{|(n+r-1)}} \frac{1}{2^{n-1+2r}}$$

$$\left[\therefore \ \lfloor(r-1 = \infty \text{ when } r = 0\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+2-1}}{\lfloor m \,\overline{|n+m+2}} \cdot \frac{x^n \cdot x^{-n}}{2^{2m+2+n-1}} \qquad [\text{On changing } m = r-1 \text{ so that } r = m+1]$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^r}{\lfloor m \,\overline{|n+m+2}} \left(\frac{x}{2}\right)^{n+1+2m}$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^r}{\lfloor m \,\overline{|n+m+2}} \left(\frac{x}{2}\right)^{n+1+2m} = -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^r}{\lfloor r \,\overline{|n+1+r+1}} \left(\frac{x}{2}\right)^{n+1+2r}$$

$$= -x^{-n} J_{n+1}(x) \qquad [\text{Changing variable from } m \text{ to } r]$$

$$\Rightarrow \qquad \boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)}$$

(III) $\boldsymbol{J'_n(x) = J_{n-1}(x) - \left(\dfrac{n}{x} J_n\right)}$ **or** $\boldsymbol{xJ'_n = -nJ_n + x J_{n-1}}$

Proof By the recurrence relation (I),

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

or $\qquad nx^{n-1} J_n(x) + x_n J_n'(x) = x_n J_{n-1}(x)$

Dividing both sides by x^{n-1},

$$nJ_n(x) + x J'_n(x) = x J_{n-1}(x)$$

or $\qquad \boxed{J'_n(x) = J_{n-1}(x) - \left(\frac{n}{x}\right) J_n(x)}$

(IV) $J'_n(x) = \left(\dfrac{n}{x}\right) J_n(x) - J_{n+1}(x)$ **or** $xJ'_n = -nJ_n + x\,J_{n-1}$

Proof By the recurrence relation (II),

$$\frac{d}{dx}\{x^{-n}\; J_n(x)\} = -x^{-n}\, J_{n+1}(x)$$

or $$-nx^{-n-1}\, J_n(x) + x^{-n}\, J'_n(x) = -x^{-n}\, J_{n+1}(x)$$

Dividing both sides by x^{-n},

$$-nx^{-1}\, J_n(x) + J'_n(x) = -J_{n+1}(x)$$

or $$\boxed{J'_n(x) = \left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x)}$$

(V) $2J'_n(x) = J_{n-1} - J_{n+1}$

Proof By recurrence relations (III) and (IV), we have

$$J'_n(x) = J_{n-1}(x) - \left(\frac{n}{x}\right) J_n(x) \tag{5}$$

and $$J'_n(x) = \left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x) \tag{6}$$

Adding (5) and (6)

$$\boxed{2\,J'_n(x) = J_{n-1}(x) - J_{n+1}(x)}$$

(VI) $J_{n-1}(x) + J_{n+1}(x) = \left(\dfrac{2n}{x}\right) J_n(x)$ **or** $2nJ_n = x(J_{n-1} + J_{n+1})$

Proof From recurrence relations (III) and (IV),

we have $$J'_n(x) = J_{n-1}(x) - \left(\frac{n}{x}\right) J_n(x) \tag{7}$$

and $$J'_n(x) = \left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x) \tag{8}$$

Subtracting (8) from (7), we get

$$0 = J_{n-1}(x) + J_{n+1}(x) - 2\left(\frac{n}{x}\right) J_n$$

or $$J_{n-1}(x) + J_{n+1}(x) = \left(\frac{2n}{x}\right) J_n(x)$$

or $$\boxed{2nJ_n = x\,(J_{n-1} + J_{n+1})}$$

6.5 GENERATING FUNCTION FOR $J_n(x)$

The function $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ is called the *generating function*.

When n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ in ascending and descending power of z.

Also, $J_n(x)$ is the coefficient of z^{-n} multiplied by $(-1)^n$ in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$,
i.e.,

(i) $$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n\ J_n(x)$$

(ii) $$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} (-1)^n\ J_n(x)\, z^{-n}$$

Proof We have $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}$

$$= \left[1+\frac{xz}{2}+\frac{1}{\underline{|2}}\left(\frac{xz}{2}\right)^2+\cdots+\frac{1}{\underline{|n}}\left(\frac{xz}{2}\right)^n+\frac{1}{\underline{|n+1}}\left(\frac{xz}{2}\right)^{n+1}+\frac{1}{\underline{|n+2}}\left(\frac{xz}{2}\right)^{n+2}+\cdots\right]$$

$$\times\left[1-\frac{xz}{2z}+\frac{1}{\underline{|2}}\left(\frac{x}{2z}\right)^2+\cdots+\frac{(-1)^n}{\underline{|n}}\left(\frac{x}{2z}\right)^n+\frac{(-1)^{n+1}}{\underline{|n+1}}\left(\frac{x}{2z}\right)^{n+1}+\frac{(-1)^{n+2}}{\underline{|(n+2)}}\left(\frac{x}{2z}\right)^{n+2}+\cdots\right] \quad (9)$$

(i) Coefficient of z^n in the above product (1)

$$= \frac{1}{\underline{|n}}\left(\frac{x}{2}\right)^n - \frac{1}{\underline{|n+1}}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{\underline{|2}\,\underline{|n+2}}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \frac{(-1)^0}{\overline{|n+1}}\left(\frac{x}{2}\right)^n + \frac{(-1)}{\underline{|1}\,\overline{|n+2}}\cdot\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{\underline{|2}\,\overline{|n+3}}\cdot\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \sum_{r=0}^{\infty}(-1)^r\,\frac{1}{\underline{|r}\,\overline{|n+r+1}}\cdot\left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

(ii) Coefficient of z^{-n} in the above product (1)

$$= \frac{(-1)^n}{\underline{|n}}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{\underline{|n+1}}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{\underline{|n+2}}\cdot\frac{1}{\underline{|2}}\cdot\left(\frac{x}{2}\right)^{n+4} + \ldots$$

$$= (-1)^n\left[\frac{(-1)^0}{\overline{|n+1}}\left(\frac{x}{2}\right)^n + \frac{(-1)}{\overline{|n+2}}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{\overline{|n+3}}\left(\frac{x}{2}\right)^{n+4} + \cdots\right] = (-1)^n J_n(x)$$

Note In the above product (1), the term independent of z is

$$1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\cdots = J_0(x)$$

Here, we observe that the coefficient of z^0, $\left(z-\frac{1}{z}\right),\left(z^2+\frac{1}{z^2}\right)\cdots\left[z^n+(-1)^n\cdot\frac{1}{z^n}\right]\cdots$ are ($J_0(x)$, $J_1(x)$, $J_2(x)$ …, $J_n(x)$… respectively.

Hence, $$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0+\left(z-\frac{1}{z}\right)J_1+\left(z^2+\frac{1}{z^2}\right)J_2+\cdots+\left[z^n+(-1)^n\frac{1}{z^n}\right]J_n+\cdots$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x)\text{, Since } J_{-n}(x)=(-1)^n J_n(x)$$

Theorem

Prove that $J_{-n}(x) = (-1)^n J_n(x)$ where n is an integer.

Proof We have

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{\lfloor r\,\Gamma(-n+r+1)}$$

Since $\Gamma(-p)$ is infinite ($p > 0$), we get terms in $J_{-n}(x)$ equal to zero till $r + 1 - n \geq 1$ so that the series begins when $r \geq n$.

Hence, we can write

$$J_{-n}(x) = \sum_{r=n}^{\infty}\frac{(-1)^r}{\lfloor r\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Putting $r = n + s$, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty}\frac{(-1)^{n+s}}{\lfloor n+s\,\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\lfloor n+r\,\Gamma(r+1)}\left(\frac{x}{2}\right)^{n+2r} \qquad \{\text{Changing the variable}\}$$

$$= (-1)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(n+r+1)\lfloor r}\left(\frac{x}{2}\right)^{n+2r}$$

$$\Rightarrow \qquad \boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

6.6 INTEGRAL FORM OF BESSEL'S FUNCTION

We know that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + \left(z - \frac{1}{z}\right)J_1 + \left(z^2 + \frac{1}{z^2}\right)J_2 + \left(z^3 \frac{1}{z^3}\right)J_3 + \cdots \tag{10}$$

Let $\qquad z = \cos\theta + i\sin\theta$ and $\dfrac{1}{z} = \cos\theta - i\sin\theta$

$\therefore \qquad z^n = \cos n\,\theta + i\sin n\theta$ and $\dfrac{1}{z^n} = \cos n\theta - i\sin n\theta$ [by de Moivre's theorem]

So that $\qquad z^n + \dfrac{1}{z^n} = 2\cos n\theta$ and $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

Substituting these values in (10), we get

$$e^{ix}\sin\theta = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\,\theta\, J_2(x) + 3i\sin 3\,\theta\, J_3(x) + \ldots$$

Since $\qquad e^{ix}\sin\theta = \cos(x\sin\theta) + i\sin(x\sin\theta)$ (11)

Equating the real and imaginary parts in (11), we get

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \ldots] \tag{12a}$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\,\theta + \ldots] \tag{12b}$$

Equations (12a) and (12b) are known as *Jacobi Series*.

Multiplying both sides of (12a) by $\cos n\theta$ and integrating w.r.t. θ between the limits 0 and π. (When n is odd, all terms on the RHS vanish. When n is even, all terms on the RHS except the one containing $\cos n\theta$ vanish). We get

$$\int_0^x \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \pi J_n(x) & \text{when } n \text{ is even} \end{cases}$$

Similarly, multiplying (12b) by $\sin n\theta$ and integrating w.r.t. θ between the limits 0 and π, we get

$$\int_0^x \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} \pi J_n(x) & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

Adding, we get

$$\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta = \pi J_n(x)$$

$$\Rightarrow \qquad J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta \text{ for all integral values of } n.$$

Example 1 Prove that:

(i) $J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

(ii) $J_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

Solution We know that

$$J_n(x) = \frac{x^n}{2^n \lceil n+1} \left[1 - \frac{x^2}{2.2\,(n+1)} + \frac{x^4}{2.4.2^2\,(n+1)(n+2)} - \cdots \right] \tag{1}$$

(i) Replacing n by $\frac{1}{2}$ in (1), we have

$$J_{\frac{1}{2}}(x) = \frac{x^{1/2}}{2^{1/2} \lceil 3/2} \left[1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \cdots \right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{\frac{1}{2}\lceil 1/2} \cdot \frac{1}{x} \left[x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots \right] \qquad \left[\because \lceil n+1 = n \lceil n \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots \right]$$

$$\boldsymbol{J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$

(ii) Replacing n by –1/2 in (1), we have

$$J_{-\frac{1}{2}}(x) = \frac{x^{-1/2}}{2^{-1/2} \lceil 1/2} \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots \right)$$

$$\boldsymbol{J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

Example 2 Prove that

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1(x) + \left(1 - \frac{24}{x^2} \right) J_0(x)$$

or express $J_n(x)$ in terms of J_0 and J_1.

Solution Using recurrence relation (VI),

$$J_{n+1}(x) = \left(\frac{2n}{x} \right) J_n(x) - J_{n-1}(x) \tag{1}$$

Putting $n = 1, 2, 3$ in Eq. (1), we have,

$$J_2(x) = \frac{1}{x}\left[2\, J_1(x) - x\, J_0(x)\right] \tag{2}$$

$$J_3(x) = \frac{1}{x}\left[4\, J_2(x) - x\, J_1(x)\right] \tag{3}$$

$$J_4(x) = \frac{1}{x}\left[6\, J_3(x) - x\, J_2(x)\right] \tag{4}$$

From (2) and (3)

$$J_3(x) = \frac{8}{x^2}\, J_1(x) - \frac{4}{x}\, J_0(x) - J_1(x) \tag{5}$$

From (4) and (5)

$$J_4(x) = \left(\frac{48 - 6x^2}{x^3}\right) J_1(x) - \frac{24}{x^2}\, J_0(x) - \frac{2}{x}\, J_1(x) + J_0(x)$$

$$\boldsymbol{J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)}$$

Example 3 Prove that

$$x^2 J_n''(x) = (n^2 - n - x^2)\, J_n(x) + x\, J_{n+1}(x), \text{ where } n = 0, 1, 2, \ldots$$

Solution Using recurrence relation (IV),

$$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x) \tag{1}$$

Differentiating (1) both sides w.r.t. 'x', we get

$$xJ''_n(x) + J'_n(x) = nJ'_n(x) - \{xJ'_{n+1}(x) + J_{n+1}(x)\}$$

or $$x^2 J''_n(x) = (n - 1)x\, J'_n(x) - x\{xJ'_{n+1}(x)\} - x\, J_{n+1}(x) \tag{2}$$

Now, using recurrence relation (III),

$$xJ'_n(x) = -n\, J_n(x) + x\, J_{n+1}(x) \tag{3}$$

Replacing n by $(n + 1)$ in (3),

$$xJ'_{n+1}(x) = -(n + 1)\, J_n(x) + x\, J_n \tag{4}$$

Putting the value of $x\, J'_n(x)$ and $x\, J'_{n+1}(x)$ from (1) and (4) in (2), we get

$$\boldsymbol{x^2\, J''_n = (n^2 - n - x^2)\, J_n + x\, J_{n+1}}$$

Example 4 Prove that

$$J_{n+3} + J_{n+5} = \frac{2}{x}\,(n + 4)\, J_{n+4}$$

Solution Using recurrence relation (VI), $2n\, J_n = x(J_{n-1} + J_{n+1})$

Replacing n by $n + 4$, we get

$$\frac{2}{x}\,(n + 4)\, J_{n+4} = J_{n+3} + J_{n+5}$$

or, $\boldsymbol{J_{n+3} + J_{n+5} = \frac{2}{x}(n+4)\, J_{n+4}}$

Example 5 Evaluate $\int x^4 J_1(x)\, dx$.

Solution Using recurrence relation (I),

$$\frac{d}{dx}\{x^n J_n\} = x^n J_{n-1}$$

$$\Rightarrow \quad \int x^n J_{n-1}\, dx = x^n J_n \tag{1}$$

Now,

$$\int x^4 J_1\, dx = \int x^2 (x^2 J_1)\, dx$$

$$= x^2(x^2 J_2) - \int 2x(x^2 J_2)\, dx \qquad \left[\because \int x^2 J_1\, dx = x^2 J_2\right]$$

$$= x^3 J_2 - 2\int x^3 J_2\, dx$$

$$\boldsymbol{\int x^4 J_1\, dx = x^4 J_2 - 2x^3 J_3 + c} \qquad \{\text{by } (1)\}$$

Example 6 Prove that $[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + 2[J_3(x)]^2 + \ldots = 1$.

Solution Using Jacobi series,

$$J_0(x) + 2\, J_2(x) \cos 2\theta + 2\, J_4(x) \cos 4\theta + \ldots = \cos (x \sin \theta) \tag{1}$$

$$2J_1(x) \sin \theta + 2\, J_3(x) \sin 3\theta + \ldots = \sin (x \sin \theta) \tag{2}$$

Squaring (1) and (2) and integrating w.r.t. θ between 0 and π,

$$[J_0(x)]^2\pi + 2[J_2(x)]^2\pi + 2[J_4(x)]^2\, \pi + \ldots = \int_0^\pi \cos^2 (x \sin \theta)\, d\theta \tag{3}$$

$$2[J_1(x)]\pi + 2[J_3(x)]\pi + \ldots = \int_0^\pi \sin^2 (x \sin \theta)\, d\theta \tag{4}$$

Since

$$\int_0^\pi \cos^2 n\theta\, d\theta = \int_0^\pi \sin^2 n\theta\, d\theta = \frac{\pi}{2}$$

Adding (3) and (4), we get

$$\pi\{[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \ldots\} = \int_0^\pi \left[\cos^2 (x \sin \theta) + \sin^2 (x \sin \theta)]\, d\theta\right] = \int_0^\pi d\theta = \pi$$

$$[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \ldots = 1$$

EXERCISE 6.1

1. Show that

   (i) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x} \sin x - \cos x\right]$

(ii) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3-x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$

(iii) $J_{7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{15-6x^2}{x^3}\right)\sin x - \left(\frac{15}{x^2}-1\right)\cos x\right]$

2. Prove that

   (i) $\frac{d}{dx}\{x\, J_n(x)\, J_{n+1}(x)\} = \{J_n^2(x) - J_{n+1}^2(x)\}$

   (ii) $x = 2\, J_0\, J_1 + 6\, J_1\, J_2 + \ldots + 2(2n+1)\, J_n\, J_{n+1} + \ldots$

3. Prove that

   (i) $J'_0 = -J_1$

   (ii) $J_2 - J_0 = 2\, J''_0$

   (iii) $J_2 = J''_0 - \left(\frac{1}{x}\right) J'_0$

   (iv) $J_2 + 3\, J'_0 + 4\, J'''_0 = 0$

4. Solve $x\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4x^2 y = 0$ in terms of Bessel function.

5. Prove that

   (i) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)d\phi$

   (ii) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)d\phi$

6. Prove that

   (i) $\cos x = J_0 - 2\, J_2 + 2\, J_4 - \ldots$

   (ii) $\sin x = 2\, J_1 - 2\, J_3 + 2\, J_5 - \ldots$

7. Prove that $J_n(x)$ is an even function when n is even and is an odd function when n is odd.

8. Prove that

$$\int_0^{\pi/2} \sqrt{\pi x}\, J_{1/2}(2x)dx = 1$$

9. Show that

   (i) $\int_0^x x^n\, J_{n-1}(x)\, dx = x^n\, J_n(x)$

   (ii) $\int_0^x x^{n+1}\, J_n(x)\, dx = x^{n+1}\, J_{n+1}(x)$

10. Show that when n is an integral,

$$\pi J_n = \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

11. Prove $\dfrac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left(\dfrac{n}{x} J_n^2 - \dfrac{n+1}{x} J_{n+1}^2\right)$

12. Prove that $J^2_0 + 2(J_1{}^2 + J_2{}^2 + J_3{}^2 + \dots) = 1$.

6.7 LEGENDRE POLYNOMIALS

Legendre's Equation

The differential equation of the form

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is known as Legendre's differential equation (or Legendre's equation) where n is a positive integer.

This equation can also be written as

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

6.8 SOLUTION OF LEGENDRE'S EQUATIONS

The Legendre's equation is

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad (13)$$

It can be solved in a series of ascending or descending powers of x. The solution in descending powers of x is more important.

Let us assume,

$$y = \sum_{r=0}^{\infty} a_r x^{k-r}$$

$$\therefore \qquad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k - r) x^{k-r-1}$$

and

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k - r)(k - r - 1)\, x^{k-r-2}$$

Substituting the value of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (13), we have

$$(1 - x^2)\sum_{r=0}^{\infty} a_r (k - r)(k - r - 1)\, x^{k-r-2} - 2x\sum_{r=0}^{\infty} a_r (k - r)\cdot x^{k-r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^{k-r} = 0$$

or $$\sum_{r=0}^{\infty} a_r\left[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1)-(k-r)(k-r-1)-2(k-r)\}x^{k-r}\right] = 0$$

or $$\sum_{r=0}^{\infty} a_r\left[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1)-(k-r)(k-r-1)\}x^{k-r}\right] = 0$$

or $$\sum_{r=0}^{\infty} a_r\left[(k-r)(k-r-1)\, x^{k-r-2} + \{n^2-(k-r)^2+n-(k-r)\}x^{k-r}\right] = 0$$

or $$\sum_{r=0}^{\infty} a_r\left[(k-r)(k-r-1)\, x^{k-r-2} + (n-k+r)(n+k-r+1)x^{k-r}\right. = 0 \quad (14)$$

which is an identity in x, we can equate to zero the coefficients of various power of x.

Therefore, equating to zero the coefficient of highest power x^k, we get the initial equation

$$a_0(n-k)(n+k+1) = 0$$

Now, $a_0 \neq 0$, as it is the coefficient of the first term

or $$\left.\begin{aligned} k &= n \\ k &= -(n+1) \end{aligned}\right\} \quad (15)$$

Equating to zero the coefficient of the next lower power of x, i.e., x^{k-1}. We have

$$a_1(n-k+1)(n+k) = 0$$

$\therefore$ $a_1 = 0$, since neither $(n-k+1)$ nor $(n+k)$ is zero for $k = n$ or $k = -(n+1)$.

Again equating to zero the coefficient of the general term, i.e., of x^{k-r}, we get the recurrence relation

$$a_{r-2}(k-r+2)(k-r+1) + (n-k+r)(n+k-r+1)\, a_r = 0$$

$\therefore$ $$a_r = -\frac{(k-r+2)(k-r+1)}{(n-k+r)(n+k-r+1)}\, a_{r-2} \quad (16)$$

Putting $r = 3$, $$a_3 = -\frac{(k-1)(k-2)}{(n-k+3)(n+k-2)}\, a_1$$

$$= 0, \text{ since } a_1 = 0$$

We have $$a_1 = a_3 = a_5 = \ldots - = 0$$

For the two values given by (15), there arise two cases.

Case I

When $k = n$, from (16), we have

$$a_r = -\frac{(n-r+2)(n-r+1)}{r\,(2n-r+1)} \cdot a_{r-2}$$

Putting $r = 2, 4, \ldots$ etc.

$$a_2 = -\frac{n\,(n-1)}{2\,(2n-1)}\, a_0$$

$$a_4 = -\frac{(n-2)(n-3)}{4\cdot(2n-3)}a_2$$

$$= \frac{n(n-1)(n-2)}{2.4\cdot(2n-1)(2n-3)}a_0\text{, etc.}$$

$\therefore$
$$y = a_0x^n + a_2x^{n-2} + a_4x^{n-4} + \ldots$$

$$\boldsymbol{y = a_0\left[x^n - \frac{n(n-1)}{2\cdot(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)(2n-3)}x^{n-4} - \cdots\right]} \tag{17}$$

which is one solution of Legendre's equation.

Case II

When $k = -(n+1)$

$$a_r = \frac{(n+r-1)(n+r)}{r(2n+r+1)}a_{r-2}$$

Putting $r = 2, 4, 6 \ldots$ etc.

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)}a_0,$$

$$a_4 = \frac{(n+3)(n+4)}{4\cdot(2n+5)}a_2$$

$$= \frac{(n+1)(n+2)(n+3)\,(n+4)}{2.4\cdot(2n+3)\,(2n+5)}\cdot a_0 \quad \text{etc.}$$

$\therefore$
$$y = \sum_{r=0}^{\infty} a_r\, x^{-n-1-r}$$

$$= a_0x^{-n-1} + a_2x^{-n-3} + a_4x^{-n-5} + \ldots$$

$$\boldsymbol{= a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}\cdot x^{-n-3} + \frac{(n+1)(n+2)(n+3)\,(n+4)}{2.4\cdot(2n+3)(2n+5)}x^{-n-5} + \cdots\right]} \tag{18}$$

which is the second solution of Legendre's equation.

Thus, two independent solution of (13) are given by (17) and (18). If we take $a_0 = \dfrac{[1.3.5\cdots(2n-1)]}{n}$. The solution (17) denoted by $P_n(x)$ and is called Legendre's function of the first kind or Legendre polynomial of degree n.

Again, if we take $a_0 = \dfrac{n}{[1.3.5\cdots(2n+1)]}$, the solution (18) is denoted by $Q_n(x)$ is called Legendre's function of second kind.

$P_n(x)$ and $Q_n(x)$ are two linearly independent solutions of (13). Hence, the general solution of (13) is

$$y = AP_n(x) + BQ_n(x)\text{, where } A \text{ and } B \text{ are arbitrary constants.} \tag{19}$$

Legendre's function of the first kind is denoted by

$$\boldsymbol{P_n(x)} = \frac{[1.3.5\cdots(2n-1)]}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)(2n-3)}x^{n-4} - \cdots\right] \quad (20)$$

Legendre's function of the second kind is denoted by

$$\boldsymbol{Q_n(x)} = \frac{\lfloor n}{1.3\cdots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4\,(2n+3)(2n+5)}x^{-(n+r)} + \cdots\right] \quad (21)$$

6.8.1 Determination of Some Legendre's Polynomials

Putting $n = 0, 1, 2, 3, 4, 5, \ldots$ in Eq. (20), we get

$$P_0(x) = \frac{1}{\lfloor 0}x^0 = 1,\ p_1(x) = \frac{1}{\lfloor 1}x^1 = x$$

$$P_2(x) = \frac{1.3}{\lfloor 2}\left[x^2 - \frac{2.1}{2.3}x^0\right] = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1.3.5}{\lfloor 3}\left[x^3 - \frac{3.2}{2.5}x^1\right] = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1.3.5.7}{\lfloor 4}\left[x^4 - \frac{4.3}{2.7}x^2 + \frac{4.3.2.1}{2.4.7.5}x^0\right] = \frac{1}{8}(35x^4 - 3x^2 + 3)$$

$$P_5(x) = \frac{1.3.5.7.9}{\lfloor 5}\left[x^5 - \frac{5.4}{2.9}x^3 + \frac{5.4.3.2}{2.4.9.7}x^1\right] = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

6.9 GENERATING FUNCTION OF LEGENDRE'S POLYNOMIALS

To show that $P_n(x)$ is the coefficient of h^n in the expansion of $(1 - 2xh + h^2)^{-1/2}$ in ascending powers of h, we have.

$$\begin{aligned}(1 - 2xh + h^2)^{-1/2} &= \{1 - h(2x - h)\}^{-1/2}\\ &= 1 + \frac{1}{2}h\,(2x - h) + \frac{1.3}{2.4}h^2\,(2x - h)^2 + \cdots\\ &\quad + \frac{1.3\cdots(2n-3)}{2.4\cdots(2n-2)}h^{n-1}\,(2x - h)^{n-1}\\ &\quad + \frac{1.3\cdots(2n-1)}{2.4\cdots(2n)}h^n\,(2x - h)^n + \cdots\end{aligned}$$

Therefore, coefficient of h^n

$$= \frac{1.3\cdots(2n-1)}{2.4\cdots 2n}(2x)^n - \frac{1.3\cdots(2n-3)}{2.4\cdots(2n-2)}\,{}^{n-1}C_1\,(2x)^{n-2} + \frac{1.3\cdots(2n-5)}{2.4\cdots(2n-4)}\,{}^{n-2}C_2\,(2x)^{n-4} - \cdots$$

$$= \frac{1.3\cdots(2n-1)}{2.4\cdots 2n} 2^n \left[x^n - \frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2} + \frac{2n(2n-2)}{(2n-1)(2n-3)} \cdot \frac{(n-2)(n-3)}{\lfloor 2} \cdot \frac{x^{n-4}}{2^4} - \cdots \right]$$

$$= \frac{1.3\cdots(2n-1)}{\lfloor n} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)\,(2n-3)} \cdot x^{n-4} - \cdots \right]$$

$$= P_n(x)$$

Hence, $\sum_{n=0}^{\infty} h^n\, P_n(x) = (1 - 2xh + h^2)^{-1/2}$, where $P_0(x) = 1$

$(1 - 2xh + h^2)^{-1/2}$ is called the *generating function of the Legendre polynomials*.

6.10 RODRIGUES' FORMULA

$P_n(x) = \dfrac{1}{2^n \lfloor n} \dfrac{d^n}{dx^n}(x^2 - 1)^n$ is known as *Rodrigues' formula*.

Proof Let $v = (x^2 - 1)^n$ then $v_1 = \dfrac{dv}{dx} = n(x^2 - 1)^{n-1} \cdot 2x$

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2 - 1)\, v_1 = 2nx(x^2 - 1)^n = 2nxv$$

or $$(1 - x^2)\, v_1 + 2nxv = 0$$

Differentiating $(n + 1)$ times by Leibinitz's theorem, we have

$$\left[(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{(n+1)\cdot n}{\lfloor 2}(-2)v_n \right] + 2n[xv_{n+1} + (n+1)v_n] = 0$$

or $$(1 - x^2)v_{n+2} - 2xv_{n+1} + n(n + 1)v_n = 0$$

or $$(1 - x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)\, v_n = 0$$

which is Legendre's equation and v_n is its solution. But the solutions of Legendre's equation are $P_n(x)$ and $Q_n(x)$.

Since $v_n = \dfrac{d^n}{dx^n}(x^2 - 1)$ contains only positive powers of x, it must be a constant multiple of $P_n(x)$,

i.e., $$v_n = C\, P_n(x)$$

or $$C\, P_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n$$

$$= \frac{d^n}{dx^n}[(x-1)^n (x+1)^n]$$ [differentiating n times by Leibnitz theorem]

$$= (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^nC_1 \cdot n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \cdots + (x+1)^n \frac{d^n}{dx^n}(x-1)^n$$

$$= (x-1)^n \cdot \underline{|n} + {}^nC_1 \cdot n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \cdots + (x+1)^n \underline{|n}$$

$$= \underline{|n}\,(x+1)^n + \text{ term containing powers of } (x-1)$$

Putting $x = 1$ on both sides,

$$C\,P_n(1) = \underline{|n} \cdot 2^n \quad \text{or} \quad C = 2^n\,\underline{|n}\text{, since } P_n(1) = 1$$

Substituting in (13), we get

$$\boxed{P_n(x) = \frac{1}{2^n\,\underline{|n}} \frac{d^n}{dx^n}(x^2-1)^n}$$

6.11 LAPLACE DEFINITE INTEGRAL FOR $P_n(x)$

(i) Laplace's First Integral of $P_n(x)$

If n is a positive integer,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{x^2-1}\cos\phi\right]^n d\phi$$

Proof We know that

$$\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}}\text{, where } a^2 > b^2$$

Putting $a = 1 - hx$ and $b = h\sqrt{(x^2-1)}$

So that $a^2 - b^2 = (1-hx)^2 - h^2(x^2-1) = 1 - 2xh + h^2$

We have

$$\pi(1-2xh+h^2)^{-1/2} = \int_0^{\pi}\left[1 - hx \pm h\sqrt{(x^2-1)}\cos\phi\right]^{-1} d\phi$$

$$= \int_0^{\pi}\left[1 - h\left\{x \mp \sqrt{x^2-1}\cos\phi\right\}\right]^{-1} d\phi$$

$$= \int_0^{\pi}[1-ht]^{-1}\,d\phi \qquad \left[\text{where } t = x \mp \sqrt{x^2-1}\cos\phi\right]$$

$$\Rightarrow \qquad \pi\sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi}(1 + ht + h^2t^2 + \cdots + h^nt^n)\,d\phi$$

Equating the coefficient of h^n on both sides, we have

$$\pi P_n(x) = \int_0^{\pi} t^n \, d\phi = \int_0^{\pi} \left[x \mp \sqrt{x^2 - 1} \cos \phi \right]^n d\phi$$

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x \pm \sqrt{x^2 - 1} \cos \phi \right]^n d\phi$$

(ii) Laplace's Second Integral for $P_n(x)$

When n is a positive integer,

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{\left[x \pm \sqrt{(x^2 - 1)} \cos \phi \right]^{n+1}}$$

Proof We have

$$\int_0^{\pi} \frac{d\phi}{a \pm b \cos \phi} = \frac{\pi}{\sqrt{a^2 - b^2}}, \text{ where } a^2 > b^2$$

Putting $a = xh - 1$ and $b = h\sqrt{x^2 - 1}$

So that $a^2 - b^2 = 1 - 2xh + h^2$

We have

$$\pi(1 - 2xh + h^2)^{-1/2} = \int_0^{\pi} \left[-1 + xh \pm h\sqrt{x^2 - 1} \cos \phi \right]^{-1} d\phi$$

or $$\frac{\pi}{h} \left[1 - 2x\frac{1}{h} + \frac{1}{h^2} \right]^{-1/2} = \int_0^{\pi} \left[h\left\{ x \pm \sqrt{(x^2 - 1)} \cos \phi \right\} \right]^{-1} d\phi$$

or $$\frac{\pi}{h} \sum_{n=0}^{\infty} \frac{1}{h^n} P_n(x) = \int_0^{\pi} (ht - 1)^{-1} \, d\phi \qquad \text{where } t = x \pm \sqrt{(x^2 - 1)} \cos \phi$$

$$= \int_0^{\pi} \frac{1}{ht} \left[1 - \frac{1}{ht} \right]^{-1} d\phi$$

$$= \int_0^{\pi} \frac{1}{ht} \left[1 + \frac{1}{ht} + \frac{1}{h^2 t^2} + \cdots + \frac{1}{h^n t^n} + \cdots \right] d\phi$$

$$= \int_0^{\pi} \left[\frac{1}{ht} + \frac{1}{h^2 t^2} + \frac{1}{h^3 t^3} + \cdots + \frac{1}{h^{n+1} t^{n+1}} + \cdots \right] d\phi$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{h^{n+1}} \int_0^{\pi} \frac{1}{t^{n+1}} \, d\phi \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{h^{n+1}} \int_0^{\pi} \frac{d\phi}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}} \right]$$

Equating the coefficient of $\frac{1}{h^{n+1}}$, we have

$$\pi P_n(x) = \int_0^{\pi} \frac{d\phi}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}}$$

or
$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}}$$

6.12 ORTHOGONAL PROPERTIES OF LEGENDRE'S POLYNOMIAL

(i) $\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0$ if $m \neq n$

(ii) $\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$

Proof (i) Legendre's equation may be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)\,y = 0$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)\,P_n = 0 \tag{22a}$$

and $\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)\,P_m = 0$ [$\therefore$ $P_n(x)$ and $P_m(x)$ are the solutions of Legendre's equation] (22b)

Multiplying (22a) by P_m and (22b) by P_n and then subtracting, we have

$$P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + \{n(n+1) - m(m+1)\}P_nP_m = 0$$

Integrating between the limits –1 to 1,

$$\int_{-1}^{1}\left[P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}\right] - \int_{-1}^{1}\left[P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}\right] dx$$

$$+ \{n(n+1) - m(m+1)\} \int_{-1}^{-1} P_m\, P_n\, dx = 0$$

Integrating by parts,

$$\left[P_m(1-x^2)\frac{d\,P_n}{dx}\right]_{-1}^{1} - \int_{-1}^{+1}\frac{d\,P_m}{dx}\left[(1-x^2)\frac{d\,P_n}{dx}\right]dx$$

$$-\left[P_n(1-x)^2\;\frac{d\,P_m}{dx}\right]_{-1}^{+1} + \int_{-1}^{-1}\frac{d\,P_n}{dx}\left\{(1-x)^2\;\frac{d\,P_m}{dx}\right\}dx$$

$$+\,[n\,(n+1)-m\,(m+1)]\times\int_{-1}^{+1}P_m P_n\;dx\;=0$$

Hence, $\displaystyle\int_{-1}^{-1}\boldsymbol{P_m(x)\,P_n(x)\,dx\;=0}$ **since** $\boldsymbol{m\neq n}$

(ii) $\displaystyle\boldsymbol{(1-2x\,h+h^2)^{-1/2}=\sum_{n=0}^{\infty}h^nP_n(x)}$

Squaring both sides, we have

$$(1-2xh+h^2)=\sum_{n=0}^{\infty}h^{2n}\{P_n(x)\}^2+\sum_{\substack{m_1n=0\\ m\neq n}}h^{m+n}\;P_m(x)\,P_n(x)$$

Integrating between the limits –1 to 1,

$$\sum_{n=0}^{\infty}\int_{-1}^{1}h^{2n}\,[P_n(x)]^2\,dx+\sum_{m,n=0}^{\infty}\int_{-1}^{1}h^{m+n}\;P_m(x)\,P_n(x)\,dx\;=\int_{-1}^{1}\frac{dx}{(1-2xh+h^2)}$$

or
$$\sum_{n=0}^{\infty}\int_{-1}^{+1}h^{2n}\,[P_n(x)]^2\,dx\;=\int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

Since the other integrals on the LHS are zero by the property $\displaystyle\int_{-1}^{1}p_m(x)\,P_n(x)\,dx\;=0$ as $m\neq n$

$$\Rightarrow\qquad \sum_{n=0}^{\infty}\int_{-1}^{1}h^{2n}\,[P_n(x)]^2\;dx\;=-\frac{1}{2h}\Big[\log\,(1-2xh+h^2\Big]_{-1}^{+1}$$

$$=-\frac{1}{2h}\Big[\log\,(1-h)^2-\log(1+h)^2\Big]$$

$$=\frac{1}{2h}\left[\log\left(\frac{1+h}{1-h}\right)^2\right]=\frac{1}{h}\log\left[\frac{1+h}{1-h}\right]$$

$$=\frac{2}{h}\left[h+\frac{h^3}{3}+\frac{h^5}{5}+\cdots\right]$$

$$\left[\because\ \log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \ldots \text{ and } \log(1-h) = -h - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4} \ldots\right]$$

$$= 2\left[1 + \frac{h^2}{3} + \frac{h^4}{5} + \cdots \frac{h^{2n}}{2n+1} + \cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{2h^{2n}}{(2n+1)}$$

Equating the coefficient of h^{2n},

$$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$$

6.13 RECURRENCE FORMULAE FOR $P_n(x)$

(i) $(2n + 1)\, xP_n = (n + 1)\, P_{n+1} + n\, P_{n-1}$

or

$$n\, P_n = (2n - 1)\, x\, P_{n-1} - (n - 1)\, P_{n-2},\ n \geq 2$$

Proof From generating function, we have

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n\, P_n(x) \tag{23a}$$

Differentiating both sides of (23a) w.r.t. 'h' we get

$$-\frac{1}{2}(1 - 2xh + h^2)^{-3/2}\, (-2x + 2h) = \sum_{n=0}^{\infty} n\, h^{n-1}\, P_n(x) \tag{23b}$$

Multiplying both sides of (23b) by $1 - 2xh + h^2$, we get

$$(x - h)\, (1 - 2xh + h^2)^{-1/2} = (1 - 2xh + h^2) \sum_{n=0}^{\infty} n\, h^{n-1}\, P_n(x)$$

or $$(x - h) \sum_{n=0}^{\infty} h^n\, P_n(x) = (1 - 2xh + h^2) \sum_{n=0}^{\infty} n\, h^{n-1}\, P_n(x), \text{ by (23a)}$$

or $$x \sum_{n=0}^{\infty} h^n\, P_n(x) - \sum_{n=0}^{\infty} h^{n+1}\, P_n(x) = \sum_{n=0}^{\infty} n\, h^{n-1} P_n - 2x \sum_{n=0}^{\infty} n\, h^n\, P_n + \sum_{n=0}^{\infty} n\, h^{n+1} P_n$$

Equating coefficients of h^n from both sides, we get

$$xP_n - \mathrm{P}_{n-1} = (n + 1)\, P_{n+1} - 2x{\cdot}nP_n + (n - 1)P_{n-1}$$

or $$\mathbf{(2\, n + 1)\, xP_n = (n + 1)\, P_{n+1} + n\, P_{n-1}} \tag{24}$$

Replacing n by $n - 1$ is (24), we have

$$\boldsymbol{n\, P_n = (2n - 1)\, x\, P_{n-1} - (n - 1)\, P_{n-2}} \tag{25}$$

(ii) $n\, P_n = x\, P'_n - P'_{n-1}$

Proof We have $$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n\, P_n(x) \tag{26}$$

Differentiating (26) w.r.t. 'h'

$$-\frac{1}{2}(1 - 2xh + h^2)^{-3/2}\,(-2x + 2h) = \sum_{n=0}^{\infty} n\, h^{n-1}\, P_n$$

or $$(x - h)\,(1 - 2xh + h^2)^{-3/2} = \sum_{n=0}^{\infty} n\, h^{n-1}\, P_n \tag{27}$$

Again differentiating (26) w.r.t. 'x' we have,

$$h(1 - 2xh + h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n\, P'_n$$

or $$h(x - h)\,(1 - 2xh + h^2)^{-3/2} = (x - h)\sum_{n=0}^{\infty} h^n\, P'_n$$ [Multiplying both sides by $(x - h)$]

or $$h\sum_{n=0}^{\infty} n\, h^{n-1} P_n = (x - h)\sum_{n=0}^{\infty} h^n P'_n$$ by (27)

or $$\sum_{n=0}^{\infty} n\, h^n P_n = x\sum_{n=0}^{\infty} h^n P'_n - \sum_{n=0}^{\infty} h^{n+1} P'_n$$

Equating coefficients of h^n on both sides, we get

$$\boldsymbol{n\, P_n = x\, P'_n - P'_{n-1}}$$

(iii) $(2n + 1)\, P_n = P'_{n+1} - P'_{n-1}$

Proof From the recurrence relation (I), we have

$$(2n + 1)\, xP_n = (n + 1)\, P_{n+1} + n\, P_{n-1}$$

Differentiating w.r.t. 'x', we get

$$(2n + 1)\, xP'_n + (2n + 1)\, P_n = (n + 1)\, P'_{n+1} + n\, P'_{n-1}$$

or $$(2n + 1)\,(n\, P_n + P'_{n-1}) + (2n + 1)\, P_n = (n + 1)\, P'_{n+1} + n\, P'_{n-1}$$

[$\because$ from recurrence (II), $x\, P'_n = n\, P_n + P'_{n-1}$]

$$(2n + 1)\,(n + 1)\, P_n = (n + 1)\, P'_{n+1} + n\, P'_{n-1} - (2n + 1)\, P'_{n-1}$$

or $$(2n + 1)\,(n + 1)\, P_n = (n + 1)\, P_{n+1} - (n + 1)\, P'_{n-1}$$

$$\boldsymbol{(2n + 1)\, P_n = P'_{n+1} - P'_{n-1}}$$

(iv) $(n+1)\,P_n = P'_{n+1} - xP'_n$

Proof From recurrence formulae (II) and (III), we have

$$nP_n = xP'_n - P'_{n-1} \tag{28}$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1} \tag{29}$$

Subtracting (28) from (29), we have

$$\boldsymbol{(n+1)P_n = P'_{n+1} - xP'_n}$$

(v) $(x^2 - 1)P'_n = nxP_n - nP_{n-1}$

Proof From the recurrence relations (II) and (IV), we have.

$$nP_n = xP'_n - P'_{n-1} \tag{30}$$

and $$(n+1)\,P_n = P'_{n+1} - xP'_n \tag{31}$$

Replacing n by $(n-1)$ in (31)

$$nP_{n-1} = P'_n - xP'_{n-1} \tag{32}$$

Multiplying both sides of (30) by x,

$$x\cdot nP_n = x^2P'_n - xP'_{n-1} \tag{33}$$

Subtracting (33) from (32), we have

$$n(P_{n-1} - xP_n) = (1 - x^2)\,P'_n$$

or $$\boldsymbol{(x^2 - 1)\,P'_n = nxP_n - n\cdot P_{n-1}}$$

(vi) $(1 - x^2)\,P'_n = (n+1)(xP_n - P_{n+1})$

Proof By the recurrence formulae (I)

$$(2n+1)\,xP_n = (n+1)\,P_{n+1} + nP_{n-1} \tag{34a}$$

which may be written as

$$(n+1)\,xP_n + nxP_n = (n+1)\,P_{n+1} + nP_{n-1}$$

or $$(n+1)(xP_n - p_{n+1}) = n(p_{n-1} - xP_n) \tag{34b}$$

From (34a) and (34b),

$$\boldsymbol{(1 - x^2)\,P'_n = (n+1)\,(xP_n - P_{n+1})}$$

Example 7 Show that

(i) $P_n(1) = 1$

(ii) $P_n(-1) = (-1)^n$

(iii) $P_n(-x) = (-1)^n\,P_n(x)$

Solution

(i) We know that

$$(1 - 2xh + h^2)^{-1/2} = 1 + hP_1(x) + h^2(P_2(x) + h^3P_3(x) + \ldots + h^nP_n(x) + \ldots$$

Putting $x = 1$ in the above equation, we get

$$(1 - 2h + h^2)^{-1/2} = 1 + hP_1(1) + h^2(P_2(1) + h^3P_3(1) + \ldots + h^nP_n(1) + \ldots$$

$$[(1-h)^2]^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1) \Rightarrow (1-h)^{-1} = \sum h^n P_n(1)$$

$$\Rightarrow \quad \sum h^n P_n(1) = (1-h)^{-1} = 1 + h + h^2 + h^2 + \ldots + h^n \ldots$$

Equating the coefficient of h^n on both sides,

we have $\quad \boldsymbol{P_n(1) = 1}$

(ii) By the generating function, we have

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{1}$$

Putting $-x$ for x in both sides,

$$(1 + 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-x) \tag{2}$$

Again putting $-h$ for h in (1), we have

$$(1 + 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x) \tag{3}$$

From (2) and (3),

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x) \tag{4}$$

Comparing the coefficient of h^n in both sides of (4),

$$P_n(-x) = (-1)^n P_n(x) \tag{5}$$

Putting $x = 1$ in (5) we get

$$\boldsymbol{P_n(-1) = (-1)^n P_n(1) = (-1)^n \cdot 1} \qquad (\because \quad P_n(1) = 1)$$

(ii) If n is even then by (5),

$P_n(-x) = P_n(x)$, hence $P_n(x)$ in even function of x

(iii) If n is odd then from (5),

$P_n(-x) = -P_n(x)$, hence $P_n(x)$ is an odd function

Example 8 Show that

(i) $P_{2n}(0) = (-1)^n \dfrac{1.3.5. -(2n-1)}{2.4.6 \cdots 2n}$

(ii) $P_{2n-1}(0) = 0$

Solution We know that

$$\sum h^{2n} P_{2n}(x) = (1 - 2xh + h^2)^{-1/2} \tag{1}$$

Putting $x = 0$ in (1), we get

$$\sum h^{2n} P_{2n}(0) = (1+h^2)^{-1/2}$$

$$= 1+\left(-\frac{1}{2}\right)h^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\lfloor 2}(h^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 3}(h^2)^3 + \cdots$$

$$\cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{\lfloor n}(h^2)^n + \cdots \quad (2)$$

Equating the coefficient of h^{2n} in (2) both sides, we have

$$P_{2n}(0) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)$$

$$= (-1)^n \frac{1.3.5\cdots(2n-1)}{2^n \lfloor n}$$

$$= (-1)^n \frac{1.3.5\cdots(2n-1)}{2.4.6.8\cdots 2n}$$

Now, coefficient of $z^{(2n+1)}$ in (2), we get $P_{2n+1}(0) = 0$

Example 9 Show that

$$\int_{-1}^{1}(1-x^2)P_m' p_n'\,dx = \begin{cases} 0 & \text{when } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{when } m = n \end{cases}$$

Solution We have

$$\int_{-1}^{1}(1-x^2)\,P_m'\,P_n'\,dx = \Big[(1-x^2)\,P_m'\,P_n\Big]_{-1}^{1} - \int_{-1}^{1}P_n\left[\frac{d}{dx}(1-x^2)\,P_m'\right]dx$$

$$= 0 - \int_{-1}^{1}P_n\frac{d}{dx}[(1-x^2)\,P_m']\,dx \qquad \text{[Integrating by parts]}$$

$$= -\int_{-1}^{1}P_n\,[-m\,(m+1)P_m]\,dx$$

$$\left[\because \frac{d}{dx}[(1-x)^2\,P_m'] = -m(m+1)P_m\right] \text{[by Legendre' equation]}$$

$$= m\,(m+1)\int_{-1}^{1}P_n\,P_m\,dx = m(m+1)\,0 = 0 \qquad \text{[by orthogonal property if } m \neq n$$

or $$= \frac{2n(n+1)}{2n+1} \qquad \text{if } m = n]$$

Example 10 Prove that: $\int_{-1}^{1} x^2 \, P_{n+1}(x)\, P_{n-1}(x)\, dx = \dfrac{2n(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$

Solution By the recurrence relation (I),

$$(2n + 1)\, x\, P_n = (n + 1)\, P_{n+1} + n\, P_{n-1} \tag{1}$$

Replacing n by $(n + 1)$ and $(n - 1)$ in (1),

$$(2n + 3)\, x\, P_{n+1} = (n + 2)\, P_{n+2} + (n + 1)\, P_n \tag{2}$$

And $$(2n - 1)x\, P_{n-1} = n \cdot P_n + (n - 1)\, P_{n-2} \tag{3}$$

Multiplying (2) and (3) and integrating from –1 to 1, we get

$$(2n+3)(2n+1) \int_{-1}^{1} x^2 \, P_{n+1}\, P_{n-1}\, dx = n(n+1) \int_{-1}^{1} P_n^2 \, dx + n(n+2) \int_{-1}^{1} P_n \cdot P_{n-2}\, dx$$

$$+ (n^2 - 1) \int_{-1}^{1} P_n \, P_{n-2}\, dx + (n-1)\,(n+2) \int_{-1}^{1} P_{n+2}\, P_{n-2} dx$$

or $$(2n+3)(2n+1) \int_{-1}^{1} x^2 \, P_{n+1}\, P_{n-1}\, dx = n(n+1) \cdot \frac{2}{2n+1} \quad \left[\because \int_{-1}^{1} P_n^2 \, dx = \frac{2}{2n+1}, \int_{-1}^{1} P_m P_n \, dx = 0\right]$$

or $$\boldsymbol{\int_{-1}^{1} x^2 \, P_{n+1}\, P_{n-1} dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}}$$

Example 11 Prove that

$$\int P_n \, dx = \frac{1}{2n+1}\left[P_{n+1} - P_{n-1}\right] + c$$

Solution From the recurrence relation (III), we have

$$(2n + 1)\, P_n = P'_{n+1} - P'_{n-1} \tag{1}$$

Integrating (1) both sides, we have

$$(2n+1) \int P_n \, dx = P_{n+1} - P_{n-1} + c$$

or $$\int P_n \, dx = \frac{P_{n+1} - P_{n-1}}{2n+1} + c$$

or $$\boldsymbol{\int P_n \, dx = \frac{1}{2n+1}\left[P_{n+1} - P_{n-1}\right] + c}$$

EXERCISE 6.2

1. Prove that $P'_{n+1} + P'_n = P_0 + 3P_1 + \dots + (2n + 1)\, P_n$
2. Expand $f(x) = x^2$ in a series of the form $\sum_{n=0}^{\infty} C_n \, P_n(x)$. Find series of Legendre’s polynomial for x^2.

3. Show that $\dfrac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)\,P_n\cdot z^n$

4. Show that $P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$

5. If $m < n$, show that $\int_{-1}^{1} x^m\, P_n(x)\,dx = \dfrac{\overline{|(1/2)}\,\overline{|(n+1}}{2^n\,\overline{|(n+3/2)}} = \dfrac{2^{n+1}(\underline{|n})^2}{\underline{|(2n+1}}$

6. Show that all the roots of $P_n(x) = 0$ are real and lie between -1 and 1.

7. If $m > n - 1$ and n is a positive integer, prove that

$$\int_0^1 x^m P_n(x)\,dx = \frac{m(m-1)(m-2)\cdots(m-n+2)}{(m+n+1)(m+n-1)\cdots(m-n+3)}$$

8. Show that $\sum_{n=0}^{\infty} P_n(x) = \dfrac{1}{\sqrt{2-2x}}$

9. Evaluate (i) $\int_{-1}^{1} x^3 P_4(x)\,dx$, (ii) $\int_{-1}^{1} x^3 P_3(x)\,dx$

10. $\int_{-1}^{1}(1-x^2)\left(\dfrac{d}{dx}P_n\right)^2 dx = \dfrac{2n(2n+1)}{2^n+1}$

11. $\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx = \dfrac{2n}{4n^2-1}$

12. Express in terms of Legendre polynomials

(i) $x^3 + 1$ (ii) $4x^3 - 3x^2 + 2x + 1$ (iii) $2 - 3x + 4x^2$

13. Prove that

$$P_n(x) = P'_{n+1}(x) - 2x\,P'_n(x) + P'_{n-1}(x)$$

14. $(n+1)^2 P_n{}^2 - (x^2-1)\,P_n{}^{-2} = P_0{}^2 + 3p_1{}^2 + 5\,p_2{}^2 + \ldots + (2n+1)\,P_n{}^2$

15. Show that $\int_{-1}^{1} x^4\,P_6(x)\,dx = 0$

16. Prove that the function $y = \dfrac{d^n}{dx^n}(x^2-1)^n$ satisfies the Legendre's differential equation $(1-x^2)\,y'' - 2xy' + n(n+1)y = 0$. Hence, obtain Rodrigues' formula for Legendre' polynomial $P_n(x)$.

17. If $x > 1$, show that $P_n(x) < P_{n+1}(x)$

18. Prove that

(i) $\int_{-1}^{1} P_n(x)\,dx = 2$ if $n = 0$

(ii) $\int_{-1}^{1} P_n(x)\,dx = 0$ if $n \ge 1$

Answers

2. $x_2 = C_0P_0(x) + C_1P_1(x) + C_2P_2(x) = (1/3)\,P_0(x) + \left(\frac{2}{3}\right) P_2(x)$

9. (i) 0 (ii) $\frac{4}{35}$

12. (i) $\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) + P_0(x)$

(ii) $\frac{8}{5} P_3(x) - 2P_2(x) + \frac{22}{5} P_1(x)$

(iii) $\frac{10}{3} P_0(x) - 3P_1(x) + \frac{8}{3} P_2(x)$

6.14 BETA FUNCTION

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for $m > 0$, $n > 0$ is called the Beta function and is denoted by $\beta(m, n)$.

Thus, $$\boxed{\boldsymbol{\beta(m, n) = \int_0^1 x^{m+1}\,(1-x)^{n-1}\,dx,\ m > 0,\ n > 0}}$$

Beta function is also called the Eulerian integral of the first kind.

6.14.1 Symmetry of Beta Function: $\beta(m, n) = \beta(n, m)$

We have

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \beta(n, m)$$

Hence, $\boldsymbol{\beta(m, n) = \beta(n, m)}$

6.14.2 Transformation of Beta Function

(i) $\boldsymbol{\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta}$

Proof We have

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \{\text{Putting } x = \sin^2\theta, \ \therefore \ dx = 2\sin\theta\cos\theta\, d\theta\}$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

(ii) $\boldsymbol{\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx}$

Proof We know that

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Put $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2}dy$

Also, when $x = 0$, $y = \infty$
And when $x = 1$, $y = 0$

Thus, $$\beta(m, n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left[1 - \frac{1}{1+y}\right]^{n-1}\left[-\frac{1}{(1+y)^2}\right]dy$$

$$= \int_\infty^0 \frac{-y^{n-1}}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx \qquad \text{[By property of definite integral]}$$

Since beta function is symmetrical in m and n, we have

$$\beta(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-n}}dx$$

But $\quad \beta(m, n) = \beta(n, m)$

$$\therefore \qquad \boxed{\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$$

6.15 GAMMA FUNCTION

The definite integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$, for $n > 0$ is called the Gamma function and is denoted by $\overline{|(n)}$

$$\text{Thus,} \qquad \overline{|(n)} = \int_0^\infty e^{-x}\, x^{n-1}\, dx, \quad \text{for} \quad n > 0$$

Gamma function is also called the Eulerian integral of the second kind.

Reduction Formula for $\overline{|(n)}$

By definition of Gamma function,

$$\overline{|(n+1)} = \int_0^\infty e^{-x}\, x^{(n+1)-1}\, dx, = \int_0^\infty x^n\, e^{-x}\, dx$$

$$= \left[-e^{-x}\, x^n\right]_0^\infty + \int_0^\infty e^{-x}\, n\, x^{n-1}\, dx$$

$$= 0 + \lim_{x\to\infty} \frac{x^n}{e^x} + n\int_0^\infty e^{-x}\, x^{n-1}\, dx \tag{35}$$

$$\text{Now,} \qquad \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^n}{1 + x + \frac{x^2}{\underline{|2}} + \cdots + \frac{x^n}{\underline{|n}} + \cdots}$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^n} + \frac{1}{x^{n-1}} + \cdots + \frac{1}{\underline{|n}} + \frac{x}{\underline{|n+1}} + \cdots} = 0$$

$$\therefore \quad \text{Eq. (35) becomes,} \quad \overline{|(n+1} = 0 + n\int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$\therefore \qquad \boxed{\overline{|(n+1)} = n\overline{|(n)}}$$ [which is the reduction formula for $\overline{|n}$]

Now, if n is a positive integer, we get

$$\overline{|(n+1)} = n\,\overline{|(n)}$$

$$= n(n-1)\,\overline{|(n-1)}$$ [By repeated application of reduction formula]

$$= n(n-1)\,(n-2)\,\Gamma(n-2)$$

$$= n(n-1)\,(n-2)\,\ldots\,3.2.1\;\Gamma(1)$$

But $$\Gamma 1 = \int_0^\infty e^{-x}\, x^{1-1}\, dx = \int_0^\infty e^{-x}\, dx = 1$$

Hence, $$\Gamma(n+1) = n(n-1)\,(n-2)\,\ldots\,3.2.1$$

$$= \underline{|n}$$

Hence, $\boxed{\Gamma(n+1) = \underline{|n}}$ when n is a positive integer.

6.16 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

Proof We know that

$$\Gamma(n) = k^n \int_0^\infty e^{-kx}\, x^{n-1}\, dx \tag{36}$$

Replacing k by z in (36), we have

$$\Gamma(n) = z^n \int_0^\infty e^{-zx}\, x^{n-1}\, dx$$

Multiplying both sides by $e^{-z}\, z^{m-1}$, we get

$$\Gamma n\, e^{-z}\, z^{m-1} = \int_0^\infty z^n\, e^{-zx}\, x^{n-1}\, e^{-z}\, z^{m-1}\, dx$$

$$= \int_0^\infty z^{n+m-1}\, e^{-z(1+x)}\, x^{n-1}\, dx$$

Integrating both sides w.r.t. z from 0 to ∞,

$$\Gamma(n) \int_0^\infty e^{-z}\, z^{m-1}\, dz = \int_0^\infty \left[\int_0^\infty z^{n+m-1}\, e^{-z(1+x)}\, x^{n-1}\, dx \right] dz$$

$$\Gamma(n)\,\Gamma(m) = \int_0^\infty x^{n-1} \frac{\Gamma(m+n)}{(1+x)^{m+n}}\, dx$$

$$\Gamma(n)\,\Gamma(m) = \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\Gamma(n)\,\Gamma(m) = \Gamma(m+n)\,\beta(m, n) \qquad \left[\because\ \beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx\right]$$

$$\therefore \qquad \boldsymbol{\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Example 12 Prove that

(i) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(ii) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

(iii) $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ where $0 < n < 1$

Solution

(i) We have

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Putting $m = n = \frac{1}{2}$, we get

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \left[\Gamma(1/2)\right]^2 \qquad [\because\ \Gamma(1) = 1]$$

or

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

Now, putting $x = \sin^2\theta$, i.e., $dx = 2\sin\theta\cos\theta\,d\theta$, when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \pi/2$.

We have

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^\pi \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cdot\cos\theta\,d\theta$$

$$\left[\Gamma\frac{1}{2}\right]^2 = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = \pi$$

$$\therefore \qquad \boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

(ii) Let $$I = \int_0^\infty e^{-x^2}\, dx \qquad (1)$$

Putting $x^2 = t$, i.e., $2x\, dx = dt$ in (1),

i.e., $$dx = \frac{1}{2} t^{-1/2}\, dt$$

We have

$$I = \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/2}\, dt \qquad [\because \text{ when } x = 0,\ t = 0 \text{ and when } x = \infty,\ t = \infty]$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\therefore \qquad \boldsymbol{\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$$

(iii) $$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \quad \text{where } 0 < n < 1$$

We have $$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

and $$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad \text{where } m > 0, n > 0$$

$$\therefore \qquad \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Putting $m + n = 1$ or $m = 1 - n$ in the above relation, we get,

$$\frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{1+x}\, dx \text{ where } 0 < n < 1 \qquad [\because \quad m > 0 \Rightarrow (1-n) > 0 \Rightarrow n < 1]$$

But $$\Gamma(1) = 1, \text{ also } \int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\,\pi}$$

$$\therefore \qquad \boldsymbol{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\,\pi}}, \text{ where } 0 < n < 1$$

6.17 DUPLICATION FORMULA

To show that $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$ where $m > 0$.

Proof We know that $\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$, where $m > 0, n > 0$

If we take $n = m$ then $\beta(m, m) = \frac{\left[\Gamma(m)\right]^2}{\Gamma(2m)}$ (37a)

Again, by definition of Beta function, we have

$$\beta(m, m) = \int_0^1 x^{m-1}\,(1-x)^{m-1}\,dx \tag{37b}$$

Putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$, we get

$$\beta(m, m) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,\cos^{2(m-1)}\theta\; 2\sin\theta\cos\theta\,d\theta$$

[when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \pi/2$]

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2m-1}\theta\,d\theta = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1} y\cdot\frac{dy}{2}, \qquad \text{Putting } 2\theta = y \Rightarrow d\theta = \frac{1}{2}\,dy$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1} y\,dy = \frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} y\cdot dy$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} y\,\cos^{\circ} y\,dy$$

$$= \frac{1}{2^{2m-1}}\,\frac{\Gamma 1/2\,(2m-1+1)\,\Gamma 1/2\,(0+1)}{2\Gamma 1/2\,(2m-1+0+2)}$$

$$= \frac{1}{2^{m-1}}\,\frac{\Gamma(m)\,\Gamma(1/2}{\Gamma\left(m+\frac{1}{2}\right)} = \frac{1}{2^{2m-1}}\cdot\frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because\ \Gamma(1/2 = \sqrt{\pi}\right]$$

Equating the two values of $\beta(m, m)$ obtained in (37a) and (37b), we get

$$\frac{\left[\Gamma(m)\right]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m) \cdot \sqrt{\pi}}{\Gamma\left(m + \frac{1}{2}\right)}$$

or $$\boxed{\boldsymbol{\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}}$$

Evaluate the following integrals,

(i) $\int_0^\infty e^{-ax} \cos bx \cdot x^{m-1}\, dx$

(ii) $\int_0^\infty e^{-ax} \sin bx \cdot x^{m-1}\, dx$

We have $$\int_0^\infty e^{-ax} e^{ibx} x^{m-1}\, dx = \int_0^\infty e^{-(a-ib)x} x^{m-1}\, dx = \frac{\Gamma(m)}{(a-ib)^m}$$

$$= (a - ib)^{-m}\, \Gamma(m) \qquad (38)$$

First of all, separate $(a - ib)^{-m}$ into real and imaginary parts putting $a = k \cos \theta$ and $b = k \sin \theta$ so that $\theta = \tan^{-1} \frac{b}{a}$ and $k = \sqrt{a^2 + b^2}$

Then $(a - ib)^{-m} = [k(\cos \theta - (\sin \theta)]^{-m} = k^{-m}(\cos \theta - i \sin \theta)^{-m}$

$= k^{-m} (\cos m\theta + i \sin m\theta)$ (by De-Moivre's theorem)

By (38), we have $\int_0^\infty e^{-ax} e^{ibx} x^{m-1}\, dx = k^{-m} (\cos m\theta + i \sin m\,\theta)\Gamma m$ $\quad [\because \quad e^{i\theta} = \cos \theta + \sin \theta]$

or $$\int_0^\infty e^{-ax} (\cos bx + i \sin bx)\, x^{m-1}\, dx = \frac{\Gamma(m)}{k^m} (\cos m\theta + i \sin m\,\theta)$$

or $$\int_0^\infty e^{-ax} \cos bx \cdot x^{m-1}\, dx + i \int_0^\infty e^{-ax} \sin bx \cdot x^{m-1}\, dx = \frac{\Gamma(m)}{k^m} \cos m\theta + i\, \frac{\Gamma(m)}{k^m} \sin m\theta \qquad (39)$$

Equating real and imaginary parts in (2), we have

$$\int_0^\infty e^{-ax} \cos bx \cdot x^{m-1}\, dx = \frac{\Gamma(m)}{k^m} \cos m\theta$$

and $$\int_0^\infty e^{-ax} \sin bx \cdot x^{m-1}\, dx = \frac{\Gamma(m)}{k^m} \sin m\theta$$

where $k = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

Some Transformations of Beta Function

(i) Prove $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \dfrac{\overline{|(m)}\,\overline{|(n)}}{2\overline{|(m+n)}}$

Proof We know that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ (40a)

and $\beta(m, n) = \dfrac{\overline{|(m)}\,\overline{|(n)}}{\overline{|(m+n)}}$ (40b)

From (40a) and (40b), we get

$$\frac{\overline{|(m)}\,\overline{|(n)}}{\overline{|(m+n)}} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\therefore \quad \boxed{\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\overline{|(m)}\,\overline{|(n)}}{2\overline{|(m+n)}}}$$ **Proved.**

(ii) Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \dfrac{\overline{\left|\left(\frac{p+1}{2}\right)\right.}\,\overline{\left|\left(\frac{q+1}{2}\right)\right.}}{2\overline{\left|\left(\frac{p+q+2}{2}\right)\right.}}$

Proof We have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \beta(m, n)$$

or

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\overline{|(m)}\,\overline{|(n)}}{\overline{|(m+n)}}$$

Putting $2m - 1 = p \quad 2n - 1 = q$, i.e., $m = \frac{1}{2}(p+1)$ and $n = \frac{1}{2}(q+1)$

We get

$$\boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\overline{\left|\left(\frac{p+1}{2}\right)\right.}\,\overline{\left|\left(\frac{q+1}{2}\right)\right.}}{\overline{\left|\left(\frac{p+q+2}{2}\right)\right.}}}$$

Example 13 Evaluate (i) $\int_0^{\infty} \dfrac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

(ii) $\int_0^{\infty} \dfrac{x^4(1-x^5)}{(1+x)^{15}}\,dx$

Solution (i) Let $I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9, 15) - \beta(15, 9)$$

$$= 0 \qquad \because \ \beta(m, n) = \beta(n, m)$$

(ii) Let $I = \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= \beta(5, 10) + \beta(10, 5) = 2\beta(5, 10)$$

$$= \frac{2\,\Gamma(5)\,\Gamma(10)}{\Gamma(5+10)} = \frac{2\cdot \lfloor 4 \, \lfloor 9}{\lfloor 15}$$

$$= \frac{1}{5005}$$

Example 14 Show that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Solution Let $I = \int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$

Putting $x^4 = \sin^2\theta$, i.e., $x = \sin^{1/2}\theta$, so that $dx = \frac{1}{2}\sin^{-1/2}\theta\cos\theta\, d\theta$, when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\frac{1}{2}\sin^{-1/2}\theta\cos\theta\, d\theta}{\cos\theta} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\cos^0\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\,\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\cdot\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\pi \Big/ \left(\sin \frac{1}{4}\pi\right)} = \frac{\sqrt{\pi}}{4} \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\pi \Big/ \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{8\sqrt{\pi}} \cdot \left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Example 15 Prove that

(i) $\displaystyle\int_0^\infty \frac{\sin b \cdot z}{z}\, dz = \frac{\pi}{2}$

(ii) $\displaystyle\int_0^2 (8 - x^3)^{-1/3}\, dx = \frac{2\pi}{3\sqrt{3}}$

Solution (i) Let $I = \displaystyle\int_0^\infty \int_0^\infty e^{-xz} \sin bz\, dx\, dz$ (1)

$$= \int_0^\infty \left[\frac{e^{-xz}}{-z}\right]_0^\infty \sin bz\, dz \qquad \text{(Integrating w.r.t. } x\text{)}$$

$$= \int_0^\infty \frac{\sin bz\, dz}{z} \tag{2}$$

Integrating (1) first w.r.t. z, we get

$$I = \int_0^\infty \left[\int_0^\infty e^{-xz} \sin bz \cdot dz\right] dx$$

$$= \int_0^\infty \frac{b}{b^2 + x^2}\, dx = \left[\tan^{-1} \frac{x}{b}\right]_0^\infty = \frac{\pi}{2} \tag{3}$$

From (2) and (3), we get

$$\boldsymbol{\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}}$$

(ii) Let $I = \displaystyle\int_0^2 (8 - x^3)^{-1/3}\, dx = \int_0^2 x^{-2}\, (8 - x^3)^{-1/3}\, x^2\, dx$

Putting $x^3 = 8y \Rightarrow 3x^2\, dx = 8dy$

Hence, $\quad I = \displaystyle\int_0^1 (8y)^{-2/3} (8 - 8y)^{-1/3} \cdot \frac{8}{3}\, dy$

$$= \int_0^1 8^{-2/3} y^{-2/3} \cdot 8^{-1/3}\, (1 - y)^{-1/3}\, \frac{8}{3}\, dy$$

$$= \frac{1}{3}\int_0^1 y^{(1/3)-1}\,(1-y)^{(2/3)-1}\,dy = \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{1}{3}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{3}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)} = \frac{1}{3}\cdot\frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{\pi}{3}\cdot\frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

Example 16 Show that $\displaystyle\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \frac{\pi}{\sqrt{2}}$

Solution

$$\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\tan\left(\frac{\pi}{2}-\theta\right)}\,d\theta = \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta \qquad \text{[By property of definite integral]}$$

$$= \int_0^{\pi/2}\cos^{1/2}\theta\cdot\sin^{-1/2}\theta\,d\theta = \frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\dfrac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)}$$

$$= \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}} \qquad \left[\because \Gamma n\cdot\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

Example 17 Show that $\displaystyle\int_0^1 x^p(1-x^q)^r\,dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$

Solution Putting $x^q = y$, $qx^{q-1}\,dx = dy$ or $q\cdot y^{\left(\frac{q-1}{q}\right)}\,dx = dy$

$$\int_0^1 x^p(1-x^q)^r dx = \int_0^1 y^{(p/q)}(1-y)^r\,\frac{1}{qy^{\left(\frac{q-1}{q}\right)}}\,dy$$

$$= \frac{1}{q}\int_0^1 y^{(p-q+1)/q}(1-y)^r\ dy = \frac{1}{q}\,\beta\left(\frac{p-q+1}{q}+1, r+1\right)$$

$$= \frac{1}{q}\,\beta\left(\frac{p+1}{q}, r+1\right)$$

Example 18 Prove that $\dfrac{\beta(m+1), n)}{\beta(m,n)} = \dfrac{m}{m+n}$

Solution By the symmetry of Beta function,

$$\beta(m+1, n) = \beta(n, m+1)$$

$$= \int_0^1 x^{n-1}(1-x)^{(m+1)-1}\ dx = \int_0^1 (1-x)^m\ x^{n-1}\ dx$$

$$= \left[(1-x)^m\ \frac{x^n}{n}\right]_0^1 - \int_0^1 m(1-x)^{m-1}(-1)\,\frac{x^n}{n}\,dx \qquad \text{[Integrating by parts]}$$

$$= \frac{m}{n}\int_0^1 x^{n-1}\,[1-(1-x)](1-x)^{m-1}\ dx \qquad \left\{\because \left[1-x)^m\ \frac{x^n}{n}\right]_0^1 = 0\right\}$$

$$= \frac{m}{n}\left[\int_0^1 x^{n-1}(1-x)^{m-1}dx - \int_0^1 x^{n-1}(1-x)^m\ dx\right]$$

$$= \frac{m}{n}\,[\beta(n,m)-\beta(n,m+1)] = \frac{m}{n}\,\beta(m,n)\ \beta(m+1,n)$$

$$\Rightarrow \quad \left(1+\frac{m}{n}\right)\beta(m+1,n) = \frac{m}{n}\,\beta(m,n)$$

$$\Rightarrow \quad \frac{\beta(m+1,n)}{\beta(m,n)} = \frac{m}{m+n}$$

Example 19 Show that $\displaystyle\int_0^{\pi/2}\sqrt{\sin\theta}\ d\theta \times \int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Solution

$$\text{LHS} = \int_0^{\pi/2}\sqrt{\sin\theta}\ d\theta \times \int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2}\sin^{1/2}\cdot\cos^0\ \theta\ d\theta \times \int_0^{\pi/2}\sin^{-1/2}\ \theta\cdot\cos^0\ \theta\ d\theta$$

$$= \frac{\Gamma\left[\frac{1}{2}\left(\frac{1}{2}+1\right)\right]\Gamma\left[\frac{1}{2}(0+1)\right]}{2\,\Gamma\left\{\frac{1}{2}\left[\frac{1}{2}+0+2\right]\right\}} \times \frac{\Gamma\left[\frac{1}{2}\left(-\frac{1}{2}+1\right)\right]\Gamma\left[\frac{1}{2}(0+1)\right]}{2\,\Gamma\left\{\frac{1}{2}\left[-\frac{1}{2}+0+2\right]\right\}}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\cdot\sqrt{\pi}\,\sqrt{\pi}}{2.2\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi$$

EXERCISE 6.3

1. Show that $\beta(m, n) = \frac{\underline{|m-1}\cdot\underline{|n-1}}{\underline{|m+n-1}}$ for m, n positive integer.

2. Evaluate

   (i) $I = \int_0^\infty x^4\, e^{-x^4}\, dx$ (ii) $\int_0^{\pi/2} \sin^6\theta \cos^7\theta\, d\theta$

3. Evaluate

   (i) $\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta$ (ii) $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$

4. Show that $\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}} = \frac{1}{n}\cdot\frac{\pi}{\sin(\pi/n)}$

5. Prove that $\int_0^1 e^{-ax}\, x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$

6. Show that $\int_0^\infty \frac{x^c}{e^x}\, dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 1$

7. Evaluate

   (i) $\Gamma(4.5)$ (ii) $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$ (iii) $\beta(2.5, 1/5)$

8. Evaluate

   (i) $\int_0^\infty (x)^{1/4}\, e^{-\sqrt{(x)}}\, dx$ (ii) $\int_0^1 (x)^{1/2}\,(1-x^2)^{1/3}\, dx$

9. (i) $\Gamma(0.1)\,\Gamma(0.2)\,\Gamma(0.3)\cdots\Gamma(0.9) = \frac{(2\pi)^{9/2}}{\sqrt{10}}$; provided $-1 < 2x < 1$

   (ii) $\Gamma\left(\frac{3}{4}-x\right)\Gamma\left(\frac{3}{4}+x\right) = \left(\frac{1}{4}-x^2\right)\pi \sec \pi x$

10. Prove that

(i) $$\int_0^1 \{(x-3)(7-x)\}^{1/4}\, dx = \frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$$

(ii) $$\int_0^{\pi/2} \left[\sqrt{\tan}\ \theta + \sqrt{\sec}\ \theta\right] d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi}\Big/\Gamma\left(\frac{3}{4}\right)\right\}$$

11. Show that $$\int_0^{\pi/2} \frac{d\theta}{(a\cos^4\theta + b + \sin^4\theta)^{1/2}} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4(ab)^{1/4}\sqrt{\pi}}$$

12. Assuming $\Gamma(n)\Gamma(1-n) = \pi \operatorname{cosec} n\pi, 0 < n < 1$, then show that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi} : 0 < p < 1$

13. Evaluate

(i) $\displaystyle\int_0^\infty \cos x^2\, dx$ (ii) $\displaystyle\int_{-\infty}^\infty \cos\frac{\pi}{2}x^2\, dx$ (iii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

14. Show that

(i) $\displaystyle\int_0^\infty x^6 e^{-2x}\, dx = \frac{45}{8}$ (ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

15. Prove that

(i) $$\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$$

(ii) $\displaystyle\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^2\, \underline{|n}}{(m+1)^{n+1}}$ where n is a positive integer and $m > -1$

16. Prove that

$$\int_0^\infty \sqrt{y}\, e^{-y^2}\, dx \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dx = \frac{\pi}{2\sqrt{2}}$$

Answers

2. (i) $\dfrac{1}{4}\Gamma\left(\dfrac{5}{4}\right)$ (ii) $\dfrac{2^4}{3\cdot 7\cdot 11\cdot 13}$

3. (i) $\dfrac{\pi}{\sqrt{2}}$ (ii) $\Gamma(n)$

7. (i) 0.1964 (ii) 0.1227 (iii) 11.629

8. (i) $\frac{3}{2}\sqrt{\pi}$ (ii) $\dfrac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{2\Gamma\frac{7}{12}}$

13. (i) $\frac{1}{2}\sqrt{\frac{\pi}{2}}$ (ii) 1 (iii) $\dfrac{1}{8\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2$

SUMMARY

1. Bessel's Equation

The differential equation of the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y = 0 \text{ or } x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-y^2)\,y=0$$

is called Bessel's equation of order 2, where n is a non-negative constant.

2. Recurrence Formulae/Relations of Bessel's Equation

(i) $\frac{d}{dx}\{x^n\, J_n(x)\} = x^n\, J_{n-1}(x)$

(ii) $\frac{d}{dx}\{x^{-n}\, J_n(x)\} = -x^{-n}\, J_{n-1}(x)$

(iii) $J'_n(x) = J_{n-1}(x) - \left(\frac{n}{x}J_n\right)$ or $xJ'_n = -nJ_n + xJ_{n-1}$

(iv) $J'_n(x) = \left(\frac{n}{x}\right)J_n(x) - J_{n+1}(x)$ or $x\,J'_n = nJ_n - xJ_{n+1}$

(v) $2J'_n(x) = J_{n-1} - J_{n+1}$

(v) $J_{n-1}(x) + J_{n+1}(x) = \left(\frac{2n}{x}\right)J_n(x)$

3. Generating Function of $J_n(x)$

The function $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ is called the generating function. When n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ in ascending and descending power of z.

Also, $J_n(x)$ is the coefficient of z^{-n} multiplied by (–1)n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$, i.e.,

(i) $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n+-\infty}^{\infty} z^n\, J_n(x)$

(ii) $$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n+-\infty}^{\infty} (-1)^n \; J_n(x)\, z^{-n}$$

4. Legendre Polynomial

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is known as Legendre's differential equation or Legendre's equation where n is a positive integer.

5. Rodrigues' Formula

$$\boxed{P_n(x) = \frac{1}{2^n \underline{|n}}\frac{d^n}{dx^n}(x^2-1)^n}$$

6. Orthogonal Properties of Legendre's Polynomial

(i) $$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0 \text{ if } m \neq n$$

(ii) $$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}, \text{ if } m = n$$

7. Recurrence Formulae for $P_n(x)$

(i) $(2n + 1)\, xP_n = (n + 1)\, P_{n+1} + n\, P_{n-1}$

or $n\, P_n = (2n - 1)\, x\, P_{n-1} - (n - 1)\, P_{n-2}$, $n \geq 2$

(ii) $n\, P_n = x\, P'_n - P'_{n-1}$

(iii) $(2n + 1)\, P_n = P'_{n+1} - P'_{n-1}$

(iv) $(n + 1)\, P_n = P'_{n+1} - xP'_n$

(v) $n(P_{n-1} - xP_n) = (1 - x^2)\, P'_n$

or $(x^2 - 1)\, P'_n = n(xP_n - P_{n-1})$

(vi) $(1 - x^2)\, P'_n = (n + 1)(xP_n - P_{n+1})$

8. Beta Function

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ for $m > 0$, $n > 0$ is called the Beta function and is denoted by $\beta(m, n)$.

Thus, $$\boxed{\beta(m,n) = \int_0^1 x^{m+1}\,(1-x)^{n-1}\, dx, m > 0, n > 0}$$

The Beta function is also called the Eulerian integral of the first kind.

9. Gamma Function

The definite integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$, for $n > 0$ is called the Gamma function and is denoted by $\Gamma(n)$.

Thus, $$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx, \quad \text{for} \quad n > 0$$

The Gamma function is also called Eulerian integral of the second kind.

10. Some Important Formulae/Relations

(i) $\Gamma(n+1) = n\,\Gamma(n)$ and $\boxed{\Gamma n+1 = \lfloor\underline{n}}$ when n is a positive integer.

(ii) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(iii) $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

(iv) $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, where $0 < n < 1$

(v) $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$

(vi) $\int_0^{\pi/2} \sin^{p}\theta \cos^{q}\theta\, d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\frac{p+q+2}{2}}$

(iv) Duplication Formula: $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2m-1}\Gamma(2m)$, where $m > 0$.

(v) Relation between Beta and Gamma Functions:

$\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

OBJECTIVE-TYPE QUESTIONS

1. Γ_n is equal to

(a) $\int_0^\infty e^{-x}x^{n-1}dx, n > 0$

(b) $\int_0^\pi e^{-x}x^{n-1}dx, n > 0$

(c) $\int_0^{\pi/2} e^{-x}x^{n-1}dx, n > 0$

(d) $\int_0^{\pi/4} e^{-x}\, dx, n > 0$

2. $\Gamma(1/2)$ is equal to

(a) π (b) $\pi/2$

(c) $\sqrt{\pi}$ (d) $\sqrt{\left(\frac{\pi}{2}\right)}$

3. The value of $\Gamma(-3/2)$ is

(a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{4\sqrt{\pi}}{3}$

(c) $\frac{-4\sqrt{\pi}}{2}$ (d) $-\frac{\sqrt{\pi}}{2}$

4. Beta function which is denoted by $\beta(m, n)$ is defined by the integral

(a) $\int_0^1 x^{m-1}(1-x)^{n+1}dx$

(b) $\int_0^1 x^{m-1}(1-x)^{n-1}dx$

(c) $\int_0^1 x^{m+1}(1-x)^{n-1}dx$

(d) $\int_0^1 x^{m+1}(1-x)^{n+1}dx$

5. The value of $\int_0^1 x^{11}(1-x)^{16}dx$ is

(a) $\beta(5, 7)$ (b) $\beta(6, 8)$

(c) $\beta(11, 15)$ (d) $\beta(15, 15)$

6. The value of $\iint_D x^{l-1}y^{m-1}dx\,dy$ where D is the domain $x \le 0$, $y \le 0$, and $x + y \le 1$ is

(a) $\frac{\Gamma l \Gamma m}{\Gamma l+m+1}$ (b) $\frac{\Gamma l \Gamma m}{\Gamma l+m}$

(c) $\frac{\Gamma l+m}{\Gamma l \Gamma m}$ (d) $\frac{\Gamma l+m+1}{\Gamma l \Gamma m}$

7. $\frac{\beta(m+1, n)}{\beta(m, n)}$ is equal to

(a) $\frac{m}{n}$ (b) $\frac{m+1}{n}$

(c) $\frac{m-1}{n}$ (d) $\frac{m}{m+n}$

8. For all positive values of $n > 0$, the value of $\overline{|(n+1)}$ is

(a) $n!$ (b) $n\Gamma n$

(c) Γn (d) $(n-1)!$

9. The value of $\int_0^\infty e^{-x^2}dx$ is equal to

(a) $\frac{\pi}{2}$ (b) $\sqrt{\left(\frac{\pi}{2}\right)}$

(c) $\sqrt{\pi}/2$ (d) $\sqrt{\pi}$

10. The value of $\int_0^\infty \frac{dx}{1+x^4}$ is

(a) $\frac{\pi}{4}$ (b) $\pi\sqrt{2}$

(c) $\frac{\pi\sqrt{2}}{4}$ (d) $\frac{\pi\sqrt{2}}{2}$

11. The value of $\beta(1, n) + \beta(m, 1)$ is

(a) $m + n$ (b) $m - n$

(c) $\frac{m-n}{mn}$ (d) $\frac{m+n}{mn}$

ANSWERS

1. (a)	2. (c)	3. (b)	4. (b)	5. (d)	6. (a)	7.(d)	8. (b)	9. (c)	10. (c)
11. (d)									

7 Vector Differential and Integral Calculus

7.1 INTRODUCTION

Vector calculus (or analysis) is a branch of mathematics concerned with differentiation and integration of vector fields. Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, fluid flow, and gravitational fields. In this chapter, we shall study the vector differential and integral calculus. We often call this study as *vector field theory* or *vector analysis*. We first introduce the some concepts.

(i) Scalar Function

A scalar function $f(x, y, z)$ is a function defined at each point in a certain domain D in space. Its value is real and depends only on the point $P(x, y, z)$ in space, but not on any particular coordinate system being used, for example, the distance function in 3D space which defines the distance between the points $P(x, y, z)$ and $Q(x_0, y_0, z_0)$, then

$$D = f(P) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \text{ defines a scalar field.}$$

(ii) Vector Function

A function $\vec{F} = \vec{F}(P) = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ defined at each point $P \in D$ is called a vector function. We say that a vector field is defined in D. In Cartesian coordinates, we can write

$$\vec{F} = F_1(z, y, z)\,\hat{i} + F_2(z, y, z)\,\hat{j} + F_3(z, y, z)\,\hat{k}$$

If the scalar and vector fields depend on time also then we denote them as $f(P, t)$ and $\vec{F}(P, t)$, respectively. Both the fields are independent of the choice of the coordinate systems.

(iii) Level Surfaces

If $f(x, y, z)$ be a single-valued continuous scalar function defined at every point $P(x, y, z)$ in the domain D. Then $f(x, y, z) = c$, a constant defines the equation of a surface and is called *a level surface of the function*. For the different values of C, we obtain the different surfaces, no two of which intersect.

7.2 PARAMETRIC REPRESENTATION OF VECTOR FUNCTIONS

The two-dimensional curve C can be parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$. Then, the position vector of a point P on the curve C can be written as follows:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

Similarly, a three-dimensional curve can be parametrized as

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}; \qquad a \le t \le b$$

Parametric Forms of the Standard Curves

(i) Straight Line $\vec{r}(t) = \vec{a} + \vec{b}t$

The parametric form is

$$\vec{r}(t) = t\,\hat{i} + (1-t)\,\hat{j} \qquad 0 \le t \le 1$$

(ii) Circle $x^2 + y^2 = a^2$, the parametric form of the circle is

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}; \qquad 0 \le t \le 2\pi$$

(iii) Parabola Consider the parabola $y^2 = 4ax$.
Then we assume $y = t$ as a parameter and the parametric form of the parabola as

$$\vec{r}(t) = \left(\frac{t^2}{4a}\right)\hat{i} + t\,\hat{j}; \qquad -\infty < t < \infty$$

(iv) Ellipse The parametric form of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be written as

$$\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j}; \qquad 0 \le t \le 2\pi$$

7.3 LIMIT, CONTINUITY, AND DIFFERENTIABILITY OF A VECTOR FUNCTION

A vector function $\vec{F}(t)$ is said to a limit 'l', when t tends to t_0, if for any given positive number $\in$, however small, there corresponds a positive number δ such that

$$|\vec{F}(t) - l| < \varepsilon, \text{ whenever } |t - t_0| < \delta$$

If $\vec{F}(t)$ tends to a limit 'l' as t tends to t_0,

$$\therefore \qquad \lim_{t \to t_0} \vec{F}(t) = l$$

A vector point function $\vec{F}(t)$ is said to be continuous for a value t_0 of t if $\vec{F}(t_0)$ is defined and for any given positive number ε, however small, there exists a positive number δ such that

$$|\vec{F}(t) - \vec{F}(t_0)| < \varepsilon, \text{ whenever } |t - t_0| < \delta.$$

A vector function $\vec{F}(t)$ is said to be differentiable at $t = t_0$, if the limit $\lim\limits_{t_0 \to 0} \frac{\overline{F}(t+t_0) - \overline{F}(t)}{t_0}$ exists.

If the limit exists then it is denoted by $\vec{F}'(t)$ or $\dfrac{d\vec{F}}{dt}$. In Cartesian coordinates, the component of $\vec{F}(t)$, $\vec{F}_1(t)$, $\vec{F}_2(t)$, and $\vec{F}_3(t)$ are differentiable at t_0, i.e., $\dfrac{d\overline{F}}{dt} = \overline{F}'(t) = \overline{F}_1'(t)\hat{i} + \overline{F}_2'(t)\hat{j} + \overline{F}_3(t)\hat{k}$. Let $\vec{F}(t) = \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the parametric representations.

Then $$\frac{d\overline{r}}{dt} = \overline{r}'(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$$

7.4 GRADIENT, DIVERGENCE, AND CURL

7.4.1 Gradient of a Scalar Field

The gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change.

The gradient of a scalar field 'f' is denoted by grad f or $\vec{\nabla} f$ and is defined as

$$\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Note The del operator $\vec{\nabla}$ operates on a scalar field and produces a vector field.

Geometrical Representation of the Gradient

Let $f(P) = f(x, y, z)$ be a differentiable scalar field. Let $f(x, y, z) = \lambda$ be a level surface and P_0 be a point on it. There are infinite numbers of smooth curves on the surface passing through the point P_0. Each of these curves has a tangent at P_0. A vector normal to this plane at P_0 is called the *normal vector* to the surface at this point.

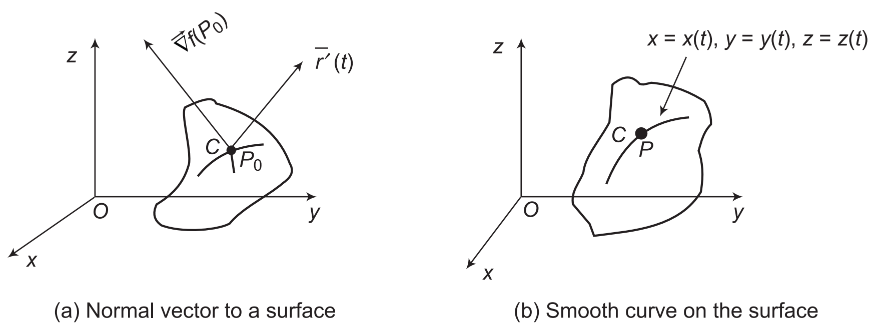


Fig. 7.1

Consider $\overline{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ be the position vector on the curve C at the point P. Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = \lambda$$

Then

$$\frac{df}{dt} = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

or
$$\left(\hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}\right)\cdot\left(\hat{i}\,\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\,\frac{dz}{dt}\right) = 0$$

or
$$\vec{\nabla}f \cdot \vec{r}'(t) = 0$$

Let $\vec{\nabla}f(P) \neq 0$ and $\vec{r}'(t) \neq 0$. Now, $\vec{r}'(t)$ is a tangent vector to C at P and lies in the tangent plane to the surface at P. Hence, $\vec{\nabla}f$ is orthogonal to every tangent vector at P.

$\therefore$ $\vec{\nabla}f$ is the vector normal to the surface $f(x, y, z) = \lambda$ at the point P.

Note **The unit normal vector** $\boldsymbol{\hat{n} = \dfrac{\textbf{grad } f}{|\textbf{grad } f|}}$

7.4.2 Divergence of a Vector Field

Let $\vec{V} = V_1(x, y, z)\,\hat{i} + V_2(x, y, z)\,\hat{j} + V_3(x, y, z)\,\hat{k}$ be a differential vector function, where x, y, z are Cartesian coordinates, and V_1, V_2, V_3 are the components of $\vec{V}$. Then the divergence of $\vec{V}$ is denoted by div $\vec{V}$ or $\vec{\nabla}\cdot\vec{V}$ and is defined as

$$\text{div } \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Another common notation for the divergence of $\vec{V}$ is

$$\text{div } \vec{V} \text{ or } \vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}\right]$$

$$\textbf{div } \vec{V} = \frac{\partial \boldsymbol{V_1}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{V_2}}{\partial \boldsymbol{y}} + \frac{\partial \boldsymbol{V_3}}{\partial \boldsymbol{z}}$$

Hence, the divergence of a vector function is a scalar function.

Note Let $\vec{V}$ denote the velocity of a fluid in a medium.

If div $\vec{V} = 0$ then the fluid is said to be *incompressible*. In electromagnetic theory, if div $\vec{V} = 0$ then the vector field $\vec{V}$ is said to be *solenoidal*.

7.4.3 Physical Interpretation of Divergence

Suppose that there is a fluid motion whose velocity at any point is $\vec{V}(x, y, z)$. Then the loss of fluid per unit volume per unit time in a small parallelepiped having centre at the point P and edges parallel to the co-ordinate axes and having lengths dx, dy, dz respectively, is given by

$$\text{div } \vec{V} = \vec{\nabla}\cdot\vec{V}$$

Let $$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}.$$

x-component of velocity V at $P = V_1(x, y, z)$.

$\therefore$ x-component of $\vec{V}$ at centre of the face $AFED$ which is perpendicular to the x-axis and is nearer to the origin O.

$$= V_1\left(x - \frac{dx}{2}, y, z\right)$$

$$= V_1(x, y, z)\frac{dx}{2}\frac{\partial V_1}{\partial x} + \cdots \quad \text{[By Taylor's theorem]}$$

$$= V_1(x, y, z) - \frac{dx}{2}\frac{\partial V_1}{\partial x} \text{ approximately.}$$

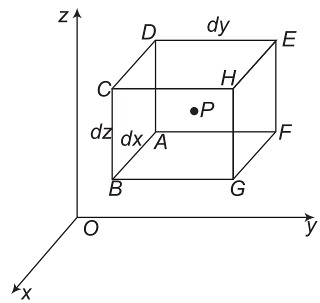


Fig. 7.2

Similarly, the x-component of $\vec{V}$ at the centre of the opposite face *GHCB*

$$= V_1 + \frac{dx}{2}\frac{\partial V_1}{\partial x} \text{ approximately}$$

$\therefore$ Volume of fluid entering the parallelepiped across *AFED* per unit time.

$$= \left(V_1 - \frac{dx}{2}\frac{\partial V_1}{\partial x}\right) dydz$$

Also, volume of fluid going out of the parallelepiped across *GHCB* per unit time

$$= \left(V_1 + \frac{dx}{2}\frac{\partial V_1}{\partial x}\right) dy\, dz$$

Therefore, loss in volume per unit time in the direction of the x-axis

$$= \left(V_1 + \frac{dx}{2}\frac{\partial V_1}{\partial x}\right) dydz - \left(V_1 - \frac{dx}{2}\frac{\partial V_1}{\partial x}\right) dydz$$

$$= \frac{\partial V_1}{\partial x} dx\, dy\, dz$$

Similarly, loss in volume per unit time in the direction of the y-axis

$$= \frac{\partial V_2}{\partial x} dx\, dy\, dz$$

And loss in volume per unit time in the direction of the z-axis

$$= \frac{\partial V_3}{\partial x} dx\, dy\, dz$$

$\therefore$ total loss of the fluid per unit volume per unit time

$$= \frac{\dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z} dx\, dy\, dz}{dx\, dy\, dz}$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \vec{\nabla} \cdot \vec{V}$$

$$= \text{div } \vec{V}$$

7.4.4 Curl of a Vector Function

The curl of a vector function $\vec{V}$ is denoted by curl $\vec{V}$ or $\vec{\nabla} \times \vec{V}$ and is defined as

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}, \text{ where } \vec{V} = V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

$$= \sum \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i}$$

where $\sum$ denotes summation obtained by the cyclic rotation of the unit vectors $\hat{i}, \hat{j}, \hat{k}$, the components V_1, V_2, V_3, and the independent variables x, y, z respectively.

Note (i) The curl of a vector function is a vector function.

(ii) *Irrotational vector*: If curl ($\vec{V}$) = 0 then the vector function $\vec{V}$ is said to be an irrotational vector.

(iii) A force field $\vec{F}$ is said to be conservative if it is derivable from a potential function f, i.e., $\vec{F}$ = grad f. Then curl $\vec{F}$ = curl (grad f) = 0.

$\therefore$ if $\vec{F}$ is conservative then curl $\vec{F}$ = 0 and there exists a scalar potential function f such that $\vec{F}$ = grad f.

7.5 PHYSICAL INTERPRETATION OF CURL

Consider a rigid body rotating with the uniform angular velocity $\Omega = a\hat{i} + b\hat{j} + c\hat{k}$ about an axis l through the origin 'O'. Let the position vector of any point $P(x, y, z)$ on the rotating body $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The linear velocity $\vec{V}$ of the point P is given by

$$\vec{V} = \Omega \times \vec{r} = (a\hat{i} + b\hat{j} + c\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (bz - cy)\,\hat{i} + (cx - az)\,\hat{j} + (ay - bx)\,\hat{k}$$

Now,

$$\text{Carl } \vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix}$$

$$= 2\,(a\,\hat{i} + b\,\hat{j} + c\,\hat{k})$$

$$= \text{Curl } \vec{V} = 2\,\Omega$$

Hence, the angular velocity of the point $P(x, y, z)$ is given by $\Omega = \frac{1}{2}(\text{curl } \vec{V})$, i.e., the angular velocity of a uniform rotating body is equal to one-half of the curl of the linear velocity. Because of this interpretation, the rotation is used.

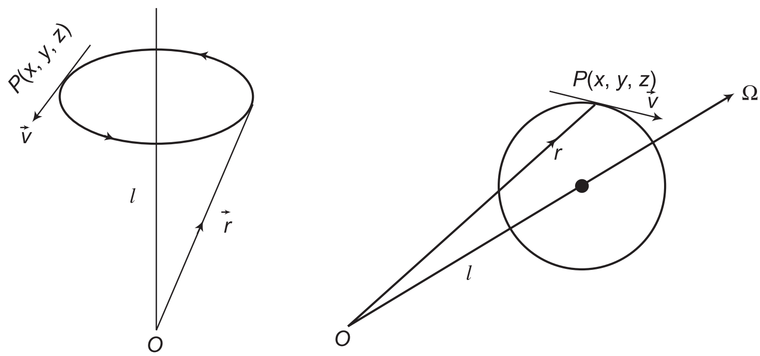


Fig. 7.3

Directional Derivative

The directional derivative of a scalar field $f(x, y, z)$ in the direction of a unit vector $\hat{a}$ is denoted by $D_a f$ and is defined as

$$D_a f = \frac{df}{ds} = \text{grad}\, f \cdot \hat{a}$$

Example 1 Find the gradients of the following scalar functions:

(i) $f(x, y, z) = x^2y^2 + xy^2 - z^2$ at (3, 1, 1)

(ii) $f(x, y, z) = xy + 2yz - 8$ at (3, –2, 1)

Solution

(i) Given $f(x, y, z) = x^2y^2 + xy^2 - z^2$

$$\text{grad}\, f = \vec{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y^2 + xy^2 - z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y^2 + xy^2 - z^2) + \hat{j}\frac{\partial}{\partial y}(x^2y^2 + xy^2 - z^2) + \hat{k}\frac{\partial}{\partial z}(x^2y^2 + xy^2 - z^2)$$

$$= \hat{i}\,(2xy^2 + y^2) + \hat{j}\,(2x^2y + 2xy) + \hat{k}\,\frac{\partial}{\partial x}(-2z)$$

$$\text{grad}\, f = (2xy^2 + y^2)\,\hat{i} + (2x^2y + 2xy)\,\hat{j} - 2z\,\hat{k}$$

$$\text{grad}\, f/(3, 1, 1) = 7\hat{i} + 24\, j5 - 2\hat{k}$$

(ii) $f(x, y, z) = xy + 2yz - 8$

$$\text{grad } f = \vec{\nabla} f = \vec{\nabla} f \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right)(xy + 2yz - 8)$$

$$= \hat{i}\,(y) + \hat{j}\,(x + 2z) + \hat{k}(2y)$$

$$= y\hat{i} + (x + 2z)\,\hat{j} + 2y\,\hat{k}$$

$$\text{grad } f/(3, -2, 1) = -2\hat{i} + 5\hat{j} - 4\hat{k}$$

Example 2 Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $P(2, 1, 3)$ in the direction of the vector $\vec{a} = \hat{i} - 2\hat{k}$.

Solution Given $f(x, y, z) = 2x^2 + 3y^2 + z^2$

$$\text{grad } f = 4x\hat{i} + 6y\,\hat{j} + 2z\,\hat{k}$$

At the point (2, 1, 3), grad $f = 8\hat{i} + 6\hat{j} + 6\hat{k}$

$\therefore$ the directional derivative of f in the direction $\hat{a}$ is

$$D_a f = \text{grad } f \cdot \hat{a}$$

$$= (8\hat{i} + 6\hat{j} + 6\hat{k}) \cdot \frac{(\hat{i} - 2\hat{k})}{\sqrt{1+4}}$$

$$= (8\hat{i} + 6\hat{j} + 6\hat{k}) \cdot \frac{(\hat{i} - 2\hat{k})}{\sqrt{5}}$$

$$= \frac{1.8 - 2.6}{\sqrt{5}}$$

$$= -\frac{4}{\sqrt{5}}$$

The minus sign indicates that f decreases at P in the direction $\vec{a}$.

Example 3 Find the directional derivative of $f(x, y) = x^2y^3 + xy$ at (2, 1), in the direction of a unit vector which makes an angle of 60° with the x-axis.

Solution grad $f = (2xy^3 + y)\,\hat{i} + (3x^2y^2 + x)\,\hat{j}$

At the point (2, 1), grad $f = 5\hat{i} + 14\hat{j}$

The unit vector $\hat{a} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$ since $\theta = 60°$.

$\therefore$ directional derivative = grad $f.\ \hat{a} = (5\hat{i} + 14\hat{j}) \cdot \left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j} \right)$

$$= \frac{5 + 14\sqrt{3}}{2}$$

Example 4 Find the unit normal vector of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point (1, 0, 2).

Solution The cone is the level surface $f = 0$ of $f(x, y, z) = 4(x^2 + y^2) - z^2$.

$$\therefore \qquad \text{grad } f = \vec{\nabla} f = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} \right)(4\,(x^2 + y^2) - z^2)$$

$$\text{grad } f = 8\,x\,\hat{i} + 8y\,\hat{j} - 2z\,\hat{k}$$

At the point (1, 0, 2), grad $f = 8\,\hat{i} - 4\,\hat{k}$

The unit normal vector $\vec{n}$ at the point (1, 0, 2) is

$$\vec{n} = \frac{\text{grad } f}{|\text{grad } f|}$$

$$= \frac{8\hat{i} - 4\hat{k}}{4\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}\,\hat{i} - \frac{1}{\sqrt{5}}\,\hat{k}$$

Example 5 Find the divergence of the function $f(x, y, z) = xy^2\,\hat{i} + 2x^2\,yz\,\hat{j} - 3\,yz^2\,\hat{k}$ at the point (1, –1, 1).

Solution Given $f(x, y, z) = xy^2\,\hat{i} + 2x^2\,yz\,\hat{j} - 3\,yz^2\,\hat{k}$

$$\text{div } f = \vec{\nabla} \cdot f = \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j} + \frac{\partial}{\partial z}\,\hat{k} \right) \cdot (xy^2\,\hat{i} + 2x^2 yz\,\hat{j} - 3yz^2\,\hat{k})$$

$$\text{div } f = y^2 + 2x^2\,z - 6yz$$

At (1, –1, 1), $\quad \text{div } f = 1 + 2 + 6 = 9$

Example 6 If $\vec{V} = x^2 z\,\hat{i} - 2y^3 z^2\,\hat{j} + xy^2\,z\hat{k}$, find curl $\vec{V}$ at (1, –1, 1).

Solution

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 z & -2y^3 z^2 & xy^2 z \end{vmatrix}$$

$$= \hat{i}\,[2xyz + 4y^3 z] - \hat{j}\,[y^2 z - x^2] + \hat{k}\,[0 - 0]$$

$$\text{Curl } \vec{V} = \hat{i}\,[2xyz + 4y^3 z] - \hat{j}\,[y^2 z - x^2]$$

At the point (1, –1, 1),

$$\text{Curl } \vec{V} = \hat{i}\,[-2 - 4] - \hat{j}\,[1 - 1]$$

$$\text{Curl } \vec{V} = -6\hat{i}$$

Example 7 Determine the constant 'a' so that the vector

$$\vec{V} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x+az)\,\hat{k} \text{ is solenoidal.}$$

Solution A vector $\vec{V}$ is solenoidal if div $\vec{V} = 0$

$$\therefore \qquad \text{div } \vec{V} = \vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[(x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x+az)\,\hat{k}\right]$$

or $\quad 1 + 1 + a = 0$

or $\quad a = -2$

7.6 IMPORTANT VECTOR IDENTITIES

1. div $(\vec{A} + \vec{B}) = \text{div } \vec{A} + \text{div } \vec{B}$
2. Curl $(\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$
3. If $\vec{A}$ is a differentiable vector function and ϕ is a differentiable scalar function then
   div $(\phi\, \vec{A}) = (\text{grad } \phi)\cdot \vec{A} + \phi \text{ div } \vec{A}$
4. Curl $(\phi\, \vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi \text{ curl } \vec{A}$
5. div $(\vec{A} \times \vec{B}) = \vec{B}\cdot \text{curl } \vec{A} - \vec{A}\cdot \text{curl } \vec{B}$
6. Curl of the gradient of ϕ is zero, i.e., curl (grad (ϕ)) = 0.
7. div curl $\vec{A} = 0$.
8. div grad $f = \nabla^2\phi$, where ∇ is a Laplace operator.

Example 8 Prove that div $(r^n\, \vec{r}) = (n+3)\, \vec{r}^{\,n}$.

Solution We know that

$$\text{div}\,(\phi\vec{A}) = \phi \text{ div } \vec{A} + \vec{A}\cdot \text{grad } \phi$$

Here, $\vec{A} = \vec{r}$ and $\phi = r^n$, we get

$$\text{div } (r^n \cdot \vec{r}) = r^n \text{ div } \vec{r} + \vec{r}\cdot \text{grad } r^n = 3r^n + \vec{r}\cdot(nr^{n-1} \text{ grad } r)$$

[$\because$ div $\vec{r} = 3$ and grad $f(u) = f'(u)$ grad u]

$$= 3r^n + \vec{r}\cdot\left(nr^{n-1}\,\frac{1}{r}\,\vec{r}\right)$$

$$= 3r^n + nr^{n-2}\,(\vec{r}\cdot\vec{r})$$

$$= 3r^n + nr^{n-2}\,r^2$$

$$= (3+n)\,r^n$$

$$\text{div }(r^n\,\vec{r}) = (n+3)\,r^n$$

Proved

EXERCISE 7.1

1. If $f(x, y, z) = 3x^2y - y^3z^2$, find grad f at the point (1, –2, –1).
2. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove the (grad u) · [(grad v) × (grad w)] = 0.
3. Find the directional derivative of $f = x^2yz + 4\,xz^2$ at the point (1, –2, –1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.
4. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q is the point (5, 0, 4).
5. For the function $f = \dfrac{y}{x^2 + y^2}$, find the value of the directional derivative making an angle of 30° with the positive x-axis at the point (0, 1).
6. Find the unit normal to the surface $z = x^2 + y^2$ at the point (–1, –2, 5).
7. Find a unit normal vector to the level surface $x^2y + 2xz = 4$ at the point (2, –2, 3).
8. Find the divergence and curl of $\vec{V}$ and verify that div (curl $\vec{V}$) = 0, where
   (i) $\vec{V} = (x^2 - y^2)\,\hat{i} + 4xy\,\hat{j} + (x^2 - xy)\hat{k}$
   (ii) $\vec{V} = (x^2 + yz)\,\hat{i} + (y^2 + zx)\,\hat{j} + (z^2 + xy)\hat{k}$
   (iii) $\vec{V} = xyz\,\hat{i} + 2x^2y\,\hat{j} + (xz^2 - y^2\,z)\hat{k}$
9. Prove that (i) $\text{div}\,\vec{r} = 3$, and (ii) $\text{curl}\,\vec{r} = 0$.
10. Show that the vector $\vec{V} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$ is irrotational.
11. If $\vec{V} = (x + y + 1)\,\hat{i} + \hat{j} + (-x - y)\,\hat{k}$, prove that $\vec{V} \cdot \text{curl}\,\vec{V} = 0$.
12. Prove that div (grad r^n) = $n(n + 1)\cdot r^{n-2}$.
13. Prove that curl (grad r^n) = 0.
14. If $\vec{u} = \dfrac{1}{r}\vec{r}$, find grad (div $\vec{u}$).
15. Prove that $\text{div}\,\hat{r} = \dfrac{2}{r}$.
16. Let $f(x, y, z)$ be a solution of the Laplace equation $\nabla^2 f = 0$. Then, show that $\vec{\nabla} f$ is a vector which is both irrotational and solenoidal.
17. If $\vec{u}$ and $\vec{v}$ are irrotational vector fields, show that $\vec{u} \times \vec{v}$ is a solenoidal vector.

Answers

1. $\text{grad}\ f = -12\,\hat{i} - 9\,\hat{j} - 16\,\hat{k}$

3. $\dfrac{37}{3}$

4. $\dfrac{4\sqrt{21}}{3}$

5. $-\dfrac{1}{2}$

6. $\frac{1}{\sqrt{21}}(2\hat{i} + 4\hat{j} + \hat{k})$

7. $-\frac{\hat{i}}{3} - \frac{2\hat{j}}{3} + \frac{2\hat{k}}{3}$

14. $-\frac{2}{r^3}\vec{r}$

7.7 VECTOR INTEGRATION

In this section, we shall define the line integral, surface integral, volume integral, and some basic engineering applications. We shall see that a line integral is a natural generalization of a definite integral, and a surface integral is a generalization of a double integral. The line integral can be transformed in to double integrals or into surface integrals and conversely.

Triple integrals can be transformed into surface integrals and vice versa. These transformations have the great importance.

Green's theorem, Gauss's divergence theorem and Stokes' theorem serve as powerful tools in many applications as well as in theoretical problems.

7.8 LINE INTEGRALS

The line integral (or path integral, curve integral, contour integral) is an integral where the function to be integrated is evaluated along a curve. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve.

Let C be a simple curve, and let the parametric representation of C be written as

$$x = x(t),\ y = y(t),\ z = z(t),\ a \leq t \leq b.$$

The position vector of a path on the curve C can be written as

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}; \qquad a \leq t \leq b$$

A line integral of a vector function $\vec{F}(r)$ over a curve C is defined as

$$\int_C F(r)\, dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt \tag{1}$$

The line integral of $F(r)$ over C with respect to the arc length s is given by

$$\int_C F(r)\, ds = \int_a^b F(r(t)) \frac{ds}{dt}\, dt = \int_a^b F(r(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{2}$$

where $$ds = \frac{ds}{dt}\, dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Note If the path of integration C is a closed path/curve then instead of $\int_C$ we also write $\oint_C$

(i) Evaluation of a Line Integral in the Plane

Example 9 Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = x^2\hat{i} + y^3\hat{j}$ and the curve C is the arc of the parabola $y = x^2$ in the xy plane from (0, 0) to (1, 1).

Solution In the xy-plane , $\vec{r} = x\hat{i} + y\hat{j}$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C (x^2\hat{i} + y^3\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j}) = \int_C (x^2dx + y^3dy)$$

Now, along the curve C, $y = x^2 \Rightarrow dy = 2x\,dx$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_{x=0}^{1} [x^2\,dx + x^6 \cdot 2x\,dx]$$

$$= \int_{x=0}^{1} [x^2 + 2x^7]\,dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}$$

Example 10 Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \hat{i}\cos y - \hat{j}(x\sin y)$ and C is the curve $y^2 = 1 - x^2$ in the xy-plane from (1, 0) to (0, 1).

Solution In the xy-plane, $\vec{F} \cdot d\vec{r} = \left[\hat{i}\cos y - \hat{j}\,(x\sin y)\right]\left[\hat{i}\,dx + \hat{j}\,dy\right]$

$= \cos y\,dx - (x\sin y)dy.$

$= d(x\cos y).$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C d\,(x\cos y)$$

$$= [x\cos y]_{(1,0)}^{(0,1)} = 0 - 1 = -1$$

(ii) Dependence of a Line Integral on a Path

Example 11 Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x^2 + y^2)\,\hat{i} - 2xy\,\hat{j}$, curve C is the rectangle in the xy-plane bounded by $y = 0$, $x = a$, $y = b$ and $x = 0$.

Solution In the xy-plane, $z = 0$, $\vec{r} = x\hat{i} + y\hat{j} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$.

Along the line OA, $y = 0$, $dy = 0$ and x varies from 0 to a,

Along the line AB, $x = a$, $dx = 0$ and y varies from 0 to b,

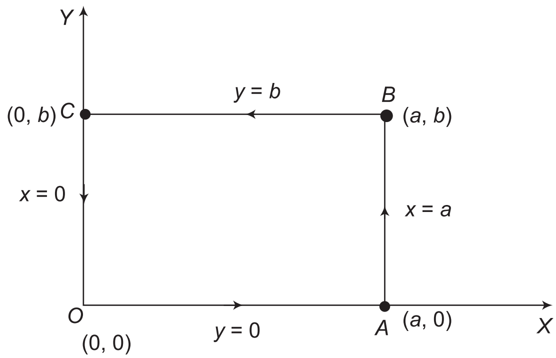


Fig. 7.4

Along the line BC, $y = b$, $dy = 0$ and x varies from a to 0,
Along the line CO, $x = 0$, $dx = 0$ and y varies from b to 0.
We have

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[(x^2+y^2)\hat{i} - 2xy\,\hat{j}\right]\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C \left[(x^2+y^2)\,dx - 2\,xy\,dy\right]$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

$$= \int_0^a x^2\,dx - \int_0^b 2\,ay\,dy + \int_a^0 (x^2+b^2)\,dx + \int_b^0 0\,dy$$

$$= \left.\frac{x^3}{2}\right|_0^a - 2a\left.\frac{y^3}{2}\right|_0^b + \left.\frac{x^3}{2} + b^2x\right|_a^0 + 0$$

$$= -2ab^2$$

Example 12 If $\vec{F} = y\hat{i} - x\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ from (0, 0) to (1, 1) along the following paths:

(i) the parabola $y = x^2$
(ii) the straight lines from (0, 0) to (1, 0) and then to (1, 1)
(iii) the straight line joining (0, 0) and (1, 1)

Solution The three paths of integration have been shown in Fig. 7.5.

We have

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (y\,\hat{i} - x\,\hat{j})\cdot(dx\,\hat{i} + d\,y\,\hat{j})$$

$$= \int_C (y\,dx - x\,dy)$$

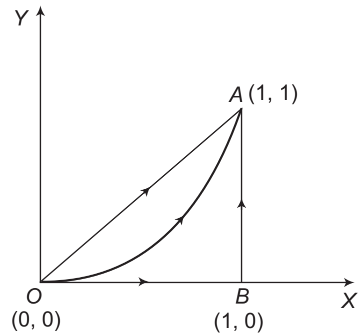


Fig. 7.5

(i) C is the arc of the parabola $y = x^2$ from (0, 0) to (1, 1). $dy = 2x\,dx$ and x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 [x^2\, dx - x\,(2x)\, dx]$$

$$= \int_0^1 [-x^2]\, dx = -\frac{x^3}{3}\bigg|_0^1 = -\frac{1}{3}$$

(ii) C is the curve consisting of straight lines OB and BA

Along OB, $y = 0$, $dy = 0$ and x varies from 0 to 1.

Along BA, $x = 1$, $dx = 0$ and y varies from 0 to 1.

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 0\, dx + \int_0^1 -1\, dy = -1$$

(iii) C is the straight line OA.

The equation of OA is $y - 0 = \dfrac{1-0}{1-0}(x - 0)$

i.e., $\quad y = x.$

$dy = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (x\, dx - x\, dx) = 0$$

7.9 CONSERVATIVE FIELD AND SCALAR POTENTIAL

The conservative field is a vector field which is the gradient of a function, known as a scalar potential. Conservative fields have the property that the line integral from one point to another is independent of the choice of path connecting the two points, it is path-independent. Conversely, path independence is equivalent to the vector field being conservative. Conservative vector fields are also irrotational. Mathematically, a vector field $\vec{F}$ is said to be conservative if there exists a scalar field ϕ such that

$$\vec{F} = \vec{\nabla}\phi$$

Here, $\vec{\nabla}\phi$ denotes the gradient of ϕ. When the above equation holds, ϕ **is called a scalar potential** for $\vec{F}$.

If $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla}\phi = 0$. In such a case, $\vec{F}$ **is called a conservative vector field**.

Example 13 Prove that $\vec{F} = (y^2 \cos x + z^2)\,\hat{i} + (2y \sin x - 4)\,\hat{j} + (3xz^2 + 2)\,\hat{k}$ is a conservative field, and find the scalar potential of $\vec{F}$.

Solution

(i) For the conservative field $\vec{F}$,

$$\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= i(0 - 0) - j(3z^2 - 2z^2) + k(2y \cos x - 2y \cos x) = 0$$

Hence, $\vec{F}$ is conservative.

(ii) Let f be the scalar potential such that $\vec{F} = \vec{\nabla}f$. Then comparing the components of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$\frac{\partial f}{\partial x} = y^2 \cos x + z^3 \tag{1}$$

$$\frac{\partial f}{\partial y} = 2y \sin x - 4 \tag{2}$$

and $$\frac{\partial f}{\partial z} = 2x\,z^2 + 2 \tag{3}$$

Integrating (1) partially w.r.t. x,

$$f = y^2 \sin x + x\,z^3 + g(y, z) \tag{4}$$

Differentiating (4) partially w.r.t. y,

$$\frac{\partial f}{\partial y} = 2y \sin x + 0 + \frac{\partial g}{\partial y} \tag{5}$$

Using (2), (5) becomes,

$$\frac{\partial g}{\partial y} = -4 \tag{6}$$

Integrating (6), w.r.t. y, we get

$$g(y, z) = -4y + g_1(z) \tag{7}$$

Substituting (7) in (4), we get

$$f = y^2 \sin x + xz^2 - 4y + g_1(z) \tag{8}$$

Differentiating (8), partially w.r.t. z and using (3)

$$0 + 3xz^2 - 0 + \frac{dg_1}{dz} = \frac{\partial f}{\partial z} = 3x\,z^2 + 2$$

Integrating w.r.t. z, we get

$$g_1(z) = 2z + C \tag{9}$$

Substituting (9) in (8), we get

$$f(x, y, z) = y^2 \sin x + xz^3 - 4y + 2z + C$$

7.10 SURFACE INTEGRALS: SURFACE AREA AND FLUX

The surface integral is a simple and natural generalization of a double integral. $\iint_R f(x,y)dx\,dy$, taken over a plane region R. In a surface integral, $f(x, y)$ is integrated over a curved surface.

The surface integral of a vector field $\vec{F}$ actually has a simpler explanation. If the vector field $\vec{F}$ represents the flow of a fluid, then the surface integral $\vec{F}$ will represent the amount of fluid flowing through the surface. The amount of the fluid flowing through the surface per unit time is also called the ***flux*** of fluid through the surface.

In the xyz-space, let S be a surface and $\vec{F}(x, y, z)$ is a vector function of the position defined and continuous over S.

Let $P(x, y, z)$ be any point on the surface S and $\hat{n}$ be the unit vector at P in the direction of outward drawn normal to the surface S at P. Then $\vec{F}\cdot\hat{n}$ is the normal component of $\vec{F}$ at P.

The integral of $\vec{F}\cdot\hat{n}$ over S $\iint_S \vec{F}\cdot\hat{n}\,dS$ is called the **flux** of $\vec{F}$ over S.

In order to evaluate surface integrals, it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on one of the coordinate planes.

Let R be the orthogonal projection of S on the xy-plane. If γ is the acute angle which the undirected normal n at the point P, to the surface S makes with the z-axis then it can be shown that

$dx\,dy = \cos\gamma\,dS$; where dS is the small element of area of the surface S at P.

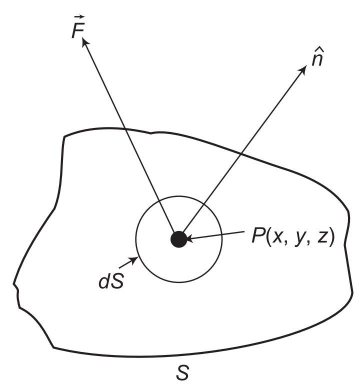


Fig. 7.6

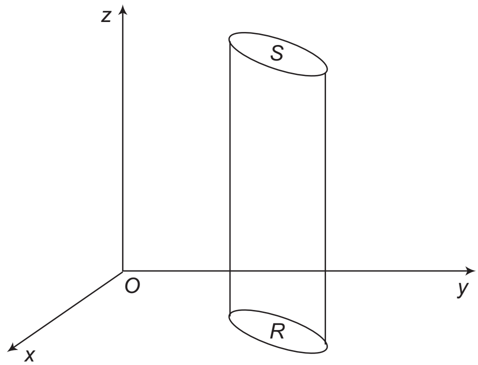


Fig. 7.7

$\therefore \quad dS = \dfrac{dx\,dy}{\cos\gamma} = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$; $\hat{k}$ is the unit vector along the z-axis.

Hence, $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Thus, the surface integral on S can be evaluated with the help of a double integral over R.

Example 14 Evaluate $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

Solution A vector normal to the surface S is

$$\vec{\nabla}(x^2 + y^2 + x^2) = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$\hat{n}$ = a unit normal to any point $P(x, y, z)$ of S

$$= \frac{2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

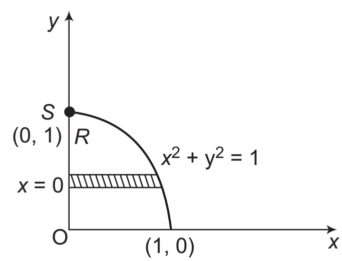


Fig. 7.8

Since $x^2 + y^2 + z^2 = 1$ on the surface S, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

where R is the projection of S on the xy-plane. The region R is bounded by the x-axis, the y-axis and the circle $x^2 + y^2 = 1$, $z = 0$.

Now, $\quad \vec{F}\cdot\hat{n} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) = 3\,xyz$

$\quad n\cdot\hat{k} = (x\,\hat{i} + y\,\hat{j} + z\,\hat{k})\cdot\hat{k} = z$

Hence,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \frac{3x\,yz}{z}\,dx\,dy$$

$$= 3\iint_R xy\,dx\,dy$$

$$= 3\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} xy\,dx\,dy$$

$$= 3\int_{y=0}^{1} y\cdot\left(\frac{x^2}{2}\right)_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_{y=0}^{1} y\cdot(1-y^2)\,dy$$

$$= \frac{3}{2}\int_{y=0}^{1} (y-y^3)\,dy$$

$$= \frac{3}{2}\cdot\frac{1}{4}$$

$$= \frac{3}{8}$$

Example 15 Evaluate $\iint_S \overline{F}\cdot\hat{n}\,dS$, where $F = y\hat{i} + 2x\hat{j} - z\hat{k}$, and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

Solution A vector normal to the surface S is given by

$$\vec{\nabla}(2x+y) = 2\hat{i} + \hat{j}$$

$\therefore$ $\hat{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\hat{i}+\hat{j}}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}(2\hat{i}+\hat{j})$$

We have

$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$, where R is the projection of S on the xz-plane. It should be noted that in this case we cannot take the projection on the xy-plane because the surface S is perpendicular to the xy-plane.

Now, $$\overline{F}\cdot\hat{n} = (y\hat{i} + 2x\hat{j} - z\hat{k})\cdot\left(\frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}\right) = \frac{2}{\sqrt{5}}y + \frac{2}{\sqrt{5}}x$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{5}}(2\hat{i} + j)\cdot\hat{j} = \frac{1}{\sqrt{5}}$$

$$\therefore \quad \iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \left(\frac{2}{\sqrt{5}}y + \frac{2}{\sqrt{5}}x\right)\cdot\frac{dx\,dz}{1/\sqrt{5}}$$

$$= 2\iint_R \frac{1}{\sqrt{5}}(y + x)\cdot\sqrt{5}\,dx\,dz$$

$$= 2\iint_R (y + x)\cdot dx\,dz$$

$$= 2\iint_R (6 - 2x + x)\,dx\,dz \quad \text{Since } 2x + y = 6 \text{ on } S$$

$$= 2\iint_R (6 - x)\,dx\,dz$$

$$= 2\int_{z=0}^{4}\int_{x=0}^{3} (6 - x)\,dx\,dz$$

$$= 2\int_{x=0}^{3} (6 - x)\cdot\left(z\right]_{z=0}^{4} dx$$

$$= 8\left[6x - \frac{x^2}{2}\right]_0^3 = 8\left[18 - \frac{9}{2}\right]$$

$$= 108$$

Example 16 Evaluate $\iint_S \bar{F}\cdot\hat{n}\,dS$, where $\bar{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution A vector normal to the surface S is given by

$$\vec{\nabla}(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

$$\therefore \quad \hat{n} = \text{a unit normal to any point on } S$$

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\hat{i} + y\hat{j}}{4} \qquad (\because\ x^2 + y^2 = 16 \text{ on the surface } S)$$

We have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{\left|\hat{n}\cdot\hat{j}\right|}$$

where R is the projection of S on the xy-plane. It should be noted that in this case, we cannot take the projection of S on the xy-plane as the surface S is perpendicular to the xy-plane.

Now,

$$\bar{F}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2 z\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j}}{4}\right)$$

$$= \frac{1}{4}(xz + xy)$$

$$\hat{n}\cdot\hat{j} = \left(\frac{x\hat{i} + y\hat{j}}{4}\right)\cdot\hat{j} = \frac{y}{4}$$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_R \frac{(xz + xy)}{4}\,\frac{dx\, dz}{y/4}$$

$$= \iint_R \frac{(xz + xy)}{y}\, dx\, dz$$

$$= \iint_R \left(x + \frac{xz}{y}\right) dx\, dz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left[x + \frac{xz}{\sqrt{16 - x^2}}\right] dx\, dz \qquad \left[\because \; y = \sqrt{16 - x^2} \text{ on } S\right]$$

$$= \int_{z=0}^{5} [4z + 8]\, dz$$

$$= 90$$

7.11 VOLUME INTEGRALS

Any integral which is evaluated along a volume is called a volume integral. Let V is a volume bounded by a surface S and $f(x, y, z)$ is a single-valued function of position defined over V. Subdivide the volume V into n elements of volumes δV_1, δV_2, … δV_n. In each part δV_k, we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k). We define $f(P_k) = f(x_k, y_k, z_k)$.

Form the sum

$$\sum_{k=1}^{n} f(P_k)\,\delta V_k$$

Now, take the limit of this sum as $n \to \infty$ in such a way that the largest of the volumes $\delta V_k \to 0$. This limit, if it exists, is called the volume integral of $f(x, y, z)$ over V and is denoted by

$$\iiint_V f(x, y, z)\, dV$$

If we subdivided the volume V into small cuboids by drawing lines parallel to the three coordinate axis then $dV = dx\, dy\, dz$ and the above volume integral becomes

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

If $\vec{F}\,(x, y, z)$ is a vector function then

$\iiint_V \vec{F}\,(x, y, z)\, dV$ or $\iiint_V \vec{F}\,(x, y, z)\, dx, dy\, dz$ is a volume integral.

Example 17 If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$ then evaluate $\iiint_V \vec{\nabla}\cdot\vec{F}\, dV$, where V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution We have $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[2x^2 - 3z)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x = 2x$$

$$\therefore \qquad \iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iiint_V 2x\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\, dz\, dy\, dx$$

$$= \int_0^2\int_0^{2-x} 2x\,[z]_0^{4-2x-2y}\, dy\, dx$$

$$= \int_0^2\int_0^{2-x} 2x\,[4-2x-2y]\, dy\, dx$$

$$= \int_0^2\int_0^{2-x} [4x\,(2-x) - 4\,xy]\, dy\, dx$$

$$= \int_0^2\left[4z\,(2-x)\, y - 2xy^2\right]_0^{2-x} dx$$

$$= \int_0^2 2x(2-x)^2\, dx = 2\int_0^2 [4x - 4x^2 + x^3]\, dx$$

$$= 2\left[2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4}\right]_0^2 = 2\left(8 - \frac{32}{3} + 4\right) = \frac{8}{3}$$

Example 18 Evaluate $\iiint\limits_V \phi\, dV$, where $\phi = 45\, x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, and $z = 0$.

Solution We have

$$\iiint\limits_V \phi\, dV = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45\, x^2 y\, dx\, dy\, dz$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2 y\, [z]_0^{8-4x-2y}\, dx\, dy$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2 y\, [8 - 4x - 2y]\, dx\, dy$$

$$= 45 \int_0^2 \left[x^2 (8 - 4x) \frac{y^2}{2} - 2x^2 \frac{y^3}{3} \right]_0^{4-2x} dx$$

$$= 45 \int_0^2 \left[\frac{x^2}{3} (4 - 2x)^3 \right] dx$$

$$= 128$$

EXERCISE 7.2

1. Evaluate $\int\limits_C \vec{F} . d\vec{r}$, where $\vec{F} = xy\hat{i} + (x^2 + y^2)\hat{j}$ and C is the x-axis from $x = 2$ to $x = 4$ and the line $x = 4$ from $y = 0$ to $y = 12$.
2. Find the circulation of $\vec{F}$ round the curve c, where $\vec{F} = (e^x \cos y)\hat{i} + (e^x \cos y)\hat{j}$ and C is the rectangle whose vertices are (0, 0), (1, 0), $\left(1, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$.
3. Calculate $\int\limits_C \left[(x^2 + y^2)\hat{i} + (x^2 - y^2)\hat{j}\right] . d\vec{r}$, where C is the curve
   - (i) $y^2 = x$ joining (0, 0) to (1, 1)
   - (ii) $x^7 = y$ joining (0, 0) to (1, 1)
   - (iii) consisting of two straight lines joining (0, 0) to (1, 0) and (1, 0) to (1, 1)
   - (iv) consisting of two straight lines joining (0, 0) to (2, –2), (2, –2) to (0, –1) and (0, –1) to (1, 1)
4. Evaluate the line integral $\int\limits_C \left[(x^2 + xy)\, dx + (x^2 + y^2)\, dy\right]$, where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

5. Evaluate $\iint_S \vec{F}.\hat{n}\, d\,S$, where $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.

6. If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, then evaluate $\iint_S \vec{F}.\hat{n}\, d\,S$.

7. Evaluate $\iint_S \vec{F}.\hat{n}\, d\,S$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

8. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ then evaluate $\iiint_V (\vec{\nabla} \times \vec{F})d\,V$, where V is the closed region bounded by the planes $x = 0$, $y = 0$ and $2x + 2y + z = 4$.

9. If V is the region in the first octant bounded by $y^2 + z^2 = 9$ and the plane $x = 2$ and $\vec{F} = 2x^2 y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ then evaluate $\iiint_V (\vec{\nabla}.\vec{F})d\,V$.

10. If $\vec{F} = (x^3 - yz)\hat{i} - 2x^2 y\hat{j} + 2\hat{k}$, evaluate $\iiint_V (\vec{\nabla}.\vec{F})\, d\,V$ over the volume of *a* cube of side b.

Answers

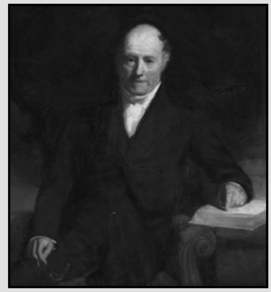
1. 768	2. 0
3. (i) $\frac{7}{20}$ (ii) $\frac{38}{45}$ (iii) 1 (iv) $-\frac{7}{3}$	4. 0
5. 24	6. 132
7. 90	8. $\frac{8}{3}(\hat{j} - \hat{k})$
9. 180	10. $\frac{b^3}{3}$

7.12 GREEN'S THEOREM IN THE PLANE: TRANSFORMATION BETWEEN LINE AND DOUBLE INTEGRAL

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R. Then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

George Green (4 July 1793–31 March 1841), a British mathematician and physicist, was born in Sneinton, Nottinghamshire, England. Green wrote the long-lived monograph. He introduced several important concepts, among them a theorem similar to modern Green's theorem, the idea of potential functions as currently used in physics, and the concept of what are now called Green's functions. Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson, and others. The Green functions play an important role in the effective and compact solution of linear ordinary and partial differential equations. They may be considered also as a crucial approach to the development of boundary integral equation methods.



The line integral is taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

Proof Suppose the equations of the curves AEB and AFB are $y = f(x)$ and $y = g(x)$ respectively,

Consider

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \int_a^b \big[M(x, g(x)) - M(x, f(x))\big]dx$$

$$= -\int_b^a M(x, g(x))\,dx - \int_a^b M(x, f(x))\,dx$$

$$= -\int_{BFA} M(x, y)\,dx - \int_{AEB} M(x, y)\,dx$$

$$= -\int_{BFAEB} M(x, y)\,dx$$

$$= \oint_C M(x, y)\,dx \tag{3}$$

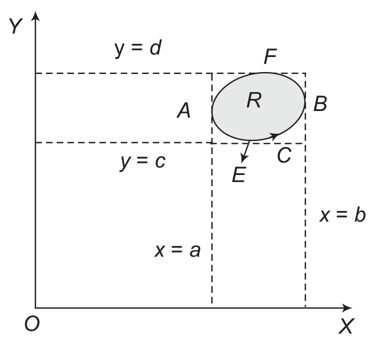


Fig. 7.9

Similarly, let the equations of curves EAF and EBF be $x = P(y)$ and $x = q(y)$ respectively.

Then

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=c}^{d}\int_{x=P(y)}^{q(y)} \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \int_c^d [N(q, y) - N(P, y)]\,dy$$

$$= \int_c^d N(q, y)\,dy + \int_d^c N(P, y)\,dy$$

$$= \oint_C N(x, y)\, dy \tag{4}$$

Adding (3) and (4), we get

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx, dy$$

Note* 1 *Vector notation of Green's Theorem

Suppose $\vec{F} = M\hat{i} + N\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j}$

Then $\vec{F} \cdot d\vec{r} = Mdx + Ndy$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k}$$

$$\therefore \quad \oint_C \vec{F} \cdot d\vec{r} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dR, \text{ where } dR = dx\, dy$$

***Note* 2** If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then by Green's theorem $\oint_C M\, dx + N\, dy = 0$

Example 19 Evaluate by Green's theorem.

$\oint_C (\cos x \cdot \sin y - xy)\, dx + (\sin x \cdot \cos y)\, dy$, where C is the circle $x^2 + y^2 = 1$.

Solution By Green's theorem in the plane, we have

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Here, $M = \cos x \sin y - xy,\ N = \sin x \cos y$

$$\frac{\partial M}{\partial y} = \cos x \cdot \cos y - x, \quad \frac{\partial N}{\partial x} = \cos x \cos y.$$

Now,

$$\oint_C M\, dx + N\, dy = \iint_R [\cos x \cos y - \cos x \cos y + x]\, dxdy$$

$$= \iint_R x\, dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r \cos\theta \cdot r\, dr\, d\theta \qquad \text{[Changing to polar coordinates]}$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^3}{3} \right]_0^1 \cos\theta\, d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{2\pi} \cos\theta \, d\theta = \frac{1}{3}[\sin\theta]_0^{2\pi}$$

$$= \frac{1}{3}(0-0)$$

$$= 0$$

Example 20 Evaluate $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ by Green's theorem, where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, and $(0, 1)$.

Solution By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

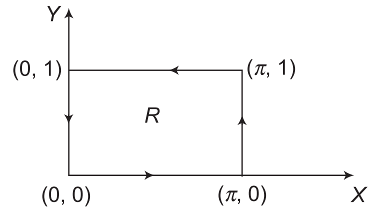


Fig. 7.10

Here, $M = x^2 - \cos hy$

$N = y + \sin x$

$\therefore$ $\dfrac{\partial M}{\partial y} = -\sin hy, \dfrac{\partial N}{\partial x} = \cos x$

Hence, the given line integral is equal to

$$\iint_R [\cos x + \sin hy]\,dx\,dy = \int_{x=0}^{\pi}\int_{y=0}^{1} [\cos x + \sin hy]\,dx\,dy$$

$$= \int_0^{\pi} (y \cos x + \cosh y)_0^1 \, dx = \int_0^{\pi} [\cos x + \cosh \cdot 1 - 1]\,dx$$

$$= [\sin x + x \cosh \cdot 1 - x]_0^{\pi}$$

$$= \pi[\cosh \cdot 1 - 1]$$

Example 21 Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution Green's theorem in plane, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

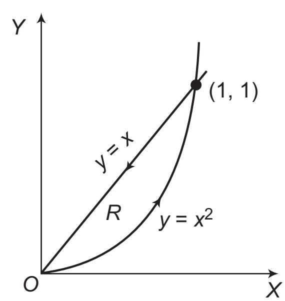


Fig. 7.11

Here $M = xy + y^2$, $N = x^2$.

To evaluate the line integral along curve C and along the curve $y = x^2$ so that $dy = 2x\,dx$.

$$\therefore \quad \int_C (xy + y^2)\,dx + x^2\,dy = \int_0^1 \left[(x \cdot x^2 + x^4)\,dx + (x^2 \cdot 2x\,dx)\right]$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left(\frac{3}{4}x^4 + \frac{x^5}{5}\right)_0^1$$

$$= \left(\frac{3}{4} + \frac{1}{5}\right) - 0 = \frac{19}{20}$$

Also, along $y = x$ so that $dy = dx$.

$$\therefore \qquad \int_1^0 \left[(x \cdot x + x^2)\, dx + x^2\, dx\right] = \int_1^0 3x^2\, dx = \frac{3}{3} x^3 \Big|_1^0 = -1$$

Therefore, the required line integral

$$= \frac{19}{20} - 1 = -\frac{1}{20}$$

Now, the curves $y = x$ and $y = x^2$ intersect at (0, 0) and (1, 1). We have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R [2x - (x + 2y)]\, dx\, dy = \iint_R (x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dx\, dy = \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - x^3 + x^4]\, dx = \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

Hence, the theorem is verified.

7.13 GAUSS'S DIVERGENCE THEOREM (RELATION BETWEEN VOLUME AND SURFACE INTEGERS)

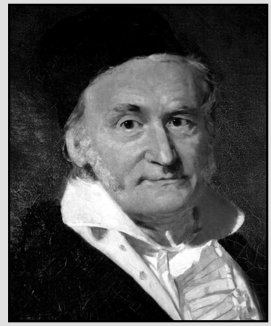
Let $\vec{F}$ be a vector function of a position which is continuous and has continuous first-order partial derivatives, in a volume V bounded by a closed surface S. Then

$$\iiint_V \vec{\nabla} \cdot \vec{\boldsymbol{F}}\, \boldsymbol{dV} = \iint_S \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, \boldsymbol{ds}$$

where $\hat{n}$ is the outwards drawn unit normal vector to S.

Proof Suppose that S is such that any straight line parallel to any one of the coordinate axis and intersecting V has one segment in common with V. If R is the orthogonal projection of S on the xy-plane then V can be represented in the form $f(x, y) \leq z \leq g(x, y)$, where (x, y) varies is R.

Johann Friedrich Carl Gauss (30 April 1777–23 February 1855), a German mathematician, was born in Brunswick, Germany. Gauss is generally regarded one of the greatest mathematicians of all time for his contributions to number theory, geometry, probability theory, geodesy, planetary astronomy, the theory of functions, and potential theory.



Gauss was the only child of poor parents. He was rare among mathematicians in that he was a calculating prodigy, and he retained the ability to do elaborate calculations in his head most of his life. Impressed by this ability and by his gift for languages, his teachers and his devoted mother recommended him to the Duke of Brunswick in 1791, who granted him financial assistance to continue his education locally and then to study mathematics at the University of Göttingen from 1795 to 1798. Gauss's first significant discovery, in 1792, was that a regular polygon of 17 sides can be constructed by ruler and compass alone. Its significance lies not in the result but in the proof, which rested on a profound analysis of the factorization of polynomial equations and opened the door to later ideas of Galois theory. His doctoral thesis of 1797 gave a proof of the fundamental theorem of algebra: every polynomial equation with real or complex coefficients has as many roots as its degree.

Let S_1 and S_2 be the upper and lower portions of S having equations $z = f(x, y)$ and $z = g(x, y)$ and having $\overline{n}_1$ and $\overline{n}_2$ normals, respectively.

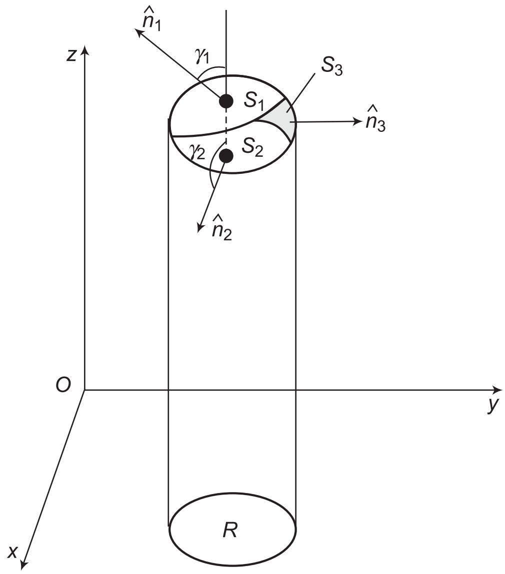


Fig. 7.12

Now,

$$\iiint\limits_V \frac{\partial F_3}{\partial z} dV = \iiint\limits_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz$$

$$= \iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z} dz \right] dx\, dy$$

$$= \iint_R \left[F_3\ (x, y, z) \right]_{z=f(x,y)}^{g(x,y)} dx\, dy$$

$$= \iint_R [(F_3(x, y, g(x, y)) - (F_3(x, y, f(x, y))] \, dx\, dy$$

$$= \iint_R F_3\ [x, y, g(x, y)]\, dx\, dy - \iint_R F_3\ [x, y, f(x, y)]\, dx\, dy \tag{1}$$

For the vertical portion S_3 of S, the normal $\overline{n}_3$ to S_3 makes a right angle γ_1 with $\hat{k}$.

$$\therefore \qquad \iint_{S_3} F_3 \hat{k}.n_3 dS_3 = 0 \ (\because \quad \hat{k}.\hat{n}_3 = 0)$$

For the upper portion S_1 of S, the normal $\overline{n}_1$ to S_1 makes an acute angle γ_1 with $\hat{k}$.

$$\therefore \qquad \hat{k}.n_1\, dS_1 = \cos\gamma_1\, d\, S_1 = dxdy$$

Hence,

$$\iint_{S_1} F_3\ \hat{k}.\overline{n}_1\, dS_1 = \iint_R F_3\ \left[x, y, g(x, y)\right] dx\, dy$$

For the lower portion S_2 of S, the normal $\overline{n}_2$ to S_2 makes an obtuse angle γ_2 with $\hat{k}$

$$\therefore \qquad \hat{k}.n_2 dS_2 = \cos\gamma_2 dS_2 = -\, dx\, dy$$

Hence,

$$\iint_{S_2} F_3\ \hat{k}.\hat{n}_2\, dS_2 = -\iint_R F_3\ \left[x, y, f(x, y)\right] dx\, dy$$

$$\therefore \qquad \iint_{s_3} F_3\ \hat{k}.n_3\, dS_3 + \iint_{S_1} F_3 \hat{k}.\hat{n}_1\, dS_1 + \iint_{S_2} F_3\ \hat{k}.\hat{n}_2\, dS_2$$

$$= 0 + \iint_R F_3\ \hat{k}.\left[x, y, g(x, y)\right] dx\, dy - \iint_R F_3\ \left[x, y, f(x, y)\right] dx\, dy \tag{2}$$

From (1) and (2), we get

$$\iint_S F_3\ \hat{k}\, \hat{n}\, dS = \iiint_V \frac{\partial F_3}{\partial z} dv \tag{3}$$

Similarly, by projecting S on the other coordinate planes, we get

$$\iint_S F_2\ \hat{j}.\hat{n}\ dS = \iiint_V \frac{\partial F_2}{\partial y} dv \tag{4}$$

and

$$\iint_S F_1\ \hat{i}.\hat{n}\ dS = \iiint_V \frac{\partial F_1}{\partial x} dv \tag{5}$$

Adding equations (3), (4), and (5), we get

$$\iint_S \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right).\hat{n}\, dS = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$$

or $$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V (\vec{\nabla}.\vec{F})\, dv$$

$$= \iiint_V \text{div}\, \vec{F}\, dv$$

Hence, the theorem.

Deductions

(i) If $\hat{n}$ is the outward-drawn unit normal vector to S them

$$\iiint_V \vec{F}.\vec{\nabla}\phi\, dV = \iint_S \phi\vec{F}.\hat{n}\, dS - \iiint_V \phi\, \text{div}\, \vec{F}\, dV$$

(ii) $$\iint_S \vec{F} \times \hat{n}\, dS = -\iiint_V \text{curl}\, \vec{F}\, dV$$

(iii) $$\iint_S \phi\, \hat{n}\, dS = \iiint_V \text{grad}\, \phi\, dV$$

Example 22 For any closed surface S, prove that

$$\iint_S \text{curl}\ \vec{F}.\hat{n}\, dS = 0$$

Solution By Gauss's divergence theorem, we have

$$\iint_S F.\hat{n}\, dS = \iiint_V div\, \vec{F}\, dV \text{ ; where } V \text{ is the volume enclosed by } S$$

$$\therefore \iint_s \text{curl}\, \vec{F}.\hat{n}\, dS = \iiint_V \text{div (curl}\, \vec{F})\, dV = 0 \qquad (\because\ \text{div curl}\ \vec{F} = 0)$$

Example 23 If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$ a, b and c are constant, show that $\iint_S \vec{F}.\hat{n}\, d\,S = \frac{4}{3}\pi(a+b+c)$, where S is the surface of a unit sphere.

Solution By Gauss's divergence theorem

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V \vec{\nabla}.\vec{F}\, dV \text{ ; where } V \text{ is the volume enclosed by } S$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right) dV$$

$$= \iiint_V (a+b+c)\, dV = (a+b+c)V$$

$$\iint_S \vec{F}.\hat{n}\, d\,S = (a+b+c)\frac{4}{3}\pi\,(1)^3$$

$$= \frac{4\pi}{3}(a+b+c)$$

[Since the volume V enclosed by a sphere of unit radius, i.e., $V = \frac{4}{3}\pi r^3$]

Here, $r = 1$.

Example 24 Evaluate $\iint_S x^2\, dy\, dz + y^2 dz dx + 2z(xy - x - y)\, dx\, dy$ where S is the surface of the cube

$$0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1.$$

Solution Using Gauss's divergence theorem, the given surface integral is equal to the volume integral.

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z\,(xy - x - y)\}\right] d\,V$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} [2x + 2y + 2xy - 2x - 2y]\, dx\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} 2xy\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^2}{2}\,y\right]_{x=0}^{1} dy\, dz = \int_{z=0}^{1}\int_{y=0}^{1} y\, dy\, dz$$

$$= \int_{z=0}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{1} dz = \frac{1}{2}\int_{z=0}^{1} dz = \frac{1}{2}[z]_0^1 = \frac{1}{2}$$

Example 25 Evaluate $\iint_S \vec{F}.\hat{n}\, d\,S$ where $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ and S is the surface of the sphere having the centre at (3 – 1, 2) and radius 3.

Solution Suppose V is the volume enclosed by the surface S. Then using Gauss's divergence theorem,

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V \text{div}\, \vec{F} d\,V$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x + 3z) - \frac{\partial}{\partial y}(xz + y) + \frac{\partial}{\partial z}(y^2 + 2z)\right] dV$$

$$= \iiint_V [2 - 1 + 2]\, dV$$

$$= \iiint_V 3\, dV$$

$$= 3\text{ V}$$

$$= 3.\frac{4}{3}\pi\,(3)^3 \qquad [\because \; V \text{ is the volume of a sphere of radius } 3]$$

$$= 108\,\pi.$$

Example 26 If $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$, find the value of $\iint_S \vec{F}.\hat{n}\,dS$, where S is the closed surface bounded, by the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 4$.

Solution Here

$$\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$$

$$\text{div } \vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1)$$

$$= 1 - 1 + 2z$$

$$\text{div } \vec{F} = 2z$$

Using Gauss's divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \text{div } \vec{F}\,dV$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2z\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2} 2z\cdot[x]_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2} 2z.\left[\sqrt{4-y^2} + \sqrt{4-y^2}\right]dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2} 4z\sqrt{4-y^2}\;dy\;dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{2} 4z\sqrt{4-y^2}\,dy\;dz$$

$$= 2\int_{z=0}^{1} 4z.\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right]_0^2 dz$$

$$= 2\int_{z=0}^{1} 4z\left[2\sin^{-1}\frac{2}{2}\right]dz$$

$$= 2 \times 8 \frac{\pi}{2} \left[\frac{z^2}{2} \right]_0^1$$

$$= 8\,\pi . \left[\frac{1}{2} \right] = 4\pi$$

Example 27 Verify Gauss's divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Solution The Gauss's divergence theorem

$$\iiint_V \operatorname{div} \vec{F} dV = \iint_S \vec{F} . \hat{n}\, dS$$

$$\text{L.H.S} = \iiint_v \operatorname{div} \vec{F}\, dV$$

$$= \iiint_V \vec{\nabla} . \vec{F}\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) \right] dV$$

$$= \iiint_V 2(x + y + z)\, dV$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x + y + z)\, dx\, dy\, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + xy + xz \right]_{x=0}^{a} dy\, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{a^2}{2} + ay + az \right] dy\, dz$$

$$= 2 \int_{z=0}^{c} \left[\frac{a^2}{2} y + \frac{ay^2}{2} + ayz \right]_{y=0}^{b} dz$$

$$= 2 \int_{z=0}^{c} \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right] dz$$

$$= 2 \left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab \frac{z^2}{2} \right]_{z=0}^{c}$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= abc\,(a + b + c).$$

$$\text{RHS} = \iint_S \vec{F}\,.\,\hat{n}\;dS$$

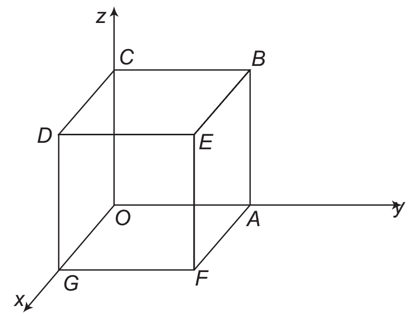


Fig. 7.13

Now, we shall evaluate the surface integral over the six faces of the rectangular parallelepiped. Over the face *DEFG*,

$$\hat{n} = \hat{i},\; x = a$$

$$\therefore \quad \iint_{DEFG} \vec{F}\,.\,\hat{n}\;dS = \int_{z=0}^{c}\int_{y=0}^{b} [(a^2 - yz)\,\hat{i} + (y^2 - az)\,\hat{j} + (z^2 - ay)\hat{k}]\cdot\hat{i}\;dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (a^2 - yz)\,dy\,dz = \left[a^2b\,z - \frac{b^2}{2}\cdot\frac{z^2}{2}\right]_{z=0}^{c}$$

$$= a^2b\,c - \frac{b^2c^2}{4} \quad \text{over the face } ABCO,\; \hat{n} = -i,\; x = 0$$

$$\therefore \quad \iint_{ABCO} \vec{F}\,.\,\hat{n}\;dS = \iint_S [(o - yz)\,\hat{i} + (y^2 - o)\,\hat{j} + (z^2 - o)\,\hat{k}]\cdot(-\hat{i})\;dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} yz\;dy\,dz$$

$$= \int_{z=0}^{c} \left[\frac{y^2\,z}{2}\right]_0^b dz$$

$$= \int_{z=0}^{c} \frac{b^2}{2}\,z\;dz = \frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c$$

$$= \frac{b^2c^2}{4} \quad \text{over the face } ABEF,\; \hat{n} = \hat{j},\; y = b$$

$$\therefore \quad \iint_{ABEF} \vec{F}\cdot\hat{n}\;ds = \int_{z=0}^{c}\int_{x=0}^{a} [(x^2 - bz)\,\hat{i} + (b^2 - zx)\,\hat{j} + (z^2 - bx)\,\hat{k}]\cdot j\;dx\,dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a} (b^2 - zx)\;dx\,dz$$

$$= b^2\,ca - \frac{a^2c^2}{4}$$

over the face *OGDC*, $\hat{n} = -\hat{j},\; y = 0$

$$\therefore \quad \iint_{OGDC} \vec{F}\cdot\hat{n}\, ds = \int_{z=0}^{c}\int_{x=0}^{a} zx\, dx\, dz = \frac{c^2 a^2}{4}$$

over the face $BCDE$, $\hat{n} = \hat{k}$, $z = c$

$$\therefore \quad \iint_{BCDE} \vec{F}\cdot\hat{n}\, ds = \int_{y=0}^{b}\int_{x=0}^{a} (c^2 - xy)\, dx\, dy = c^2\, ab - \frac{a^2 b^2}{4}$$

over the face $AFGO$, $\hat{n} = \hat{k}$, $z = 0$

$$\therefore \quad \iint_{AFGO} \vec{F}\cdot\hat{n}\, ds = \int_{y=0}^{b}\int_{x=0}^{a} xy\, dx\, dy = \frac{a^2 b^2}{4}$$

Adding the six surface integrals, we obtain

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \left(a^2\, bc - \frac{c^2 b^2}{4} + \frac{c^2 b^2}{4}\right) + \left(b^2\, ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4}\right) + \left(c^2\, ab - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4}\right)$$

$$= abc\,(a + b + c)$$

Hence,

$$\iiint_V div\, \vec{F}\, dv = \iint_S \vec{F}\cdot\hat{n}\, ds$$

The theorem is verified.

7.14 STOKES' THEOREM (RELATION BETWEEN LINE AND SURFACE INTEGRALS)

Sir George Gabriel Stokes (13 August 1819–1 February 1903) was born in Ireland. Stokes is most well-known for his study of viscous fluids, for Stokes' theorem, and for his law of viscosity. Stokes' theorem is the basic theorem of vector analysis. The law of viscosity is what describes the motion of a solid sphere in a fluid. It is also thought that his major advance was in the wave theory of light. In 1841, Stokes graduated as the top First Class degree in the Mathematical Tripos and he was the first Smith's prizeman. Pembroke College immediately gave him a Fellowship. In 1851, Stokes was elected to the Royal Society and was awarded the Rumford medal of that society in 1852. In 1854, he was appointed the Secretary.

Suppose S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $\vec{F}\,(x, y, z)$ be a continuous vector function which has continuous partial derivatives. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds = \iint_S \mathbf{curl}\, \vec{F}\cdot\hat{n}\, ds$$

where C is a simple closed curve and $\hat{n}$ is the outward unit normal vector drawn to the surface S.

Proof Suppose S be a surface, which is such that its projections on xy, yz, and zx planes are regions bounded by simple closed curves and the vector function $\vec{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$. Let S be represented simultaneously in the forms $z = f(x, y)$, $y = g(x, z)$ and $x = h(y, z)$, where f, g, h are continuous functions having continuous first-order partial derivatives.

Consider the integral

$$\iint_S (\vec{\nabla} \times F_1 \hat{i}) \cdot \hat{n}\, dS$$

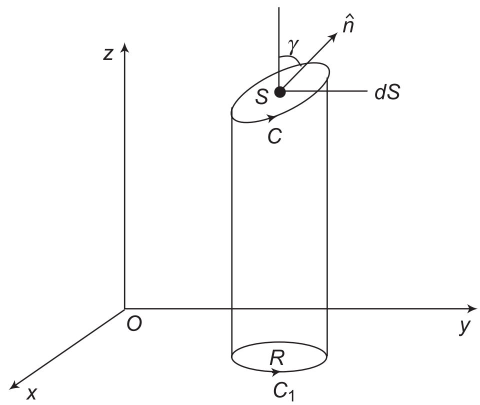


Fig. 7.14

Now,

$$\vec{\nabla} \times (\vec{F}_1\, \hat{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$$

$$= \frac{\partial F_1}{\partial z}\hat{j} - \frac{\partial F_1}{\partial y}\hat{k}$$

$$\therefore \quad [\vec{\nabla} \times (\vec{F}_1\, \hat{i})] \cdot \hat{n} = \left[\frac{\partial F_1}{\partial z}\hat{j} \cdot \hat{n} - \frac{\partial F_1}{\partial y}\hat{k} \cdot \hat{n}\right]$$

$$= \frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma$$

Now,
$$\iint_S [\vec{\nabla} \times (\vec{F}_1\, \hat{i})] \cdot \hat{n}\, dS = \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS$$

Our aim is to show that

$$\iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] ds = \oint_C F_1\, dx$$

Let R be the orthogonal projection of S on the xy-plane and $z = f(x, y)$ of S

$\therefore$ the line integral over C as a line integral over C_1 then

$$\oint_C F_1\,(x, y, z)\, dx = \oint_{C_1} F_1\,[x, y, f(x, y)]\, dx$$

$$= \oint_{C_1} \{F_1\,[x, y, f(x, y)]\, dx + 0\, dy\}$$

$$= -\iint_R \frac{\partial F_1}{\partial y}\, dx\, dy \qquad \text{[By Green's theorem in the plane for the region } R\text{].}$$

But
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \cdot \frac{\partial f}{\partial y} \qquad [\because \; z = f(x, y)].$$

$$\therefore \quad \oint_C F_1\, dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \cdot \frac{\partial f}{\partial y}\right) dx\, dy \qquad (5)$$

Now, the equation $z = f(x, y)$ of the surface S can be written as

$$\phi(x, y, z) \equiv z - f(x, y) = 0 \ldots \qquad (6)$$

$$\therefore \qquad \text{grad } \phi = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$$

$$\hat{n} = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{\text{grad } \phi}{a} \quad \text{say } |\text{grad } \phi| = a$$

$$(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{j} + (\cos\gamma)\,\hat{k} = -\frac{1}{a}\frac{\partial f}{\partial x}\hat{i} - \frac{1}{a}\frac{\partial f}{\partial y}\hat{j} + \frac{1}{a}\hat{k}$$

$$\therefore \qquad \cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x}, \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y} \text{ and } \cos\gamma = \frac{1}{a}$$

Now, $$dS = \frac{dxdy}{\cos\gamma} = a\cdot dxdy$$

$$\therefore \qquad \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS$$

$$= \iint_R \left[\frac{\partial F_1}{\partial z}\cdot\left(\frac{-1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\cdot\left(\frac{+1}{a}\right)\right] dx\, dy$$

$$= -\iint_R \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y}\right] dx\, dy \tag{7}$$

From (5) and (7), we get

$$\oint_C F_1\, dx = \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS = \iint_S [\vec{\nabla}\times(\vec{F}_1\,\hat{i})]\cdot\hat{n}\, dS \tag{8}$$

Similarly, by the projections on the other coordinate planes, we get

$$\oint_C F_2\, dy = \iint_S [\vec{\nabla}\times(\vec{F}_2\,\hat{i})]\cdot\hat{n}\, dS \tag{9}$$

$$\oint_C F_3\, dz = \iint_S [\vec{\nabla}\times(\vec{F}_3\,\hat{i})]\cdot\hat{n}\, dS \tag{10}$$

Adding equations (8), (9) and (10) we obtain

$$\oint_C (F_1\, dx + F_2\, dy + F_3\, dz) = \iint_S [\vec{\nabla}\times(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})]\cdot\hat{n}\, dS$$

$$\boldsymbol{\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS = \iint_S \textbf{curl } \vec{F}\cdot\hat{n}\, dS}$$

Example 28 Prove that $\oint_C \vec{r}\cdot d\vec{r} = 0.$

Solution Using Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds$$

$$\therefore \qquad \oint_C \vec{r}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{r}\cdot\hat{n}\,ds$$

$$= \iint_S 0\cdot\hat{n}\,ds \qquad [\because \quad \text{curl}\,\vec{r} = 0]$$

$$= 0$$

Hence, proved.

Example 29 Evaluate $\oint_C [e^x\,dx + 2y\,dy - dz]$; by Stokes' theorem where C is the curve $x^2 + y^2 = 4,\ z = 2$.

Solution Using Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

Now,

$$\oint_C e^x\,dx + 2y\,dy - dz = \oint_C (e^x\,\hat{i} + 2y\,\hat{j} - \hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint_C \vec{F}\cdot d\vec{r}\text{ ; where } \vec{F} = e^x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

$$\text{curl}\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = 0$$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= 0\text{; since curl } \vec{F} = 0.$$

Example 30 Apply Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)\,\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0. 0), (1. 0, 0), (1, 1, 0).

Solution We have $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)\,\hat{k}$

$$\text{curl}\ \vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\,\hat{i} + \hat{j} + 2(x-y)\,\hat{k}$$

Since the triangle lies in the xy-plane so that $\hat{n} = \hat{k}$,

$$\therefore \qquad \text{curl}\ \vec{F}\cdot\hat{n} = [\hat{j} + 2(x-y)\,\hat{k}]\cdot\hat{k} = 2(x-y)$$

Using Stokes' theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, dS$$
$$= \iint_S 2(x-y)\, dS$$
$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\, dxdy$$
$$= 2\int_{x=0}^{1}\left[xy - \frac{y^2}{2}\right]_{y=0}^{x} dx$$
$$= 2\int_{x=0}^{1}\left[\left(x^2 - \frac{x^2}{2}\right) - 0\right] dx$$
$$= 2\int_{x=0}^{1}\frac{x^2}{2}\, dx = \int_{x=0}^{1} x^2\, dx$$
$$= \left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{3}$$

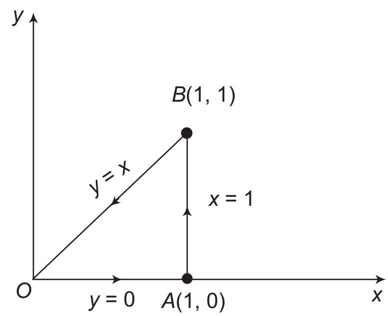


Fig. 7.15

Example 31 Verify Stokes' theorem for the vector field $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$, integrated around the rectangle $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$, and $y = b$.

Solution We have

$$\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$$

$$\therefore \quad \vec{F}\cdot d\vec{r} = [(x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}]\cdot[dx\,\hat{i} + dy\,\hat{j}]$$
$$= (x^2 - y^2)\, dx + 2xy\, dy$$

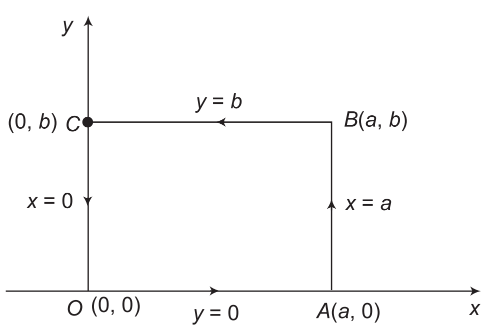


Fig. 7.16

Stokes' theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds \tag{1}$$

Now,

$$\oint_C \vec{F}\cdot \vec{d}\, r = \oint_C (x^2 - y^2)\, dx + 2\, xy\, dy$$

$$= \oint_{OA} + \oint_{AB} + \oint_{BC} + \oint_{CO} \tag{2}$$

Along the line OA; $y = 0$ so that $dy = 0$ and x varies from 0 to a.
Along the line AB; $x = a$ so that $dx = 0$ and y varies from 0 to b.
Along the line BC; $y = b$ so that $dy = 0$ and x varies from a to 0.
Along the line CO; $x = 0$ so that $dx = 0$ and y varies from b to 0.

$\therefore$ Eq. (2), becomes

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^a x^2 dx + \int_0^b 2ay\, dy + \int_a^0 (x^2 - b^2)\, dx + \int_b^0 O$$

$$= \left[\frac{x^3}{3}\right]_0^a + 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} - b^2\, x\right]_0^a + 0$$

$$= \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3}$$

$$\oint_C \vec{F}\cdot d\vec{r} = 2\, ab^2$$

On the other hand,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x^2 - y^2) & 2\, xy & O \end{vmatrix} = 4\, y\, \hat{k}$$

$$\therefore \quad \iint_S \text{Curl } \vec{F}\cdot\hat{n}\, ds = 4\int_{x=0}^{a}\int_{y=0}^{a} y\, dy\, dx \qquad [\because \ \hat{n} = \hat{k}]$$

$$= 4\int_0^a \left[\frac{y^2}{2}\right]_0^b dx = 2\, b^2 \int_0^a dx = 2\, ab^2$$

Hence,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds\,.$$

Stokes' theorem is verified.

Example 32 Verify Stokes' theorem for $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$, where S is the upper-half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution The boundary C of S is a circle in the xy-plane of unity radius and centre at the origin. The equations of the curve C are $x^2 + y^2 = 1$, $z = 0$.

Let $x = \cos t$, $y = \sin t$, $z = 0$; $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (y\,\hat{i} + z\,\hat{j} + x\,\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint_C y\,dx + z\,dy + x\,dz$$

$$= \oint_C y\,dx \qquad \text{[Since on the boundary } C,\ z = 0 \text{ so } dz = 0]$$

$$= \int_0^{2\pi} \sin t \cdot \frac{dx}{dt}\,dt$$

$$= \int_0^{2\pi} -\sin^2 t\,dt = -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)\,dt$$

$$= -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_o^{2\pi} = -\pi$$

Now, we evaluate $\iint_S \text{Curl } \vec{F}\cdot\hat{n}\,ds$

We have curl $\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\hat{i} + \hat{j} + \hat{k})$

If R is the plane region bounded by the circle C,

$$\therefore \quad \iint_S \text{curl } \vec{F}\cdot\hat{n}\,dS = \iint_R \text{curl } \vec{F}\cdot\hat{k}\,dS$$

$$= -\iint_R (i + j + k)\cdot\hat{k}\,dS = -\iint_R dS = -R$$

But R = area of a circle of radius unity $= \pi(1)^2 = \pi$

$$\therefore \quad \iint_S \text{curl } \vec{F}\cdot\hat{n}\,dS = -\pi$$

Hence,
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\,dS$$

Stokes' theorem is verified.

EXERCISE 7.3

1. Evaluate $\int_C [e^{-x} \sin y\, dx + e^{-x} \cos y\, dy]$ by Green's theorem in plane where C is the rectangle with vertices (0, 0). (π, 0), $\left(\pi, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$.

2. If $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$ and $\vec{r} = x\,\hat{i} + y\,\hat{j}$, find the value of $\oint_C \vec{F}\cdot d\vec{r}$ around the rectangular boundary $x = 0$, $x = a$, $y = 0$ and $y = b$.

3. Apply Green's theorem in the plane to evaluate $\int_C [(y - \sin x)\, dx + \cos x\, dy]$; where C is the triangle enclosed by the lines $y = 0$, $x = \pi$, $\pi y = 2x$.

4. Apply Green's theorem in the plane to evaluate $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, where C is the boundary of the surface enclosed by the x-axis and the semicircle $y = \sqrt{(1 - x^2)}$.

5. Verify Green's theorem in the plane for $\int_C [(2xy - x^2)\, dx + (x^2 + y^2)\, dy]$, where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$.

6. Verify Green's theorem in the plane for $\int_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region defined by $y^2 = x$ and $x^2 = y$.

7. Apply Gauss's divergence theorem to evaluate $\iint_S \vec{F}\cdot\hat{n}\, dS$; where $\vec{F} = x\,\hat{i} - y\,\hat{j} + (z^2 - 1)\,\hat{k}$ and S is the closed surface bounded by the planes $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 4$.

8. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, dS$ by Gauss's divergence theorem, where $\vec{F} = x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

9. Evaluate by Gauss's divergence theorem of the integral $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS$, where $\vec{F} = [xy\, e^z + \log(z + 1) - \sin x]\,\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

10. Verify Gauss's divergence theorem for $\vec{F} = 4xy\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ taken over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

11. Verify Gauss's divergence theorem for $\vec{F} = 4x\,\hat{i} - 2y^2\,\hat{j} + z^2\,\hat{k}$ taken over the region bounded by the surfaces $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

12. Verify Gauss's divergence theorem to show that $\iint_S [(x^3 - yz)\,\hat{i} - 2x^2 y\,\hat{j} + 2\,\hat{k}]\cdot\hat{n}\, dS = \frac{a^5}{3}$, where S denotes the surface of the cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$.

13. Evaluate the integral $\oint_C [e^x\,dx + 2y\,dy - dz]$, by Stokes' theorem where C is the curve $x^2 + y^2 = 4$ and $z = 2$.

14. Verify Stoke's theorem for the function $\vec{F} = x^2\,\hat{i} + xy\,\hat{j}$, integrated around the square in the plane $z = 0$, whose sides are along the lines $x = 0$, $x = a$, $y = 0$, and $y = a$.

15. Evaluate by Stoke's theorem of the integral $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$.

16. Verify Stokes' theorem for the function $\vec{F} = z\,\hat{i} + x\,\hat{j} + y\,\hat{k}$, where the curve is the unit circle in the xy-plane bounded by the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$.

17. Evaluate $\iint_S (\vec{\nabla} \times \vec{F})\cdot\hat{n}\,dS$, where $\vec{F} = (y - z + 2)\,\hat{i} + (yz + 4)\,\hat{j} - xz\,\hat{k}$ and S is the surface of the cube $x = y = z = 0$, $x = y = z = 2$ above the xy-plane.

18. Verify Stokes' theorem for the function $\vec{F} = xy\,\hat{i} + xy^2\,\hat{j}$, integrated round the square with vertices (1, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 0, 0).

19. Verify Stokes' theorem for $\vec{F} = -y^3\,\hat{i} + x^3\,\hat{j}$, where S is the circular disc $x^2 + y^2 \le 1$, $z = 0$.

Answers

1. $2(e^{-\pi} - 1)$

2. $2\,ab^2$

3. $\left(-\dfrac{\pi}{4} - \dfrac{2}{\pi}\right)$

4. $\dfrac{4}{3}$

5. $\oint_C M dx + N dy = \iint_S \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) dx\,dy = 0$

6. $\oint_C M dx + N dy = \iint_S \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) dx\,dy = \dfrac{3}{2}$

7. 4π

8. $4\pi a^5$

9. 0

10. $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\,\vec{F}\,dv = \dfrac{3}{2}$

11. Each of the two integrals is $84\,\pi$

13. 0

15. 2

16. Each side of integrals is π

17. –4

19. Each side of the integrals is $\dfrac{3\pi}{2}$

SUMMARY

1. Gradient of a Scalar Field

The gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change.

The gradient of a scalar field 'f' is denoted by grad f or $\vec{\nabla} f$ and is defined as

$$\vec{\nabla} f = \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}.$$

The unit normal vector $\hat{n} = \dfrac{\text{grad } f}{|\text{grad } f|}$.

2. Divergence of a Vector Field

Let $\vec{V} = V_1(x, y, z)\,\hat{i} + V_2(x, y, z)\,\hat{j} + V_3(x, y, z)\,\hat{k}$ be a differential vector function, where x, y, z are Cartesian coordinates, and V_1, V_2, V_3 are the components of $\vec{V}$. Then the divergence of $\vec{V}$ is denoted by div $\vec{V}$ or $\vec{\nabla} \cdot \vec{V}$ and is defined as

$$\text{div } \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Another common notation for the divergence of $\vec{V}$ is

$$\text{div } \vec{V} \text{ or } \vec{\nabla} \cdot \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k})$$

$$\text{div } \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Hence, the divergence of a vector function is a scalar function.

Note Let $\vec{V}$ denote the velocity of a fluid in a medium.

If div $\vec{V} = 0$ then the fluid is said to be incompressible. In electromagnetic theory, if div $\vec{V} = 0$ then the vector field $\vec{V}$ is said to be solenoidal.

3. Curl of a Vector Function

The vector function $\vec{V}$ is denoted by curl $\vec{V}$ or $\vec{\nabla} \times \vec{V}$ and is defined as

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}, \text{ where } \vec{V} = V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial z} \right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)\hat{k}$$

$$= \sum \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right)\hat{i}$$

where $\sum$ denotes summation obtained by the cyclic rotation of the unit vectors $\hat{i}, \hat{j}, k$, the components V_1, V_2, V_3, and the independent variables x, y, z respectively.

Note **(i)** The curl of a vector function is a vector function.

(ii) **Irrotational Vector:** If curl ($\vec{V}$) = 0 then the vector function $\vec{V}$ is said to be an irrotational vector.

(iii) A force field $\vec{F}$ is said to be conservative if it is derivable from a potential function f, i.e., $\vec{F}$ = grad f. Then curl $\vec{F}$ = curl (grad f) = 0.

$\therefore$ if $\vec{F}$ is conservative then curl $\vec{F} = 0$ and there exists a scalar potential function f such that $\vec{F}$ = grad f.

4. Important Vector Identities

(i) $\text{div } (\vec{A}+\vec{B}) = \text{div } \vec{A} + \text{div } \vec{B}$

(ii) $\text{Curl } (\vec{A}+\vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$

(iii) If $\vec{A}$ is a differentiable vector function and ϕ is a differentiable scalar function then $\text{div } (\phi \vec{A}) = (\text{grad } \phi)\cdot \vec{A} + \phi \text{ div } \vec{A}$

(iv) $\text{Curl } (\phi \vec{A}) = (\text{grad } \phi)\times \vec{A} + \phi \text{ curl } \vec{A}$

(v) $\text{div } (\vec{A}\times\vec{B}) = \vec{B}\cdot \text{curl } \vec{A} - \vec{A}\cdot \text{curl } \vec{B}$

(vi) Curl of the gradient of ϕ is zero, i.e., curl (grad (ϕ) = 0

(vii) div curl $\vec{A} = 0$.

(viii) div grad $f = \nabla^2\phi$, where ∇ is a Laplace operator.

5. Conservative Field and Scalar Potential

Conservative field is a vector field which is the gradient of a function, known as a scalar potential. Conservative fields have the property that the line integral from one point to another is independent of the choice of path connecting the two points. It is path-independent. Conversely, path independence is equivalent to the vector field being conservative. Conservative vector fields are also irrotational. Mathematically, a vector field $\vec{F}$ is said to be conservative if there exists a scalar field ϕ such that $\vec{F} = \vec{\nabla}\phi$.

Here, $\vec{\nabla}\phi$ denotes the gradient of ϕ. When the above equation holds, ϕ is called a *scalar potential* for $\vec{F}$.

If $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}\phi = 0$. In such a case, $\vec{F}$ is called a *conservative vector field.*

6. Green's Theorem in the Plane: Transformation between Line and Double Integral

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ in R. Then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

The line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

7. Gauss's Divergence Theorem (Relation between Volume and Surface Integers)

Let $\vec{F}$ be a vector function of position which is continuous and has continuous first-order partial derivatives, in a volume V bounded by a closed surface S. Then

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dv = \iint_S \vec{F}.\hat{n}\,ds$$

where $\hat{n}$ is the outward drawn unit normal vector to S.

8. Stokes' Theorem (Relation between Line and Surface Integrals)

Suppose S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $\vec{F}(x, y, z)$ be a continuous vector function. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds = \iint_S \text{curl }\vec{F}\cdot\hat{n}\,ds$$

where C is a simple closed curve and $\hat{n}$ is the outward unit normal vector drawn to the surface S.

OBJECTIVE-TYPE QUESTIONS

1. The magnitude of the gradient of the function $f(x, y, z) = xyz^3$ at (1, 0, 2) is

(a) 0 (b) 3
(c) 8 (d) ∞

[GATE (ME) 1998]

2. If the velocity vector in a two-dimensional fluid is given by $\vec{v} = 2xy\hat{i} + (2y^2 - x^2)\hat{j}$, the curl $\vec{v}$ will be

(a) $2y^2\hat{j}$ (b) $6y\hat{k}$
(c) 0 (d) $-4x\hat{k}$

[GATE (ME) 1999]

3. In a flow fluid in xy-plane, the variation of velocity with limit is given by

$$\vec{v} = (x^2 + yt)\hat{i} \text{ and } \vec{v}(x^2 + y^2)\hat{i}$$

The acceleration of the particle in this field, occupying the point (1, 1) at time $t = 1$ will be

(a) $\hat{i}$ (b) $2\hat{i}$
(c) $3\hat{i}$ (d) $5\hat{i}$

[GATE (ME) 1999]

4. The maximum value of the directional derivative of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at a point (1, 1, –1) is

(a) 10 (b) –4
(c) $\sqrt{152}$ (d) 152

[GATE (ME) 2000]

5. The divergence of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is

(a) $\hat{i} + \hat{j} + \hat{k}$ (b) 3
(c) 0 (d) 1

[GATE (ME) 2001]

6. Gauss' divergence theorem relates certain

(a) surface integrals to volume integrals
(b) surface integrals to line integrals
(c) vector quantities to other vector quantities
(d) line integral to volume integrals

[GATE (ME) 2001]

7. Given a vector field $\vec{F}$, the divergence theorem states that

(a) $\int_S \vec{F}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{F}\,dV$

(b) $\int_S \vec{F} \cdot d\vec{S} = \int_V \vec{\nabla} \times \vec{F}\, dV$

(c) $\int_S \vec{F} \times d\vec{S} = \int_V \vec{\nabla} \cdot \vec{F}\, dV$

(d) $\int_S \vec{F} \times d\vec{S} = \int_V \vec{\nabla} \times \vec{F}\, dV$

[GATE (EE) 2002]

8. The vector field $\vec{F} = x\hat{i} - y\hat{j}$ is

(a) divergence free, but not irrotational
(b) irrotational, but not divergence free
(c) divergence free and irrorational
(d) neither divergence free nor irrotational

[GATE (ME) 2003]

9. The velocity field is given by $\vec{v} = 2y\hat{i} + 3x\hat{j}$, where x and y are in metres. The acceleration of a fluid particle at $(x, y) = (1, 1)$ in the x-direction is

(a) 0 m/s^2 (b) 5.00 m/s^2
(c) 6.00 m/s^2 (d) 8.40 m/s^2

[GATE (CE) 2004]

10. Value of the integral $\oint_C (xy\,dy - y^2 dx)$, where C is the square cut from the first quadrant by the line $x = 1$ and $y = 1$ will be (use Green's theorem to change the line integral with double integral)

(a) $\frac{1}{2}$ (b) 1

(c) $\frac{3}{2}$ (d) $\frac{5}{2}$

[GATE (ME) 2005]

11. Stokes' theorem connects

(a) a line integral and a surface integral
(b) a surface integral and a volume integral
(c) a line integral and a volume integral
(d) gradient of a function and its surface integral **[GATE (ME) 2005]**

12. The line integral $\int \vec{V} \cdot d\vec{r}$ of the vector field $\vec{V}(r) = 2xyz\,\hat{i} + x^2 z\,\hat{j} + x^2 y\,\hat{k}$ from the origin to the point $P(1, 1, 1)$ is

(a) 1 (b) 0
(c) –1
(d) cannot be determined without specifying the path **[GATE (ME) 2005]**

13. For a scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1, 3) is

(a) $\sqrt{\frac{13}{9}}$ (b) $\sqrt{\frac{9}{2}}$

(c) $\sqrt{5}$ (d) $\frac{9}{2}$

[GATE (EE) 2005]

14. If a vector $\vec{R}(t)$ has a constant magnitude then

(a) $\vec{R} \cdot \frac{d\vec{R}}{dt} = 0$ (b) $\vec{R} \times \frac{d\vec{R}}{dt} = 0$

(c) $\vec{R} \cdot \vec{R} = \frac{d\vec{R}}{dt}$ (d) $\vec{R} \times \vec{R} = \frac{d\vec{R}}{dt}$

[GATE (IE) 2005]

15. $\nabla \times \nabla \times \vec{P}$ where $\vec{P}$ is a vector, is equal to

(a) $\vec{P} \times \nabla \times P$ (b) $\nabla^2 \vec{P} + \nabla(\nabla \cdot \vec{P})$

(c) $\nabla P^2 + \nabla \times \vec{P}$ (d) $\nabla(\nabla \cdot \vec{P}) - \nabla^2 P$

[GATE (ECE) 2006]

16. $\iint (\nabla \times \vec{P}) \cdot dS$, where $\vec{P}$ is a vector, is equal to

(a) $\oint \vec{P} \cdot dl$ (b) $\oint \nabla \times \nabla \times \vec{P} \cdot dl$

(c) $\oint \nabla \times \vec{P} \cdot dl$ (d) $\iiint \nabla \cdot \vec{P}\, dl$

[GATE (ECE) 2006]

17. Divergence of the field

$$V(x, y, z) = -(x\cos xy + y)\hat{i} + (y\cos xy)\hat{j} + (\sin z^2 + x^2 + y^2)\hat{k}$$ is

(a) $2z\cos z^2$ (b) $\sin xy + 2z\cos z^2$
(c) $x\sin xy - \cos z$ (d) none

[GATE (EE) 2007]

18. A velocity vector is given as

$$\vec{V} = 5xy\,\hat{i} + 2y^2\,\hat{j} + 3yz^2\,\hat{k}$$

The divergence of the velocity vector at (1, 1, 1) is

(a) 9 (b) 10
(c) 14 (d) 15

[GATE (CE) 2007]

19. The inner (dot) product of two vectors $\vec{P}$ and $\vec{Q}$ is zero. The angle (degrees) between the two vectors is

(a) 0 (b) 5
(c) 90 (d) 120

[GATE (CE) 2008]

20. The divergence of the vector field $(x-y)\hat{i}+(y-x)\hat{j}+(x+y+z)\hat{k}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[GATE (ME) 2008]

21. The directional derivative of the scalar function $f(x, y, z) = x^2 + 2y^2 + z$ all at the point $P(1, 1, 2)$ in the direction of the vector $\vec{a} = 3\hat{i} + 4\hat{j}$ is

(a) –4 (b) –2
(c) –1 (d) 1

[GATE (ME) 2008]

22. The divergence of a vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1, 1, 1) is equal to

(a) 7 (b) 4
(c) 3 (d) 0

[GATE (ME) 2009]

23. For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point $P(1, 2, -1)$ is

(a) $2\hat{i}+6\hat{j}+4\hat{k}$ (b) $2\hat{i}+12\hat{j}-4\hat{k}$
(c) $2\hat{i}+12\hat{j}+4\hat{k}$ (d) $\sqrt{56}$

[GATE (CE) 2009]

24. For scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the directional derivative at the point $P(1, 2, -1)$ in the direction of a vector $\hat{i}-\hat{j}+2\hat{k}$ is

(a) –18 (b) $-3\sqrt{6}$
(c) $3\sqrt{6}$ (d) 18

[GATE (CE) 2009]

25. The line integral of the vector function $\vec{F} = 2x\hat{i} + x^2\hat{j}$ along the X-axis from $x = 1$ to $x = 2$ is

(a) 0 (b) 2.33
(c) 3 (d) 5.33

[GATE (IPE) 2009]

26. Consider a closed surface S surrounding volume V. If $\vec{r}$ is the position vector of a point inside S, with $\hat{n}$ the unit normal on S, the value of the integral $\oiint_S 5\hat{r}\cdot n dS$ is

(a) 3 V (b) 5 V
(c) 10 V (d) 15 V

[GATE (EC) 2011]

27. The direction of the vector A is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is constant. The value of n for which $\nabla\cdot A = 0$ is

(a) –2 (b) 2
(c) 1 (d) 0

[GATE (EE) 2012]

28. For the spherical surface, $x^2 + y^2 + z^2 - 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is given by

(a) $\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$

(b) $\frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}$

(c) $\hat{k}$

(d) $\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$

[GATE (ME) 2012]

29. For a vector E, which one of the following statements is NOT TRUE?

(a) If $\nabla\cdot E = 0$, E is called solenoidal
(b) If $\nabla\times E = 0$, E is called conservative
(c) If $\nabla\times E = 0$, E is called irrotational
(d) If $\nabla\cdot E = 0$, E is called irrotational

[GATE (IN) 2013]

30. The curl of the gradient of the scalar field defined by $V = 2x^2y + 3y^2z + 4z^2x$ is

(a) $4xya_x + 6yza_y + 8zxa_z$
(b) $4a_x + 6a_y + 8a_z$
(c) $(4xy+4z^2)a_x+(2xy^2+6yz)a_y+(3y^2+8zx)a_z$
(d) 0

[GATE (EE) 2013]

31. Given a vector field field $F = y^2xa_x - yza_y = x^2a_z$, the line integral $\int F\cdot dl$ evaluated along a segment on the x-axis from $x = 1$ to $x = 2$ is

(a) –2.33 (b) 0

(c) 2.33 (d) 7

[GATE (EE) 2013]

32. The divergence of the vector field $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is

(a) 0 (b) 1/3

(c) 1 (d) 3

[GATE (EC) 2013]

33. The following surface integral is to be evaluated over a sphere for the given steady velocity vector field, $F = xi + yj + zk$ defined with respect to a Cartesian coordinate system having i, j, and k as a unit base vectors.

$$\iint_S \frac{1}{4}(F \cdot n)\,dA$$

where S is the sphere, $x^2 + y^2 + z^2 = 1$ and n is the outward unit normal vector to the sphere. The value of the surface integral is

(a) π (b) 2π

(c) $3\frac{\pi}{4}$ (d) 4π

[GATE (ME) 2013]

34. Consider a vector field $\vec{A}(\vec{r})$. The closed loop line integral $\oint \vec{A} \cdot d\vec{l}$ can be expressed as

(a) $\oiint (\nabla \times \vec{A}) \cdot d\vec{s}$ over the closed surface bounded by the loop

(b) $\oiint (\nabla \cdot \vec{A}) \cdot dv$ over the closed volume bounded by the loop

(c) $\iiint (\nabla \cdot \vec{A})\,dv$ over the open volume bounded by the loop

(d) $\iint (\nabla \times \vec{A}) \cdot d\vec{s}$ over the closed surface bounded by the loop

[GATE (EC) 2013]

35. Divergence of the vector field $x^2 z\hat{i} + xy\hat{j} - yz^2\hat{k}$ at $(1, -1, 1)$ is

(a) 0 (b) 3

(c) 5 (d) 6

[GATE (ME) 2014]

36. In a steady incompressible flow, the velocity distribution is given by $\bar{V} = 3x\hat{i} - Py\hat{j} + 5z\hat{k}$, where, V is in m/s and x, y, and z are in m. In order to satisfy the mass conservation, the value of the constant P(in s^{-1}) is ______.

[GATE (CH) 2014]

37. Given the vector $A = (\cos x)(\sin y)\hat{a}_x + (\sin x)(\cos y)\hat{a}_y$, where $\hat{a}_x, \hat{a}_y$ denote unit vectors along x, y directions, respectively. The magnitude of curl of A is ______.

[GATE (EC) 2014]

38. The directional derivative of $f(x, y) = \frac{xy}{\sqrt{2}}(x + y)$ at (1, 1) in the direction of the unit vector at an angle of $\frac{\pi}{4}$ with y-axis, is given by ______.

[GATE (EC) 2014]

39. The magnitude of the gradient for the function $f(x, y, z) = x^2 + 3y^2 + z^3$ at the point (1, 1, 1) is ______.

[GATE (EC) 2014]

40. If $r = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and $|\vec{r}| = r$, then div $(r^2\nabla(\ln r))$ = ______.

[GATE (EC) 2014]

41. Given $\bar{F} = z\hat{a}_x + x\hat{a}_y + y\hat{a}_z$. If S represents the portion of the sphere $x^2 + y^2 + z^2 = 1$ for $z \geq 0$, then $\int_S \nabla \times \vec{F} \cdot d\vec{s}$ is ______.

[GATE (EC) 2014]

42. If $E = -(2y^3 - 3yz^2)\hat{x} - (6xy^2 - 3xz)\hat{y} + (6xyz)\hat{z}$ is the electric field in a source free region, a valid expression for the electrostatic potential is

(a) $xy^3 - yz^2$ (b) $2xy^3 - xyz^2$

(c) $y^3 + xyz^2$ (d) $2xy^3 - 3xyz^2$

[GATE (EC) 2014]

43. The line integral of function $F = yzi$, in the counterclockwise direction, along the circle $x^2 + y^2 = 1$ at $z = 1$ is

(a) -2π (b) $-\pi$

(c) π (d) 2π

[GATE (EE) 2014]

44. Which one of the following describes the relationship among the three vectors, $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $5\hat{i} + 6\hat{j} + 4\hat{k}$?

(a) The vectors are mutually perpendicular

(b) The vectors are linearly dependent

(c) The vectors are linearly independent
(d) The vectors are unit vectors

[GATE (ME) 2014]

45. Curl of vector $\vec{F} = x^2z^2\,\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\,\hat{j} - 2y^2z\,\hat{k}$

(b) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\,\hat{j} - 2y^2z\,\hat{k}$

(c) $2xz^2\,\hat{i} - 4xyz\,\hat{j} + 6y^2z^2\,\hat{k}$

(d) $2xz^2\,\hat{i} + 4xyz\,\hat{j} + 6y^2z^2\,\hat{k}$

[GATE (ME) 2014]

46. Gradient of a scalar variable is always

(a) a vector (b) a scalar
(c) a dot product (d) zero

[GATE (CH) 2014]

47. For an incompressible flow field $\vec{v}$, which one of the following conditions must be satisfied?

(a) $\nabla \cdot \vec{v} = 0$ (b) $\nabla \times \vec{v} = 0$

(c) $(\vec{v} \cdot \nabla) \times \vec{v} = 0$ (d) $\dfrac{\partial V}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = 0$

[GATE (ME) 2014]

ANSWERS

1. (c)	2. (d)	3. (d)	4. (c)	5. (b)	6. (a)	7. (a)	8. (c)	9. (d)	10. (b)
11. (a)	12. (a)	13. (c)	14. (a)	15. (d)	16. (a)	17. (a)	18. (d)	19. (c)	20. (d)
21. (b)	22. (c)	23. (b)	24. (b)	25. (c)	26. (d)	27. (a)	28. (a)	29. (d)	30. (d)
31. (b)	32. (d)	33. (a)	34. (d)	35. (c)	36. (7.99 to 8.01)		37. (0)	38. (3)	39. (7)
40. (3)	41. (3.14)	42. (d)	43. (b)	44. (b)	45. (a)	46. (a)	47.(a)		

8 Infinite Series

8.1 SEQUENCE

Definition

A function whose domain is the set of natural numbers N and range, a subset of real numbers R is called a *sequence*.

A sequence is of the form $\{(1, x_1), (2, x_2), (3, x_3), \ldots, (n, x_n)\}$, where $x_1, x_2, \ldots, x_n$ are real numbers.

The real number x_n that the sequence associates with the positive integer n is called the image of n under the sequence.

Generally, it is denoted by $\{x_1, x_2, \ldots, x_n\}$.

Here, $x_1, x_2, \ldots, x_n$ are the terms of the sequence, and so x_n is the n^{th} term of the sequence.

A sequence has its n^{th} term denoted by $\{x_n\}$.

8.2 THE RANGE

The range set is the set consisting of all distinct elements of a sequence, without repetition and without regard to the position of a term. Thus, the range may be a finite set or an infinite set.

8.3 BOUNDS OF A SEQUENCE

(i) Bounded-above Sequences

A sequence $\{x_n\}$ is said to be bounded above if there exists a real number M such that $x_n \le M \; \forall \; n \in N$.

(ii) Bounded-below Sequences

A sequence $\{x_n\}$ is said to be bounded below if there exists a real number m such that $x_n \ge m, \; \forall \; n \in N$.

(iii) Bounded Sequence

A sequence is said to be bounded if it is bounded above and below. M and m are the upper and the lower bounds of the sequence, for example,

$$x_n = \{(-1)^n; \; n \in N\} \text{ is a bounded sequence.}$$

8.4 CONVERGENCE OF A SEQUENCE

A sequence $\{x_n\}$ is said to converge to a real number 'l' if for each $\in\ > 0$, there exists a positive integer m (depending on $\in$) such that $|x_n - l| < \in$, for all $n \geq m$.

Mathematically, we write

$$x_n \to l \text{ as } n \to \infty \quad \text{or} \quad \lim_{n\to\infty} x_n = l$$

*A sequence $\{x_n\}$ is said to be divergent if $\lim_{n\to\infty} x_n$ is not a finite quantity, i.e., if $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$.

Examples: (i) Let a sequence $\{x_n\} = \{n^2\}$

Then $\lim_{n\to\infty} n^2 = \infty$ (which is not a finite)

Hence, $\{x_n\}$ is divergent.

(ii) Let $\{x_n\} = \left[\frac{1}{2^n}; n \in N\right]$

Then $\lim_{n\to\infty} \frac{1}{2^n} = 0$ (a finite quantity)

Hence, the sequence $\{x_n\}$ is convergent.

Oscillatory Sequence

A sequence $\{x_n\}$ which neither converges to a finite number nor diverges to ∞ or $-\infty$ is said to be an oscillatory sequence.

Example: $\{x_n\} = \{(-1)^n\}$ oscillates finitely between -1 and $+1$ and the sequence $\{x_n\} = \{n(-1)^n\}$ oscillates infinitely between $-\infty$ and $+\infty$.

Note

(i) Every convergent sequence has a unique limit.

(ii) Every convergent sequence is bounded.

8.5 MONOTONIC SEQUENCE

A sequence $\{x_n\}$ is said to be monotonic increasing if $x_{n+1} \geq x_n$, $\forall\ n$

A sequence $\{x_n\}$ is said to be monotonic decreasing if $x_{n+1} \leq x_n$, $\forall\ n$

Thus, a sequence $\{x_n\}$ is said to be monotonic if it is either monotonic increasing or decreasing.

A sequence $\{x_n\}$ is strictly increasing if $x_{n+1} > x_n$, $\forall\ n$ and strictly decreasing if $x_{n+1} < x_n$, $\forall\ n$

Examples: (i) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ is a monotonic decreasing sequence.

(ii) $\{x_n\} = \left\{\frac{n}{n+1}\right\}$ is a monotonic increasing sequence.

(iii) $\{x_n\} = \{n\}$ is monotonic sequence.

8.6 INFINITE SERIES

If $< u_n >$ be a sequence of real numbers then the sum of the infinite number of terms of this sequence, i.e., the expression $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is defined as an infinite series and is denoted by

$$\sum_{n=1}^{\infty} u_n \text{ or } \sum u_n$$

A sequence $< S_n >$ where S_n denotes the sum of the first n terms of the series.

Thus, $S_n = u_1 + u_2 + u_3 + \cdots + u_n; \forall\, n.$

The sequence $< S_n >$ is called the *sequence of partial sums of the series* and the partial sums, $S_1 = u_1, S_2 = u_1 + u_2, S_3 = u_1 + u_2 + u_3$ and so on.

8.6.1 Positive-term Series

The series $\sum u_n$ is called a *positive-term series* if each term of this series is positive, i.e.,

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

8.6.2 Alternating Series

A series whose terms are alternatively positive and negative, i.e., $\sum u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

8.6.3 Convergent Series

A series $\sum u_n$ is said to be convergent if the sum of the first n terms of the series tends to a finite and unique limit as n tends to infinity, i.e., if $\lim_{n\to\infty} S_n$ = finite and unique then the series $\sum u_n$ is convergent.

8.6.4 Divergent Series

A series $\sum u_n$ is said to be divergent if the sum of the first n terms of the series tends to $+\infty$ or $-\infty$ as $n \to \infty$, i.e., if $\lim_{n\to\infty} S_n = +\infty \text{ or } -\infty$ then the series $\sum u_n$ is divergent.

8.6.5 Oscillatory Series

The oscillatory series are two types:

(i) Oscillate Finitely

A series $\sum u_n$ is said to oscillate finitely if the sum of its first n terms tends to a finite but not unique limit as n tends to infinity, i.e., if $\lim_{n\to\infty} S_n$ = finite but not unique then $\sum u_n$ oscillates finitely.

(ii) Oscillate Infinitely

A series $\sum u_n$ is said to oscillate infinitely if the sum of its first n terms oscillates infinitely, i.e., if $\lim_{n\to\infty} S_n = +\infty \text{ or } -\infty$ both then the series $\sum u_n$ oscillates infinitely.

Note 1 The convergency or divergency of a series is not affected by altering, adding, or neglecting a finite number of its terms.

Note 2 The convergency or divergency of a series is not affected by the multiplication of all the terms of the series by a fixed nonzero number.

8.6.6 A Necessary Condition for Convergence

A necessary condition for a positive-term series $\sum u_n$ to converge is that $\lim_{n\to\infty} u_n = 0$.

Proof Let $S_n = u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n \cdots$

$$S_{n-1} = u_1 + u_2 + u_3 + \cdots + u_{n-1}$$

$$\therefore \qquad u_n = S_n - S_{n-1} \tag{1}$$

Now, the given series $\sum u_n$ is convergent if $\lim_{n\to\infty} S_n$ is finite and unique,

$\therefore$ $\lim_{n\to\infty} S_n = S$ be a finite and unique quantity.

Then Eq. (1) gives

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= S - S \\ &= 0\end{aligned}$$

Hence, $\sum u_n$ is convergent.

Note 1 The above condition is not sufficient.

Example:

$$\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \left(\frac{1}{n}\right) + \cdots$$

where $u_n = \dfrac{1}{n}$

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

But $\sum u_n$ is divergent.

Note 2 If $\lim_{n\to\infty} u_n = 0$, we are not sure whether the series $\sum u_n$ is convergent or not but if $\lim_{n\to\infty} u_n \neq 0$ then the series $\sum u_n$ is divergent.

8.6.7 Cauchy's Fundamental Test for Divergence

If $\lim_{n\to\infty} u_n \neq 0$ the series $\sum u_n$ is divergent.

Example 1 Test the convergence of the series $1 + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{n}{n+1} + \cdots \infty$.

Solution

$$u_n = \frac{n}{n+1}$$

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$$

Hence, by Cauchy's test, $\sum u_n$ is divergent.

Example 2 Test the convergence of $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution

Here, $$u_n = \frac{n}{n+1}$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$

$$= 1 \neq 0$$

Hence, by Cauchy's test, $\sum u_n$ is divergent.

Example 3 Discuss the convergence of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\cdots$

Solution Let u_n be the n^{th} term of this series

Then $$u_n = \frac{1}{n(n+2)}$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n(n+2)} = 0$$

Hence, by necessary condition for convergence, the series $\sum u_n$ is convergent.

Example 4 Discuss the convergence of the series $\sum_{n=0}^{\infty} (-1)^n$.

Solution Here, $\sum u_n = \sum (-1)^n$

$$\therefore \quad S_n = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + \ldots \text{ to } n \text{ terms}$$

$$= 1 - 1 + 1 - 1 + \ldots \text{ to } n \text{ terms}$$

$$= 1 \text{ or } 0 \text{ accordingly as } n \text{ is odd or even.}$$

$$\therefore \quad \lim_{n\to\infty} S_n = 1 \text{ or } 0, \text{ i.e., finite but not unique.}$$

Hence, the series $\sum u_n$ is a finitely oscillating series.

EXERCISE 8.1

Test the convergence of the following series:

1. $1 + 3 + 5 + 7 + \ldots$
2. $2 + 4 + 6 + 8 + \ldots + 2n + \ldots$
3. $1^3 + 2^3 + 3^3 + \ldots + n^3 + \ldots$
4. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\infty$

5. $2 - 2 + 2 - 2 + 2 - \dots \infty$
6. $6 - 5 - 1 + 6 - 5 - 1 + 6 - 5 - 1 \dots \infty$
7. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \infty$
8. $\sum (\sqrt{n+1} - \sqrt{n})$
9. $\sum \frac{n}{(n+1)(n+2)(n+3)}$
10. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
11. $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$
12. $\sum (-2^n)$
13. $\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \cdots \infty$
14. $1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \cdots + \frac{2^{n-1}}{2^{n+1}} + \cdots \infty$
15. $\sum (6 - n^2)$

Answers

1. Divergent
2. Divergent
3. Divergent
4. Convergent
5. Oscillatory
6. Oscillatory
7. Convergent
8. Divergent
9. Convergent
10. Divergent
11. Divergent
12. Divergent
13. Divergent
14. Divergent
15. Divergent

8.7 GEOMETRIC SERIES

The series $1 + x + x^2 + x^3 + x^4 + \dots \infty$ is

(i) Convergent if $|x| < 1$ **(ii)** Divergent if $x \geq 1$ **(iii)** Oscillatory if $x \leq -1$

Proof

$$S_n = 1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

(i) When $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1-0}{1-x} = \left(\frac{1}{1-x}\right)$$

$$= \text{a finite quantity}$$

Hence, the series of convergent.

(ii)(a) When $x > 1$, $\lim_{n\to\infty} x^n = \infty$ then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{x^n - 1}{x-1} = \infty$$

Hence, the series is divergent.

(b) When $x = 1$, the series becomes $1 + 1 + 1 + 1 + \dots \infty$

$$S_n = 1 + 1 + 1 + 1 + \dots n \text{ times} = n$$

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

Hence, the series is divergent.

(iii)(a) When $x < -1$, let $x = -r$; $r > 1$

$$x^n = (-r)^n = (-1)^n r^n$$

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \lim_{n\to\infty} \frac{1-(-1)^n r^n}{1-(-r)}$$

$$= +\infty \text{ if } n \text{ is odd}$$
$$= -\infty \text{ if } n \text{ is even}$$

Hence, the series is oscillatory.

(b) When $x = -1$, the series becomes $1 - 1 + 1 - 1 + 1 - \dots \infty$

$$\therefore \quad S_n = 1 - 1 + 1 - 1 + \dots n \text{ terms}$$

$$\lim_{n\to\infty} S_n = 0 \text{ if } n \text{ is even}$$
$$= 1 \text{ if } n \text{ is odd}$$

Hence, the series is oscillatory finitely.

EXERCISE 8.2

Test the convergence of the following series:

1. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots \infty$
2. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots \infty$
3. $2 + 3 + \frac{9}{2} + \frac{27}{4} + \cdots \infty$
4. $1 - 2 + 4 - 8 + \dots \infty$
5. $1 - 1 + 1 - 1 + \dots \infty$

Answers

1. Convergent
2. Convergent
3. Divergent
4. Oscillatory
5. Oscillatory

8.8 ALTERNATING SERIES

A series whose terms are alternatively positive and negative is called an alternating series.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$

8.9 LEIBNITZ TEST

If the alternating series $u_1 - u_2 + u_3 - u_4 + \dots$ $(u_n > 0 \; \forall \; n)$ is such that

(i) $u_{n+1} \le u_n \; \forall \; n$ and

(ii) $\lim_{n\to\infty} u_n = 0$

then the series converges.

Example 5 Test the series $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots$

Solution In the given series, we have

(i) the terms are alternately, +ve and –ve
(ii) the terms are continually decreasing

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n^3}$

$$= \lim_{n\to\infty} \frac{1+1/n}{n^2} = \frac{1+0}{\infty} = 0 \text{ (finite)}$$

Hence, the given alternating series is **convergent**.

Example 6 Test the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

Solution The given series can be written as

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots$$

In this series, we find that

(i) the terms are alternatively +ve and –ve
(ii) the terms are continually decreasing

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$

Hence, the given series is **convergent**.

Example 7 Test the convergency of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

Solution

$$u_n = \frac{1}{\sqrt{n}}, u_{n-1} = \frac{1}{\sqrt{(n-1)}}$$

In the given series, we have

(i) the terms are alternatively +ve and –ve
(ii) $u_n < u_{n-1}$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Hence, the alternating series is **convergent**.

Example 8 Test the convergency of the series $\frac{\sqrt{2}-\sqrt{1}}{1} - \frac{\sqrt{3}-\sqrt{2}}{2} + \frac{\sqrt{4}+\sqrt{3}}{3} - \frac{\sqrt{5}-\sqrt{4}}{4} + \cdots$

Solution Here, $u_n = \left[\dfrac{\sqrt{n+1}-\sqrt{n}}{n}\right]$

In the given series, we have

(i) the terms of the series are alternatively +ve and –ve

(ii) the terms are continually decreasing as $u_n > u_{n+1}$ for all n

(iii) $$\lim_{n\to\infty}\left[\frac{\sqrt{n+1}-\sqrt{n}}{n}\right] = \lim_{n\to\infty}\frac{1}{n}\left[\sqrt{n}\sqrt{1+\frac{1}{n}}-\sqrt{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\left(1+\frac{1}{n}\right)^{1/2}-1\right]$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\left(1+\frac{1}{2}\cdot\frac{1}{n}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\cdot\frac{1}{n^2}+\cdots\right)-1\right]$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\frac{1}{2n}-\frac{1}{8n^2}+\cdots\right]=0$$

Hence, the alternating series is convergent.

Example 9 Test the convergence of the series $2-\dfrac{3}{2}+\dfrac{4}{3}-\dfrac{5}{4}+\cdots$

Solution Here, $u_n = \dfrac{n+1}{n}$

In the given series, we have

(i) the terms are alternately +ve and –ve

(ii) the terms are in decreasing order, i.e., $u_n < u_{n-1}$ and

(iii) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$$

Hence, the third condition of the alternating series test is not satisfied, so the series is not convergent.

However, we can write the given series as

$$(1+1)-\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)-\left(1+\frac{1}{4}\right)+\cdots$$

or $$(1-1+1-1+1+\cdots)+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)$$

The series in the II$^{\text{nd}}$ bracket is convergent, since its value is log (1 + 1), i.e., log 2.

But the series in the I$^{\text{st}}$ bracket is an oscillating series whose value is either 0 or 1 accordingly as n is even or odd.

$\therefore$ the sum of n terms of the given series as $n \to \infty$ is (0 + log 2) or (1 + log 2) accordingly as n is even or odd, i.e., log 2 or (1 + log 2) accordingly is n is even or odd.

Hence, by the definition, the given series is oscillating.

Example 10 Test the convergency of the series $1 - 2x + 3x^2 - 4x^3 + \ldots$ when $x < 1$.

Solution In the given series, we have

(i) the terms are alternately +ve and –ve

(ii) the terms are continually decreasing as $x < 1$

(iii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} nx^{n-1} = \frac{1}{x}\lim\limits_{n\to\infty} nx^n = 0$ $\left[\because \lim\limits_{n\to\infty} nx^n = 0 \text{ if } x < 1\right]$

Hence, all the three conditions of the alternating series test are satisfied and so the given series is **convergent**.

EXERCISE 8.3

Test the convergence of the following series:

1. $\sum \frac{(-1)^{n-1}}{n}$

2. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)}$

3. $\log\left(\frac{2}{1}\right) - \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) - \cdots\infty$

4. $\sum \log\left(\frac{n}{n+1}\right)$

5. $\frac{1}{(\sqrt{2}-1)} - \frac{1}{(\sqrt{3}-1)} + \frac{1}{(\sqrt{4}-1)} - \infty$

6. $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \cdots\infty$

7. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots\infty$

8. $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$

Answers

1. Convergent
2. Convergent
3. Convergent
4. Convergent
5. Convergent
6. Convergent
7. Convergent
8. Convergent if $p > 0$

8.10 POSITIVE-TERM SERIES

Series with positive terms are the simplest and the most important type of series one comes across. The simplicity arises mainly from the sequence of its partial sums being monotonic increasing. Let $\sum u_n$ be an infinite series of positive terms and $< S_n >$ be the sequence of its partial terms. Then,

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n; \ \forall\, n, \text{ and}$$

$$S_{n+1} = u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1}$$

$\therefore$ $S_{n+1} - S_n = u_{n+1} > 0$, since $u_n > 0$

i.e., $S_{n+1} > S_n$

Hence, the sequence $<S_n>$ is monotonic increasing. Now, two cases arise:

Case I: When the sequence $<S_n>$ is bounded above then the series $\sum u_n$ is convergent.

Case II: When the sequence $<S_n>$ is not bounded above then the series $\sum u_n$ is divergent.

Note A positive-term series $\sum u_n$ always diverges to $+\infty$ provided $\lim_{n\to\infty} u_n \neq 0$

Example 11 Test the series $\sum u_n$ whose n^{th} term is $\left(1+\frac{1}{n}\right)$.

Solution Here, $u_n = 1+\frac{1}{n}$

The given series is a series of positive terms and so it is either a convergent or a divergent series.

But $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$$

Hence, the given series $\sum u_n$ is **divergent**.

Example 12 Test the series $\sum u_n$, whose n^{th} term is $\left(\frac{n}{n+1}\right)$.

Solution The given series of +ve terms is either convergent or a divergent series.

Here, $$u_n = \frac{n}{n+1}$$

But $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{1+0} = 1 \neq 0$$

Hence, the series $\sum u_n$ is **divergent**.

8.11 *p*-SERIES TEST

The infinite series $\sum \frac{1}{n^p}$, i.e., $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots\infty$ is

(i) Convergent, if $p > 1$ (ii) Divergent, $p \leq 1$

8.12 COMPARISON TEST

If $\sum u_n$ and $\sum v_n$ be two given series and if $\lim_{n\to\infty}\frac{u_n}{v_n}$ is a finite and nonzero quantity then $\sum u_n$ and $\sum v_n$ are either both convergent or both divergent.

Working Rule

First of all, we compare the given infinite $\sum u_n$ with an auxiliary series $\sum v_n$ for convergency or divergency of the series $\sum v_n$. This should already be known to us and we then find $\lim_{n\to\infty}\frac{u_n}{v_n} = l$ (a finite and nonzero quantity). Then the series $\sum u_n$ and $\sum v_n$ converge and diverge together.

Example 13 Test the convergency of the series $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+\cdots\infty$

Solution The given series can be written as

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}}+\cdots$$

Here, $$\sum u_n = \frac{1}{\sqrt{n}} \qquad (1)$$

Comparing (1) with $\sum \dfrac{1}{n^p}$, we get $p=\dfrac{1}{2}<1$

Hence, the given series is **divergent**.

Example 14 Test the convergency of the series $1+\dfrac{1}{2^2}+\dfrac{2^2}{3^3}+\dfrac{3^3}{4^4}+\dfrac{4^4}{5^5}\cdots$

Solution Omitting the first term of the given series, we have

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)^n}$$

$$= \frac{n^n}{n\left(1+\frac{1}{n}\right)\times n^n\left(1+\frac{1}{n}\right)^n}$$

$$u_n = \frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}$$

Let $v_n = \dfrac{1}{n}$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \text{ (a finite quantity)}$$

But $$\sum v_n = \sum\frac{1}{n} \approx \sum\frac{1}{n^p} \Rightarrow p=1$$

Hence, the given series is **divergent**.

Example 15 Test the convergency of the series $\dfrac{2}{1^p}+\dfrac{3}{2^p}+\dfrac{4}{3^p}+\dfrac{5}{4^p}+\cdots$

Solution The n^{th} term of the given series is

$$u_n = \frac{n+1}{n^p} = \frac{n(1+1/n)}{n^p} = \frac{(1+1/n)}{n^{p-1}}$$

Let $$v_n = \frac{1}{n^{p-1}}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \text{ (a finite quantity)}$$

But $\sum v_n$ is convergent if $p - 1 > 1$ or $p > 2$ and it is divergent if $p - 1 \le 1$ or $p \le 2$.
Hence, the given series is convergent if $p > 2$ and divergent if $p \le 2$.

Example 16 Test the convergence of the series $\sqrt{\dfrac{1}{2^3}} + \sqrt{\dfrac{2}{3^3}} + \sqrt{\dfrac{3}{4^3}} + \sqrt{\dfrac{4}{5^3}} + \cdots$

Solution Let the given series be denoted by $\sum u_n$

Then

$$u_n = \sqrt{\frac{n}{(n+1)^3}} = \sqrt{\frac{1}{n^2(1+1/n)^3}} = \frac{1}{n\sqrt{\left(1+\dfrac{1}{n}\right)^3}}$$

Let $$v_n = \sqrt{\frac{1}{n^2}} = \frac{1}{n}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{\left(1+\dfrac{1}{n}\right)^3}} = 1 \qquad \text{[which is a finite number]}$$

Now, $\sum v_n \approx \sum \dfrac{1}{n^p} \quad \Rightarrow \sum \dfrac{1}{n} \approx \sum \dfrac{1}{n^p}$

$\therefore \quad p = 1$, the given series is **divergent**.

Example 17 Test the convergence of the series $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{3}{2\cdot 3\cdot 4} + \dfrac{5}{3\cdot 4\cdot 5} + \cdots$

Solution Let the given series be denoted by $\sum u_n$

where $$u_n = \frac{(2n-1)}{n(n+1)(n+2)} = \frac{n\left(2-\dfrac{1}{n}\right)}{n^3\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)}$$

$$u_n = \frac{\left(2-\dfrac{1}{n}\right)}{n^2\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)}$$

Let $v_n = \dfrac{1}{n^2}$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2-\dfrac{1}{n}\right)}{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)} = 2 \qquad \text{[which is a finite number]}$$

But $$\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$$

$\therefore$ $p = 2 > 1$; hence the given series $\sum u_n$ is **convergent**.

Example 18 Test the convergence of the series whose general term is given by

$$u_n = \sqrt{(n^2+1)} - \sqrt{(n^2-1)}$$

Solution

$$u_n = \sqrt{(n^2+1)} - \sqrt{(n^2-1)}$$

$$= n\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}} - n\left(1-\frac{1}{n^2}\right)^{\frac{1}{n}} = n\left[\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}} - \left(1-\frac{1}{n^2}\right)^{\frac{1}{n}}\right]$$

$$= n\left[\left\{1+\frac{1}{2n^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\cdot\frac{1}{n^4}+\cdots\right\} - \left\{1-\frac{1}{2n^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\cdot\frac{1}{n^4}-\cdots\right\}\right]$$

$$= 2n\left[\frac{1}{2n^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{3!}\cdot\frac{1}{n^6}+\cdots\right]$$

$$= \frac{1}{n}+\frac{1}{8n^5}$$

Retaining only the highest power of n,

Let $$\sum v_n = \frac{1}{n}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{8n^5}+\cdots}{1/n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{8n^4}+\cdots\right) = 1 \text{ (which is finite)}$$

But $$\sum v_n = \frac{1}{n} \approx \sum \frac{1}{n^p}$$

$\therefore$ $p = 1$

Hence, the given series $\sum u_n$ is divergent.

Example 19 Test the convergence of the series, whose general term is $\sqrt{(n^4+1)} - \sqrt{(n^4-1)}$.

Solution Do same as Example 18. (It is a convergent series.)

Example 20 Test the series $\sum \sin\frac{1}{n}$

Solution Let the given series be $\sum u_n$, where

$$u_n = \sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{3!}\cdot\frac{1}{n^3} + \frac{1}{5!}\cdot\frac{1}{n^5} - \cdots$$

Retaining only the highest power of n in u_n, we have the auxiliary series

$$\sum v_n = \sum\frac{1}{n}, \text{ where } v_n = \frac{1}{n}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n} - \frac{1}{3!}\cdot\frac{1}{n^3} + \frac{1}{5!}\cdot\frac{1}{n^5} - \cdots}{1/n}$$

$$= \lim_{n\to\infty}\left[1 - \frac{1}{6n^2} + \frac{1}{120n^4} - \cdots\right] = 1 \qquad \text{[which is a finite number]}$$

Hence, $\sum u_n$ and $\sum v_n$ converge or diverge together. But the auxiliary series

$$\sum v_n = \sum\frac{1}{n} \approx \sum\frac{1}{n^p}$$

$$\therefore \quad p = 1$$

Hence, the given series is divergent.

Example 21 Test the series $\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \cdots$

Solution Neglecting the first term, denote the series by $\sum u_n$.

Then,
$$\sum u_n = \left(\frac{1}{x-1} + \frac{1}{x+1}\right) + \left(\frac{1}{x-2} + \frac{1}{x+2}\right) + \cdots$$

$$\sum u_n = \frac{2x}{x^2-1^2} + \frac{2x}{x^2-2^2} + \frac{2x}{x^2-3^2} + \cdots$$

$$\therefore \quad u_n = \frac{2x}{x^2-n^2}$$

Let the auxiliary series $\sum v_n = \sum\frac{1}{n^2}$, i.e., $v_n = \frac{1}{n^2}$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2x}{\left(\frac{x^2}{n^2} - 1\right)} = -2x \quad \text{[a finite quantity]}$$

Hence, $\sum u_n$ and $\sum v_n$ converge or diverge together.

But the auxiliary series $\sum v_n = \frac{1}{n^2} \approx \frac{1}{n^p}$

$$p = 2 > 1$$

Hence, the given series is **convergent**.

EXERCISE 8.4

Test the convergence of the following series:

1. $\sum \frac{n+1}{n^2}$
2. $\frac{\sqrt{1}}{1+\sqrt{1}} + \frac{\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{3}}{3+\sqrt{3}} + \cdots$
3. $\sum \left[\sqrt{n+1} - \sqrt{n}\right]$
4. $\sum \left[\sqrt{n^2+1} - n\right] \cdot x^{2n}$
5. $\sum \left[\sqrt{n^3+1} - \sqrt{n^3}\right]$
6. $\sum \left[(n^3+1)^{1/3} - n\right]$
7. $\sum \left[\sqrt{n^3+1} - \sqrt{n^3-1}\right]$
8. $\sum \left[\sqrt{n} + \sqrt{n+1}\right]^{-1}$
9. $\sum \frac{1}{n} \sin\left(\frac{1}{n}\right)$
10. $\sum \left[\frac{\sqrt{n+1} - \sqrt{n-1}}{n}\right]$
11. $\frac{14}{1^3} + \frac{24}{2^3} + \frac{34}{3^3} + \cdots + \frac{10n+4}{n^3} + \cdots$
12. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$
13. $\sum \frac{1}{2n-1}$
14. $\sum \frac{n}{(2+3n^3)}$
15. $\sum \frac{(2n^2+1)}{3n^3+5n^2+6}$

Answers

1. Divergent
2. Divergent
3. Divergent
4. Convergent if $x^2 < 1$ and divergent $x^2 \geq 1$
5. Convergent
6. Convergent
7. Convergent
8. Divergent
9. Convergent
10. Divergent
11. Convergent
12. Convergent
13. Divergent
14. Divergent
15. Divergent

8.13 D'ALEMBERT'S RATIO TEST

Let $\sum u_n$ be a series of positive terms, such that

(I) If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, then the series is

(a) convergent if $\lambda < 1$, (b) divergent if $\lambda > 1$, and (c) fails if $\lambda = 1$

(II) If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$ then the series $\sum u_n$ is divergent

OR

(I) If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda$ then

(a) convergent if $\lambda > 1$, (b) divergent if $\lambda < 1$, and (c) fails if $\lambda = 1$

(II) If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$ then $\sum u_n$ is convergent.

Example 22 Test the series $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \infty$

Solution

Here, $$u_n = \frac{1}{n!},\ u_{n+1} = \frac{1}{(n+1)!}$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = (n+1)$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} (n+1) = \infty \text{ which is } > 1$$

Hence, by ratio test, the series is **convergent**.

Example 23 Test the series $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots + \frac{(n+1)!}{3^n} + \cdots$

Solution

Here, $$u_n = \frac{(n+1)!}{3^n}, \text{ so } u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{3^n} \cdot \frac{3^{n+1}}{(n+2)!} = \frac{(n+1)! \cdot 3^n \cdot 3^1}{3^n \cdot (n+2)(n+1)!}$$

$$= \frac{3}{n+2}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3}{n+2} = 0 < 1$$

Hence, the given series is **divergent**.

Example 24 Test the series $\frac{x^1}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \cdots$

Solution

Here, $$u_n = \frac{x^n}{n(n+1)} \text{ so } u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{x^n}{n(n+1)} \cdot \frac{(n+1)(n+2)}{x^{n+1}} = \frac{(n+2)}{n \cdot x}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \left(1 + \frac{2}{n}\right)$$

$\therefore$ $$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{n} \cdot \left(1 + \frac{2}{n}\right) = \frac{1}{x}$$

From the ratio test, we conclude that the given series $\sum u_n$ is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$, and

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$.

(iii) If $x = 1$ then this test fails and the given series becomes $\sum u_n$, whose n^{th} term

$$u_n = \frac{1}{n(n+1)} = \frac{1}{n^2\left(1 + \frac{1}{n}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$\therefore$ $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{1 + 1/n}\right) = 1, \text{ which is a finite quantity.}$$

Now, $$\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p} \Rightarrow p = 2 > 1,$$ the given series is **convergent**.

Example 25 Test the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots \infty$

Solution If $\sum u_n$ be the given series then the n^{th} term is

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \text{ and so } u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^n}$$

$$= \left(\frac{n+2}{n+1}\right)\sqrt{\frac{n+1}{n}} \cdot \frac{1}{x^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1 + \frac{2}{n}\right)}{\left(1 + \frac{1}{n}\right)} \sqrt{\left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x^2}$$

$$\therefore \quad \lim_{n\to\infty} \cdot \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x^2}\right] = \frac{1}{x^2}$$

From the ratio test, we conclude that the given series $\sum u_n$ is convergent or divergent accordingly as $\frac{1}{x^2} > 1$ or < 1, i.e., $x^2 < 1$ or > 1.

If $x^2 = 1$; then this test fails and the given series reduces to $\sum u_n$ whose n^{th} term

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}(1+1/n)}$$

Let $v_n = \frac{1}{n^{3/2}}$; then by comparison test

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = 1, \text{ which is a finite quantity}$$

Hence, $\sum u_n$ and $\sum v_n$ converge or diverge together but $\sum v_n = \sum \frac{1}{n^{3/2}} \approx \sum \frac{1}{n^p}$ so $p = \frac{3}{2} > 1$. Hence, the series is **convergent**. Thus, the given series is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Example 26 Test the series $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$

Solution Here, $u_n = \frac{n}{1+2^n}$, so $u_{n+1} = \frac{n+1}{1+2^{n+1}}$

Now,

$$\frac{u_n}{u_{n+1}} = \frac{n}{1+2^n}\cdot\frac{1+2^{n+1}}{(n+1)} = \left(\frac{n}{n+1}\right)\cdot\left(\frac{1+2^{n+1}}{1+2^n}\right)$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{1}{1+\frac{1}{n}}\right)\cdot\left(\frac{1/2^n+2}{1/2^n+1}\right)$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\left(\frac{1}{1+\frac{1}{n}}\right)\cdot\left(\frac{\frac{1}{2^n}+2}{\frac{1}{2^n}+1}\right)\right] = 2 > 1$$

Hence, by ratio test, the given series is **convergent**.

Example 27 Show that the series $1 + \frac{x}{1!}(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \cdots\infty$ is convergent.

Solution If $\sum u_n$ be the given series then we have the n^{th} term $u_n = \frac{x^{n-1}}{(n-1)!}\cdot(\log a)^{n-1}$ and so $u_{n+1} = \frac{x^n}{n!}(\log a)^n$.

Now, $$\frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} \cdot \frac{n!}{x^n \cdot (\log a)^n}$$

$$= \frac{n}{x(\log a)}$$

$\therefore$ $$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{n}{x(\log a)}\right] = \infty > 1 \text{ for all } x$$

Hence, by ratio test, the given series is **convergent**.

Some Important Remarks

(i) To test for absolute convergence, we have to apply only the test for series with positive terms.

(ii) Every absolutely convergent series is also convergent.

(iii) For an absolutely convergent series, the series formed by positive terms only is convergent and the series formed by negative terms only is also convergent.

(iv) If $\sum u_n$ is conditionally convergent then the series of its positive terms and the series of its negative terms are both divergent.

EXERCISE 8.5

Test the convergence of the following series:

1. $\sum \frac{n^{2n}}{n!}$
2. $\sum \frac{3n+1}{5n+1} \cdot x^n$
3. $1 + 3x + 5x^2 + 7x^3 + \ldots$
4. $1 + \frac{2x}{5} + \frac{6x^2}{9} + \cdots + \frac{2^{n+1}+2}{2^{n+1}+1} x^n + \cdots$
5. $\sum \left[\sqrt{n^2+1} - n\right] \cdot x^{2n}$
6. $1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \frac{5!}{5^2} + \cdots$
7. $\sum \frac{n!}{n^n}$
8. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
9. $\sum \frac{n^3+a}{2^n+a}$
10. Show that the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ is convergent if x is numerically less than 1.

Answers

1. Divergent
2. Convergent if $x < 1$ and divergent $x \geq 1$
3. Convergent if $x < 1$ and divergent if $x \geq 1$
4. Convergent if $x < 1$ and divergent if $n \geq 1$
5. Convergent of $x < 1$ and divergent if $x \geq 1$
6. Divergent
7. Convergent
8. Convergent
9. Convergent
10. Convergent

8.14 CAUCHY'S ROOT (OR RADICAL) TEST

Let $\sum u_n$ be an infinite series of positive terms and let $\lim_{n\to\infty} [u_n]^{\frac{1}{n}} = \lambda$

Then the series is

(i) Convergent if $\lambda < 1$, and

(ii) Divergent if $\lambda > 1$.

(iii) If $\lambda = 1$, this test fails.

Example 28 Test the convergence of $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

Solution Here, $u_n\left(1+\frac{1}{n}\right)^{-n^2}$

$$\therefore \quad \lim_{n\to\infty}\{u_n\}^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)^{n}}\right] \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e\right]$$

$$= \frac{1}{e} < 1 \qquad [\because \ e = 2.718,\ 2 < e < 3]$$

Hence by Cauchy's root test, the given series is **convergent**.

Example 29 Test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$$

Solution The n^{th} term of the given series is

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$$

$$\therefore \quad \lim_{n\to\infty}[u_n]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left\{\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right\}^{-n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\times\left\{\left(\frac{n+1}{n}\right)^n - 1\right\}\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\cdot\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$= 1\cdot(e-1)^{-1}$$

$$= \frac{1}{(e-1)} < 1 \qquad [\because \ 2 < e < 3 \text{ and we take } e = 2.718]$$

Hence, by Cauchy's root test, the given series is **convergent**.

Example 30 Test the convergence of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \infty$$

Solution Neglecting the first term, we obtain the n^{th} term of the given series is

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\therefore \qquad \lim_{n\to\infty}[u_n]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)\cdot x\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\cdot x\right] = x$$

From Cauchy's root test, the given series is convergent if $x < 1$, and divergent if $x > 1$, and if $x = 1$, this test fails.

Now, putting $x = 1$ in the given series, we get

$$u_n = \left(\frac{n+1}{n+2}\right)^n$$

and $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

Hence, the series is **divergent**.

Thus, the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 31 Test the convergence of $\sum\left(a+\frac{x}{n}\right)^n$

Solution Here $u_n = \left(a+\frac{x}{n}\right)^n$

$$\therefore \quad \lim_{n\to\infty}[u_n]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(a+\frac{x}{n}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[a+\frac{x}{n}\right] = a$$

By Cauchy's root test if $a > 1$ then the given series is divergent, if $a < 1$, it is convergent, and if $a = 1$ then $u_n = \left(1+\frac{x}{n}\right)^n$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^{\frac{1}{x}}\ \forall\ x$

$$\neq 0$$

Hence, the given series is **divergent** for $a = 1$. Thus, the given series is convergent if $a < 1$ and divergent if $a \geq 1$.

Example 32 Test the convergence of the series $\sum\frac{n^{n^2}}{(n+1)^{n^2}}$.

Solution Here, $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\therefore \quad \lim_{n\to\infty}[u_n]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\frac{n^n}{(n+1)^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = \frac{1}{e} < 1$$

Hence, by Cauchy's test, the given series is **convergent**.

EXERCISE 8.6

Test the convergence of the following series:

1. $\sum_{1}^{\infty}\left(\frac{nx}{n+1}\right)^n$

2. $\sum_{1}^{\infty}\left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

3. $\sum_{1}^{\infty}\left[2^{-n-(-1)^n}\right]$

4. $\sum_{1}^{\infty}\left[\frac{\log n}{\log(1+n)}\right]^{n^2 \log n}$

5. $\sum_{1}^{\infty}\frac{1}{n^n}$

6. $\sum\left[n^{\frac{1}{n}}-1\right]^n$

7. $\sum\left[1-\frac{1}{n}\right]^{n^2}$

8. $1 + a + ab + a^2b + a^2b^2 + \ldots$ where $0 < ab < 1$

9. $\sum\frac{(n+\sqrt{n})^n}{2^n \cdot n^{n+1}}$

10. $\sum\left[n^{\frac{1}{n}}+x\right]^n; \forall x$

Answers

1. Convergent if $n < 1$, divergent if $x \geq 1$
2. Convergent
3. Convergent
4. Convergent
5. Convergent
6. Convergent
7. Convergent
8. Convergent
9. Convergent
10. Divergent

8.15 RAABE'S TEST

Let $\sum u_n$ be an infinite series of positive terms and let

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \lambda$$

Then the series is

(i) Convergent if $\lambda > 1$, and

(ii) Divergent if $\lambda < 1$.

(iii) $\lambda = 1$, this test fails.

Working Rule

Find the n^{th} term of the given series and denote it by u_n. Then find u_{n+1}, calculate $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lambda$ (say).

Then series is convergent if $\lambda > 1$ or divergent if $\lambda < 1$ and if $\lambda = 1$ then this test fails. Then calculate $\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]$ and apply Raabe's test.

Example 33 Test the convergence of the series

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\cdots$$

Solution The n^{th} term of the given series (neglected first term) is

$$u_n=\frac{3.6.9.12\cdots(3n)}{7.10.13.16...(3n+4)}\cdot x^n$$

$$u_{n+1}=\frac{3.6.9.12\cdots(3n)(3n+3)}{7.10.13.16...(3n+4)(3n+7)}x^{n+1}$$

Now, $$\frac{u_n}{u_{n+1}}=\left(\frac{3n+7}{3n+3}\right)\cdot\frac{1}{x}\text{ (other terms canceling out)}$$

$$\therefore\quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\frac{3n+7}{3n+3}\cdot\frac{1}{x}\right]$$

$$=\lim_{n\to\infty}\left[\frac{3+7/n}{3+3/n}\cdot\frac{1}{x}\right]=\frac{1}{x}$$

By ratio test, the given series is convergent if $\frac{1}{x}>1$, i.e., $x<1$ and divergent if $\frac{1}{x}<1$ or $x>1$

If $x = 1$, this test fails. Then

$$\frac{u_n}{u_{n+1}}=\left(\frac{3+7/n}{3+3/n}\right)$$

$$\therefore\quad n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{3+7/n}{3+3/n}-1\right]$$

$$=n\left[\frac{\frac{4}{3n}}{1+1/n}\right]$$

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=\lim_{n\to\infty}\left[\frac{4/3}{1+1/n}\right]=\frac{4}{3}>1$$ [Convergent (by Raabe's test)]

Thus, the given series is convergent if $x\le 1$ and divergent if $x > 1$.

Example 34 Test the convergence of the series $1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1\cdot 2\cdot 3}+\cdots$

Solution Leaving the first term, the n^{th} term of the series is

$$u_n=\frac{a(a+1)(a+2)...\,(a+n-1)}{1\cdot 2\cdot 3\cdots n}\text{ and}$$

$$u_{n+1}=\frac{a(a+1)(a+2)...\,(a+n-1)(a+n)}{1\cdot 2\cdot 3\cdots n(n+1)}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \left(\frac{n+1}{a+n}\right)$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[\frac{n+1}{n+a}\right] = \lim_{n\to\infty} \left[\frac{1+1/n}{1+a/n}\right] = 1$$

Hence, the ratio test is fails.

Now, $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{n+1}{a+n} - 1\right)$$

$$n\left(\frac{1-a}{a+n}\right) = \left(\frac{1-a}{1+a/n}\right)$$

$$\therefore \quad \lim_{n\to\infty} \left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n\to\infty} \left[\frac{1-a}{1+a/n}\right] = (1-a)$$

From Raabe's test if $(1 - a) > 1$ or $a < 0$, the series is convergent, if $(1 - a) < 1$ or $a > 0$, the series is divergent and if $1 - a = 1$ or $a = 0$, this test fails and the given series becomes $1 + 0 + 0 + 0 + \ldots$, the sum of whose first n terms is always equal to 1. Hence, when $a = 0$, the series is convergent.

Thus, the given series is convergent if $a \le 0$ and the series is divergent if $a > 0$.

EXERCISE 8.7

Test the convergence of the following series:

1. $\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots$
2. $\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \cdots$
3. $1 + \frac{2}{3\cdot 4} + \frac{2\cdot 4}{3\cdot 5\cdot 6} + \cdots + \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)(2n+2)} + \cdots$
4. $\frac{1^2}{4^2} + \frac{1^2\cdot 5^2}{4^2\cdot 8^2} + \frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2} + \frac{1^2\cdot 5^2\cdot 9^2\cdot 13^2}{4^2\cdot 8^2\cdot 12^2\cdot 16^2} + \cdots$
5. $\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots$
6. $1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$
7. $\sum_{n=1}^{\infty} \frac{x^n}{a+\sqrt{n}}$

Answers

1. Divergent
2. Divergent
3. Convergent
4. Convergent
5. Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$
6. Convergent if $x < 1$ and divergent if $x \geq 1$
7. Divergent

8.16 ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

(i) Absolute Convergent Series

A series $\sum u_n$ is said to be absolutely convergent if the positive-term series $\sum |u_n|$ is convergent.

Example: The series $\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots$ is absolutely convergent, because $\sum | u_n | = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$ is an infinite geometric series of positive terms with common ratio $\frac{1}{2} < 1$.

Hence, it is **convergent**.

(ii) Conditionally Convergent Series

A series $\sum u_n$ is said to be conditionally convergent if the series $\sum |u_n|$ is divergent. It is also known as *semi-convergent* or *accidentally convergent*.

Example: The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$ is conditionally convergent, because $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is not convergent by the p-series test, $\sum |u_n| = \sum \frac{1}{n} \approx \sum \frac{1}{n^p}$ so $p = 1$.

Hence, it is **divergent**.

8.17 TEST FOR ABSOLUTE CONVERGENCE

Testing whether or not a series $\sum u_n$ is absolutely convergent reduces to testing for convergence the series of positive term $\sum |u_n|$. This can be done by making use of various tests shown in previous sections.

8.17.1 Some Important Remarks

(i) To test for absolute convergence, we have to apply only the test for series with positive terms.
(ii) Every absolutely convergent series is also convergent.
(iii) For an absolutely convergent series, the series formed by positive terms only is convergent and the series formed by negative terms only is also convergent.

(iv) If $\sum u_n$ is conditionally convergent then the series of its positive terms and the series of its negative terms are both divergent.

Example 35 Test the absolute convergence of the series $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots$

Solution Let $\sum u_n = 1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots$

Then $$\sum |u_n| = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = \sum \frac{1}{2^{n-1}} = \sum v_n \text{ (say)}$$

Here, $$v_n = \frac{1}{2^{n-1}} \text{ and } v_{n+1} = \frac{1}{2^n}$$

$$\therefore \quad \lim_{n\to\infty} \frac{v_n}{v_{n+1}} = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}}\cdot 2^n\right) = \lim_{n\to\infty}(2) = 2 > 1$$

From the ratio test, the series $\sum |u_n|$ is convergent. Also, in $\sum u_n$, we find that its terms are alternatively positive and negative, its terms are continually decreasing and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}}\right) = 0$.

Thus, all the above conditions of Leibnitz's test are satisfied and $\sum u_n$ is convergent. Hence, the given series $\sum u_n$ is **absolutely convergent**.

Example 36 Test the absolute convergence of the series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$

Solution Let $\sum u_n = 1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$

Then $$\sum |u_n| = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots = \sum \frac{1}{\sqrt{n}} = \sum v_n \text{ (say)}$$

$$\therefore \quad \sum v_n = \sum \frac{1}{n^{1/2}} \approx \sum \frac{1}{n^p} \text{ so that } p = \frac{1}{2} < 1, \text{ Hence } \sum |u_n| \text{ is divergent series.}$$

Also in the series $\sum u_n$, we find that its terms are alternatively positive and negative, its terms are continually decreasing and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Thus, all the three conditions of Leibnitz's test are satisfied and as such, $\sum u_n$ is convergent.

But $\sum |u_n|$ is divergent; hence, the given series $\sum u_n$ is **conditionally convergent**.

Example 37 Is the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n\sqrt{n}}\right)$ absolutely convergent?

Solution Let $\sum u_n = \sum (-1)^{n-1}\left(\frac{1}{n\sqrt{n}}\right)$

Then $$\sum |u_n| = \sum \frac{1}{n\sqrt{n}} = \sum v_n \text{ (say)}$$

Here, $$\sum u_n = \sum \frac{1}{n^{3/2}} \text{ so that } p = \frac{3}{2} > 1 \text{ convergent}$$

Also, in $\sum u_n$, we find that the terms are alternately +ve and –ve, the terms are continually decreasing and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$, $\sum u_n$ is convergent.

Hence, the given series is **absolutely convergent**.

Example 38 Test the absolute convergence of the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$

Solution Let $\sum u_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$

Then
$$u_n = \frac{(-1)^{n-1} x^n}{n} \text{ and } u_{n+1} = \frac{(-1)^n x^{n+1}}{(n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(-1)^{n-1} \cdot x^n}{n} \times \frac{(n+1)}{(-1)^n x^{n+1}} = -\left(\frac{n+1}{nx}\right)$$

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{n+1}{nx}\right| = \left|1 + \frac{1}{n}\right| \cdot \left|\frac{1}{x}\right|$$

$$\therefore \quad \lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \left[\left|1 + \frac{1}{n}\right| \cdot \frac{1}{|x|}\right] = \frac{1}{|x|}$$

By ratio test, we find that the series $\sum |u_n|$ is convergent or divergent accordingly as $\frac{1}{|x|} > 1$ or < 1, i.e., $|x| < 1$ or > 1.

Also, we know that every absolutely convergent series is convergent so $\sum u_n$ converges if $|x| < 1$.

If $x < 0$ then $\sum (u_n) = -u_n$

$\therefore$ $\sum u_n$ is convergent or divergent iff $\sum |u_n|$ is convergent or divergent.

Since $\sum |u_n|$ is divergent if $x < -1$, so $\sum u_n$ is divergent if $x < -1$.

If $x = 1$, we have $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$ and is a convergent series.

If $x = -1$, we have $\sum u_n = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = -\sum \frac{1}{n}$

which is a divergent series as $\sum \frac{1}{n^p}$ is divergent when $p = 1$.

If $x > 1$, u_n does not tend to zero as $n \to \infty$ and so $\sum u_n$ is not a convergent series if $x > 1$.

Hence, $\sum u_n$ converges if and only if $-1 < x \le 1$.

Example 39 Show that the series $1-\frac{1}{2^k}+\frac{1}{3^k}-\frac{1}{4^k}+\cdots$ is absolutely convergent if $k > 1$ and conditionally convergent if $0 < k \le 1$.

Solution Let $\sum u_n = 1-\frac{1}{2^k}+\frac{1}{3^k}-\frac{1}{4^k}+\cdots$

Then $$|u_n| = 1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots=\sum\frac{1}{n^k}$$

which is a convergent series if $k > 1$ and divergent series if $k \le 1$. [$\because$ by p-series test]

Also, by Leibnitz's test, we know that $\sum u_n$ is convergent if $k > 1$ or $0 < k \le 1$.

[$\because$ $\sum u_n$ is divergent if $k \le 0$]

$\therefore$ If $k > 1$, $\sum u_n$ and $\sum |u_n|$ are both convergent. Hence the given series $\sum u_n$ is absolutely convergent.

If $0 < k \le 1$, $\sum u_n$ is convergent while $\sum |u_n|$ is divergent.

Hence, the given series $\sum u_n$ is **conditionally convergent**.

EXERCISE 8.8

1. Is the series $1 - 2x + 3x^2 - 4x^3 + \ldots$ where $x > 1$ absolutely convergent?
2. Show that the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\infty$ is conditionally convergent.
3. Show that the series $1+n+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ converges absolutely for all values of x.
4. Test the absolutely convergence of the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots\infty$
5. Test for absolute convergence of the series $\frac{2}{1^2}-\frac{3}{2^2}+\frac{4}{3^2}-\frac{5}{4^2}+\cdots$
6. Prove that the series $\sum u_n$ is conditionally convergent where $u_n=(-1)^n\frac{1}{n}$.
7. Test for absolute convergence the series $\sum(-1)^{n-1}\frac{n^2}{(n+1)!}$.
8. Show that the series $\sum\frac{(-1)^{n+1}}{\log(n+1)}$ conditionally converges.

Answers

4. Absolutely convergent
5. Conditionally convergent
7. Absolutely convergent

8.18 POWER SERIES

A series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n \tag{2}$$

is said to be a *power series*. The numbers a_r are called *coefficients of the power series*.

Now, if we apply the Cauchy's root test for convergence of the power series, we conclude that the series is absolutely convergent if $|x| < \dfrac{1}{\lambda}$ and divergent if $|x| > \dfrac{1}{\lambda}$.

Let $\dfrac{1}{\lambda} = R$; then the power series is absolutely convergent if $|x| < R$ and divergent if $|x| > R$ where R is called the radius of convergence and is defined as $\dfrac{1}{R} = \lim\limits_{n\to\infty} |a_n|^{1/n}$ OR $R = \dfrac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}}$

Note A power series is absolutely convergent within its interval of convergence and divergent outside it.

Example 40 Find the interval of absolute convergence for the series $\sum_{1}^{\infty} \dfrac{x^n}{n^n}$.

Solution Let $a_n = \dfrac{1}{n^n}$

Since the given series is a power series, it will be absolutely convergent within its interval of convergence.

Now,
$$R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \left|\dfrac{1}{n^n}\right|^{1/n}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \dfrac{1}{(n^n)^{1/n}}} = \infty$$

Hence, the series converges absolutely, $\forall\ x$.

Example 41 Test for absolute convergence of the series $\sum_{n=1}^{\infty} nx^{n-1}$.

Solution Here, $a_n = n$

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^n = \lim_{n\to\infty} |n|^{1/n} = 1$$

Hence, $R = 1$; so the series converges absolutely on the interval $]-1, 1[$, but does not converge at $x = \pm 1$.

Example 42 Test for absolute convergence of the series $\sum_{1}^{\infty} \frac{x^{n-1}}{n}$.

Solution Here, $a_n = \frac{1}{n}$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} = 1$$

$\therefore$ $R = 1$ so the series converges at $x = -1$ but not at $x = 1$ and it converges absolutely on $]-1, 1[$.

Example 43 Find the radius of convergence of the series $\frac{x}{2} + \frac{1.3}{2.5}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}x^3 + \cdots$

Solution Radius of convergence $(R) = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot 2\cdot 5\cdot 8\cdots(3n+2)}{2\cdot 5\cdot 8\cdots(3n-1)1\cdot 3\cdot 5\cdots(2n+1)} = \frac{3}{2}$

Hence, the series converges absolutely for all x, where $|x| < \frac{3}{2}$

8.19 UNIFORM CONVERGENCE

Consider a sequence or a series whose terms are function of a variable, say x, in the interval $[a, b]$. The convergence of such a sequence or a series in the given interval is called uniform convergence and is denoted as $\{S_n(x)\}$.

A sequence $\{S_n(x)\}$ is said to converge uniformly to the limit $S(x)$ in the interval $[a, b]$, if for any $\in > 0$, there exists a positive integer m such that $|S_n(n) - S(x)| < \in$ for all $n > m$ and $x \in [a, b]$.

Theorem 1: Cauchy's General Principle of Uniform Convergence

The necessary and sufficient condition for the sequence $\{S_n(n)\}$ to converge uniformly in $[a, b]$ is that, for any given $\in > 0$, there exists a positive integer (I_+) independent of x, such that

$|S_n(x) - S_m(x)| < \in$ for all $n > m \geq I_+$ and all $x \in [a, b]$.

Theorem 2: (M_n Test)

Let $\{S_n(n)\}$ be a sequence and let $M_n = Sup\ \{|S_n - S|: x \in [a, b]\}$. Then $\{S_n(x)\}$ converges uniformly to S if and only if $\lim_{n\to\infty} M_n = 0$.

8.19.1 Uniform Convergence of a Series of Functions

The series $\sum u_n(x)$ is said to converge uniformly on $[a, b]$ if the sequence $\{S_n(n)\}$ of its partial sums converges uniformly on $[a, b]$.

Theorem 3: Weierstrass's M-Test

A series $\sum u_n(x)$ converges uniformly and absolutely on $[a, b]$ if $|u_n(x)| \leq M_n$ for all n and $x \in [a, b]$, where M_n is independent of x and $\sum M_n$ is convergent.

8.19.2 Properties of Uniformly Convergent Series

We know that the fundamental properties of the functions f_n do not in general hold for the pointwise limit function f, we shall now show that roughly speaking these properties hold for the limit function f when the convergence is uniform. In this connection, we now give some theorems which become particularly important in application.

A. If a sequence $\{S_n\}$ converges uniformly in $[a, b]$, and x_0 is a point of $[a, b]$ such that $\lim\limits_{n\to\infty} S_n(x) = a_n$; $n = 1, 2, 3, 4, \ldots$ then

(i) $\{a_n\}$ converges, and (ii) $\lim\limits_{n\to\infty} S(x) = \lim\limits_{n\to\infty} a_n$

B. If a series $\sum\limits_{n=1}^{\infty} S_n$ converges uniformly to S in $[a, b]$, and x_0 is a point in $[a, b]$ such that $\lim\limits_{x\to x_0} S_n(x) = a_n$; $n = 1, 2, 3, \ldots$ then

(i) $\sum\limits_{n=1}^{\infty} a_n$ converges, and (ii) $\lim\limits_{x\to x_0} S(x) = \sum\limits_{n=1}^{\infty} a_n$

Note The limit of the sum function of a series is equal to the sum of the series of limits of function, i.e.,

$$\lim_{x\to x_0} \sum_{1}^{\infty} S_n(x) = \sum_{1}^{\infty} \lim_{x\to x_0} S_n(x)$$

C. If a sequence $\{f_n\}$ of continuous functions on an interval $[a, b]$ and if $f_n \to f$ uniformly on $[a, b]$ then f is continuous on $[a, b]$.

D. If a series $\sum\limits_{1}^{\infty} f_n$ converges uniformly to f in $[a, b]$ and its terms f_n are continuous at a point x_0 of the interval then the sum function f is also continuous at x_0.

E. If a sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ and each function f_n is integrable then f is integrable on $[a, b]$ and the sequence $\left\{\int_a^x f_n\, dt\right\}$ converges uniformly to $\int_a^x f\, dt$ on $[a, b]$, i.e.,

$$\int_a^x f\, dt = \lim_{n\to\infty} \int_a^x f_n\, dt \;\forall\; x \in [a, b]$$

F. If a series $\sum f_n$ converges uniformly to f on $[a, b]$ and each term $f_n(x)$ is integrable then f is integrable on $[a, b]$ and the series $\sum\left(\int_a^x f_n\, dt\right)$ converges uniformly to $\int_a^x f\, dt$ on $[a, b]$, i.e.,

$$\int_a^x f\, dt = \sum_{1}^{\infty}\left(\int_a^x f_n\, dt\right), \forall\; x \in [a, b]$$

G. If $\sum f_n(x)$ converges to the sum f in $[a, b]$, $\sum f'_n(x)$ converges uniformly to $F(x)$ and $f'_n(x)$ is a continuous function of x in $[a, b]$ $\forall\; n$, then $F(x) = f'(x)$.

Example 44 Show that the sequence $\{S_n\}$, where $S_n(x) = \dfrac{nx}{1+n^2x^2}$, is not uniformly convergent on any interval containing zero.

Solution Here, $S_n(x) = \dfrac{nx}{1+n^2x^2}$

$$\therefore \quad \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0, \forall\ x$$

Now, $\dfrac{nx}{1-n^2x^2}$ attains the maximum value $\dfrac{1}{2}$ at $x = \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ as $n \to \infty$

Now, $M_n = Sup|S_n(x) - S(x)|;\ x \in [a, b]$

$$= Sup\left|\frac{nx}{1+n^2x^2}\right|$$

$$= \frac{1}{2}, \text{ which does not tend to zero as } n \to \infty$$

Hence, the sequence $\{f_n(x)\}$ is not uniformly convergent in any interval containing zero.

Example 45 Is the sequence $\{S_n(n)\}$, where $S_n(x) = \dfrac{x}{1+nx^2};\ x \in R$ converge uniformly in any closed interval I?

Solution Given $S_n(x) = \dfrac{x}{1+nx^2};\ x \in R$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0, \forall\ x$$

Now,

$$M_n = Sup\left|S_n(x) - S(x)\right| = Sup\left|\frac{x}{1+nx^2}\right|;\ x \in I$$

$$= \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

Hence, $\{S_n(x)\}$ converges uniformly on I.

$$\left[\because \frac{x}{1+nx^2} \text{ attains the maximum value } \frac{1}{2\sqrt{n}} \text{ at } x = \frac{1}{\sqrt{n}}\right]$$

Example 46 Test for uniform convergence, the series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots;\ -\frac{1}{2} \le x \le \frac{1}{2}$$

Solution Here, $S_n(x) = \dfrac{2^n x^{2n-1}}{1+x^{2n}}$

$$S_n(x) = \left|\frac{2^n x^{2n-1}}{1+x^{2n}}\right| \le 2^n (\alpha)^{2n-1}; |x| \le \alpha \le \frac{1}{2}$$

The series $\sum 2^n (\alpha)^{2n-1}$ converges and hence, by *M*-test the given series converges uniformly on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Example 47 The series $\sum_{1}^{\infty} \frac{(-1)^n}{n} |x|^n$ is uniformly convergent in [–1, 1].

Solution Since $|x|^n$ is a +ve monotonic decreasing and bounded for [–1, 1], and the series $\sum_{1}^{\infty} \frac{(-1)^n}{n}$ is uniformly convergent.

$\therefore \quad \sum_{1}^{\infty} \frac{(-1)^n}{n} |x|^n$ is also uniformly convergent in [–1, 1].

Example 48 Show that the series $\frac{\cos x}{1^p} + \frac{\cos 2x}{2^p} + \frac{\cos 3x}{3^p} + \cdots + \frac{\cos nx}{n^p}$ converges uniformly on the real line for $p > 1$.

Solution Let $\quad S_n = \frac{\cos nx}{n^p}$

$$\therefore \quad |S_n| = \left|\frac{\cos nx}{n^p}\right| \le \frac{1}{n^p}, \forall\ x \in R$$

But the series $\sum \frac{1}{n^p}$ converges for $p > 1$. Hence, by Weierstrass's *M*-test, the given series $\sum S_n(x)$ converges uniformly on R for $p > 1$.

Example 49 Test for uniformly convergence of the series $\sum_{n=D}^{\infty} a^n \cos nx$.

Solution Do the same as in Example 48.

8.20 BINOMIAL, EXPONENTIAL, AND LOGARITHMIC SERIES

(i) Convergence of Binomial Series

The series is $1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r + \cdots \infty$

Here, $u_r = \frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-r)!} x^{r-1}$ and $u_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r$

Applying the ratio test, we have

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{n-r+1}{r}\right| |x| = \left|\frac{n+1}{r} - 1\right| |x|$$

$$\therefore \qquad \lim_{r\to\infty} \frac{u_{r+1}}{u_r} = |x|$$

So that the binomial series is absolutely convergent if $|x| < 1$. This means that for this range of values of x, the series is equal to $(1 + x)^n$; n is a +ve integer.

(ii) Convergence of Logarithmic Series

The series $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + (-1)^{n-1}\dfrac{x^n}{n} + \cdots$ is convergent for $-1 < x \le 1$.

Here $u_n = (-1)^{n-1}\dfrac{x^n}{n}$ and $u_{n+1} = (-1)^n \dfrac{x^{n+1}}{n+1}$

Applying ratio test, we have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{(-1)^n x^{n+1}}{n+1} \times \frac{n}{(-1)^{n-1} x^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{-x}{1+1/n}\right] = -x$$

Thus, the series is convergent for $|x| < 1$ and divergent for $|x| > 1$.

At $x = 1$, the series becomes

$1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots \infty = \sum (-1)^n \dfrac{1}{n}$ which is convergent and at $x = -1$, the series becomes

$-\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots \infty\right) = \sum\left(-\dfrac{1}{n}\right)$ which is divergent.

(iii) Convergence of Exponential Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty \text{ is convergent for all values of } x.$$

Here, $\qquad u_n = \dfrac{x^{n-1}}{(n-1)!}$ and $u_{n+1} = \dfrac{x^n}{n!}$

Applying ratio test, we have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{x}{n}\right] = 0 < 1$$

Hence, the series is convergent, $\forall\, x$.

EXERCISE 8.9

1. Find the radius of convergence of the following series:

   (i) $x+\dfrac{x^2}{2^2}+\dfrac{2!}{3^3}x^3+\dfrac{3!}{4^4}x^4+\cdots$

   (ii) $1+x^2+x^4+x^6+\ldots$

   (iii) $1+x+2!\,x^2+3!\,x^3+\ldots$

2. Show that the series $\sum_1^{\infty} f_n$; where $f_n(x)=\dfrac{x^2}{(1+x^2)^n}$, does not converge uniformly for $x \ge 0$.

3. Show that the series $\sum_1^{\infty}\dfrac{1}{n^4+n^2x^2}$ is uniformly convergent in [–K, K], for real K.

4. Show that the series $\sum_1^{\infty}\dfrac{x}{n(1+nx^2)}$ converges uniformly for all real x.

5. Show that $\sum\dfrac{a_n}{n^x}$ converges uniformly in [0, 1] if $\sum a_n$ converges.

Answers

1. (i) (e) **(ii)** (1) **(iii)** (0)

SUMMARY

1. Infinite Series

If $< u_n>$ be a sequence of real numbers then the sum of the infinite number of terms of this sequence, i.e., the expression $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is defined as an infinite series and is demoted by

$$\sum_{n=1}^{\infty} u_n \text{ or } \sum u_n$$

2. Positive-term Series

The series u_n is called a positive-term series if each term of this series is positive, i.e.,

$$u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

3. Alternating Series

It is a series whose terms are alternatively positive and negative, i.e., $u_n = u_1 - u_2 + u_3 - u_4 + \ldots$

4. Convergent Series

A series u_n is said to be convergent if the sum of the first n terms of the series tends to a finite and unique limit as n tends to infinity,

i.e., if $\lim\limits_{n\to\infty} S_n$ = finite and unique then series u_n is convergent.

5. Divergent Series

A series u_n is said to be divergent if the sum of first n terms of the series tends to $+\infty$ or $-\infty$ as $n \to \infty$, i.e., if $\lim_{n\to\infty} S_n = +\infty \text{ or } -\infty$, then the series u_n is divergent.

6. Oscillatory Series

The oscillatory series are two types:

(i) Oscillate Finitely A series u_n is said to oscillate finitely if the sum of its first n terms tends to a finite but not unique limit as n tends to infinity, i.e., if $\lim_{n\to\infty} S_n$ = finite but not unique then u_n oscillates finitely.

(ii) Oscillate Infinitely A series u_n is said to oscillate infinitely if the sum of its first n terms oscillates infinitely, i.e., if $\lim_{n\to\infty} S_n = +\infty \text{ or } -\infty$ both then the series u_n oscillates infinitely.

7. A Necessary Condition for Convergence

A necessary condition for a positive-term series u_n to converge is that $\lim_{n\to\infty} u_n = 0$.

8. Geometric Series

The series $1 + x + x^2 + x^3 + x^4 + \ldots \infty$ is

(i) Convergent if $|x| < 1$ (ii) Divergent if $r \geq 1$ (iii) Oscillatory if $x \leq -1$.

9. *p*-series Test

The infinite series $\sum \frac{1}{n^p}$ i.e., $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \infty$ is

(i) Convergent if $p > 1$ (ii) Divergent if $p \leq 1$

10. Comparison Test

If u_n and v_n be two given series and if $\lim_{n\to\infty} \frac{u_n}{v_n} = a$ finite and non-zero quantity then u_n and v_n are either both convergent or both divergent.

11. D'Alembert Ratio's Test

Let u_n be a series of positive terms, such that

I. If $\frac{u_{n+1}}{u_n} = \lambda$ then the series is

(a) convergent if $\lambda < 1$, (b) divergent if $\lambda > 1$, and (c) if $\lambda = 1$, this test fails.

II. If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$ then the series u_n is divergent.

OR

I. If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda$ then

(a) if $\lambda > 1$, it is convergent, (b) if $\lambda < 1$, it is divergent, and (c) if $\lambda = 1$, this test fails.

II. If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$ then u_n is convergent.

12. Cauchy's Root (or Radical) Test

Let u_n be an infinite series of positive terms and let $\lim\limits_{n\to\infty} [u_n]^{\frac{1}{n}} = \lambda$.

Then the series is

(i) Convergent if $\lambda < 1$,

(ii) Divergent if $\lambda > 1$, and

(iii) If $\lambda = 1$, this test fails.

13. Raabe's Test

Let u_n be an infinite series of positive terms and let $\lim\limits_{n\to\infty} \left[n\left(\frac{u_n}{u_{n+1}} - 1 \right) \right] = \lambda$. Then the series is

(i) convergent if $\lambda > 1$,

(ii) divergent if $\lambda < 1$, and

(iii) if $\lambda = 1$, this test fails.

14. Logarithmic Test

If u_n be an infinite series of positive terms and let

$$\lim_{n\to\infty} \left[n \log\left(\frac{u_n}{u_{n+1}} \right) \right] = \lambda$$

Then the series is

(i) convergent if $\lambda > 1$,

(ii) divergent if $\lambda < 1$, and

(iii) if $\lambda = 1$, this test fails.

15. Absolute Convergence and Conditional Convergence

(i) Absolute Convergent Series A series u_n is to be absolutely convergent if the positive-term series $|u_n|$ is convergent.

(ii) Conditionally Convergent Series A series u_n is said to be conditionally convergent if the series $|u_n|$ is divergent. It is also known as semi-convergent or accidentally convergent.

16. Power Series

A series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x_+^n = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

is said to be a power series. The numbers a_r are called *coefficients* of the power series.

Now, if we apply the Cauchy's root test for convergence of the power series, we conclude that the series is absolutely convergent if $|x| < \frac{1}{\lambda}$ and divergent if $|x| > \frac{1}{\lambda}$

Let $\frac{1}{\lambda} = R$; then the power series is absolutely convergent if $|x| < R$ and divergent if $|x| > R$.

where R is called the radius of convergence and is defined as $\frac{1}{R} = \lim\limits_{n\to\infty} |a_n|^{1/n}$ OR $R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}}$

Note A power series is absolutely convergent within its interval of convergence and divergent outside it.

17. Uniform Convergence

Consider a sequence or a series whose terms are function of a variable, say x, in the interval $[a, b]$. The convergence of such a sequence or a series in the given interval is called uniform convergence and is denoted as $\{S_n(x)\}$.

A sequence $\{S_n(x)\}$is said to converge uniformly to the limit $S(x)$ in the interval $[a, b]$, if for any $\in > 0$, there exists a positive integer m such that $|S_n(n) - S(x)| < \in$ for all $n > m$ and $x \in [a, b]$.

Theorem I: Cauchy's General Principle of Uniform Convergence

The necessary and sufficient condition for the sequence $\{S_n(n)\}$ to converge uniformly in $[a, b]$ is that, for any given $\in > 0$, there exists a positive integer (I_+) independent of x, such that

$$|S_n(x) - S_m(x)| < \in \text{ for all } n > m \geq I_+ \text{ and all } x \in [a, b].$$

Theorem II: (M_n-Test)

Let $\{S_n(n)\}$ be a sequence and let $M_n = S^{up}\ \{1S_n - S|: x \in\ [a, b]\}$. Then $\{S_n(n)\}$ converges uniformly to S if and only if $\lim\limits_{n\to\infty} M_n = 0$.

18. Uniform Convergence of a Series of Functions

The series $u_n(x)$ is said to converge uniformly on $[a, b]$ if the sequence $\{S_n(n)\}$ of its partial sums converges uniformly on $[a, b]$.

Theorem III: Weierstrass's M-test

A series $u_n(x)$ converges uniformly and absolutely on $[a, b]$ if $|u_n(x)| \leq M_n$ for all n and $x \in\ [a, b]$, where M_n is independent of x and M_n is convergent.

OBJECTIVE-TYPE QUESTIONS

1. The series $\dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dfrac{1}{\log 5} + \cdots$ is

(a) divergent
(b) absolutely convergent
(c) conditionally convergent
(d) convergent

2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$ is

(a) divergent
(b) absolutely convergent
(c) convergent
(d) oscillatory

3. The series $\dfrac{2}{1^2} - \dfrac{3}{2^2} + \dfrac{4}{3^2} - \dfrac{5}{4^2} + \cdots \infty$ is

(a) convergent
(b) divergent
(c) conditionally convergent
(d) absolutely convergent

4. The series $1 + \dfrac{1}{4^{2/3}} + \dfrac{1}{9^{2/3}} + \dfrac{1}{16^{2/3}} + \cdots$ is

(a) convergent
(b) divergent
(c) absolutely convergent
(d) conditionally convergent

5. The sum of the alternating harmonic series $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots$ is

(a) zero (b) infinite
(c) 2 (d) log 2

6. The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \log\frac{n+1}{n}\right)$ is

(a) convergent
(b) divergent
(c) oscillatory
(d) absolutely convergent

7. The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is

(a) divergent
(b) oscillatory
(c) absolutely convergent
(d) convergent

8. The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $x > 0$ is

(a) convergent for $x < 1$, divergent for $x \geq 1$
(b) convergent for $x = 1$, divergent for $x < 1$
(c) convergent for $x > 1$, divergent for $x \leq 1$
(d) convergent for $x \leq 1$, divergent for $x = 1$

9. The series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$ is

(a) absolutely convergent
(b) divergent
(c) convergent
(d) oscillatory

10. The series $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is

(a) convergent
(b) divergent
(c) absolutely convergent
(d) oscillatory

11. The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to

(a) 2 ln2 (b) $\sqrt{2}$
(c) 2 (d) e

[GATE (EC) 2014]

ANSWERS

1. (d)	2. (c)	3. (b)	4. (a)	5. (d)	6. (a)	7. (d)	8. (a)	9. (c)	10. (a)
11. (d)									

9 Fourier Series

9.1 INTRODUCTION

Fourier series introduced in 1807 by Fourier was one of the most important developments in applied mathematics. It is very useful in the study of heat conduction, electrostatics, mechanics, etc. The Fourier series is an infinite series representation of periodic functions in terms of trigonometric sine and cosine functions.

The Fourier series is a very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as nonhomogeneous terms.

The Fourier series constructed for one period is valid for all values. Harmonic analysis is the theory of expanding functions in Fourier series.

Joseph Fourier (21 March 1768–16 May 1830), born in Auxerre, Bourgogne, France, was a mathematician and physicist, Joseph Fourier is renowned for showing how the conduction of heat in solid bodies could be analyzed in terms of infinite mathematical series, which is fondly called the *Fourier Series*. Joseph Fourier played an important role in the Egyptian expeditions as well as in the discovery of the greenhouse effect.

9.2 PERIODIC FUNCTION

A function $f(x)$ is said to be periodic if $f(x + T) = f(x)$ for all x and T is the smallest positive number for which this relation holds. Then T is called period of $f(x)$.

If T is the period of $f(x)$ then $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \dots = f(x + nT)$

Also, $\quad f(x) = f(x - T) = f(x - 2T) = f(x - 3T) = \dots = f(x - nT)$

$\therefore \quad f(x) = f(x \pm nT)$, where n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

For example, $\sin x$, $\cos x$, $\sec x$, and $\operatorname{cosec} x$, are periodic functions with period 2π while $\tan x$ and $\cot x$ are periodic functions with period π.

The functions $\sin nx$ and $\cos nx$ are periodic with period $\dfrac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic.

If T_1 and T_2 are the periods of $f(x)$ and $g(x)$ then the period of a $f(x) + b\,g(x)$ is the least common multiple of T_1 and T_2.

For example, cos x, cos $2x$, cos $3x$ are periodic functions with periods 2π, π, and $\frac{2\pi}{3}$ respectively.

$\therefore\ f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic with period 2π, the LCM of 2π, π, and $\frac{2\pi}{3}$.

9.3 FOURIER SERIES

The Fourier series is an infinite series which is represented in terms of the trigonometric sine and cosine functions of the form

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \ldots$$

or
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin n, x\right]$$

where the constants a_0, a_n, and b_n are called Fourier coefficients.

Result

Let m and n be integers, $m \neq 0$, $n \neq 0$ for $m \neq n$.

1. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos mx \cdot \cos nx\, dx = 0$
2. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin mx \cdot \cos nx\, dx = 0$
3. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin mx \cdot \cos nx\, dx = 0$
4. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos mx\, dx = 0$
5. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin mx\, dx = 0$

for $m = n$

1. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos mx \cdot \cos nx\, dx = \int_{\alpha}^{\alpha+2\pi} \cos^2 mx\, dx = \pi$
2. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin^2 mx\, dx = \pi$
3. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos mx \cdot \sin mx\, dx = 0$

9.4 EULER'S FORMULAE

Let $f(x)$ be a periodic function with period 2π defined in the interval $(\alpha, \alpha, + 2\pi)$, of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

where the constants a_0, a_n, b_n, are called the *coefficients*.

(i) To Find the Coefficient a_0

Integrate both sides of (1) with respect to x in the interval α to $\alpha + 2\pi$. Then,

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2}\,dx + \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big)\right] dx$$

$$= \frac{a_0}{2}[x]_{\alpha}^{\alpha+2\pi} + \sum_{n=1}^{\infty}\left[a_n \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx\right]$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{2}\cdot 2\pi + 0 = a_0 \cdot \pi$$

[Using results (4) and (5), the last two integrals for all n will be zero]

Hence,
$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \tag{2}$$

(ii) To Find the Coefficient a_n for n = 1, 2, 3, ...

Multiplying both sides of (1) by $\cos nx$ and integrating w. r. t. 'x' in the interval $(\alpha, \alpha + 2\pi)$, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx = \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2}\cos nx \cdot dx + \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big)\right] \cos nx\,dx$$

$$= \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2}\cos nx\,dx + \sum_{n=1}^{\infty}\left[\int_{\alpha}^{\alpha+2\pi} a_n \cos^2 nx\,dx + \int_{\alpha}^{\alpha+2\pi} b_n \sin nx . \cos nx\,dx\right]$$

$$= \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty}\left[\int_{\alpha}^{\alpha+2\pi} a_n \cos^2 nx\,dx\right] + \sum_{n=1}^{\infty}\left[\int_{\alpha}^{\alpha+2\pi} b_n \cos nx \cdot \sin nx\,dx\right]$$

$= 0 + a_n \cdot \pi + 0$ [Using the result (1) for $m = n$ and (3) for $m = n$]

$= \pi a_n$

or
$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot \cos nx\,dx \tag{3}$$

Similarly, $$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \tag{4}$$

The formulae (2), (3), and (4) are known as *Euler formulae*.

9.5 DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician).

All the functions that normally arise in engineering problems satisfy these conditions and, hence, they can be expressed as a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),\ \text{where } a_0,\ a_n, \text{ and } b_n \text{ are Fourier coefficients,}$$

***Provided*:**

(i) Function $f(x)$ is periodic, single valued and finite.

(ii) Function $f(x)$ has a finite number of discontinuities in any one period.

(iii) Function $f(x)$ has at the most of finite number of maxima and minima.

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity.

At a point of discontinuity, the sum of the series is equal to the mean of the right- and left-hand limits, i.e., $\frac{1}{2}[f(x+0)+f(x-0)]$, where $f(x + 0)$ and $f(x - 0)$ denote the limit on the right and the limit on the left respectively.

9.6 FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In Article 9.4, we derived Euler's formula for a_0, a_n, and b_n on the assumption that $f(x)$ in continuous in $(\alpha, \alpha + 2\pi)$.

However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for a_0, a_n, and b_n are to be evaluated by breaking up the range of integration.

Let $f(x)$ be defined by $f(x) = f_1(x);\ \alpha < x < x_0$

$$= f_2(x);\ x_0 < x < \alpha + 2\pi$$

where x_0 is the point of finite discontinuity in the interval $(\alpha, \alpha + 2\pi)$.

The values of a_0, a_n, and b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\cdot\cos nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cdot\sin nx\,dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0 - 0)$ and $f(x_0 + 0)$ exist but are unequal. The sum of the Fourier series

$$= \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]$$

Example 1 Find the Fourier series expansion for the periodic function $f(x) = x;\ 0 < x < 2\pi$

Solution Consider the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

The Fourier coefficients a_0, a_n, b_n are obtained as follows:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}\left[(-1)^2 - 1\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}(-1)^{2n} \qquad [\because \quad \cos 2n\pi = (-1)^{2n}]$$

$$= -\frac{2}{n}$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$x = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)\sin nx = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

Example 2 Find the Fourier series for the function $f(x) = \dfrac{(\pi - x)^2}{4}$; $0 < x < 2\pi$.

Solution Consider the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

The Fourier coefficients are obtained as follows:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{12\pi}\left[-\pi^3-\pi^3\right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - \{-2(\pi-x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0+\frac{2\pi\cos 2nx}{n^2}+0\right)-\left(0-\frac{2\pi\cos 0}{n^2}+0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right] = \frac{4\pi}{4\pi\, n^2} = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 - \{-2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n}-0+\frac{2\cos 2\,n\pi}{n^3}\right)-\left(-\frac{\pi^2}{n}-0+\frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{b}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right] = 0 \qquad [\because \cos 2n\pi = 1]$$

$$\therefore \qquad f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

Example 3 Find the Fourier series for the function $f(x) = x - x^2$ in the interval $(-\pi, \pi)$. Hence, deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

Solution Consider the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (1)

The Fourier coefficients are obtained as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 - 2\pi)\frac{\cos n\pi}{n^2} - (1 + 2\pi)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left(-4\pi\frac{\cos n\pi}{n^2}\right) = -4\frac{(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cdot\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - \pi)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3} + (-\pi - \pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\frac{\cos n\pi}{n}\right] = 2\frac{(-1)^n}{n}$$

$$\therefore \qquad (x - x^2) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right] - 2\left[-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \cdots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[+\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right] + 2\left[+\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right] \tag{2}$$

Putting $x = 0$ in (2), we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$ **Hence, proved**.

Example 4 Find a Fourier series for the function $(x + x^2)$ in the interval $-\pi < x < \pi$. Hence, show that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$.

Solution Consider the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{-\pi}^{\pi}[a_n \cos nx + b_n \sin nx] \tag{1}$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx + \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

[Since $x \cos nx$ is an odd function of x, and $x^2 \cos nx$ is an even function of x]

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left\{\frac{\sin nx}{n}\right\} - (2x)\cdot\left\{\frac{-\cos nx}{n^2}\right\} + 2\left\{-\frac{\sin nx}{n^3}\right\}\right]_0^{\pi} = \frac{2}{\pi}\left[2\pi\frac{\cos\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

[$\because$ $x \sin nx$ is an even function and $x^2 \sin nx$ is an odd function]

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}+0\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}(-1)^x\right]=\frac{2}{n}(-1)^n$$

Now, putting the values of a_0, a_n and b_n in (1), we get

$$x+x^2 = \frac{\pi^3}{3}+\sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\cos nx+\frac{2}{n}(-1)^n\sin nx\right]$$

$$= \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x+x^2 = \frac{\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\cdots\right]+2\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\cdots\right] \tag{2}$$

Putting $x=\pi$ and $x=-\pi$ in (2), we have

$$\pi+\pi^2 = \frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \tag{3}$$

and

$$-\pi+\pi^2 = \frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \tag{4}$$

Adding equations (3) and (4), we get

$$2x^2 = \frac{2\pi^2}{3}+8\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

or

$$= \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

Hence, proved.

Example 5 Find a Fourier series expansion for the function $f(x)$ in the interval $(-\pi, \pi)$, where

$$f(x) = \begin{cases} 0 & -\pi < x \le 0 \\ \dfrac{\pi x}{4} & 0 < x < \pi \end{cases}$$

Solution Consider $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}\left[a_n\cos nx+b_n\sin nx\right]$ (1)

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx+\frac{1}{\pi}\int_0^{\pi}\frac{\pi x}{4}\,dx$$

$$= \frac{1}{4}\int_0^{\pi} x\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0.\cos nx\,dxf + \frac{1}{\pi}\int_0^{\pi}\frac{\pi x}{4}\cos nx\,dx$$

$$= 0 + \frac{1}{4}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{1}{n^2}(\cos n\pi - 1)\right]$$

$$= \frac{1}{4n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}\frac{\pi x}{4}\sin nx\,dx$$

$$= \frac{1}{4}\int_0^{\pi} x\sin nx\,dx = \frac{1}{4}\left[x\cdot\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = -\frac{\pi}{4n}\cos n\pi = -\frac{\pi}{4n}(-1)^n$$

Putting the values of a_0, a_n, b_n in (1),

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{1}{4n^2}\{(-1)^n - 1\}\cos nx + \sum_{n=1}^{\infty}\left[-\frac{\pi}{4n}(-1)^n\right]\sin nx\right.$$

$$= \frac{\pi^2}{16} + \frac{1}{4}\left(-2\cos x - \frac{2}{3^2}\cos 3x - \cdots\right) - \frac{\pi}{4}\left(-\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

$$= \frac{\pi^2}{16} + \left(-\frac{1}{2}\cos x + \frac{\pi}{4}\sin x\right) + \left(-\frac{\pi}{4.2}\sin 2x\right) + \left(-\frac{1}{2.3^2}\cos 3x + \frac{\pi}{4.3}\sin 3x\right) + \cdots$$

Example 6 Find the Fourier series expansion of $f(x) = e^{ax}$ in the interval $(0, 2\pi)$.

Solution Consider

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx] \tag{1}$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$$

$$= \frac{1}{\pi a}(e^{2\pi a} - 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{(a^2+n^2)}(a\cos nx + b\sin nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[ae^{2\pi a}\cos 2n\pi - 1\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[ae^{2\pi a} - 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cdot\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{n}{\pi(a^2+n^2)}\left[-e^{-2\pi a}\cos 2n\pi + 1\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left[1 - e^{2a\pi}\right]$$

Putting the values of a_0, a_n, b_n in (1), we get

$$e^{ax} = \left(\frac{e^{2\pi a}-1}{\pi a}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{\pi(a^2+n^2)}(ae^{2\pi a}-1)\cos nx + \frac{n}{\pi(a^2+n^2)}(1-e^{2\pi a})\sin nx\right]$$

$$= \left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left[\frac{1}{a} - \sum_{n=1}^{\infty}\left(\frac{n}{a^2+n^2}\right)\sin nx\right] + \sum_{n=1}^{\infty}\frac{(ae^{2\pi a}-1)}{\pi(a^2+n^2)}\cdot\cos nx$$

9.7 FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

(i) Even Function

A function $f(x)$ is said to be an even function, if

$$f(-x) = f(x) \text{ for all } x$$

For example, x^4, $\cos x$, $\sec x$ are even functions.

Notes

(a) The graph of $f(x)$ is symmetric about the y-axis.

(b) $f(x)$ contains only even powers of x and may contain only $\cos x$, $\sec x$.

(c) The product of two even functions is an even function.

(d) The sum of two even functions is an even function.

(e) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(-x) = f(x)$

Graphs of Even Functions

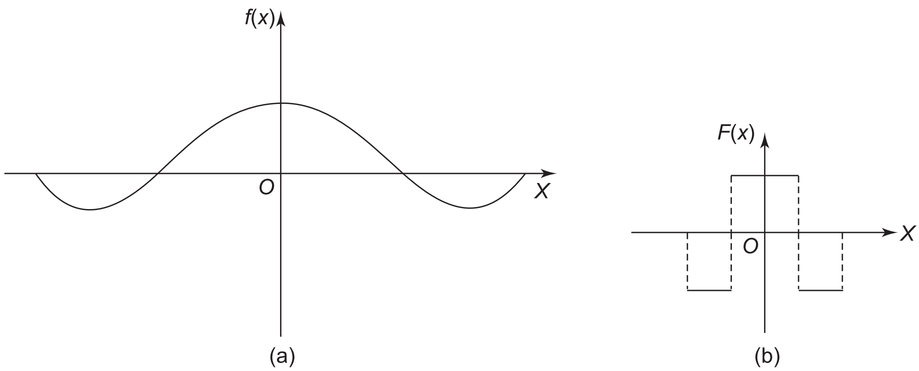


Fig. 9.1

(ii) Odd Function

A function $f(x)$ is said to be an odd function if

$$f(-x) = -f(x) \text{ for all } x$$

For example, x, x^3, $\sin x$, $\tan x$ are odd functions.

Notes

(a) The graph of $f(x)$ is symmetric about the origin.

(b) $f(x)$ contains only odd powers of x and may contain only $\sin x$, $\operatorname{cosec} x$.

(c) The sum of two odd functions is odd.

(d) The product of an odd function and an even function is an odd function.

(e) Product of two odd functions is an even function.

(f) $\int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f(-x) = -f(x)$

Graphs of Odd Functions

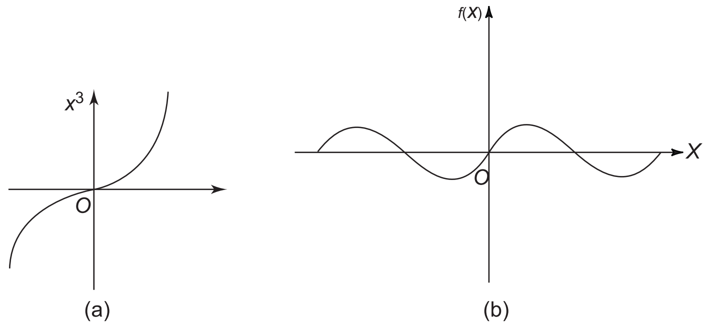


Fig. 9.2

(iii) Fourier Series for Even and Odd Functions

Let the Fourier series of $f(x)$ in $(-\pi, \pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{5}$$

where
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\, dx \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx\, dx$$

Case I Consider $f(x)$ is an even function in $(-\pi, \pi)$. Then all b_n's will be zero. Thus, the Fourier series of an even function contains only cosine terms and is known as *Fourier cosine series* given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{6}$$

where
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx \text{, and}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx$$

Case II Consider $f(x)$ is an odd function in $(-\pi, \pi)$, Then all a_n's will be zero. Also a_0 is zero since $f(x)$ is an odd function. Thus, the Fourier series of an odd function contains only sine terms and is known as *Fourier sine series* given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Example 7 Find the Fourier series for the function $f(x) = |x|$ in $-\pi < x < \pi$. Hence, show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Solution $f(x) = |x| = |-x| = f(-x)$

Therefore, $f(x)$ is an even function and, hence, $b_n = 0$,

Let the Fourier series

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

Then, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)^2$$

$$a_0 = \pi$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{x}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ $$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] \tag{2}$$

Putting $x = 0$ in (2), we get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 8 Find the Fourier series for the function $f(x) = x + x^2, -\pi < x < \pi$. Hence, show that

(i) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$

(ii) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$

Solution Let the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

Then
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = 0 + \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx$$

$$= \frac{2}{3}[x^2]_0^{\pi} = \frac{2}{3}\pi^2$$

and
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2\cos nx)\,dx$$

$$= 0 + \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \qquad [\because \ x\cos nx \text{ is an odd function}]$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n}\bigg|_0^{\pi} - \int_0^{\pi} 2x\cdot\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{-4}{n\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right)\bigg|_0^{\pi} - \int_0^{\pi} 1\cdot -\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{-4}{n\pi}\left[-\frac{\pi\cos nx}{n} + \frac{\sin n\pi}{n^2}\bigg|_0^{\pi}\right]$$

$$= \frac{-4}{n\pi}\left[-\frac{\pi\cos n\pi}{n} + 0\right] = \frac{4}{n^2}(-1)^n \qquad [\because \ \cos n\pi = (-1)^n]$$

Now, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx + 0 \qquad [\because\ x^2\sin nx \text{ is an odd function}]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\pi} - \int_0^{\pi} 1.\left(-\frac{\cos nx}{n^2}\right)dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos nx}{n} + \frac{\sin nx}{n^2}\Bigg|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n} + 0\right] = -\frac{2}{n}\cos n\pi$$

$$b_n = -\frac{2}{n}(-1)^n$$

Putting the values of a_0, a_n, and b_n in (1), we get

$$f(x) = (x + x^2) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[\frac{-1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right] \qquad (2)$$

Putting $x = -\pi$ in (2), we get

$$\frac{1}{2}\left[f(-\pi + 0) + f(-\pi - 0)\right] = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

or $$\frac{1}{2}\left[\pi + \pi^2 - \pi + \pi^2\right] = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

or $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Again putting $x = 0$ in (2), we get

$$\frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2}\cdots\right] = 0$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\cdots = \frac{\pi^2}{12}$$ **Hence, proved**

Example 9 Find the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$. Hence, show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\cdots = \frac{\pi^2}{12}$$

Solution Since $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ $f(x) = x^2$ is an even function and $b_n = 0$

Let the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)\Bigg|_0^{\pi} - \int_0^{\pi} 2x\cdot \frac{\sin nx}{n}\,dx\Bigg\}\right]$$

$$= \frac{2}{\pi}\left[-\frac{2}{n}\left\{x\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\pi} - \int_0^{\pi} 1\cdot\left(-\frac{\cos nx}{n}\right)dx\right\}\right]$$

$$= \frac{2}{\pi}\left[-\frac{2}{n}\left\{-\frac{\pi\cos n\pi}{n} + \frac{\sin nx}{n^2}\Bigg|_0^{\pi}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4}{n^2}(-1)^n$$

Putting the values of a_0 and a_n in (1), we get

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\,x\,.$$

$$= \frac{\pi^2}{3} + 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2\,x}{2^2} - \frac{\cos 3\,x}{3^2} + \frac{\cos 4\,x}{4^2} - \ldots\right)$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2\,x}{2^2} + \frac{\cos 3\,x}{3^2} - \frac{\cos 4\,x}{4^2} + \ldots\right) \tag{2}$$

Putting $x = 0$ in (2), we get

$$\frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right) = 0$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$ **Hence, proved**

Example 10 Find the Fourier series for the function $f(x) = x^3$ in $(-\pi, \pi)$.

Solution Since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

$\therefore$ $f(x)$ an odd function.

Then the Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\,x$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin n\,x\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin n\,x\,dx$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos n\,x}{n}\right) - 3\,x^2\left(-\frac{\sin n\,x}{n^2}\right) + 6\,x(-1)\left(-\frac{\cos n\,x}{n^3}\right) - 6\frac{\sin n\,x}{n^4}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3}{n}\cos n\,\pi + \frac{6\,\pi}{n^3}\cos n\,\pi\right]$$

$$= 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$\therefore$ $$f(x) = x^3 = 2\sum_{n=1}^{\infty}\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)(-1)^n \cdot \sin nx$$

$$= 2\left[\left(\frac{\pi^2}{1} - \frac{6}{1^3}\right)\sin x - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin 2\,x + \ldots\right]$$

Example 11 Find the Fourier-series expansion of the periodic function $f(x) = x;\ -\pi < x < \pi;$ $f(x + 2\pi) = f(x)$

Solution Since $f(-x) = -x = -f(x)$

$f(x) = x$ is an odd function.

Then the Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n x$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin n x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos n x}{n}\right)+\frac{\sin n x}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\cdot\pi\left(-\frac{\cos n\pi}{n}\right)$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$x = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin n x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

EXERCISE 9.1

1. Prove that in the interval $-\pi < x < \pi$,

$$x \cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin n x\,.$$

2. Find the Fourier series of $f(x) = |\cos x|$ is the interval $-\pi < x < \pi$.
3. Find the Fourier series of $f(x) = x \sin x$. in $-\pi < x < \pi$.
4. Find the Fourier series of $f(x) = x \cos x$ in $-\pi < x < \pi$.
5. Find the Fourier series to represent the function $f(x) = |\sin x|$ in $-\pi < x < \pi$.
6. Find the Fourier-series expansion of the function $f(x) = (\pi + x),\ -\pi < x < \pi$,

   Hence, show that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$
7. Find the Fourier-series expansion of the function

$$f(x) = \cos x - \sin\left(\frac{x}{2}\right),\ -\pi < x < \pi.$$

8. Find the Fourier series for the function $f(x) = x^3$ in $-\pi < x < \pi$.

9. Find the Fourier series for the function $f(x) = 1 - |x|;\ -\pi < x < \pi$.

10. Expand the function $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ in Fourier series in the interval $(-\pi, \pi)$.

Answers

2. $f(x) = \dfrac{2}{\pi} + \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos\frac{n\pi}{2}}{1-n^2} \cos nx$

3. $f(x) = 1 - \dfrac{1}{2}\cos x + \sum_{n=2}^{\infty} \dfrac{2(-1)^{n+1}}{n^2-1} \cos nx$

4. $f(x) = -\dfrac{1}{2}\sin x + \sum_{n=2}^{\infty} \dfrac{(-1)^n}{(n^2-1)} \sin nx$

5. $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\left(\dfrac{\cos 2x}{3} + \dfrac{\cos 4x}{15} + \ldots + \dfrac{\cos 2nx}{4n^2-1} + \ldots\right)$

7. $f(x) = \cos x + \dfrac{8}{\pi} \sum_{n=2}^{\infty} \left[\dfrac{n\cos n\pi}{(4n^2-1)} \sin nx\right]$

8. $x^3 = \dfrac{2}{\pi} \sum_{n=1}^{\infty} \left[\left(\dfrac{6\pi}{n^3} - \dfrac{\pi^3}{n}\right)(-1)^n \sin nx\right]$

9. $f(x) = \left(\dfrac{2-\pi}{2}\right) + \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n^2} P_n \cos(nx)$

10. $f(x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} \cdot \cos(nx)$

9.8 CHANGE OF INTERVAL

Suppose we want to represent the function $f(x)$ defined in the closed interval $(-c, c)$ by a Fourier series, c being any positive real number. We consider this interval as a result of elongating (or compressing) the interval $[-\pi, \pi]$. The interval $-c \le x \le c$ is transformed into the interval $-\pi \le z \le \pi$ by the transformation

$$z = \frac{\pi x}{c}$$

Then $$f(x) = f\left(\frac{cz}{\pi}\right) = F(z), \text{ (say)}$$

Let $$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nz + b_n \sin nz\right]$$

where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\,dz$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\cdot\cos nz\,dz\,;\; n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\cdot\sin nz\,dz\,;\; n = 1, 2, 3, \ldots$$

Applying the transformation $z = \dfrac{\pi x}{c}$ so that $dz = \dfrac{\pi}{c}dx$, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right]$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cdot\cos\left(\frac{n\pi x}{c}\right)dx\,;\; n = 0, 1, 2, 3, ..$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\cdot\sin\left(\frac{n\pi x}{c}\right)dx\,;\; n = 1, 2, 3, \ldots$$

9.9 FOURIER HALF-RANGE SERIES

Suppose a function $f(x)$ is defined on some finite interval. It may also be the case that a periodic function $f(x)$ of period $2l$ is defined only on a half-interval $[0, l]$. It is possible to extend the definition of $f(x)$ to the other half $[-l, 0]$ of the interval $[-l, l]$ so that $f(x)$ is either an even or an odd function. In the first case, we call it an even periodic extension of $f(x)$ and in the second case, we call it an odd periodic extension of $f(x)$. If $f(x)$ is given and an even periodic extension is done then $f(x)$ is an even function in $[-l, l]$. Hence, $f(x)$ has a Fourier cosine series. If $f(x)$ is given and an odd periodic extension is done then $f(x)$ is an odd function in $[-l, l]$. Hence, $f(x)$ has now a Fourier sine series.

Therefore, if a function $f(x)$ is defined only on a half-interval $[0, l]$ then it is possible to obtain a Fourier cosine or a Fourier sine-series expansion depending on the requirements of a particular problem, by suitable periodic extensions. Now, we define the Fourier cosine and sine series.

(i) Fourier Cosine Series

The Fourier cosine series expansion of $f(x)$ on the half-range interval $[0, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx, \text{ and}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

(ii) Fourier Sine Series

The Fourier sine-series expansion of $f(x)$ on the half range interval $[0, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

Example 12 Find the Fourier cosine series of the functions

$$f(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4 \end{cases}$$

Solution Note that $f(x)$ is to be extended as an even function. Let the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \tag{1}$$

$$a_0 = \frac{2}{4}\int_0^4 f(x)\,dx = \frac{1}{2}\left[\int_0^2 x^2\,dx + \int_2^4 4\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{x^3}{3}\right)_0^2 + 4(x)_2^4\right] = \frac{1}{2}\left[\frac{8}{3}+8\right] = \frac{16}{3}$$

$$a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\left[\int_0^2 x^2\cos\left(\frac{n\pi x}{4}\right)dx + \int_2^4 4\cos\left(\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{1}{2}\left[\left\{\frac{x^2\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} + \frac{2x\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2} - \frac{2\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^3}\right\}_0^2 + 4\left\{\frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right\}_2^4\right]$$

$$= \frac{1}{2}\left[\frac{4\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{4}} + \frac{4\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)^2} - \frac{2\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)^3} - \frac{4\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{4}\right)}\right]$$

$$= \frac{32}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{4}\right) - \left(\frac{2}{n\pi}\right)\cdot\sin\left(\frac{n\pi}{2}\right)\right]$$

Putting the values of a_0 and a_n in (1), we get

$$f(x) = \frac{8}{3} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)\cdot\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{4}\right)$$

Example 13 Find the half-range Fourier cosine series of the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$$

Solution Let the half-range Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \tag{1}$$

where
$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 2x\,dx + \int_1^2 2(2-x)\,dx$$

$$= 2\frac{x^2}{2}\Big|_0^1 + 2\frac{(2-x)^2}{2}\Big|_1^2$$

$$= 2 - 1 = 1$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 2x\cdot\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 2(2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= 2\left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_0^1$$

$$+ 2\left[\frac{2}{n\pi}(2-x)\sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_1^2$$

$$= 2\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\cdot\cos\left(\frac{n\pi}{4}\right) - \frac{4}{n^2\pi^2}\right]$$

$$+ 2\left[-\frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{16}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{8}{n^2\pi^2}\cdot\cos n\pi - \frac{8}{n^2\pi^2}$$

$$= \frac{8}{n^2\pi^2}\cdot\left[2\cos\left(\frac{n\pi}{2}\right) - \cos n\pi - 1\right]$$

$$\therefore \quad a_1 = \frac{8}{\pi^2}\cdot[0+1-1] = 0,\ a_2 = -\frac{8}{\pi^2},\ a_3 = a_4 = a_5 = 0,$$

$$a_6 = -\frac{8}{9\pi^2}, \ldots$$

$$\therefore \quad f(x) = 1 + 0 - \frac{8}{\pi^2}\cdot\cos\pi x - \frac{8}{9\pi^2}\cdot\cos 3\pi x - \cdots$$

$$= 1 - \frac{8}{\pi^2}\cdot\cos\pi x - \frac{8}{9\pi^2}\cdot\cos 3\pi x - \cdots$$

Example 14 Expand $f(x) = \pi x - x^2$ in a half-range sine series in the interval $(0, \pi)$ up to the first three terms.

Solution Let the Fourier half-range sine series is

$$(\pi x - x^2) = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1}$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{4}{\pi n^3}[1 - \cos n\pi]$$

$$= \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$= 0 \text{ if } n \text{ is even}$$

$$= \frac{8}{\pi n^3} \text{ if } n \text{ is odd}$$

$\therefore$ Eq. (1) becomes

$$(\pi x - x^2) = \frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right]$$

Example 15 Find a half-range cosine series of $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in $0 < x < l$.

Solution Let a half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{1}$$

Then
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)dx = \frac{2}{l}\left[-\frac{l}{\pi}\cos\left(\frac{\pi x}{l}\right)\right]_0^l$$

$$= -\frac{2}{\pi}[\cos\pi - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{nx}{l}\right)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l 2\sin\left(\frac{nx}{l}\right)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l\left[\sin\frac{(n+1)\pi x}{l} - \sin\frac{(n-1)\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{l}{(n+1)\pi}\cos\frac{(n+1)\pi x}{l} + \frac{l}{(n-1)\pi}\cos\frac{(n-1)\pi x}{l}\right]_0^l$$

$$= \frac{1}{l}\left[\left(-\frac{l}{(n+1)\pi}\cos(n+1)\pi + \frac{l}{(n-1)\pi}\cos(n-1)\pi\right) - \left(-\frac{l}{(n+1)\pi} + \frac{l}{(n-1)\pi}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)}+\frac{(-1)^{n-1}}{(n-1)}+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]$$

$$= 0 \text{ if } n \text{ is odd}$$

$$= \frac{-4}{\pi(n+1)(n-1)} \text{ if } n \text{ is even and } n \neq 1$$

Now,
$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cdot\cos\frac{\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,dx = \frac{1}{l}\cdot\left(\frac{-l}{2\pi}\cos\frac{2\pi x}{l}\right)_0^l$$

$$= \frac{1}{2\pi}\left(-\cos 2\pi+1\right) = \frac{1}{2\pi}(-(-1)^2+1) = 0$$

Hence,
$$\sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3}+\frac{\cos\frac{4\pi x}{l}}{3\cdot 5}+\ldots\right]$$

Example 16 Find the Fourier half-range sine series for e^x in $0 < x < 1$.

Solution Let the half-range Fourier sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \tag{1}$$

Then,
$$b_n = \frac{2}{1}\int_0^1 f(x)\cdot\sin n\pi x\,dx$$

$$= 2\int_0^1 e^x\cdot\sin n\pi x\,dx$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}(\sin n\pi x - n\pi\cos n\pi x)\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-n\pi\cos n\pi) - \frac{1}{1+n^2\pi^2}\cdot(-n\pi)\right]$$

$$= \frac{2}{(n^2\pi^2+1)}\left[-en\pi(-1)^n + n\pi\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[1-e(-1)^n\right]$$

Thus, $$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1-e(-1)^n]}{1+n^2\pi^2} \sin n\pi x$$

Example 17 Obtain a half-range Fourier cosine series for $f(x) = x \sin$ in $0 < x < \pi$, and, hence, show that $\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots$.

Solution Let the half-range Fourier cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx$$

$$= \frac{2}{\pi}[x(-\cos x) - (-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi\right] = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cdot\left[2\cos nx\cdot\sin x\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{(n+1)} + \frac{\pi\cos(n-1)\pi}{(n-1)}\right] n \neq 1$$

$$a_n = -\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}$$

$$= (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{n^2-1}, n \neq 1$$

When $n = 1$, we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

Thus, $$x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} - \dots\right) \quad (2)$$

Putting $x = \dfrac{\pi}{2}$ in (2), we get

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \dots\right)$$

or $$\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right]$$

or $$\frac{\pi - 2}{4} = \left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right]$$ **Hence, proved**.

EXERCISE 9.2

1. Find the half-range Fourier sine series for the function $f(x) = x^2$ in $0 < x < 3$.
2. Find the half-range Fourier sine series for the function $f(x) = x - x^2$ in $0 < x < 1$.
3. Find the half-range sine series for the function

$$f(x) = \begin{cases} \left(\frac{1}{4} - x\right), & 0 < x < \frac{1}{2} \\ \left(x - \frac{3}{4}\right), & \frac{1}{2} < x < 1 \end{cases}$$

4. Obtain a half-range cosine series for the functions

$$f(x) = \begin{cases} kx & \text{if } 0 \le x \le \dfrac{l}{2} \\ k(l-x) & \text{if } \dfrac{l}{2} \le x \le l \end{cases}$$

Deduce the sum of the series $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

5. Find the half-range sine series for the functions

$f(x) = (2x - 1); 0 < x < 1.$

6. Obtain the half-range cosine and sine series for

$F(x) = x$ in $0 \le x \le \pi.$

7. Show that the series $\dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{2n\pi x}{l}$ represents $\left(\dfrac{l}{2} - x\right)$ when $0 \le x \le l.$

8. Find the Fourier series of the given functions on the given intervals.

(i) $f(x) = x^2; -\pi < x < \pi$

(ii) $f(x) = \begin{cases} \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$

(iii) $f(x) = \begin{cases} x^2, & -\pi < x < 0 \\ -x^2, & 0 \le x < \pi \end{cases}$

(iv) $f(x) = 1 - |x|, -\pi < x < \pi$

(v) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

(vi) $f(x) = \cos x, -\pi < x < \pi$

(vii) $f(x) = x, -\pi \le x \le \pi, f(x + 2\pi) = f(x)$

(viii) $f(x) = |\sin x|, \; -\pi < x < \pi$

(ix) $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

(x) $f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ -x + \pi, & 0 < x < \pi \end{cases}$

9. Find the Fourier series expansion of the following periodic function with period $2\pi.$

$$F(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ 0, & 0 \le x < \pi, \; F(x + 2\pi) = f(x) \end{cases}$$

10. Find the Fourier series of the function

$F(x) = x \cdot \sin x, 0 < x < 2\pi$

11. Obtain the Fourier series of the function $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi.$ Prove that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

12. Find a Fourier series expansion for the function defined by

$$F(x) = \begin{cases} -1 & ; \quad -\pi < x < 0 \\ 0 & ; \quad x = 0 \\ 1 & ; \quad 0 < x < \pi \end{cases}$$

Hence, deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4}$.

13. Find a Fourier-series expansion for the function $F(x)$ defined by

$$F(x)=\begin{cases}1+\dfrac{2x}{\pi} & ; \quad -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi} & ; \quad 0 \le x \le \pi\end{cases}$$

Hence, deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

14. Find a Fourier series expansion of the periodic function $F(x)$ of period 2π, where

$$F(x)=\begin{cases}0 & ; \quad -\pi < x < 0 \\ x^2 & ; \quad 0 \le x \le \pi\end{cases}$$

Hence, show that $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$ and $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$.

15. Obtain the Fourier-series expansion of the following periodic function of period 4, $F(x) = 4 - x^2; -2 \le x \le 2$. Hence, show that $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$.

16. Obtain the Fourier-series for the function $F(x) = |x|, -\pi < x < \pi$. Hence, show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

17. Obtain the Fourier-series for the function

$$F(x) = x \cos x, -\pi < x < \pi$$

18. Obtain the Fourier-series expansion of the periodic function

$$F(x) = e^x, -\pi < x < \pi, f(x + 2\pi) = f(x)$$

19. Find the Fourier cosine and sine series of the function $F(x) = 1, 0 \le x \le 2$.
20. Obtain the Fourier cosine series of the function

$$F(x)=\begin{cases}x^2; & 0 \le x \le 2 \\ 4\ ; & 2 \le x \le 4\end{cases}$$

21. Express $F(x) = x$ as a half-range sine series is $0 < x < 2$.
22. Obtain the Fourier cosine-series expansion of the periodic function defined by

$$F(x) = \sin\left(\frac{\pi x}{l}\right), 0 < x < l.$$

23. Obtain the half-range sine series for $F(x) = e^x$ is $(0, 1)$.
24. Find the Fourier sine series and the Fourier cosine series corresponding to the function $f(x) = (\pi - x)$ is $0 < x < \pi$.

25. Obtain the Fourier half-range sine and cosine series for the function $f(x) = K$ in $0 < x < 2$.
26. Find the half-period sine series for the $f(x)$ given in the range $(0, l)$ by the graph ABC as shown in Fig. 9.3.

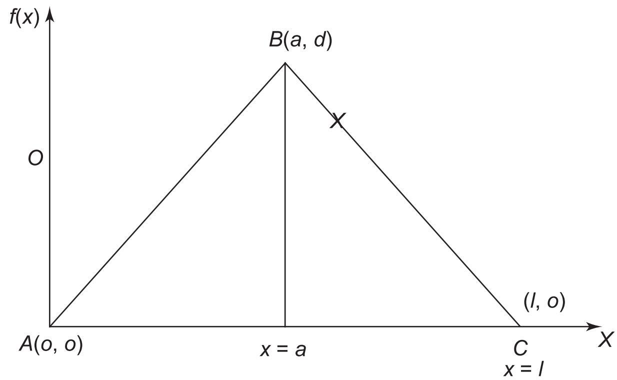


Fig. 9.3

Answers

1. $$x^2 = \sum_{n=1}^{\infty}\left[\frac{18}{n\pi}(-1)^{n+1} + \frac{36}{n^3\pi^3}[(-1)^n - 1]\right]\cdot \sin\left(\frac{n\pi x}{3}\right)$$

2. $$(x - x^2) = \frac{8}{\pi^3}\left[\sin \pi t + \frac{1}{3^3}\sin 3\pi t + \frac{1}{5^3}\sin 5\pi t + \ldots\right]$$

3. $$f(x) = \begin{cases} \left(\frac{1}{4} - x\right), & 0 < x < \frac{1}{2} \\ \left(x - \frac{3}{4}\right), & \frac{1}{2} < x < 1 \end{cases}$$

$$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots$$

4. $$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\cdot\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\right]$$

and $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

5. $$f(x) = -\frac{2}{\pi}\left[\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \ldots\right]$$

6. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$ and

$f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$

19. $F_c\{f(x)\} = 1,\ F_s\{f(x)\} = \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \cdots\right]$

20. $F(x) = \frac{8}{3} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)\right]\cdot\cos\left(\frac{n\pi x}{4}\right)$

21. $F(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right]$

22. $F(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi x}{l} + \frac{1}{15}\cos\frac{4\pi x}{l} + \frac{1}{35}\cos\frac{6\pi x}{l} + \cdots\right].$

23. $F(x) = 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + \cdots\right]$

24. $f(x) = 2\sum_{n=1}^{\infty}\frac{\sin \pi x}{x},\ F(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

25. $f(x) = \frac{4K}{\pi}\left[\frac{\sin\left(\frac{\pi x}{2}\right)}{1} + \frac{\sin\left(\frac{3\pi x}{2}\right)}{3} + \frac{\sin\left(\frac{5\pi x}{2}\right)}{5} + \cdots\right],\ f(x) = K$

26. $F(x) = \frac{2dl^2}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi a}{l}\right)}{n^2}\cdot\sin\frac{n\pi x}{l}$

9.10 MORE ON FOURIER SERIES

(i) Trigonometric Series

Trigonometric series is a functional series of the form

$$\frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + a_2 \sin 2x) + \ldots$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, and b_n are the called the *coefficients*.

(ii) Convergence of Fourier Series

Here we shall discuss the conditions under which Fourier-series expansion is possible and also find the function to which the series converges.

Consider $f(x)$ to be a piecewise continuous function on $[-a, a]$

(a) The function $f(x)$ is defined and continuous for all x in $(-a, a)$ except, may be, at a finite number of points in $(-a, a)$.

(b) At any point, say $x_0 \in (-a, a)$, where the function $f(x)$ is not continuous, both the right-hand limit and left hand limit, i.e., RHL = $\lim\limits_{x \to x_0^+} f(x)$ and LHL = $\lim\limits_{x \to x_0^-} f(x)$ exist and are finite.

If both $f(x)$ and $f'(x)$ are piecewise continuous then the function $f(x)$ is also called *piecewise smooth*.

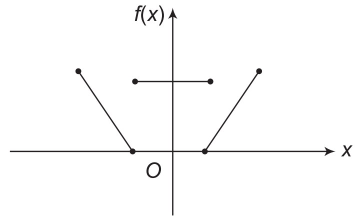


Fig. 9.4 *Typical piecewise function*

(iii) Parseval's Theorem and Identities

If the Fourier series of the function $f(x)$ over an interval $c < x < c + 2l$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_a \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ then } \frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof The Fourier-series expansion of $f(x)$ in $c < x < c + 2l$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \tag{7}$$

where $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot\cos\frac{n\pi x}{l}\,dx$$

and $$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot\sin\frac{n\pi x}{l}\,dx$$

Now, multiplying Eq. (7) on both sides by $f(x)$, we have

$$[f(x)]^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty}a_n f(x)\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}b_n f(x)\sin\frac{n\pi x}{l} \tag{8}$$

Integrating (8) on both sides w.r.t. 'x' from c to $c + 2l$, we get

$$\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0}{2}\times\int_c^{c+2l} f(x)\,dx + \sum_{n=1}^{\infty}a_n\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx + \sum_{n=1}^{\infty}b_n\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{a_0}{2}\cdot la_0 + \sum_{n=1}^{\infty} a_n(la_n) + \sum_{n=1}^{\infty} b_n(lb_n)$$

$$= \frac{la_0^2}{2} + l\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\therefore \quad \frac{1}{2l}\int_c^{c+2l} [f(x)]^2\, dx = \frac{1}{2l}\left[\frac{la_0^2}{2} + l\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

$$\text{or} \quad \frac{1}{2l}\int_c^{c+2l} [f(x)]^2\, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Hence, proved.

(iv) Parseval's Identities for Different Intervals

(a) If $f(x)$ is an even function in $(-l, l)$ then

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

(b) If $f(x)$ is an odd function in $(-l, l)$ then

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^2$$

(c) If $f(x) = \dfrac{a_0}{2} + \sum_1^{\infty} a_n \cos\dfrac{n\pi x}{l}$ in $(0, l)$ then

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_1^{\infty} a_n^2$$

(d) If $f(x) = \sum_1^{\infty} b_n \sin\dfrac{n\pi x}{l}$ in $(0, l)$ then

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^2$$

Example 18 Find Fourier-series expansion of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^2}{90}$$

Solution Given $f(x) = x^2$ and range $= (-\pi, \pi)$ then the Fourier series of $f(x) = x^2$ in $(-\pi, \pi)$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx \tag{1}$$

Here, $a_0 = \dfrac{2\pi^2}{3}, a_n = \dfrac{4(-1)^x}{n^2}, b_n = 0$

By Parseval's identity,

$$\int_{-\pi}^{\pi} (x^2)^2\, dx = 2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{1}^{\infty}\frac{16}{n^4}\right]$$

$$\left(\frac{x^5}{5}\right)_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^2}{90}$$ **Hence, proved**.

(v) Complex Form of Fourier Series

The Fourier series of a periodic function $f(x)$ of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{9}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}}}{2i}\right)$$

$$\left[\because\ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ibn}{2}\right)e^{\frac{in\pi x}{l}} + \left(\frac{a_n + ib_n}{2}\right)e^{-\frac{in\pi x}{l}}\right]$$

$$= C_0 + \sum_{n=1}^{\infty}\left[C_n e^{\frac{in\pi x}{l}} + C_{-n} e^{-\frac{in\pi x}{l}}\right] \tag{10}$$

Now, $$C_n = \frac{1}{2l}\left[\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx - i\int_{-l}^{l} f(x)\cdot\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\,dx$$

and $$C_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{in\pi x}{l}}\,dx$$

Combining these, we have $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\,dx \qquad (11)$$

where $n = 0, \pm 1, \pm 2, \ldots$

Then, the Eq. (10), can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi n}{l}}$$

which is called *complex form of Fourier series* and its coefficients are given by Eq. (11).

Example 19 Find the complex form of the Fourier series of the function $f(x) = e^{-ax}$ in $-1 \le x \le 1$.

Solution We know that

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \qquad (\because\ l = 1)$$

where $$C_n = \frac{1}{2}\int_{-1}^{1} e^{-ax}e^{-in\pi x}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(a+in\pi)x}\,dx = \frac{1}{2}\left[\frac{e^{-(a+in\pi)x}}{-(a+in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{(a+in\pi)} - e^{-(a+in\pi)}}{(a+in\pi)}\right]$$

$$= \frac{e^{a}(\cos n\pi + i\sin n\pi) - e^{-a}(\cos n\pi - i\sin n\pi)}{2(a+in\pi)}$$

$$= \left(\frac{e^{a} - e^{-a}}{2}\right)\cdot(-1)^n \cdot \frac{a - in\pi}{a^2 + n^2\pi^2} = \frac{(-1)^n(a - in\pi)\cdot\sinh a}{(a^2 + n^2\pi^2)}$$

Hence, $$e^{-ax} = \sum_{-\infty}^{\infty} \frac{(-1)^n(a - in\pi)\sinh a}{(a^2 + x^2\pi^2)}\, e^{in\pi x}$$

9.11 SPECIAL WAVEFORMS

Some special waveforms are described here.

(i) Sawtoothed Waveform

The function $f(x)$ in the interval $(-\pi, \pi)$, with period 2π representing the discontinuous function is called a sawtoothed waveform showing in Fig. 9.5.

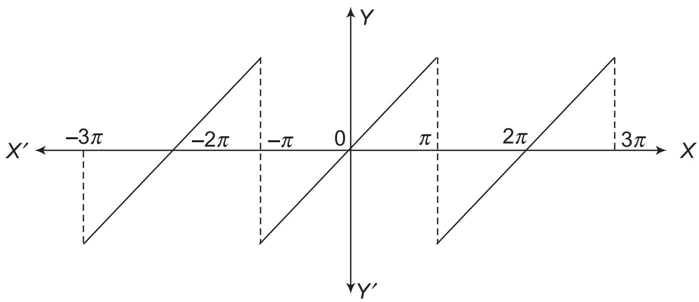


Fig. 9.5

(ii) Modified Sawtoothed Waveform

A function $f(x)$ is defined in the interval $(-\pi, \pi)$ and is defined in the period 2π.

$$\begin{aligned} f(x) &= 0; -\pi < x \le 0 \\ &= x; 0 \le x < \pi \end{aligned}$$

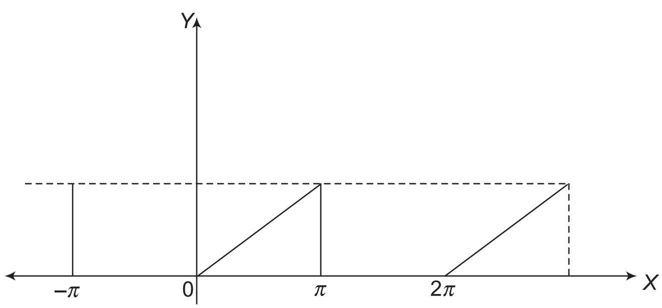


Fig 9.6

Its Fourier-series expansion is

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right] + \frac{a}{\pi}\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

(iii) Half-wave Rectifier

The half-wave rectifier is defined in the period 2π and is given as

$$f(x) = i = \begin{cases} I_0 \sin x; & 0 \le x \le \pi \\ 0; & \pi \le x \le 2\pi \end{cases}$$

where I_0 is the maximum current.

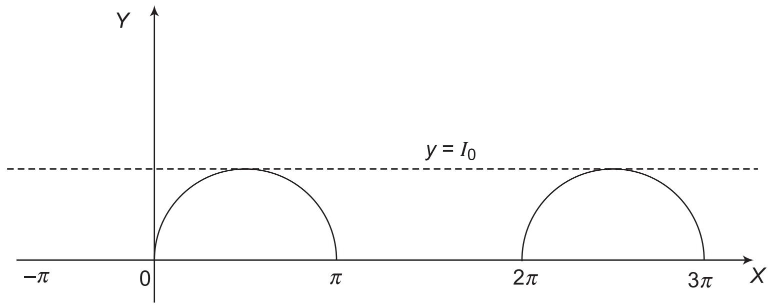


Fig. 9.7

Its Fourier-series expansion is

$$i = \frac{I_0}{\pi} + \frac{I_0 \sin x}{2} - \frac{2I_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)}$$

(iv) Full-wave Rectifier

The full-wave rectifier is an extension of the function $f(x) = a \sin x$; $0 \le x \le \pi$. See Fig. 9.8.

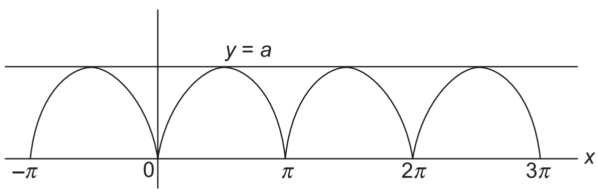


Fig. 9.8

Its Fourier-series expansion is

$$f(x) = \frac{4a}{\pi}\left[\frac{1}{2} - \frac{1}{1.3}\cos 2x - \frac{1}{3.5}\cos 4x - \frac{1}{5.7}\cos 6x - \cdots\right]$$

(v) Triangular Waveform

The triangular waveform is an extension of the function

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}; & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}; & 0 \le x \le \pi \end{cases}$$

Its Fourier-series expansion is

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

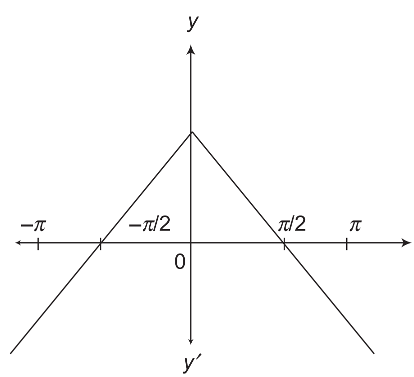


Fig. 9.9

(vi) Square Wave

The periodic function $f(x)$, with period $f(x + 2\pi) = f(x)$ is given by

$$f(x) = -k;\ -\pi < x < 0$$
$$= k;\ 0 < x < \pi$$

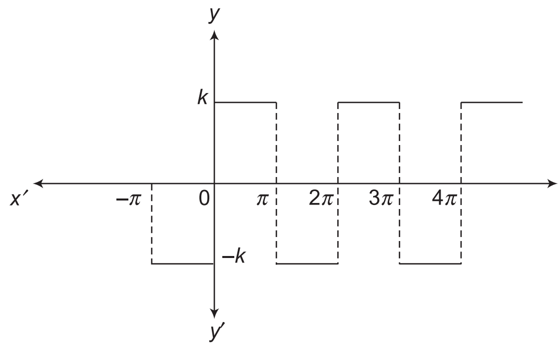


Fig. 9.10

Its Fourier series is

$$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right]$$

9.12 HARMONIC ANALYSIS AND ITS APPLICATIONS

Harmonic analysis is a branch of mathematics concerned with the representation of functions or signals as the superposition of basic waves. It has become a vast subject with applications in areas as diverse as signal processing, quantum mechanics and neuroscience.

Harmonic analysis is the theory of expanding a given function $f(x)$ in the range $(-\pi, \pi)$ or $(0, 2\pi)$ with period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \tag{12}$$

where the Fourier coefficients a_0, a_n and b_n are given by

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \tag{13}$$

when the function has been defined by an explicit function of an independent variable. When the function the $f(x)$ is not given in analytical form, i.e., given by a graph or by a table of corresponding values, in such cases, the integral in (13) cannot be evaluated and instead the following alternative forms of (13) are employed.

Then Eq. (13) gives

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= 2\cdot\frac{1}{(2\pi-0)}\int_{-\pi}^{\pi} f(x)\,dx \qquad \left[\because \text{ Mean} = \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

$\therefore$ $a_0 = 2\cdot$[mean value of $f(x)$ in $(-\pi, \pi)$ or $(0, 2\pi)$]

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = 2\cdot\frac{1}{(2\pi-0)}\int_0^{2\pi} f(x)\cos nx\,dx$$

= 2[mean value of $f(x)$ cos nx in $(-\pi, \pi)$ or $(0, 2\pi)$]

The term (a_1 cos x + b_1 sin x) in the Fourier series (12) is called the *fundamental* or *first harmonic*, and the term (a_2 cos $2x$ + b_2 sin $2x$) in (1) is called the second *harmonic*, and so on.

Example 20 Determine the first three coefficients in the Fourier cosine series for $f(x)$, where $f(x)$ is given in the following table:

x	0	1	2	3	4	5
$y(x)$	4	8	15	7	6	2

Solution Let the Fourier cosine series in the range $(0, 2\pi)$ be

$$f(x) = \frac{a_0}{2} + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \cdots \tag{1}$$

Taking the interval as 60º, we have

$\theta =$	0º	60º	120º	180º	240º	300º
$x =$	0	1	2	3	4	5
$f(x) =$	4	8	15	7	6	2

θº	cos θ	cos 2θ	cos 3θ	$f(x)$	$f(x)$ cos θ	$f(x)$ cos 2θ	$f(x)$ cos 3θ
0	1	1	1	4	4	4	4
60	$\frac{1}{2}$	$-\frac{1}{2}$	–1	8	4	–4	–8
120	$-\frac{1}{2}$	$-\frac{1}{2}$	1	15	$-\frac{15}{2}$	$-\frac{15}{2}$	15
180	–1	1	–1	7	–7	7	–7
240	$-\frac{1}{2}$	$-\frac{1}{2}$	1	6	–3	–3	6
300	$\frac{1}{2}$	$-\frac{1}{2}$	–1	2	1	–1	–2
			$\Sigma=$	42	–8.5	–4.5	8

$$a_0 = 2\cdot\text{ (Mean of } f(x)) = 2\times\frac{42}{6} = 14$$

$$\left.\begin{aligned} a_1 &= 2\cdot(\text{Mean of } f(x)\cos\theta) = +2\times-\frac{8.5}{6} = -2.8 \\ a_2 &= 2\cdot(\text{Mean of } f(x)\cos 2\theta) = 2\times-\frac{4.5}{6} = -1.5 \\ a_3 &= 2\cdot(\text{Mean of } f(x)\cos 3\theta) = 2\times\frac{8}{6} = 2.7 \end{aligned}\right\}$$

Using the values in the above table, we have

$$a_0 = 2[\text{Mean of } y \text{ in } (0, 2\pi)] = 2\left(\frac{8.700}{6}\right) = 2.9$$

$$a_1 = 2[\text{Mean of } y\ \cos x \text{ in } (0, 2\pi)] = 2\left(-\frac{1.100}{6}\right) = -0.37$$

$$a_2 = 2[\text{Mean of } y \cos 2x \text{ in } (0, 2\pi)] = 2\left(-\frac{0.300}{6}\right) = -0.10$$

$$a_3 = 2[\text{Mean of } y \cos 3x \text{ in } (0, 2\pi)] = 2\left(\frac{0.100}{6}\right) = 0.03$$

$$b_1 = 2[\text{Mean of } y \sin x \text{ in } (0, 2\pi)] = 2\left(\frac{3.117}{6}\right) = 1.04$$

$$b_2 = 2[\text{Mean of } y \sin 2x \text{ in } (0, 2\pi)] = 2\left(-\frac{0.173}{6}\right) = -0.06$$

$$b_3 = 2[\text{Mean of } y \sin 3x \text{ in } (0, 2\pi)] = 2\left(\frac{0}{6}\right) = 0$$

$$\therefore\quad y = \frac{2.9}{2} + (-0.37 \cos x + 1.04 \sin x) + (-0.10 \cos 2x - 0.06 \sin 2x) + (0.03 \cos 3x + 0 \sin 3x)$$

$$= 1.45 + (-0.37 \cos x + 1.04 \sin x) + (-0.10 \cos 2x - 0.06 \sin 2x) + (0.03 \cos 3x)$$

Example 21 Analyze harmonically the data given below and express $y = f(x)$ in Fourier series up to the third harmonic.

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution Since the last value of y is a repetition of the first, only the six values will be used and the period of the given interval $(0, 2\pi)$ is 2π.

Consider the Fourier series as

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) \tag{1}$$

The desired values are tabulated as follows:

x	y	cos x	cos $2x$	cos $3x$	sin x	sin $2x$	sin $3x$	y cos x	y cos $2x$	y cos $3x$	y sin x	y sin $2x$	y sin $3x$
0	1.0	1	1	1	0	0	0	1	1	1	0	0	0
$\frac{\pi}{3}$	1.4	0.5	–0.5	–1	0.866	0.866	0	0.700	–0.700	–1.4	1.212	1.212	0
$\frac{2\pi}{3}$	1.9	–0.5	–0.5	1	0.866	–0.866	0	–0.950	–0.950	1.9	+1.645	–1.645	0
π	1.7	–1	1	–1	0	0	0	–1.7	1.7	–1.7	0	0	0
$\frac{4\pi}{3}$	1.5	–0.5	–0.5	1	0.866	0.866	0	–0.750	–0.750	1.5	1.299	1.299	0
$\frac{5\pi}{3}$	1.2	0.5	–0.5	–1	–0.866	–0.866	0	0.600	–0.600	–1.2	–1.039	–1.039	0
	$\Sigma y =$ 8.7							$\Sigma y \cos x$ = –1.100	$\Sigma y \cos 2x$ = –0.300	$\Sigma y \cos 3x$ = 0.100	$\Sigma y \sin x$ = 3.117	$\Sigma y \sin 2x$ = –0.173	Σy sin = 0

SUMMARY

1. Periodic Function

A function $f(x)$ is said to be periodic if $f(x + T) = f(x)$ for all x and T is the smallest positive number for which this relation holds. Then T is called the period of $f(x)$.

If T is the period of $f(x)$ then $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots f(x + nT)$

Also, $f(x) = f(x - T) = f(x - 2T) = f(x - 3T) = \ldots f(x - nT)$

$\therefore$ $f(x) = f(x \pm nT)$, when n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

2. Fourier Series

Fourier series is an infinite series which is represented in terms of the trigonometric sine and cosine functions of the form

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos x + b_2 \sin 2x + a_3 \cos x + b_3 \sin 3x + \ldots$$

or $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin n, x\right]$$

where the constants a_0, a_n, and b_n are called *Fourier coefficients*.

3. Euler's Formulae

Let $f(x)$ be a periodic function with period 2π defined in the interval $(\alpha, \alpha + 2\pi)$ of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

where the constants a_0, a_n, b_n are called the coefficients, where

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot\cos nx\,dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$$

The above formulae are known as Euler formulae.

4. Fourier Series for Discontinuous Functions

Let $f(x)$ be defined by $f(x) = f_1(x)$; $\alpha < x < x_o$.
$= f_2(x)$; $x_0 < x < \alpha + 2\pi$.
where x_0 is the point of finite discontinuity in the interval $(\alpha, \alpha + 2\pi)$.
The values of a_0, a_n, and b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cdot\sin nx\,dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0 - 0)$ and $f(x_0 + 0)$ exist but are unequal. The sum of the Fourier series $= \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]$.

5. Even and Odd Functions

A function $f(x)$ is said to be even, if
$f(-x) = f(x)$ for all x. For example, x^4, $\cos x$, $\sec x$ are even functions.
A function $f(x)$ is said to be odd if
$f(-x) = -f(x)$ for all x. For example, x, x^3, $\sin x$, $\tan x$ are odd functions.

6. Fourier Series for Even and Odd Functions

Case I Consider $f(x)$ is an even function in $(-\pi, \pi)$. Then all b_n's will be zero. Thus, the Fourier series of an even function contains only cosine terms and is known as *Fourier cosine series* given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

Case II Consider $f(x)$ is an odd function in $(-\pi, \pi)$. Then all a_n's will be zero. Also, a_0 is zero since $f(x)$ is an odd. Thus, the Fourier series of an odd function contains only sine terms and is known as *Fourier sine series* given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

7. Fourier Cosine Series

The Fourier cosine series expansion of $f(x)$ on the half-range interval $[0, l]$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

8. Fourier Sine Series

The Fourier sine series expansion of $f(x)$ on the half-range interval $[0, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

OBJECTIVE-TYPE QUESTIONS

1. e^x is periodic, with a period of
(a) 2π (b) $2\pi i$
(c) π (d) πi
[GATE (CE) 1997]

2. The function $f(x) = e^x$ is
(a) even
(b) odd
(c) neither even nor odd
(d) none of the above **[GATE (CE) 1999]**

3. The trigonometric Fourier series of an even function of time does not have a
(a) dc terms
(b) cosine terms
(c) sine terms
(d) odd harmonic terms
[GATE (CE) 1996, 1998]

4. The Fourier series expansion of a symmetric and even function $f(x)$, where

$$f(x) = 1 + \frac{2x}{\pi};\ -\pi < x < 0$$
$$= 1 - \frac{2x}{\pi};\ 0 < x < \pi$$

will be

(a) $\sum_{1}^{\infty} \frac{4}{x^2\pi^2}(1 + \cos n\pi)$

(b) $\sum_{1}^{\infty} \frac{4}{n^2\pi^2}(1-\cos n\pi)$

(c) $\sum_{1}^{\infty} \frac{4}{n^2\pi^2}(1+\sin n\pi)$

(d) $\sum_{1}^{\infty} \frac{4}{n^2\pi^2}(1-\sin n\pi)$

[GATE (CE) 2003]

5. The Fourier series of a odd periodic function contains only

(a) odd harmonic (b) even harmonic
(c) cosine terms (d) sine terms

[GATE (EC) 1994]

6. A discontinuous real function can be expressed as

(a) Taylor's and Fourier series
(b) Taylor's series and not by Fourier series
(c) neither by Taylor's nor by Fourier series
(d) not by Taylor's, but by Fourier series

[GATE (CE) 1998]

7. The Fourier series of a real periodic function has only

P: cosine terms if it is even
Q: sine terms if it is even
R: cosine terms if it is odd
S: sine terms if it is odd

Which of the above statements are true?

(a) P and S (b) P and R
(c) Q and S (d) Q and R

[GATE (EC) 2009]

8. The Fourier coefficient a_0 of a function $f(x)$ in the interval $(0, 2\pi)$ is:

(a) $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

(b) $\frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$

(c) $\frac{1}{\pi}\int_{-\pi}^{2\pi} f(x)\sin nx\,dx$

(d) $\int_{0}^{2\pi} f(x)\cos nx\,dx$

9. The Fourier coefficient a_n of a function $f(x)$ in the interval $(0, 2\pi)$ is

(a) $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

(b) $\frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx$

(c) $\frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,dx$

(d) $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

10. The half-range sine series for the function $f(x)$ in $(0, \pi)$ is

(a) $\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

(b) $\frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

(c) $\frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

(d) $\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

11. The Fourier coefficient a_0 is equal to for the function $f(x) = \begin{cases} -\pi; & -\pi < x < 0 \\ x; & 0 < x < \pi \end{cases}$

(a) $\pi/2$ (b) $-\pi/2$
(c) π (d) $\pi/3$

12. Value of the Fourier coefficient a_0 of the function $f(x) = x$ in $(0, 2\pi)$ is

(a) -2π (b) 2π
(c) π (d) $\pi/2$

13. For the function $f(x) = x^2$ in $(0, 2\pi)$, the value of the Fourier coefficient b_n is

(a) $\frac{-4\pi}{n}$ (b) $\frac{4\pi}{n}$

(c) $\frac{\pi}{n}$ (d) $\frac{2\pi}{n}$

14. For the function $f(x) = \begin{cases} 0; & -\pi < x < 0 \\ x; & 0 < x < \pi \end{cases}$ the value of the Fourier coefficient a_0 is

(a) $\pi/2$ (b) $-\pi/2$
(c) π (d) $2\pi/3$

ANSWERS

1. (a)	2. (b)	3. (c)	4. (b)	5. (d)	6. (d)	7. (a)	8. (b)	9. (b)	10. (c)
11. (b)	12. (b)	13. (a)	14. (a)						

10 Ordinary Differential Equations: First Order and First Degree

10.1 INTRODUCTION

Many practical problems in science and engineering are formulated by finding how one quantity is related to, or depends upon, one or more quantities defined in the problem. Often, it is easier to model a relation between the rates of changes in the variables rather than between the variables themselves.

The study of this relationship gives rise to differential equations. Derivatives can always be interpreted as rates.

In this chapter, we consider the simplest of these differential equations which is of the first order and first degree. We study the solutions of differential equations which are variable-separable, homogeneous, non-homogeneous, exact, non-exact using the integrating factor, linear and Bernoulli's equations.

10.2 BASIC DEFINITIONS

A differential equation is an equation which contains the derivatives of dependent variables with respect to independent variables.

Differential equations are classified into two categories, ***ordinary*** and ***partial***, depending on the number of independent variables contained in the equation.

(i) Ordinary Differential Equation (ODE)

The ODE is an equation, which contains the derivatives of a dependent variable w.r.t. only one independent variable.

Example: $\dfrac{dy}{dx} + 2y = e^x$

(ii) Partial Differential Equation (PDE)

A PDE is an equation which contains the derivatives of a dependent variable w.r.t. two or more independent variables.

Example: $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 4;$ here, the dependent variable z depends on two independent variables, x and y.

(iii) Order of a Differential Equation

The order of a differential equation is the order of the highest derivatives appearing in the equation.

Example: $\dfrac{d^2y}{dx^2} + y = 0$; here, the highest derivative is $\dfrac{d^2y}{dx^2}$, so the order is 2.

(iv) Degree of a Differential Equation

The degree of a differential equation is the power of the highest order derivative appearing in the equation, when the equation is made free from radical signs and fractions.

Example: $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + y = 0$; here, the degree is 1.

(v) Solution, or Integral, or Primitive

The solution or integral, or primitive, of a differential equation is any function which satisfies the equation, i.e., reduces it to an identity.

A differential equation may have a unique solution or many solutions or no solution.

In explicit functions, the dependent variable can be expressed explicitly in terms of the independent variable, say $y = f(x)$. Otherwise, the solution is said to be an implicit solution where $f(x, y) = 0$, where $f(x, y)$ is an implicit function.

The general (or complete) solution of an n^{th} order differential equation will have n arbitrary constants.

(vi) Particular Solution

The particular solution is a solution obtained from the general solution by choosing particular values of an arbitrary constant. Integral curves of differential equations are the graphs of the general or particular solutions of differential equations.

(vii) Initial (Boundary) Value Problem

The initial (boundary) value problem is one in which a solution to a differential equation is obtained subject to conditions on the unknown function and its derivative specified at one, two, or more values of the independent variable.

Such conditions are called *initial* (*boundary*) *conditions*.

The general (or complete) integral of a differential equation is an implicit function $f(x, y, c) = 0$.

(viii) Particular Integral

Particular integral is one obtained from the general integral for a particular value of the constant c.

Singular solutions of a differential equation are (unusual or odd) solutions of a differential equation, which cannot be obtained from the general solution.

10.3 FORMATION OF AN ORDINARY DIFFERENTIAL EQUATION

Consider y and x as the dependent and the independent variables respectively.

The equation: $f(x, y, c) = 0$ (1)

where c is an arbitrary constant.

Equation (1) represents a family of curves.

For example, the equation $x^2 + y^2 = r^2$ where r is arbitrary, represents a circle with center at the origin and radius r.

Let $\quad g(x, y, c, d) = 0 \qquad (2)$

Equation (2) containing two arbitrary constants c and d also represents a family of curves. We often say that it represents a two-parameter family of curves. For example, the equation $y = mx + k$, where m and k are arbitrary constants, represents a two-parameter family of straight lines having the slope m and passing through the point $(0, k)$.

To eliminate the arbitrary constant c in Eq. (1), we need two equations. One equation is given by (1) itself and the second equation is obtained by differentiating Eq. (1) w.r.t. x. On eliminating c from the two equations, we obtain an equation containing x, y, and $\frac{dy}{dx}$, which is a first-order ODE.

For example, consider $y = cx^2$. Differentiating, we get $\frac{dy}{dx} = 2cx$; eliminating c, we get

$$\frac{dy}{dx} = 2x\left(\frac{y}{x^2}\right) = 2\frac{y}{x}$$

or
$$x\frac{dy}{dx} - 2y = 0; x \neq 0$$

Hence, $y = cx^2$ satisfied the differential equation $x\frac{dy}{dx} - 2y = 0$

Similarly, to eliminate the arbitrary constants c and d in Eq. (2), we need three equations. One equation is given by Eq. (2) and the remaining two equations are obtained by differentiating Eq. (2) w.r.t. x two times. On eliminating c and d from these equations, we obtain a second-order ODE.

10.4 FIRST-ORDER AND FIRST-DEGREE DIFFERENTIAL EQUATIONS

A first-order and first-degree ODE is given by

$$\frac{dy}{dx} = f(x, y) \qquad (3)$$

Equation (3) has certain standard types of first-order first-degree differential equations for which solutions can be readily obtained by standard methods such as

(i) Variable-separable
(ii) Homogeneous differential equation
(iii) Non-homogeneous differential equation reducible to homogeneous equation
(iv) Linear first-order differential equation (Leibnitz's equation)
(v) Bernoulli's differential equation
(vi) Exact differential equation
(vii) Non-exact differential equations that can be made exact with the help of integrating factors

10.4.1 Variable-Separable

Consider an equation of the form

$$f_1(x)\, g_1(y)dx + f_2(x)\, g_2(y)dy = 0 \qquad (4)$$

Step 1

The variables x and y can be separated into the form of

$$\frac{f_1(x)}{f_2(x)}\,dx + \frac{g_1(y)}{g_2(y)}\,dy = 0$$

or $$R(x)dx + S(y)dy = 0 \tag{5}$$

where $R(x) = \dfrac{f_1(x)}{f_2(x)}$ is a function of x only and $S(y) = \dfrac{g_1(y)}{g_2(y)}$ is a function of y only.

Step 2

Integrating (5) both sides, we get

$$\int R(x)dx + \int S(y)dx = c$$

where c is an arbitrary constant. The arbitrary constant can be chosen in any form suitable for the answer, i.e., we can replace it by $\sin c$, $\cos c$, $\tan^{-1}c$, e^c, or $\log c$, etc.

Example 1 Solve $\dfrac{dy}{dx} = e^{2x-y} + x^3e^{-y}$.

Solution The given equation can be written into the form of $\dfrac{dy}{dx} = (e^{2x} + x^3)e^{-y}$

Separating the variables, we get

$$e^y\,dy = (e^{2x} + x^3)dx$$

Integrating both sides, we get

$$\int e^y\,dy = \int (e^{2x} + x^3)\,dx + c$$

$$e^y = \frac{1}{2}e^{2x} + \frac{x^4}{4} + c$$

Example 2 Solve $(1 + x^2)dy = (1 + y^2)dx$.

Solution The given equation is $(1 + x^2)dy = (1 + y^2)\,dx$

Separating the variables,

$$\frac{1}{1+y^2}\,dy = \frac{1}{1+x^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx + \tan^{-1} c$$

or $$\tan^{-1}y - \tan^{-1}x = \tan^{-1}c$$

$$\tan^{-1}\left(\frac{y-x}{1+yx}\right) = \tan^{-1} c$$

or $$\left(\frac{y-x}{1+xy}\right) = c$$

or $$(y - x) = c(1 + xy)$$

Example 3 Solve $(1 + e^x)\, y\, dy = (1 + y)\, e^x\, dx$

Solution Separating the variables in the given equation,

$$\frac{y}{1+y}dy = \frac{e^x}{1+e^x}dx$$

Integrating both sides,

$$\int \frac{y}{1+y}dy = \int \frac{e^x}{1+e^x}\, dx + \log c$$

or $$\int \left[1 - \frac{1}{1+y}dy\right] = \int \frac{e^x}{1+e^x}\, dx + \log c$$

$$y - \log(1 + y) = \log(1 + e^x) + \log c$$

or $$y = \log(1 + y) + \log(1 + e^x) + \log c$$

or $$y = \log[(1 + y)(1 + e^x)\cdot c]$$

or $$e^y = c(1 + y)(1 + e^x)$$

Example 4 Solve $\tan x \cdot \sin^2 y\, dx + \cos^2 x \cdot \cot y\, dy = 0$

Solution Separating the variables in the given equation,

$$\tan x \cdot \sec^2 x\, dx + \cot y \cdot \operatorname{cosec}^2 y\, dy = 0$$

Integrating both sides,

$$\int \tan x \cdot \sec^2 x\, dx + \int \cot y \cdot \operatorname{cosec}^2 y\, dy = c$$

or $$\frac{\tan^2 x}{2} - \frac{\cot^2 y}{2} = c$$

or $$\tan^2 x - \cot^2 y = c_1$$

where $$c_1 = 2c$$

Example 5 Solve $x\sqrt{1+y^2}\, dx + y\sqrt{1+x^2}\, dy = 0$

Solution Separating the variables in the given equation,

$$\frac{x}{\sqrt{1+x^2}}dx + \frac{y}{\sqrt{1+y^2}}dy = 0$$

Integrating both sides, we get

$$\int x\cdot(1+x^2)^{-1/2}\, dx + \int y\cdot(1+y^2)^{-1/2}\, dy = c$$

or $$\frac{1}{2}\int 2x\cdot(1+x^2)^{-1/2}\, dx + \frac{1}{2}\int 2y\cdot(1+y^2)^{-1/2}\, dy = c$$

or $$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

EXERCISE 10.1

Solve the following differential equations:

1. $(xy + x)\, dx = (x^2y^2 + x^2 + y^2 + 1)dy$
2. $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$
3. $\left(y - x\dfrac{dy}{dx}\right) = a\left(y^2 + \dfrac{dy}{dx}\right)$
4. $(1 - x^2)\,(1 - y)\, dx = xy\,(1 + y)dy$
5. $3e^x \cdot \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$
6. $\dfrac{dy}{dx} + x^2 = x^2 e^{3y}$
7. If $\dfrac{dy}{dx} = e^{x+y}$ and it is given that for $x = 1$, $y = 1$, find y when $x = -1$
8. $a\left(x\dfrac{dy}{dx} + 2y\right) = xy\dfrac{dy}{dx}$
9. $(x^2 - x^2y)\, dy + (y^2 + xy^2)\, dx = 0$
10. $\log\left(\dfrac{dy}{dx}\right) = ax + by$

Answers

1. $\log(x^2 + 1) = y^2 - 2y + 4\log[c(y + 1)]$
2. $e^y = e^x + \dfrac{x^3}{3} + c$
3. $y = c(x + a)(1 - ay)$
4. $\log[x(1-y)^2] = \dfrac{x^2}{2} - \dfrac{y^2}{2} - 2y + c$
5. $\tan y = c(1 - e^x)^3$
6. $(e^{3y} - 1) = c_1 e^{(3y + x^2)}$, where $c_1 = e^{3c}$
7. $y = -1$
8. $\tan x \cdot \tan y = c$
9. $\log x/y - (y + x)/xy = c$
10. $\sec y = c - 2\cos x$

10.4.2 Homogeneous Equations

Consider a differential equation of the form

$$\frac{dy}{dx} = f(y/x) \tag{6}$$

Step 1

Put $\qquad v = \dfrac{y}{x}$ or $y = vx$

And putting $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in 1, we get

$$v + x\frac{dv}{dx} = f(v) \tag{7}$$

Step 2

Separating the variables in (7) and integrating, the solution is obtained as

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c$$

Step 3

Replace v by y/x, in the solution obtained in Step 2.

10.4.3 Homogeneous Functions

A function $f(x, y)$ is said to be homogeneous of degree n in the variables x and y if for any t,

$$f(tx, ty) = t^n f(x, y)$$

Example 6 Solve $(x^2 - y^2)\, dx + 2xy\, dy = 0$

Solution The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2\, xy} \tag{1}$$

Equation (1) is a homogeneous differential equation.

Step 1: Put $v = y/x$ or $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{x^2(v^2 - 1)}{2x^2 v}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \tag{2}$$

Step 2: Separating the variables in (2) and integrating,

$$\int \frac{2v}{v^2 + 1}\, dv + \int \frac{1}{x}\, dx = \log c$$

$$\log(1 + v^2) + \log x = \log c$$

or $$\log[(1 + v^2)x] = \log c$$

or $$x(1 + v^2) = c \tag{3}$$

Step 3: Replacing v by y/x in (3), we get

$$(y^2 + x^2) = cx$$

Example 7 Solve $x^2\, ydx - (x^3 + y^3)\, dy = 0.$

Solution The given differential can be written as

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \tag{1}$$

Step 1: Put $v = y/x$ or $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + v^3x^3}$$

$$v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

or $$x\frac{dv}{dx} = -\frac{v^4}{1+v^3} \tag{2}$$

Step 2: Separating the variables and integrating,

$$\int\frac{dx}{x} = -\int\frac{1+v^3}{v^4}\,dv - \log c$$

$$\log x + \log c = \frac{1}{3v^3} - \log v$$

$$\log (c\,x\,v) = \frac{1}{3v^3}$$

or $$3\log (c\,x\,v) = \frac{1}{v^3} \tag{3}$$

Step 3: Replacing v by y/x in (3), we get

$$y^3 = k\,e^{x^3/y^3}$$

where $$k = 1/c^3$$

Example 8 Solve $x(\cos y/x)\,(ydx + xdy) = y(\sin y/x)\,(xdy - ydx)$.

Solution The given equation can be written as

$$x\left(\cos\frac{y}{x}\right)\left(y + x\frac{dy}{dx}\right) = y\left(\sin\frac{y}{x}\right)\left(x\frac{dy}{dx} - y\right)$$

or $$x\frac{dy}{dx}\left(x\cos\frac{y}{x} - y\sin\frac{y}{x}\right) = -y^2\cdot\sin\frac{y}{x} - xy\cos\frac{y}{x}$$

or $$\frac{dy}{dx} = \frac{y}{x}\left[\frac{y\sin\left(\frac{y}{x}\right) + x\cos\left(\frac{y}{x}\right)}{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)}\right] \tag{1}$$

Step 1: Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{vx(vx\sin v + x\cos v)}{x\,(vx\sin v - x\cos v)}$$

or $$v + x\frac{dv}{dx} = \frac{v(vx\sin v + x\cos v)}{(vx\sin v - x\cos v)} \tag{2}$$

Step 2: Separating the variables in (2), we get

$$-\left[\frac{\cos - v\sin v}{v\cos v}\right]dv = \frac{2}{x}\,dx$$

On integrating both sides, we get

$$-\log(v\cos v) = 2\log x + \log c$$

or $$cx^2 = \frac{1}{v\cos v} \tag{3}$$

Step 3: Replacing v by y/x in (3), we get

$$cx^2(y/x)\cdot\cos(y/x) = 1$$

or $$cxy\cos(y/x) = 1$$

Example 9 Solve $(1 + 2e^{x/y}) + 2e^{x/y}(1 - x/y)\dfrac{dy}{dx} = 0$.

Solution We can write the given equation in the form of

$$\left(1+2e^{\frac{x}{y}}\right)\left(dx+2e^{\frac{x}{y}}\right)\left(\frac{1-x}{y}\right)dy = 0 \tag{1}$$

Equation (1) is a homogeneous differential equation.

Step 1: Putting $v = x/y$, so that $dx = ydv + vdy$ in (1), we get

$$(1 + 2e^u)(ydv + vdy) + 2e^u(1 - u)dv = 0$$

or $$(v + 2e^v)\,dy + y(1 + 2e^v)\,dv = 0 \tag{2}$$

Step 2: Separating the variables,

$$\frac{dy}{y} + \frac{1+2e^v}{v+2e^v}\,dv = 0 \tag{3}$$

Integrating (3) on both sides, we get

$$\log y + \log(v + 2e^v) = \log e$$

or $$y(v + 2e^v) = e \tag{4}$$

Step 3: Replacing v by x/y in (4), we get

$$y\left(\frac{x}{y} + e^{x/y}\right) = c$$

EXERCISE 10.2

Solve the following differential equations:

1. $(2xy + 3y^2)dx - (2xy + x^2)dy = 0$
2. $y - x\left(\dfrac{dy}{dx}\right) = x + y\left(\dfrac{dy}{dx}\right)$

3. $\dfrac{dy}{dx} = \dfrac{y^2}{(xy - x^2)}$

4. $xdy - ydx = \sqrt{(x^2 + y^2)}\, dx$

5. $x \cdot \sin\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = y \sin\left(\dfrac{y}{x}\right) + x$

6. $(2x - 5y)dx + (4x - y)dy = 0$, given $y(1) = 4$

7. $\left[y + \sqrt{(x^2 + y^2)}\right] dx - xdy = 0;\ y(1) = 0$

8. $x\dfrac{dy}{dx} = y - x\cos^2\left(\dfrac{y}{x}\right)$

9. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}(1 - x)dy = 0$

10. $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)dy$

Answers

1. $y^2 + xy = cx^3$

2. $\dfrac{1}{2}\log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right) = \log c$

3. $y = k \cdot e^{y/x}$

4. $y + \sqrt{(y^2 + x^2)} = cx^2$

5. $\cos\left(\dfrac{y}{x}\right) + \log c\ x = 0$

6. $(2x + y)^2 = 12(y - x)$

7. $y + \sqrt{(x^2 + y^2)} = x^2$

8. $\tan(y/x) = \log(c/x)$

9. $x + y e^{\frac{x}{y}} = c$

10. $c^2 (y^2 + x^2)^2 = (y^2 - x^2)$

10.4.4 Non-homogeneous Differential Equations Reducible to Homogeneous Form

Consider a differential equation of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \tag{8}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are all constants.

Case 1

If $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \text{ i.e., } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Then Eq. (8) can be reduced to homogeneous form by taking new variables X and Y such that

$$x = X + h \text{ and } y = Y + k$$

where h and k are constants to be chosen as to make the given equation homogeneous. With the above substitution, we get $dx = dX$ and $dy = dY$, so that

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Hence, the given equation becomes

$$\frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+k)+c_1}{a_2(X+h)+b_2(Y+k)+c_2}$$

$$= \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$$

Now, choose h and k such that $a_1h+b_1k+c_1=0$ and $a_2h+b_2k+c_2=0$

Then the differential equation becomes

$$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y} \tag{9}$$

which is homogeneous.

Now, Eq. (9) can be solved as in (homogeneous form) by substituting $Y = VX$. Finally, by replacing X by $(x-h)$ and Y by $(y-k)$, we shall get the solution in original variables x and y.

Case 2

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, i.e., $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

In this case, consider $\frac{a_1}{a_2} = \frac{b_1}{b_2} = r$; then Eq. (8) becomes of the form.

$$\frac{dy}{dx} = \frac{r(a_2x+b_2y)+c_1}{(a_2x+b_2y)+c_2} \tag{10}$$

To solve Eq. (10) by putting $z = a_2x + b_2y$ reduces (8) to a separable equation in the variables x and z.

Example 10 Solve $\frac{dy}{dx} = \frac{2x+2y-2}{3x+y-5}$

Solution Here, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, i.e., $\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = 2-6 = -4 \neq 0$

Then the given differential equation is reduced to homogeneous form by putting $x = X + h$ and $y = Y + K$.

$$\therefore \quad \frac{dY}{dX} = \frac{2(X+h)+2(Y+k)-2}{3(X+h)+(Y+k)-5}$$

$$\frac{dY}{dX} = \frac{2X+2Y+(2h+2k-2)}{3X+Y+(3h+k-5)}$$

Choose for h and k such that

$$2h+2k-2=0 \text{ and } 3h+k-5=0$$

i.e.

$$h+k=2 \tag{1}$$

$$3h+k=5 \tag{2}$$

Solving equations (1) and (2), we get

$$h = 2 \text{ and } k = -1$$

With these values of h and k, the given differential equation, becomes

$$\frac{dY}{dX} = \frac{2X + 2Y}{3X + Y} \tag{3}$$

which is a homogeneous equation.

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$ in Eq. (3)

$$V + X\frac{dV}{dX} = \frac{2X + 2VX}{3X + VX} = \frac{2 + 2V}{3 + V'}$$

or $$X\frac{dV}{dX} = \frac{2 + 2V}{3 + V} - V = \frac{2 - V - V^2}{3 + V}$$

Separating the variables, we get

$$-\frac{dX}{X} = \frac{3 + V}{V^2 + V - 2}\,dV$$

$$= \frac{3 + V}{(V + 2)(V - 1)}$$

or $$3\frac{dX}{X} = \left[\frac{1}{V + 2} - \frac{4}{V - 1}\right]dV \qquad \text{[By resolving into partial factions]} \tag{4}$$

Integrating (4) both sides, we get

$$3 \log X = \log (V + 2) - 4 \log (V - 1) + \log C$$

or $$\log \{X^3 (V - 1)^4\} = \log \{C(V + 2)\}$$

or $$X^3 (V - 1)^4 = C\,(V + 2) \tag{5}$$

Replacing V by Y/X in (5), we get

$$X^3\,\frac{(Y - X)^4}{X^4} = C\frac{(Y + 2X)}{X}$$

or $$(Y - X)^4 = C(Y + 2X) \tag{6}$$

Replacing X by $x - 2$ and Y by $y + 1$ in (6) we get

$$[(y + 1) - (x - 2)]^4 = C[(y + 1) + 2\,(x - 2)]$$

or $$[y - x + 3]^4 = C[y + 2x - 3]$$

Example 11 Solve $\frac{dy}{dx} = -\frac{(x - 2y + 1)}{(4x - 3y - 6)}$

Solution The given differential equation is non-homogeneous.

Here, $$\frac{a_1}{a_2} = \frac{1}{4} \neq \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3}$$

i.e., $$\begin{vmatrix} 1 & -2 \\ 4 & -3 \end{vmatrix} = 5 \neq 0$$

The given differential equation is reduced to a homogeneous equation by putting

$$x = X + h \text{ and } y = Y + k \text{ so that } dx = dX \text{ and } dy = dY$$

$$\therefore \quad \frac{dY}{dX} = -\left[\frac{X + h - 2(Y + k) + 1}{4(X + h) - 3(Y + k) - 6}\right]$$

$$\frac{dY}{dX} = -\left[\frac{X - 2Y + h - 2k + 1}{4X - 3Y + 4h - 3k - 6}\right]$$

Choose for h and k such that

$$h - 2k + 1 = 0 \text{ and } 4h - 3k - 6 = 0$$

Thus, the homogeneous equation in the new variables X and Y is

$$\frac{dY}{dX} = \frac{X - 2Y}{3Y - 4X} \tag{1}$$

Putting $Y = VX$ and $\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$ in (1), we get

$$V + X\frac{dV}{dX} = \frac{X - 2VX}{3VX - 4X} = \frac{1 - 2V}{3V - 4}$$

or $$X\frac{dV}{dX} = \frac{1 - 2V}{3V - 4} - V = \frac{1 - 2V - 3V^2 + 4V}{3V - 4}$$

$$= \frac{1 + 2V - 3V^2}{3V - 4}$$

Separating the variables,

$$\left(\frac{3V - 4}{3V^2 - 2V - 1}\right)dV = \frac{-dX}{X} \tag{2}$$

Integrating (2) on both sides, we get

$$\frac{1}{2}\log(3V^2 - 2V + 1) - \frac{3}{4}\log\left(\frac{3V - 3}{3V + 1}\right) = -\log X + \log C$$

or $$\log(3V^2 - 2V + 1)^2 - \log\left(\frac{3V - 3}{3V + 1}\right)^3 = \log\left(\frac{C}{X}\right)^4$$

$$\log\left[\frac{(3V + 1)^5}{3(V - 1)}\right] = \log\left(\frac{C}{X}\right)^4$$

$$\frac{(3V + 1)^5}{3(V - 1)} = \frac{C^4}{X^4} \tag{3}$$

Replacing V by $\dfrac{Y}{X}$,

$$\frac{(3Y+X)^5}{3X^5.(Y-X)}\cdot X = \frac{C^4}{X^4}$$

Or
$$(3Y+X)^5 = 3C^4\,(Y-X)$$
$$(3Y+X)^5 = c_1\,(Y-X)$$

Where $c_1 = 3C^4$.

Replacing $Y = y - 2$ and $X = x - 3$, we get

$$(x+3y-9)^5 = c_1\,(y-x+1)$$

Example 12 Solve $(2x + y + 1)dx + (4x + 2y - 1)\,dy = 0$

Solution We can write the given equation into the form

$$\frac{dy}{dx} = -\frac{2x+y+1}{4x+2y-1} \tag{1}$$

Here,
$$\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2} = \frac{b_1}{b_2} = \frac{1}{2}$$

$\therefore$ Consider $z = 2x + y$, so that $\dfrac{dz}{dx} = 2 + X\,\dfrac{dy}{dx}$

With these substitutions, the given equation reduces to

$$\frac{dz}{dx} - 2 = -\frac{z+1}{2z-1}$$

or
$$\frac{dz}{dx} = 2 - \frac{z+1}{2z-1} = \frac{3z-3}{2z-1}$$

Separating the variables,

$$\frac{2z-1}{3(z-1)}\,dz = dx$$

or
$$\frac{2z-1}{z-1}dz = 3\,dx$$

or
$$\left(2+\frac{1}{z-1}\right)dz = 3\,dx$$

On integrating both sides, we get

$$2z + \log\,(z-1) = 3x + C$$

Replace z by $2x + y$

$$2(2x+y) + \log(2x+y-1) = 3x + C$$
$$4x + 2y + \log(2x+y-1) = 3x + C$$

or
$$(x+2y) + \log\,(2x+y-1) = C$$

Example 13 Solve $\dfrac{dy}{dx} = \dfrac{y-x}{y-x+2}$.

Solution Here, $\dfrac{a_1}{a_2} = \dfrac{1}{1} = \dfrac{b_1}{b_2} = \dfrac{-1}{-1} = \dfrac{1}{1}$

i.e., $$\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -1 + 1 = 0$$

Let $$z = y - x$$

$$\frac{dz}{dx} = \frac{dy}{dz} - 1 \quad \text{or} \quad \frac{dy}{dx} = 1 + \frac{dz}{dx}$$

With these substitutions, the given equation reduces to

$$\frac{dz}{dx} + 1 = \frac{z}{z+2}$$

or $$\frac{dz}{dx} = \frac{z}{z+2} - 1 = \frac{z-z-2}{z+2}$$

$$\frac{dz}{dx} = \frac{-2}{z+2}$$

Separating the variables, we get

$$-2dx = (z+2)dz$$

or $$2dx + (z+2)dz = 0$$

On integrating both sides, we get

$$2x + \frac{z^2}{2} + 2z = C$$

Replacing z by $y - x$, we get

$$2x + \frac{(y-x)^2}{2} + 2(y-x) = C$$

or $$2y + \frac{(y-x)^2}{2} = C$$

or $$4y + (y-x)^2 = c_1$$

where $$c_1 = 2c$$

EXERCISE 10.3

Solve the following differential equations:

1. $(2x + y - 3)dy = (x + 2y - 3)dx$
2. $(2x - 5y + 3)dx - (2x + 4y - 6)dy = 0$
3. $\dfrac{dy}{dx} = \dfrac{2x+3y+1}{3x-2y-5}$
4. $(x + 2y - 2)dx + (2x - y + 3)dy = 0$

5. $(3x - y - 9)\dfrac{dy}{dx} = (10 - 2x + 2y)$

6. $\dfrac{dy}{dx} = \dfrac{ax + by - a}{bx + ay - b}$

7. $\dfrac{dy}{dx} = \dfrac{2x + y - 1}{4x + 2y + 5}$

8. $\dfrac{dy}{dx} = \dfrac{x + y + 7}{2x + 2y + 3}$

9. $\dfrac{dy}{dx} = \dfrac{x - y + 3}{2x - 2y + 5}$

10. $\dfrac{dy}{dx} = \dfrac{(x - 2y + 3)}{(2x - 4y + 5)}$

Answers

1. $(x + y - 2) = c^2 (x - y)^3$

2. $(4y - x - 3)(y + 2x - 3)^2 = c$

3. $\log[(x - 1)^2 + (y + 1)^2] - 3\tan^{-1}\left(\dfrac{y + 1}{x - 1}\right) = C$

4. $x^2 + 4xy - y^2 - 4x + 6y = C$

5. $(y - 2x + 7) = c(x + y + 1)^4$

6. $(y - x + 1)^{\frac{(a+b)}{a}} (y + x - 1)^{\left(\frac{a-b}{a}\right)} = C$

7. $(10y - 5x + 7)\log(10x + 5y + 9) = C$

8. $\dfrac{2}{3}(x + y) - \dfrac{11}{9}\log(3x + 3y + 10) = x + c$

9. $x - 2y + \log(x - y + 2) = C$

10. $x^2 - 4xy + 4y^2 + 6x - 10y = C$

10.4.5 Linear Differential Equations or (Liebnitz's Linear Equation)

(i) Definition

A differential equation is said to be linear if the degree of dependent variable and its derivatives occur first only and not any product of the dependent variable and its derivative in the equation.

The standard form of the linear equation of first order is

$$\frac{dy}{dx} + Py = Q \tag{11}$$

where P and Q are functions of x only.

To find the solution of (11), multiply (11) throughout by $e^{\int P dx}$. We get

$$e^{\int P dx}\frac{dy}{dx} + Pye^{\int P dx} = Qe^{\int P dx}$$

Rewriting $\quad d\left(y.e^{\int P dx}\right) = Q e^{\int P dx}$

Integrating both sides, we get

$$y.e^{\int P dx} = \int Q.e^{\int P dx} dx + C \tag{12}$$

which is the solution of the equation (11).

(ii) Method of Solving Linear Equations

Step 1 Write the given differential equation in the standard form (11).

Step 2 Find the integrating factor (IF) = $e^{\int P dx}$

Step 3 The solution of the given differential equation is

$$y(\text{IF}) = \int Q(\text{IF})\,dx + C$$

Note 1 The LHS of Eq. (12) is always the product of the dependent variable (y) and the (IF)

Note 2 When the given differential equation is nonlinear in y, it would be much convenient to treat x as the dependent variable instead of y and solve the equation $\frac{dx}{dy} + P(y)x = Q(y)$, which is linear in x. In this case integrating factor (IF) is $e^{\int P dy}$, the complete solution is

$$x(\text{IF}) = \int Q(\text{IF})\,dy + C$$

Example 14 Solve $(x^2 - 1)\frac{dy}{dx} + 2xy = 1.$

Solution

Step 1: The given equation can be written in the standard form

$$\frac{dy}{dx} + \left(\frac{2x}{x^2 - 1}\right)y = \frac{1}{(x^2 - 1)} \tag{1}$$

Here, $P = \frac{2x}{x^2 - 1}, Q = \frac{1}{x^2 - 1}$

Step 2: Integrating factor (IF) = $e^{\int P dx} = e^{\int \frac{2x}{x^2 - 1} dx} = e^{\log(x^2 - 1)}$

$$\text{IF} = (x^2 - 1)$$

Step 3: The Solution of the given equation is

$$y(\text{IF}) = \int Q(\text{IF})\,dx + C$$

$$y(x^2 - 1) = \int \frac{1}{(x^2 - 1)}.(x^2 - 1)\,dx + C = \int dx + C$$

$$y(x^2 - 1) = (x + C)$$

Example 15 Solve $\cos^2 x\left(\frac{dy}{dx}\right) + y = \tan x.$

Solution

Step 1: The given equation can be written in the standard form

$$\frac{dy}{dx} + (\sec^2 x)\,y = \tan x \cdot \sec^2 x \tag{1}$$

Here, $P = \sec^2 x$, $Q = \tan x.\sec^2 x$

Step 2: IF = $e^{\int P dx} = e^{\int \sec^2 x dx}$

$$\text{IF} = e^{\tan x}$$

Step 3: The solution of (1) is

$$y \cdot e^{\tan x} = \int (\tan x . \sec^2 x) . e^{\tan x} dx + C \qquad [\text{Putting } \tan x = t, \sec^2 x\, dx = dt]$$

$$= \int te^t dt + C$$

$$= te^t - e^t + C$$

$$y \cdot e^{\tan x} = (\tan x - 1)\, e^{\tan x} + C$$

or $$y = (\tan x - 1) + Ce^{-\tan x}$$

Example 16 Solve $(1 + y^2)dx = (\tan^{-1} y - x)dy$.

Solution

Step 1: The given equation can be written in the standard form

$$\frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{1 + y^2}$$

or $$\frac{dx}{dy} + \left(\frac{1}{1 + y^2}\right) x = \frac{\tan^{-1} y}{1 + y^2} \qquad (1)$$

Here, $P = \dfrac{1}{1 + y^2}$, $Q = \dfrac{\tan^{-1} y}{1 + y^2}$

Step 2: IF = $e^{\int P dy}$

$$= e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1} y}$$

Step 3: The complete solution of (1) is

$$x . e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2} . e^{\tan^{-1} y} dy + C \qquad \left[\text{Putting } \tan^{-1} y = t, \frac{1}{1 + y^2} dy = dt\right]$$

$$= \int te^t\, dt + C$$

$$= (t - 1)\, e^t + C$$

$$x . e^{\tan^{-1} y} = (\tan^{-1} y - 1) e^{\tan^{-1} y} + C$$

Example 17 $\dfrac{dy}{dx} + (2 \tan x)\, y = \sin x$; given that $y = 0$ when $x = \dfrac{\pi}{3}$.

Solution

Step 1: The given equation is in the the standard form.

Here, $P = 2\tan x$, $Q = \sin x$,

Step 2:
$$\text{IF} = e^{\int P dx}$$
$$= e^{2\int \tan x dx}$$
$$= e^{2\log \sec x}$$
$$= \sec^2 x$$

Step 3: The complete solution of the given differential equation is

$$y\cdot\sec^2 x = \int \sin x . \sec^2 x\, dx + C$$
$$= \int \tan x . \sec dx + C$$
$$y\cdot\sec^2 x = \sec x + C \qquad (1)$$

$y = 0$ at $\pi = \dfrac{\pi}{3}$, then Eq. (1) becomes

$$0 = \sec\frac{\pi}{3} + C \Rightarrow C = -\sec\frac{\pi}{3} = -2$$

$\therefore$ $\quad y \sec^2 x = \sec x - 2$ or $y = \cos x - 2\cos^2 x$

Example 18 Solve $(1+x^2)\dfrac{dy}{dx} = (e^{\tan^{-1} x} - y)$.

Solution

Step 1: The given equation can be written in the standard form.

$$\frac{dy}{dx} + \left(\frac{1}{1+x^2}\right)y = \frac{e^{\tan^{-1}x}}{1+x^2} \qquad (1)$$

Here, $P = \dfrac{1}{1+x^2}, Q = \dfrac{e^{\tan^{-1}x}}{1+x^2}$

Step 2:
$$\text{IF} = e^{\int P dx}$$
$$= e^{\int \frac{1}{1+x^2} dx}$$
$$\text{IF} = e^{\tan^{-1}x}$$

Step 3: The complete solution of (1) is

$$y . e^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x.}}{1+x^2} . e^{\tan^{-1} x}\, dx + C \qquad \left[\text{Putting } e^{\tan^{-1} x} = t, \frac{e^{\tan^{-1} x}}{1+x^2} dx = dt\right]$$

$$= \int t\, dt + C$$

$$= \frac{t^2}{2} + C$$

$$y.e^{\tan^{-1} x} = \frac{(e^{\tan^{-1} x})^2}{2} + C$$

Example 19 Solve $\frac{dy}{dx} + \frac{y \log y}{x - \log y} = 0.$

Solution The given equation is not linear in y because of the presence of the term log y. It is neither separable nor homogeneous nor exact. But with x taken as dependent variable, the equation can be rewritten as

$$\frac{dx.}{dy} + \frac{1}{y \log y} \cdot x = \frac{1}{y} \qquad (1)$$

Here, $P = \frac{1}{y \log y} . Q = \frac{1}{y}$

Step 1: $\text{IF} = e^{\int P dy}$

$$= e^{\int \frac{1}{y} \cdot \frac{1}{\log y} dy}$$

$$= e^{\log(\log y)}$$

$$\text{IF} = \log y$$

Step 2: The complete solution of (1) is

$$x \cdot \log y = \int \frac{1}{y} . \log y\, dy + C \qquad \left[\text{Putting } \log y = t \Rightarrow \frac{1}{y} dy = dt\right]$$

$$= \int t\, dt + C$$

$$= \frac{t^2}{2} + C$$

$$x \cdot \log y = \frac{(\log y)^2}{2} + C$$

Example 20 Solve $y^2\, dx + (3xy - 1)\, dy = 0$.

Solution

Step 1: The given equation can be written in the standard form

$$\frac{dx}{dy} + \left(\frac{3}{y}\right) x = \frac{1}{y^2} \qquad (1)$$

Here, $P = \frac{3}{y},\ Q = \frac{1}{y^2}$

Step 2: $\text{IF} = e^{\int P dy}$

$$= e^{3\int \frac{1}{y} dy}$$

$$= e^{3\log y}$$

$$= y^3$$

Step 3: The complete solution of (1) is

$$x \cdot y^3 = \int \frac{1}{y^2} . y^3\, dy + C$$

$$= \int y\, dy + C$$

$$= x . y^3 = \frac{y^2}{2} + C$$

EXERCISE 10.4

Solve the following differential equations:

1. $y\, dx - (x + 2y^3)\, dy = 0$
2. $y^2\, dx + (xy - 2y^2 - 1)dy = 0$
3. $dx + (3y - x)\, dy = 0$
4. $\frac{dy}{dx} + y = e^{e^x}$
5. $\frac{dy}{dx} + y = \frac{1}{1 + e^{2x}}$
6. $2(y - 4x^2)\, dx + x\, dy = 0$
7. $dx - (x + y + 1)dy = 0$
8. $\frac{dy}{dx} + 2y = e^x\,(3\sin 2x + 2\cos 2x)$
9. $(1 + x^2)\, dy + 2\, xy\, dx = \cot x\, dx$
10. $\frac{dy}{dx} + y \cot x = 2x . \operatorname{cosec} x$
11. $\frac{dy}{dx} + \frac{y}{x} = \sin x^2$
12. $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$
13. $\cos x\left(\frac{dy}{dx}\right) + y \sin x = \sec^2 x$

Answers

1. $x = y^3 + cy$
2. $xy = y^2 + \log y + C$
3. $x - 3y - 3 = ce^y$
4. $y.e^x = e^{e^x} + C$

5. $y = e^{-x}. \tan^{-1}(e^x) + ce^{-x}$
6. $x^2y = 2x^4 + C$
7. $x = -(y + 2) + ce^y$
8. $y = ce^{-2x} + e^x. \sin 2x$
9. $y = \dfrac{\log(\sin x) + C}{(1 + x^2)}$
10. $y = (x^2 + c) \operatorname{cosec} x$
11. $xy = -\dfrac{1}{2}\cos x^2 + C$
12. $y(1 + x^2) = \sin x + C$
13. $y \sec x = \tan x + C$

10.4.6 Nonlinear Equation Reducible to Linear Form (Bernoulli's Equation)

A first-order and first-degree differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)\cdot y^n \tag{13}$$

is known as Bernoulli equation which is nonlinear for any value of the real number n (except for $n = 0$ and $n = 1$).

If $n = 0$ then Eq. (13) converts to linear first-order differential equation.

If $n = 1$ then Eq. (13) converts to linear separable differential equation.

For any n (except $n = 0$ and $n = 1$), Eq. (13) can be reduced to linear differential equation dividing (13) both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \tag{14}$$

Now, putting $y^{1-n} = z$ and $\dfrac{dz}{dx} = (1 - n)y^{-n}\dfrac{dy}{dx}$ in (14), we get

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)Z = Q(x)$$

or
$$\frac{dz}{dn} + (1-n)P(x)z = (1-n)Q(x) \tag{15}$$

which is a linear first-order differential equation in z and x discussed in Section 10.4.5.

Method of Finding Solution to Bernoulli Equations

Step 1 First of all, rewrite the given differential equation in the standard Bernoulli equation

Step 2 Dividing both sides by y^n

Step 3 Put $z = y^{1-n}$ and obtain the first-order linear differential equation in z.

Step 4 Solve linear differential equation in z by the method discussed in Section 10.4.5.

Example 21 Solve $x\dfrac{dy}{dx} + y = y^2 \log x$.

Solution

Step 1: The given differential equation can be written in the standard form.

$\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = \dfrac{\log x}{x}\cdot y^2$ (dividing throughout by x) (1) [This is a Bernoulli equation with $n = 2$.]

Step 2: Dividing (1) both sides by y^2, we get

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1} = \left(\frac{\log x}{x}\right) \tag{2}$$

Step 3: Putting $z = y^{-1}$, $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$ in (2), we get

$$-\frac{dz}{dx}+\frac{1}{x}z = \frac{\log x}{x}$$

or
$$\frac{dz}{dx}+\left(\frac{-1}{x}\right)z = -\left(\frac{\log x}{x}\right) \tag{3}$$

which is a linear first-order equation in z and x

Step 4: Here, $P = \frac{-1}{x}$, $Q = -\left(\frac{\log x}{x}\right)$

$$\begin{aligned} \text{IF} &= e^{\int P dx} \\ &= e^{-\int \frac{1}{x}dx} \\ &= e^{-\log x} \\ \text{IF} &= \frac{1}{x} \end{aligned}$$

The complete solution is

$$\begin{aligned} z.\frac{1}{x} &= \int -\left(\frac{\log x}{x}\right).\frac{1}{x}dx + C \\ &= -\int \frac{1}{x^2}.\log x\, dx + C \\ &= -\left[-\frac{1}{x}\log x - \int\left(\frac{1}{x}\right).\left(\frac{-1}{x}\right)dx\right] + C \end{aligned}$$

[Integration by parts taking $\frac{1}{x^2}$ as the second function]

$$\begin{aligned} &= \frac{1}{x}\log x - \int \frac{1}{x^2}dx + C \\ z\cdot\frac{1}{x} &= \frac{1}{x}\log x + \frac{1}{x} + C \end{aligned}$$

Replacing z by $\frac{1}{y}$,

$\therefore$
$$\frac{1}{xy} = \frac{1}{x}(\log x + 1) + C$$

or $$y(1 + \log x) + cxy = 1$$

Example 22 Solve $3\dfrac{dy}{dx} + xy = xy^{-2}$.

Solution

Step 1: The given equation can be written as

$$\frac{dy}{dx} + \left(\frac{x}{3}\right)y = \frac{x}{3}y^{-2} \tag{1}$$

This is a Bernoulli equation with $n = -2$.

Step 2: Dividing (1) both sides by y^{-2}, we get

$$y^2\frac{dy}{dx} + \frac{x}{3}y^3 = \frac{x}{3} \tag{2}$$

Step 3: Putting $z = y^3$ and $\dfrac{dz}{dx} = 3y^2\dfrac{dy}{dx}$ in (2), we get

$$\frac{1}{3}\frac{dz}{dx} + \frac{x}{3}z = \frac{x}{3}$$

or $$\frac{dz}{dx} + xz = x \tag{3}$$

which is a linear equation in z and x.

Step 4: Here, $P = x$, $Q = x$

$$\text{IF} = e^{\int P dx}$$

$$= e^{\int x dx}$$

$$= e^{\frac{x^2}{2}}$$

Complete solution of (3) is

$$z \cdot e^{\frac{x^2}{2}} = \int x \cdot e^{\frac{x^2}{2}} \cdot dx + C \qquad \left[\text{Putting } \frac{x^2}{2} = t \Rightarrow xdx = dt\right]$$

$$= \int e^t dt + C$$

$$= e^t + C$$

$$z \cdot e^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} + C$$

$$z = 1 + Ce^{\frac{-x^2}{2}}$$

Replacing z by y^3,

$\therefore$ $$y^3 = 1 + ce^{\frac{-x^2}{2}}$$

Example 23 Solve $(xy^5 + y)\, dx - dy = 0$.

Solution

Step 1: The given equation can be written as

$$\frac{dy}{dx} - y = xy^5 \tag{1}$$

This is a Bernoulli equation with $n = 5$

Step 2: Dividing (1) both sides by y^5, we get

$$y^{-5}\frac{dy}{dx} - y^{-4} = x \tag{2}$$

Step 3: Putting $z = y^{-4}$ and $\dfrac{dz}{dx} = -4y^{-5}\dfrac{dy}{dx}$ in (2), we get

$$-\frac{1}{4}\frac{dz}{dx} - z = x$$

or
$$\frac{dz}{dx} + 4z = -4x \tag{3}$$

which is a linear equation is z and x.

Step 4: Here, $P = 4$, $Q = -4x$

$$\text{IF} = e^{\int P dx}$$

$$\text{IF} = e^{\int 4 dx}$$

$$\text{IF} = e^{4x}$$

Complete solution of (3) is.

$$z\cdot e^{4x} = \int(-4x)\, e^{4x}\, dx + C$$

$$z\cdot e^{4x} = -x\,e^{4x} + \frac{1}{4}e^{4x} + C$$

Replacing z by y^{-4}, we get

$$\frac{e^{4x}}{y^4} = -xe^{4x} + \frac{1}{4}e^{4x} + C$$

Example 24 Solve $\dfrac{dy}{dx} - 2\cos x\cdot\cot y + \sin^2 x\cdot\operatorname{cosec} y\cdot\cos x = 0$.

Solution

Step 1: The given equation can be written as

$$\frac{dy}{dx} = 2\cos x\cdot\cot y - \sin^2 x\cdot\cos x\cdot\operatorname{cosec} y \tag{1}$$

Step 2: Dividing (1) both sides by cosec y,

$$\sin y \frac{dy}{dx} - (2\cos x)\cos y = -\sin^2 x \cdot \cos x$$

or
$$-\sin y \frac{dy}{dx} + (2\cos x)\cos y = \sin^2 x \cdot \cos x \tag{2}$$

Step 3: Putting $\cos y = z$ and $\dfrac{dz}{dx} = -\sin y \dfrac{dy}{dx}$, in (2), we get

$$\frac{dz}{dx} + (2\cos x) z = \sin^2 x \cdot \cos x \tag{3}$$

which is a linear equation is z and x

Step 4: Here, $P = 2\cos x$, $Q = \sin^2 x \cdot \cos x$

$$\text{IF} = e^{\int P dx}$$

$$= e^{2\int \cos x\, dx}$$

$$= e^{2\sin x}$$

Complete solution of (5) is

$$z \cdot e^{2\sin x} = \int \sin^2 x \cdot \cos x \cdot e^{2\sin x}\, dx + C$$

$$ze^{2\sin x} = \int t^2 \cdot e^{2t}\, dt + C \qquad [\text{Put } \sin x = t; \text{ then } \cos x dx = dt]$$

$$= \left[t^2 \frac{e^{2t}}{2} - \int 2t \cdot \frac{e^{2t}}{2} dt + C \right]$$

$$= \left[\frac{1}{2} t^2 e^{2t} - \int t \cdot e^{2t}\, dt + C \right]$$

$$= \left[\frac{1}{2} t^2 e^{2t} - \left\{ t \frac{e^{2t}}{2} - \int \frac{e^{2t}}{t} dt \right\} + C \right]$$

$$= \left[\frac{1}{2} t^2 e^{2t} - \frac{te^{2t}}{2} + \frac{e^{2t}}{4} + C \right]$$

$$= \frac{1}{2} \sin^2 x\, e^{2\sin x} - \frac{\sin x\, e^{2\sin x}}{2} + \frac{e^{2\sin x}}{4} + C$$

or
$$z = \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x + \frac{1}{4} + c\, e^{-2\sin x}$$

Replacing z by $\cos y$, we get

$$\cos y = \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x + \frac{1}{4} + c\,e^{-2\sin x}$$

Example 25 Solve $\cos x\, dy = y(\sin x - y)dx$.

Solution

Step 1: The given equation can be written as

$$\cos x \frac{dy}{dx} = y \sin x - y^2$$

or
$$\cos x \frac{dy}{dx} - (\sin x)\, y = -y^2$$

or
$$\frac{dy}{dx} - (\tan x)\, y = -\sec x \cdot y^2 \tag{1}$$

Step 2: Dividing (1) both sides by y^2, we get

$$y^{-2}\frac{dy}{dx} - (\tan x)\, y^{-1} = -\sec x \tag{2}$$

Step 3: Putting $z = y^{-1}$ and $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$ in (2), we get

$$-\frac{dz}{dx} - (\tan x)\, z = -\sec x$$

or
$$\frac{dz}{dx} + (\tan x)\, z = \sec x \tag{3}$$

This is a linear in z and x.

Step 4: Here, $P = \tan x$, $Q = \sec x$

$$\text{IF} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

Complete solution of (3) is

$$z \sec x = \int \sec x \cdot \sec x\, dx + C$$

$$= \int \sec^2 x\, dx + C$$

$$z \sec x = \tan x + C$$

Replacing z by $\frac{1}{y}$, we get

$$\frac{1}{y}\sec x = \tan x + C$$

or
$$\sec x = y(\tan x + C)$$

EXERCISE 10.5

Solve the following differential equations:

1. $\dfrac{dy}{dx} + x\sin 2y = x^3 .\cos^2 y$
2. $(4e^{-y}\sin x - 1)\, dx - dy = 0$
3. $dx - (x^2 y^3 + xy)dy = 0$
4. $x\dfrac{dy}{dx} + y = x^3 y^6$
5. $\dfrac{dy}{dx} - \cot y + x\cot y = 0$
6. $\dfrac{dy}{dx} + y + y^2\,(\sin x - \cos x) = 0$
7. $(y - y^2 x^2 \sin x)\, dx + xdy = 0$
8. $\dfrac{dy}{dx} + \dfrac{1}{2x} y = \dfrac{x}{y^3}$; $y(1) = 2$
9. $\dfrac{dz}{dx} + \left(\dfrac{z}{x}\right)\log z = \dfrac{z}{x}(\log z)^2$
10. $2xy\dfrac{dy}{dx} = y^2 - 2x^3$; $y\,(1) = 2$
11. $\dfrac{dy}{dx} + \left(\dfrac{x}{1-x^2}\right) y = x\sqrt{y}$
12. $(1 - x^2)\dfrac{dy}{dx} + xy = xy^2$

Answers

1. $2y = (x^2 - 1) + 2C\,e^{-x^2}$
2. $e^y = 2\,(\sin x - \cos x) + Ce^{-x}$
3. $x\,(2 - y^2) + \;C\,xe^{\frac{-y^2}{2}} = 1$
4. $\left(Cx^2 + \dfrac{5}{2}\right)x^3 y^5 = 1$
5. $\sec y = x + 1 + Ce^x$
6. $y\,(ce^x - \sin x) = 1$
7. $xy\,(C + \cos x) = 1$
8. $x^2 y^4 = x^4 + 15$
9. $(1 + Cx)\log z = 1$
10. $y^2 = x\,(5 - x^2)$
11. $\dfrac{\sqrt{y}}{(1-x^2)^{1/4}} = -\dfrac{1}{3}(1-x^2)^{3/4} + C$
12. $Cy = (1-y).\sqrt{(1-x^2)}$

10.4.7 Exact Differential Equations

The differential of a function $f(x, y)$ is denoted by $d\,f$ and is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{16}$$

Consider

$$M(x, y)dx + N(x, y)dy = 0$$ (17) is the differential equation.

Suppose $$\frac{\partial f}{\partial x} = M(x, y) \tag{18}$$

and $$\frac{\partial f}{\partial y} = N(x, y) \tag{19}$$

Using equations (18) and (19), then the given equation (17) becomes

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x, y)\,dx + N(x, y)\,dy = 0$$

i.e., $df = 0$

On integration, $f(x, y) = C$ = arbitrary constant.

Therefore, the expression of (17) $M(x, y)dx + N(x, y)dy$ is said to be an exact differential and the equation (17) is called an exact differential equation.

(i) Necessary Condition for Exactness

Differentiating (18) and (19) partially w.r.t. y and x respectively, we get

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the necessary condition for the differential equation (17) to be an exact differential equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(ii) Method of Finding the Solution to an Exact Differential Equation

To solve a differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

Step 1 Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$.

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given equation is an exact differential equation.

Step 2 The solution of the given equation is

$$\int M(x,y)\,dx + \int N(x,y)\,dy = C$$

In the first integral treating y as a constant and in second integral take only those terms in N which do not contain x.

Example 26 Solve $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$

Solution

Step 1: $M(x, y) = (ax + hy + g)$ and $N(x, y) = (hx + by + f)$

Find $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\,(ax + hy + g)$

$$\frac{\partial M}{\partial y} = h \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(hx + by + f) = h$$

$$\therefore \quad \frac{\partial M}{\partial y} = h = \frac{\partial N}{\partial x}$$

The given equation is an exact differential equation.

Step 2: The solution of the given equation is

$$\underset{\text{Treating } y \text{ as a constant}}{\int M\,dx} + \underset{\text{take those terms in } N \text{ that do not contain } `x\text{'}}{\int N\,dy} = C \text{ (arbitrary constant)}$$

$$\int (ax + hy + g)\,dx + \int (by + f)\,dy = C \text{ (arbitrary constant)}$$

$$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = C \text{ (arbitrary constant)}$$

Example 27 Solve $e^y\,dx + (xe^y + 2y)dy = 0$.

Solution

Step 1: $M = e^y$, $N = (xe^y + 2y)$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} e^y = e^y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xe^y + 2y) = e^y$$

$$\therefore \quad \frac{\partial m}{\partial y} = e^y = \frac{\partial N}{\partial x}$$

Hence, the given equation is exact.

Step 2: The complete solution is

$$\int e^y\,dx + \int 2y\,dy = C$$

$$xe^y + y^2 = C$$

Example 28 Solve $\left(3x^2 y + \frac{y}{x}\right)dx + (x^3 + \log x)\,dy = 0$.

Solution

Step 1: $M = 3x^2 y + y/x$, $N = (x^3 + \log x)$

$$\frac{\partial M}{\partial y} = 3x^2 + \frac{1}{x}, \quad \frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, the given differential equation is exact.

Step 2: The solution of given equation is

$$\int M\,dx + \int N\,dy = C$$

$$\int (3x^2 y + y/x)\,dx + \int 0\,dy = C$$

$$x^3 y + y \log x = C$$

Example 29 Solve $(\cos x - x \cos y)\, dy - (\sin y + y \sin x)dx = 0.$

Solution

Step 1: Here, $M = -(\sin y + y \sin x)$, $N = \cos x - x \cos y$.

$$\frac{\partial M}{\partial y} = -(\cos y + \sin x.), \quad \frac{\partial N}{\partial x} = (-\sin x - \cos y) = -(\cos y + \sin x)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, the equation is exact.

Step 2:

$$\underset{\text{Treating } y \text{ as a constant}}{\int M\,dx} + \underset{\text{take those terms in } N \text{ that do not contain `}x\text{'}}{\int N\,dy} = C$$

$$-\int (\sin y + y \sin x)\,dx + \int 0\,dy = C$$

$$-x \sin y + y \cos x = C$$

EXERCISE 10.6

Solve the following differential equations:

1. $(\cos x \cdot \cos y - \cot x)\, dx - (\sin x \cdot \sin y)dy = 0$
2. $(x^2 - ay)\, dx - (ax - y^2)\, dy = 0$
3. $(1 + e^{x/y})\, dx + e^{x/y}(1 - x/y)dy = 0$
4. $(\sin x.\sin y - x e^y)\, dy = (e^y + \cos x.\cos y)\, dx$
5. $(2x^3 - xy^2 - 2y + 3)\, dx - (x^2 y + 2x)\, dy = 0$
6. $(x^2 + y^2 - a^2)\, x\, dx + (x^2 - y^2 - b^2)\, y\, dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$; $y(1) = 2$
8. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$
9. $(y - x^3)\, dx + (x + y^3)\, dy = 0$
10. $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy e^{xy^2} - 3y^2)\, dy = 0$
11. $(1 + 4xy + 2y^2)\, dx + (1 + 4xy + 2x^2)\, dy = 0$

12. $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left[x+\log x-x\sin y\right]dy=0$

13. $\left[\frac{y}{(x+y)^2}-1\right]dx+\left[1-\frac{x}{(x+y)^2}\right]dy=0$

Answers

1. $\sin x \cdot \cos y = \log (c \sin x)$
2. $x^3 + y^3 - 3\, axy + C = 0$
3. $x + y\, e^{x/y} = C$
4. $x\, e^y + \sin x \cdot \cos y = C$
5. $x^4 - x^2 y^2 - 4xy + 6x = C$
6. $x^4 + 2x^2 y^2 - y^4 - 2\, a^2 x^2 - 2\, b^2 y^2 = C$
7. $x^2 - xy + y^2 = 3$
8. $x^2 - 2 \tan^{-1} (x/y) + y^2 = K$
9. $4\, xy - x^4 + y^4 = C$
10. $x^4 - y^3 + e^{xy^2} = C$
11. $x + 2x^2y + 2xy^2 + y = C$
12. $y(x + \log x) + x \cos y = C$
13. $x - x^2 + y^2 = C\,(x + y)$

10.4.8 To Convert non-exact Differential Equations into Exact Differential Equations Using Integrating Factors

Consider a differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{20}$$

which is not exact.

Suppose there exists a function $F(x, y)$ such that

$$F(x, y)[M(x, y)dx + N(x, y)dy] = 0 \tag{21}$$

is an exact equation, then $F(x, y)$ is called an integrating factor (IF) of the differential equations (20).

Methods to Find an Integrating Factor to a Non-exact Differential Equation

$M\,dx + N\,dy = 0$ are

Case 1 If $\frac{1}{N}[M\cdot y - N\cdot x] = f(x)$ then IF $= e^{\int f(x)dx}$

Case 2 If $\frac{1}{M}[N\cdot x - M\cdot y] = g(y)$ then IF $= e^{\int g(y)dy}$

Case 3 If the differential equation of the form $y\, f_1(xy)\, dx + x\, f_2(x,\ y)\, dy = 0$

Then $\qquad$ IF $= \frac{1}{M.x - N.y}$ provided $M\cdot x - N\cdot y \neq 0$

Case 4 If the given differential equation $M\ dx + N\ dy = 0$ is a homogeneous equation and $M\cdot x + N\cdot y \neq 0$ then

$$\text{IF} = \frac{1}{M.x + N.y}$$

Case 5 Consider if the differential equation of the form

$$x^a y^b (m_1 y\, dx + n_1 x\, dy) + x^c y^d (m_2 y\, dx + n_2 x\, dy) = 0$$

where $a, b, c, d, m_1, n_1, m_2, n_2$ are all constants and $m_1 m_2 - n_1 n_2 \neq 0$

Then the integrating factor is $x^h . y^k$.

where $$\frac{a+h+1}{m_1} = \frac{b+k+1}{n_1} \text{ and } \frac{c+h+1}{m_2} = \frac{d+k+1}{n_2}$$

Example 30 Solve $2y\, dx + (2x \log x - xy)\, dy = 0.$

Solution $M = 2\, y, N = 2x \log x - xy$

$$\frac{\partial M}{\partial y} = 2, \frac{\partial N}{\partial x} = 2\,(1 + \log x) - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, the given equation is not exact.

Since $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(2x \log x - xy)}(2 - 2 - 2\log x - y)$$

$$= \frac{-2\log x + y}{x(2\log x - y)}$$

$$= \frac{-2\log x + y}{-x(-2\log x + y)} = -\frac{1}{x} = f(x)$$

$$\text{IF} = e^{\int f(x)\,dx}$$

$$= e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

Multiply the given differential equation by (IF) $\frac{1}{x}$, we get

$$\frac{2y}{x} dx + (2 \log x - y)\, dy = 0 \text{ which is exact equation.}$$

On integrating, we get

$$\int \frac{2y}{x} dx + \int -y\, dy = c \Rightarrow 2y \log x - \frac{y^2}{2} = C$$

Example 31 Solve $y(2x^2 - xy + 1)dx + (x - y)dy = 0$

Solution $M = 2yx^2 - xy^2 + y, N = x - y$

$$\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1, \frac{\partial N}{\partial x} = 1$$

$$\therefore \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, the given equation is not exact.

Since $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(x-y)}[2x^2 - 2xy + 1 - 1]$$
$$= \frac{1}{(x-y)} 2x(x-y)$$
$$= 2x = f(x)$$
$$\text{IF} = e^{\int f(x)dx} = e^{\int 2x dx} = e^{x^2}$$

Multiplying the given differential equation by (IF) e^{x^2}, we get

$e^{x^2} y(2x^2 - xy + 1)dx + e^{x^2}(x - y)dy = 0$ which is an exact equation.

Integrating both sides, we get

$$\int e^{x^2} y(2x^2 - xy + 1)dx + \int o dy = 0$$
$$e^{x^2}(2xy - y^2) = C$$

Example 32 Solve $(x - y)dx - dy = 0$; $y(0) = 2$.

Solution $M = (x - y)$, $N = -1$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 0$$
$$\therefore \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, the given equation is not exact

Now, $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-1}[-1 \quad -0] = 1 \text{ is a function of } x \text{ (say)}$$
$$\text{IF} = e^{\int 1 dx} = e^x$$

Multiplying the given equation by e^x, we get

$e^x(x - y)dx - e^x dy = 0$ which is exact equation.

Integrating both sides, we get

$$\int e^x(x-y)dx - \int 0 dy = C$$
$$(x - 1)e^x - ye^{-x} = C$$

Put $x = 0$, $y = 2$ so that $C = -3$

$$\therefore \qquad (x - 1)e^x - ye^{-x} = -3$$

Example 33 Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

Solution $M = 3x^2y^4 + 2xy,\ N = 2x^3y^3 - x^2$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x,\quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\therefore \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given equation is not exact.

Now,
$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{(3x^2y^4 + 2xy)}[6x^2y^3 - 2x - 12x^2y^3 - 2x]$$

$$= \frac{1}{(3x^2y^4 + 2xy)}\left[-6x^2y^3 - 4x\right]$$

$$= \frac{-2}{y(3x^2y^3 + 2x)}\left[3x^2y^3 + 2x\right]$$

$$= \frac{-2}{y} = g(y)$$

$$\text{IF} = e^{\int g(y)dy} = e^{-2\int \frac{1}{y}dy} = \frac{1}{y^2}$$

Multiplying the given equation by $\frac{1}{y^2}$, we get

$$\frac{1}{y^2}(3x^2y^4 + 2xy)dx + \frac{1}{y^2}(2x^3y^2 - x^2)dy = 0 \text{ which is an exact equation.}$$

Integrating both sides, we get

$$\int\left[3x^2y^2 + 2\frac{x}{y}\right]dx + \int o\cdot dy = C$$

$$x^3y^2 + \frac{x^2}{y} = C$$

Example 34 Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$.

Solution The given equation is $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$ (1)

Dividing (1) both sides by xy, we get

$$y(1 + 2xy)dx + x(1 - xy)dy = 0 \tag{2}$$

$$M = y\,f_1(xy),\ N = x\,f_2(xy)$$

$$\text{IF} = \frac{1}{M\cdot x - N\cdot y} = \frac{1}{xy(1+2xy) - xy(1-xy)}$$

$$= \frac{1}{3x^2y^2} = f(x, y)$$

Multiplying (2) both sides by $\frac{1}{3x^2y^2}$, we get

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

On integrating,

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}dy = C$$

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = C$$

or
$$-\frac{1}{xy} + 2\log x - \log y = 3C$$

or
$$-\frac{1}{xy} + 2\log x - \log y = C_1$$

Example 35 Solve $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$.

Solution $(x^3 + y^3)dx - xy^2dy = 0$ (1)

which is a homogeneous equation of degree 3.

$M = x^3 + y^3$ and $N = -xy^2$ both are the homogeneous with degree 3.

Since $M \cdot x + N \cdot y = x^4 \neq 0$, then

$$\text{IF} = \frac{1}{M \cdot x + N \cdot y} = \frac{1}{x^4 + xy^3 - xy^3} = \frac{1}{x^4}$$

Multiplying (1) by $\frac{1}{x^4}$, we get

$$\frac{1}{x^4}(x^3 + y^3)dx + \frac{1}{x^4}(-xy^2) = 0 \text{ which is an exact equation.}$$

$$\Rightarrow \quad \int\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = C$$

$$\Rightarrow \quad \log x - \frac{y^3}{3x^3} = C$$

Example 36 Solve $y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$.

Solution Here, $M = y(y^2 - 2x^2)$ and $N = x(2y^2 - x^2)$ are both homogeneous functions of degree 3.

Since $\quad M \cdot x + N \cdot y = xy(y^2 - 2x^2) + xy(2y^2 - x^2)$

$$= 3xy(y^2 - x^2) \neq 0, \text{ then}$$

$$\text{IF } \frac{1}{M + N \cdot y} = \frac{1}{3xy(y^2 - x^2)}$$

Multiplying the given differential equation by (IF), we get

$$\frac{y(y^2 - 2x^2)}{3xy(y^2 - x^2)} dx + \frac{x(2y^2 - x^2)}{3xy(y^2 - x^2)} dy = 0$$

or $$\frac{(y^2 - 2x^2)}{3x(y^2 - x^2)} dx + \frac{(2y^2 - x^2)}{3y(y^2 - x^2)} dy = 0$$

$\Rightarrow$ $$\int \frac{y^2 - 2x^2}{3x(y^2 - x^2)} dx = C$$

$\Rightarrow$ $$x^2y^2(y^2 - x^2) = C_1 \text{ where } C_1 = e^C$$

Example 37 Solve $2y^2\,dx + 4x^2\,ydx + 4xy\,dy + 3x^3\,dy = 0$.

Solution The given equation can be written as

$$x^2(4ydx + 3xdy) + y(2ydx + 4xdy) = 0 \tag{1}$$

Comparing (1) with $x^ay^b(m_1ydx + n_1xdy) + x^cy^d(m_2yds + n_2xdy) = C$

Here, $a = 2$, $b = 0$, $m_1 = 4$, $n_1 = 3$, $c = 0$, $d = 1$, $m_2 = 2$, $n_2 = 4$

Also, $m_1m_2 - n_1n_2 = 8 - 12 = -4 \neq 0$

The unknown constants (h, k) in the integrating factor are determined from the following:

$$\frac{a + h + 1}{m_1} = \frac{b + k + 1}{n_1} \text{ and } \frac{c + h + 1}{m_2} = \frac{d + k + 1}{n_2}$$

$\therefore$ $$\frac{2 + h + 1}{4} = \frac{0 + k + 1}{3} \Rightarrow 4k - 3h = 5$$

and $$\frac{0 + h + 1}{2} = \frac{1 + k + 1}{4} \Rightarrow k - 2 = 0$$

Solving for h, k, we get $k = 2$, $h = 1$

The integrating factor (IF) $= x^hy^k = x^1y^2$

Multiplying the given equation by (IF) xy^2, we get

$$xy^2(2y^2 + 4x^2y)dx + xy^2(4xy + 3x^3)dy = 0$$

or $\quad 2xy^4dx + 4x^3y^3\,dx + 4x^2y^3\,dy + 3x^4y^2dy = 0$

or $$d(x^2y^4) + d(x^4y^3) = 0$$

On integrating, we get

$$x^2y^4 + x^4y^3 = C$$

10.4.9 Integrating Factors by Inspection (Grouping of Terms)

1. If the differential equation contains the group of terms $xdx + ydy$ then $\frac{1}{x^2 + y^2}$ or some function of these expression can be tried as an integrating factor.

   We have $\frac{xdx + ydy}{x^2 + y^2} = \frac{1}{2} d[\log(x^2 + y^2)]$

2. If the differential equation contains the group of terms $xdy + ydx$ then $\frac{1}{xy}$ can be tried as an integrating factor.

   We have $\frac{xdy + ydx}{xy} = d[\log(xy)]$

3. If the differential equation contains the group of terms $xdy + xdx$ then it can be tried as an integrating factor, we have $d(x \cdot y)$ is an exact differential.
4. If the differential equation contains the group of terms $xdy - ydx$ then $\frac{1}{x^2}$ can be tried as an integrating factor. We have

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

5. If the differential equation contains the group of terms $xdy - ydx$ then $\frac{1}{xy}$ can be tried as an integrating factor. We have

$$\frac{dy}{y} - \frac{dx}{x} = d\left(\log \frac{y}{x}\right)$$

6. If the differential equations contains the group of terms $xdy - ydx$ then $\frac{1}{x^2 + y^2}$ can be tried as an integrating factor. We have

$$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$$

Example 38 Solve $y(y^3 - x)dx + x(y^3 + x)dy = 0$.

Solution $$y^4dx - xy\,dx + xy^3dy + x^2dy = 0 \tag{1}$$

We can write the given into the form

$$y^3(ydx + xdy) + x(xdy - ydx) = 0$$

$$y^3\, d(xy) + x \cdot x^2 \cdot d\left(\frac{y}{x}\right) = 0$$

or $$d(xy) + \left(\frac{x}{y}\right)^3 \cdot d\left(\frac{y}{x}\right) = 0$$

or $$d(xy) + \left(\frac{x}{y}\right)^{-3} \cdot d\left(\frac{y}{x}\right) = 0$$

On integrating, we get

$$xy - \frac{1}{2}\left(\frac{y}{x}\right)^{-2} = C_1$$

Example 39 Solve $(y + x)\,dy = (y - x)dx$.

Solution $ydy + xdy - ydx + xdx = 0$

Regrouping

$$(ydy + xdx) + (xdy - ydx) = 0$$

$$\frac{1}{2}\,d(x^2 + y^2) + xdy - ydx) = 0$$

Dividing both sides by $(x^2 + y^2)$, we get

$$\frac{1}{2(x^2 + y^2)}\,d(x^2 + y^2) + \left(\frac{xdy - ydx}{x^2 + y^2}\right) = 0$$

$$\frac{1}{2}\,d\,[\log(x^2 + y^2)] + \frac{\dfrac{xdy - ydx}{x^2}}{1 + \dfrac{y^2}{x^2}} = 0$$

$$\frac{1}{2}\,d\,[\log(x^2 + y^2)] + d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = 0$$

On integrating, we get

$$\log\sqrt{(x^2 + y^2)} + \tan^{-1}\left(\frac{y}{x}\right) = C$$

Example 40 Solve $y(x^3e^{xy} - y)dx + x(y + x^3e^{xy})dy = 0$.

Solution $yx^3e^{xy}dx - y^2dx + xydy + x^4e^{xy}dy = 0$

Regrouping

$$x^3e^{xy}(ydx + xdy) + y(xdy - ydx) = 0$$

or $$x^3 d\,(e^{xy}) + y \cdot x^2 \cdot \left(\frac{xdy - ydx}{x^2}\right) = 0$$

Dividing both sides by x^3, we get

$$d\,(e^{xy}) + \left(\frac{y}{x}\right) \cdot d\left(\frac{y}{x}\right) = 0$$

On integrating, we get

$$e^{xy} + \frac{1}{2}\left(\frac{y}{x}\right)^2 = C$$

EXERCISE 10.7

Solve the following differential equations:

1. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$
2. $y(x^2y + e^x)dx - e^x dy = 0$

3. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
4. $y(x + y)dx + (x + 2y - 1)dy = 0$
5. $2xydy - (x^2 + y^2 + 1)dx = 0$
6. $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$
7. $y(1 + xy)dx + x(1 + xy + x^2y^2)dy = 0$
8. $(y + xy^2)\, dx - xdy = 0$
9. $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$
10. $2xy\, dx + (y^2 - x^2)\, dy = 0$; $y\,(2) = 1$
11. $3\,(x^2 + y^2)\, dx + x\,(x^2 + 3y^2 + 6y)\, dy = 0$
12. $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0$
13. $(x^4 + y^4)dx - xy^3dy = 0$
14. $y^2dx + (x^2 - xy - y^2)dy = 0$
15. $(y - x)dx + (y + x)dy = 0$
16. $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$
17. $(8ydx + 8xdy) + x^2y^3(4ydx + 5xdy) = 0$
18. $x^3y^3(2ydx + xdy) - (5ydx + 7xdy) = 0$
19. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$
20. $x\,(3y\, dx + 2x\, dy) + 8y^4\,(y\, dx + 3x\, dy) = 0$
21. $y\,(3x + 2y^2)\, dx + 2x\,(2x + 3y^2)\, dy = 0$
22. $(x^3\, y^3 + 1)\, dx + x^4\, y^2\, dy = 0$
23. $(4\, x^3\, y^3 - 2xy)\, dx + (3x^4\, y^2 - x^2)\, dy = 0$
24. $3x^2\, y\, dx + (y^4 - x^3)\, dy = 0$
25. $y\, dx + (x + x^3\, y^2)\, dy = 0$

Answers

1. $x^3y^3 + \dfrac{x^2}{y} = C$
2. $\dfrac{x^3}{3} + \dfrac{e^x}{y} = C$
3. $\dfrac{x^2y^4}{2} + xy^2 + \dfrac{y^6}{3} = C$
4. $y(x + y - 1) = ce^{-x}$
5. $y^2 - x^2 + 1 = cx.$
6. $2\log x - \log y - \dfrac{1}{xy} = C$
7. $\dfrac{1}{2x^2y^2} + \dfrac{1}{xy} - \log y = c$
8. $\dfrac{x}{y} + \dfrac{x^2}{2} = C$
9. $(y^3 + 2)\, x + y^4 = Cy^2$
10. $x^2 + y^2 = 5y$
11. $xe^y(x^2 + 3y^2) = C$
12. $\dfrac{x}{y} - 2\log x + 3\log y = C$
13. $y^4 = 4x^4 \log x + Cx^4$
14. $(x - y)y^2 = c(x + y)$
15. $\log\,(x^2 + y^2)^{1/2} - \tan^{-1}\left(\dfrac{x}{y}\right) = C$
16. $x^2y^4 - x^4y^2 = C$
17. $4x^2y^2 + x^4y^5 = C$
18. $x^3y^3 + 2 = Cx^{5/3} \,.\, y^{7/3}$
19. $4\sqrt{(xy)} - \dfrac{2}{3}\left(\dfrac{y}{x}\right)^{3/2} = C$
20. $x^3\, y^2 + 4x^2\, y^6 = C$
21. $x^2\, y^4\,(x + y^2) = C$
22. $\dfrac{(xy)^3}{3} + \log x = C$
23. $x^4\, y^3 - x^2y = C$
24. $3x^3 + y^4 = Cy$
25. $2\, x^2\, y^2 \log\,(C.\, y) = 1$

APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

10.5 PHYSICAL APPLICATIONS

Consider a body of mass m start from a fixed point O along a straight line OX under the action of a force F.

Fig. 10.1

Let P be the position of the body at any time 't', where $OP = x$, then the velocity of the body is $\frac{dx}{dt}$ and the acceleration of the body $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$.

Also, by Newton's second law of motion,

$$\begin{aligned} F &= ma \\ &= m\frac{dv}{dt} \\ &= m\frac{d^2x}{dt^2} \end{aligned}$$

where F is the force.

Example 41 A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that $2\frac{kx}{m} = \log\frac{a^2}{a^2 - v^2}$; where $mg = ka^2$

Solution The forces acting on the body are (i) its weight (mg) acting vertically downwards, and (ii) the resistance kv^2 of the air acting vertically upwards.

$\therefore$ The accelerating force on the body $= mg - kv^2$

$$\begin{aligned} &= ka^2 - kv^2 \qquad [\because \; mg = ka^2] \\ &= k(a^2 - v^2) \end{aligned}$$

By Newton's second law, the equation of motion of the body is

$$mv\frac{dv}{dx} = k(a^2 - v^2)$$

or

$$\frac{v}{a^2 - v^2}dv = \frac{k}{m}dx$$

Integrating both sides, we get

$$\int\frac{v}{a^2 - v^2}dv = \frac{k}{m}\int dx$$

or

$$-\frac{1}{2}\int\frac{2v}{a^2 - v^2}dv = \frac{k}{m}x + C$$

or $$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x + C \tag{1}$$

Initially, when $x = 0$, $v = 0$

$\therefore$ $$C = -\frac{1}{2}\log a^2$$

From (1), $$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} - \frac{1}{2}\log a^2$$

or $$\frac{kx}{m} = \frac{1}{2}[\log a^2 - \log(a^2 - v^2)]$$

or $$\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}$$ **Hence, proved**.

Example 42 A boat is rowed with a velocity V across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the equation of the path of the boat and the distance downstream to the point, where it lands.

Solution Let O be the initial point, where the boat starts and at any time 't', on point $P(x, y)$, its distance from the two banks are y and $(a - y)$.

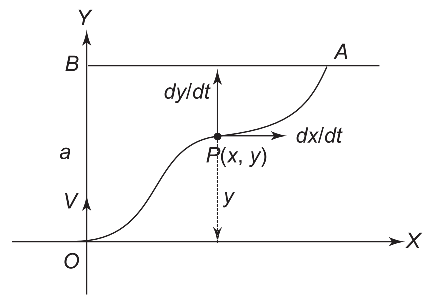


Fig. 10.2

$\therefore$ $$\frac{dx}{dt} \propto y(a-y)$$

or $$\frac{dx}{dt} = ky(a-y)$$

$$\frac{dy}{dt} = \text{velocity of the boat} = V$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{V}{ky(a-y)} \tag{1}$$

Equation (1) represents the resultant velocity of the boat at P. From (1), we have

$$y(a-y)\,dy = \frac{V}{k}dx$$

Integrating, we get

$$\frac{ay^2}{2} - \frac{y^3}{3} = \frac{V}{k}x + C \tag{2}$$

When $x = 0$, $y = 0$, then (2) becomes $C = 0$

$\therefore$ The equation of the path of the boat is $x = \dfrac{k}{6V}y^2(3a - 2y)$ (3)

The distance downstream to the point where the boat lands is obtained by putting $y = a$ in (3)

$\therefore$ $$x = \frac{ka^3}{6V}$$

10.6 RATE OF GROWTH OR DECAY

If the rate of change of a quantity 'y' is proportional to the quantity present at any instant 't', i.e.,

$$\frac{dy}{dt} \propto y$$

or $$\frac{dy}{dt} = ky \tag{22}$$

where k is the proportionality constant.

If $k > 0$, i.e., positive then the problem is one of growth and if $k < 0$, i.e., negative then the problem is one of decay.

10.7 NEWTON'S LAW OF COOLING

If the rate of change of the temperature 'T' of a body is proportional to the difference between T and the temperature T_A of the ambient medium,

then $$\frac{dT}{dt} \propto (T - T_A)$$

$$\frac{dT}{dt} = -k(T - T_A)$$

where $(-k)$ is the proportionality constant.

Example 43 Uranium disintegrates at a rate proportional to the amount present at any instant. If P_1 and P_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.

Solution Let P gram of uranium be present at any time 't'. Therefore, the equation of disintegration of uranium is

$$\frac{dP}{dt} = -kP, \text{ where } k \text{ is a constant}$$

or $$\frac{dP}{P} = -kdt$$

Integrating both sides, we get

$$\log P = -kt + C \tag{1}$$

Initially, when $t = 0$, $P = P_0$ (say)

Then (1) becomes

$$\log P_0 = C$$

$\therefore$ $$\log P = \log P_0 - kt$$

or $$\log P_0 - \log P = kt \tag{2}$$

When $t = T_1$, $P = P_1$ and $t = T_2$, $P = P_2$

$\therefore$ from (2), we have $\log P_0 - \log P_1 = kT_1$ (3)

and $$\log P_0 - \log P_2 = kT_2 \tag{4}$$

Subtracting (3) from (4), we get $k(T_2 - T_1) = \log P_1 - \log P_2$

or $$k(T_2 - T_1) = \log\left(\frac{P_1}{P_2}\right)$$

or $$k = \frac{\log(P_1/P_2)}{(T_2 - T_1)}$$

Consider T as the half-life of uranium i.e., $t = T$, $P = P_0/2$

$\therefore$ from (2), we get $kT = \log P_0 - \log\frac{P_0}{2}$

$$kT = \log 2$$

$\therefore$ $$T = \frac{(T_2 - T_1)\log 2}{\log\left(\frac{P_1}{P_2}\right)}$$

Example 44 If a substance cools from 370 K to 330 K in 10 minutes, when the temperature of the surrounding air is 290 K, find the temperature of the substance after 40 minutes.

Solution Using Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T - T_A)$$

$$\frac{dT}{T - T_A} = -kdt$$

Integrating both sides, we get

$$\log(T - T_A) = -kt + C_0 \text{ or } T - T_A = e^{-kt + C_0}$$
$$= e^{-kt}e^{C_0}$$

or $$T - T_A = Ce^{-kt}, \text{ where } c = e^{C_0}$$

or $$T = T_A + Ce^{-kt} \tag{1}$$

$\therefore$ $$T_{(t)} = 290 + Ce^{-kt} \tag{2}$$

Using $T(0) = 370$ K, so that

$$370 = 290 + Ce^{-k\cdot 0} \Rightarrow c = 80$$

$\therefore$ Eq. (2) becomes

$$T(t) = 290 + 80\,e^{-kt} \tag{3}$$

Again, using $T(10) = 330$ K in (3),

$\therefore$ $$330 = 290 + 80\,e^{-10\,k}$$

$$40 = 80\,e^{-10\,k} \text{ or } e^{-10\,k} = \frac{1}{2}$$

or $$-10\,k = \log\frac{1}{2} = -\log 2$$

or $$k = \frac{\log 2}{10} = 0.06931$$

Putting the values of C and k in (2), we have

$$T(t) = 290 + 80\, e^{-0.06931t}$$

$\therefore$ the temperature of the substance $t = 40$ minutes is

$$T(40) = 290 + 80\, e^{-0.06931 \times 40}$$

$$T(40) = 295$$

Example 45 If the temperature of the air is 30ºC and a substance cools from 100ºC to 70ºC in 15 minutes, find the time when the temperature is 40ºC.

Solution Using Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T - T_A)$$

or $$\frac{dT}{dt} = -k(T - 30) \text{ or } \frac{dT}{T - 30} = -\,kdt$$

Integrating, we get

$$\log(T - 30) = -kt + C_0$$

or $$T(t) = 30\, e^{-kt + C_0} = 30 + Ce^{-kt}, \text{ where } C = e^{C_0} \tag{1}$$

Initially, when $t = 0$, $T = 100$ºC

$$100 = 30 + C \Rightarrow C = 70; \text{ then}$$

Equation (1) becomes

$$T(t) = 30 + 70\, e^{-kt} \tag{2}$$

Again, when $t = 15$, $T = 70$ºC

$\therefore$ $$70 = 30 + 70\, e^{-15k}$$

$$40 = 70\, e^{-15k} \text{ or } e^{-15k} = \frac{4}{7}$$

or $$-15\,k = \log\frac{4}{7} = -\log\left(\frac{7}{4}\right)$$

or $$15k = \log\left(\frac{7}{4}\right)$$

$$k = \frac{1}{15}\log\left(\frac{7}{4}\right) = 0.0162$$

$\therefore$ Eq. (2), becomes

$$T(t) = 30 + 70\, e^{(-0.0162)t} \tag{3}$$

When $T = 40$ºC, find t

Using (3), we get

$$40 = 30 + 70\, e^{(-0.0162)t}$$

or $$10 = 70\, e^{(-0.0162)t}$$

or $$e^{-0.0162t} = \frac{1}{7}$$

or $$-0.0162\, t = \log\frac{1}{7} = -\log 7$$

or $$t = \frac{\log 7}{0.0162} = 52.16 \text{ minutes}$$

Hence, the temperature will be 40ºC after $t = 52.16$ minutes

10.8 CHEMICAL REACTIONS AND SOLUTIONS

Example 46 A tank contains 5000 litres of fresh water. Salt water which contains 100 g of salt per litre flows into it at the rate of 10 litres per minute and the mixture kept uniform by stirring, runs out at the same rate. When will the tank contain 20,0000 g of salt? How long will it take for the quantity of salt in the tank of increase from 15,0000 gm to 25,0000 gm?

Solution Let Q g be the quantity of salt present in the tank at time 't'. The rate of change of quality Q with respect to time t is

$$\frac{dQ}{dt} = \text{Rate of salt entering the tank} - \text{Rate of salt leaving the tank} \tag{1}$$

Now, the rate of salt entering the tank

$$= 100 \times 10 = 1000 \text{ g/min}$$

Let C g be the concentration of salt at time 't'.

∴ the rate at which the salt content decreases due to the outflow $= C \times 10 = 10\, C$ g/min

∵ the rate of inflow (entering) is equal to the rate of outflow (leaving), there is no change in the volume of water at any instant.

$$C = \frac{Q}{5000}$$

∴ the rate of outflow $= 10 \times \frac{Q}{5000} = \frac{Q}{500}$ g/min

Now, Eq. (1) becomes

$$\frac{dQ}{dt} = 1000 - \frac{Q}{500} = \frac{500000 - Q}{500}$$

or $$\frac{dQ}{500000 - Q} = \frac{dt}{500}$$

On integrating, $$-\log(500000 - Q) = \frac{t}{500} + C_1 \tag{2}$$

Initially, when $t = 0$, $Q = 0$, then

$$C = -\log 500000$$

∴ Eq. (2) becomes, $$\frac{t}{500} = \log 500000 - \log(500000 - Q)$$

or $$t = 500 \log \frac{500000}{(500000 - Q)} \tag{3}$$

When $Q = 200000$, $t = T$

$\therefore$ $$T = 500 \log \frac{500000}{500000 - 200000}$$

$$= 500 \log \frac{500000}{300000}$$

$$= 500 \log\left(\frac{5}{3}\right) = 500 \times 2.303 \log_{10}\left(\frac{5}{3}\right)$$

$$= 255.41 \text{ minutes}$$

When $Q = 150000$, $t = T_1$ and $Q = 250000$, $t = T_2$

Then (3) becomes

$$T_1 = 500 \log \frac{500000}{350000} = 500 \log_e\left(\frac{10}{7}\right)$$

and $$T_2 = 500 \log \frac{500000}{350000} = 500 \log_e(2)$$

$\therefore$ Required time $(T) = T_2 - T_1$

$$= 500\left[\log_e 2 - \log_e\left(\frac{10}{7}\right)\right]$$

$$= 500 \log_e\left(\frac{7}{5}\right)$$

$$= 500 \times 2.303 \log_{10}\left(\frac{7}{5}\right) = 168.23 \text{ minutes}$$

10.9 SIMPLE ELECTRIC CIRCUITS

Let q be the electric charge on a capacitance C and i be the current. Then

(i) $i = \frac{dq}{dt}$

(ii) the potential drop across the resistance R is iR

(iii) the potential drop across the capacitance C is $\frac{q}{C}$

(iv) the potential drop across the inductance L is $L\frac{di}{dt}$ and by Kirchhoff's law, the total potential drop (or voltage drop) in the circuit is equal to the applied voltage (emf), i.e.,

$$\boxed{L\frac{di}{dt} + iR = E}$$

Example 47 A simple electrical circuit containing a resistance R, inductance L, and capacitance C in series. If the electromotive force $E = v \sin \omega t$. Find the current (i).

Solution By Kirchhoff's law,

$$L\frac{di}{dt} + iR = E = v \sin \omega t$$

or
$$\frac{di}{dt} + \frac{R}{L}i = \frac{v}{L}\sin \omega t \tag{1}$$

Equation (1) is Leibnitz's linear equation; Then

$$\text{IF} = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

$\therefore$ The solution of (1) is

$$i(\text{IF}) = \int \frac{V}{L}\sin \omega t \cdot (\text{IF})\, dt + C$$

$$i \cdot e^{\frac{Rt}{L}} = \int \frac{V}{L}\sin \omega t \cdot e^{\frac{Rt}{L}}\, dt + C$$

$$ie^{\frac{Rt}{L}} = \frac{V}{L}\left[\frac{R\sin \omega t - \omega L\cos \omega t}{R^2 + \omega^2 L^2}\right] Le^{\frac{Rt}{L}} + C$$

or
$$i = V\left[\frac{R\sin \omega t - \omega L\cos \omega t}{(R^2 + \omega^2 L^2)}\right] + C e^{-\frac{Rt}{L}} \tag{2}$$

Initially, when $t = 0$, $i = i_0$

$$C = i_0 + \frac{V\omega L}{R^2 + \omega^2 L^2}$$

$\therefore$
$$i = V\left[\frac{R\sin \omega t - \omega L\cos \omega t}{(R^2 + \omega^2 L^2)}\right] + \left[i_0 + \frac{V\omega L}{R^2 + \omega^2 L^2}\right] e^{-\frac{Rt}{L}}$$

Example 48 The equations of electromotive force in terms of current i for an electrical circuit having resistance R and capacity C in series, is $E = Ri + \int \frac{i}{C}dt$. Find the current i at any time t, when $E = E_0 \sin \omega t$.

Solution Do as same Example 47.

10.10 ORTHOGONAL TRAJECTORIES AND GEOMETRICAL APPLICATIONS

The concept of a orthogonal trajectories is very useful in the field of mathematics like, in fluid flow, the stream lines and the equipotential lines of constant velocity potential curves of electric force and equipotential lines, lines of heat flow and isothermal curves. The orthogonal trajectories of a given family of curves depends on the solution of the first-order differential equations. The general solution of a first-order differential equation represents a one-parameter family of curves and we find out another one-parameter family of curves which is orthogonal to the given one-parameter family of curves. If one family of the curve is orthogonal (i.e., perpendicular) to the other family of curves, i.e., the tangents to the curves at the point of intersection are perpendicular, they are called orthogonal trajectories.

(i) Method of Finding the Orthogonal Trajectories of a Family of Curves $f(x, y, c) = 0$

Step 1 Find the differential equation of the family of curves $f(x, y, c) = 0$. By eliminating the arbitrary constant c, we obtain $\frac{dy}{dx} = g(x, y)$ (23)

Step 2 Let $\frac{dy}{dx} = -\frac{1}{g(x,y)}$ (24)

Equation (24) represents the family of orthogonal trajectories (since the product of (23) and (24) gives –1, i.e., the curves of the first family are at right angles to the curves of the second family).

Step 3 Solving Eq. (24), we obtain $h(x, y, d) = 0$ (25)

The family of curves (25) is the required orthogonal trajectories of the family of curves $f(x, y, c) = 0$ and d is the parameter.

(ii) Method of Finding the Orthogonal Trajectories of a Family of Curves $f(r, \theta, c) = 0$

Step 1 Find the differential equation of the family of curves $f(r, \theta, c) = 0$. By eliminating the arbitrary constant c, we obtain

$$g\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \tag{26}$$

Step 2 Replace $\frac{dr}{d\theta}$ *by* $-r^2\frac{d\theta}{dr}$ in (26), and we get

$$g\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0 \tag{27}$$

Step 3 Solving Eq. (27) and we obtained the required other orthogonal trajectories curve.

Example 49 Find the orthogonal trajectories of the hyperbolas $x^2 - y^2 = c$

Solution The given curve $x^2 - y^2 = c$ (1)

Step 1: Differentiating (1) both sides w.r.t. 'x', we obtain

$$2x - 2yy' = 0 \tag{2}$$

Equation (2) is governs the family of hyperbolas.

Step 2: The differential equation governing the orthogonal trajectories is

$$2x - 2y\left(-\frac{1}{y'}\right) = 0$$

or $$x+\frac{y}{y'}=0$$

or $$xy'+y=0$$

or $$x\frac{dy}{dx}+y=0 \tag{3}$$

Step 3: We now find the solution of (3), separating the variables, we get

$$xdy+ydx=0$$

or $$d(xy)=0$$

On integrating,

$$xy=d \tag{4}$$

Thus, the rectangular hyperbolas $xy = d$ are the orthogonal trajectories of the hyperbolas $x^2-y^2=C$.

Example 50 Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$, where λ is the parameter.

Solution Given the family of confocal conics

$$\frac{x^2}{a^2}+\frac{y^2}{c^2+\lambda}=1 \tag{1}$$

Differentiating (1) both sides w.r.t. 'x', we get

$$\frac{2x}{a^2}+\frac{2y}{a^2+\lambda}y'=0$$

or $$\frac{x}{a^2}+\left(\frac{y}{a^2+\lambda}\right)y'=0 \tag{2}$$

Equation (2) governs the family of conics.

$\therefore$ The differential equation governing the orthogonal trajectories is replaced by $y'=-\frac{1}{y}$, in (2). We get

$$\frac{x}{a^2}+\left(\frac{y}{a^2+\lambda}\right)\left(-\frac{1}{y'}\right)=0$$

or $$\left(\frac{-y}{a^2+\lambda}\right)\frac{1}{y'}=\frac{-x}{a^2}$$

or $$\left(\frac{y^2}{a^2+\lambda}\right)\frac{1}{y'}=\frac{xy}{a^2}$$

or $$\frac{y^2}{a^2+\lambda}=\left(\frac{xy}{a^2}\right)y'$$

Putting the value of $\dfrac{y^2}{a^2+\lambda}$ in (1), we get

$$\frac{x^2}{a^2}+\left(\frac{xy}{a^2}\right)y'=1 \text{ or } \left(\frac{xy}{a^2}\right)y'=1-\frac{x^2}{a^2}$$

or
$$\frac{(xy)}{a^2}y'=\frac{a^2-x^2}{a^2}$$

or
$$(xy)\frac{dy}{dx}=(a^2-x^2)$$

Separating the variables, we get

$$\left(\frac{a^2-x^2}{x}\right)dx = ydy$$

Integrating, we obtain

$$\int\left(\frac{a^2-x^2}{x}\right)dx=\int ydy+C$$

$$a^2\log x-\frac{x^2}{2}=\frac{y^2}{2}+C$$

or
$$(x^2+y^2)=2a^2\log x+C'$$

which is the required orthogonal trajectories.

Example 51 Find the orthogonal trajectories of the cardioids

$$r=a(1-\cos\theta)$$

Solution Given the cardioids curve

$$r=a(1-\cos\theta) \tag{1}$$

Differentiating (1) both sides w.r.t. 'θ', we get

$$\frac{dr}{d\theta}=a\sin\theta \tag{2}$$

Eliminating 'a' from (1) and (2), we get

$$\frac{dr}{d\theta}=\frac{r\sin\theta}{1-\cos\theta}$$

or
$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\sin\theta}{1-\cos\theta}=\cot\frac{\theta}{2} \tag{3}$$

which is the differential equation governing the given family.

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (3), we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right)=\cot\frac{\theta}{2}$$

or $$-r\frac{d\theta}{dr} = \cot\frac{\theta}{2}$$

or $$\frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0 \qquad (4)$$

which is the differential equation governing the orthogonal trajectories.

Equation (4) can be rewritten as

$$\frac{dr}{r} = -\frac{\sin\theta/2}{\cos\theta/2}\,d\theta$$

Integrating $$\int\frac{1}{r}\,dr = -\int\frac{\sin\theta/2}{\cos\theta/2}\,d\theta + \log C$$

or $$\log r = 2\log\cos\frac{\theta}{2} + \log C$$

or $$r = C\cos^2\frac{\theta}{2}$$

$$= \frac{C}{2}(1+\cos\theta)$$

$$r = C'(1 + \cos\,\theta)$$

which is the required orthogonal trajectories.

10.11 VELOCITY OF ESCAPE FROM THE EARTH

Example 52 Find the initial velocity of a particle which is fixed in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.

Solution Let the distance of the particle from the earth's centre is r.

According to Newton's law of gravitation, the acceleration a of the particle is proportional to $\frac{1}{r^2}$.

Thus, $$a = v\frac{dv}{dr} = -\frac{\mu}{r^2} \qquad (1)$$

where v is the velocity of the particle when at a distance r from the earth's centre. The acceleration is –ve because velocity (v) is decreasing.

When $r = R$, the earth's radius then $a = -g$

∴ the acceleration of gravity at the surface is

$$-g = -\frac{\mu}{R^2} \text{ or } \mu = gR^2$$

∴ $$v\frac{dv}{dr} = -\frac{gR^2}{r^2}$$

Separating the variables and integrating, we obtain

$$\int v\,dv = -gR^2\int\frac{dr}{r^2} + C$$

or $$v^2 = \frac{2gR^2}{r} + 2C \tag{2}$$

On the earth's surface, $r = R$ and $v = v_0$ (initial velocity),

Then $v_0^2 = 2gR + 2C$

or $2C = v_0^2 - 2gR$

$\therefore$ Eq. (2) becomes

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2\,gR$$

When $v \to 0$, the particle stops and the velocity will change from +ve to –ve and particle will return to the earth.

Thus, the velocity will remain +ve iff $v_0^2 \geq 2\,gR$.

Hence, the minimum velocity of projection $v_0 = \sqrt{2\,gR}$ is called the velocity of escape from the earth.

Example 53 Find the least velocity with which a particle must be projected vertically upwards so that it does not return to the earth. Assume that it is acted upon by the gravitational attraction of the earth only.

Solution Do same as Example 52.

EXERCISE 10.8

Physical Applications

1. A particle falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.
2. A paratrooper and her parachute weigh 50 kg. At the instant the parachute opens, she is travelling vertically downward at the speed of 20 m/s. If the air resistance varies directly as the instantaneous velocity and it is 20 newtons when the velocity is 10 m/s, find the limiting velocity, the position and the velocity of the paratrooper at any time t.
3. A body falling from rest is subjected to the force of gravity and an air resistance of $\frac{n^2}{g}$ times the square of velocity. Show that the distance travelled by the body in t seconds is $\frac{g}{n^2}$ log cos $h(t)$.

Rate of Growth or Decay

4. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple?
5. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 1.5 hours?
6. If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear?

Newton's Law of Cooling

7. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference of a temperature between the object and the surrounding. Supposing water

at a temperature 100°C cools to 80°C in 10 minutes, in a room maintained at a temperature of 30°C, find when the temperature of water will become 40°C.

8. Water at temperature 100°C cools in 10 minutes to 80°C in a room of temperature 25°C. Find
   (i) the temperature of water after 20 minutes, and
   (ii) the time when the temperature is 40°C.

Chemical Reaction

9. In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation dx/dt at any time t is known to be equal to the product $(a - x)$ $(b - x)$ of the amounts of the two substances then remaining untransformed. Find t in terms of x if $a = 0.7$, $b = 0.6$ and $x = 0.3$ when $t = 300$ seconds.
10. A tank is initially filled with 100 gallons of salt solution containing 1 lb of salt per gallon. Fresh brine containing 2 lb of salt per gallon runs into the tank at the rate of 5 gallons per minute are the mixture assumed to be kept uniform by stirring runs out at the same time. Find the amount of salt in the tank at any time, and find how long it will take for this amount to reach 150 lb.

Simple Electric Circuits

11. When a resistance R Ω is connected in series with an inductance L henries, an emf of E volts, the current i amperes at time t is given by $L\dfrac{di}{dt} + Ri = E.$ If $E = 10 \sin t$ volts and $i = 0$, when $t = 0$, find i as a function of t.
12. Find the solution of the equation $L\dfrac{di}{dt} + Ri = 200 \times \cos 300t$, when $R = 100$, $L = 0.05$, and find i given that $i = 0$ when $t = 0$, what value does i approach after a long time?
13. When a switch is closed in a circuit containing a battery E, a resistance R, and an inductance L, the current i builds up at rate given by $L\dfrac{di}{dt} + Ri = E$, find i in terms of t.
    How long will it be before the current has reached one-half its maximum value if $E = 6$ volts, $R = 100$ Ω and $L = 0.1$ henry?

Orthogonal Trajectories

14. Show that the one-parameter family of curves $y^2 = 4C(C + x)$ are self-orthogonal (a family of curves is self-orthogonal if it is its own orthogonal family).
15. Find the orthogonal trajectories of the family of
    (i) Parabolas $y = ax^2$
    (ii) Parabolas $y^2 = 4\,ax$
    (iii) Semicubical parabolas $ay^2 = x^3$
16. Find the orthogonal trajectories of a given family of curves
    (i) $e^x + e^{-y} = C$
    (ii) $y^2 = Cx$
    (iii) $x^2 - y^2 = Cx$
    (iv) $x^2 + y^2 = C^2$
17. Find the orthogonal trajectories of each of the following family of curves:
    (i) $r^2 = a \sin 2\theta$
    (ii) $r = a(1 + \cos\theta)$
    (iii) $r = a(1 + \sin^2\theta)$

(iv) $r = a\cos^2\theta$
(v) $r = 2a(\cos\theta + \sin\theta)$
(vi) $r^2 = a^2\cos 2\theta$

Answers

1. $x = \dfrac{g}{k^2}\log\left(\dfrac{g}{g-kv}\right) - \dfrac{v}{k}$

2. $x = 25t - \dfrac{125}{g}(1 - e^{-gt/25})$

4. 3.17 hours

5. 604.9

6. 64.5 day's

7. 57.9 minutes

8. (i) 65.5ºC (ii) 51.9 minutes

9. $t = 300 - 5\log 2 + 5\log\left(\dfrac{0.7-x}{0.5-x}\right)$

10. $100(2 - e^{-t/20})$; 13.9 minutes

11. $i = \dfrac{10}{L^2+R^2}(R\sin t - L\cos t + Le^{-R/L})$

12. $i = \dfrac{40}{409}[20\cos 300t + 3\sin 300t] - \dfrac{800}{409}e^{-200t}$; and $\dfrac{40}{\sqrt{409}}$

13. 0.000693 seconds

15. (i) $x^2 + 2y^2 = c^2$ (ii) $2x^2 + y^2 = c$ (iii) $3y^2 + 2x^2 = C^2$

16. (i) $e^y - e^{-x} = \lambda$ (ii) $x^2 + \dfrac{y^2}{2} = \lambda$ (iii) $y(y^2 + 3x^2) = \lambda$ (iv) $y = \lambda x$

17. (i) $r^2 = \lambda\cos 2\theta$ (ii) $r = \lambda(1 - \cos\theta)$ (iii) $r^2 = \lambda\cos\theta\cot\theta$
(iv) $r^2 = \lambda\sin\theta$ (v) $r = \lambda(\cos\theta - \sin\theta)$ (vi) $r^2 = \lambda^2\sin 2\theta$

SUMMARY

First-Order and First-Degree Differential Equations

An equation of the form

$$\frac{dy}{dx} = (x, y) \tag{1}$$

Equations like (1) are certain standard types of first-order first-degree differential equations for which solutions can be readily obtained by standard methods such as the following:

(a) Variable-Separable

An equation of the form

$$f_1(x)\,g_1(y)\,dx + f_2(x)\,g_2(y)\,dy = 0$$

The variables x and y can be separated and then integrating both sides, we get the solution of the given equation.

(b) Homogeneous Equations

A differential equation of the form

$$\frac{dy}{dx} = f(y/x) \tag{2}$$

Step 1: Putting $v = \dfrac{y}{x}$ or $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (2), we get

$$v + x\frac{dv}{dx} = f(v) \tag{3}$$

Step 2: Separating the variables in (3) and integrating, the solution is obtained as

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c$$

Step 3: Replace v by y/x, in the solution obtained in Step 2.

(c) Linear Differential Equations or (Leibnitz's Linear Equation)

The standard form of the linear equation of first order is

$$\frac{dy}{dx} + Py = Q - (1)$$

where P and Q are functions of x only then the method of solving the linear equation is as follows:

Step 1: Write the given differential equation in the standard form (1)

Step 2: Find the integrating factor (IF) = $e^{\int pdx}$

Step 3: The solution of the given differential equation is

i.e., $$y(\mathrm{IF}) = \int Q(\mathrm{IF})\,dx + C$$

(d) Nonlinear Equation Reducible to Linear Form (Bernoulli's Equation)

A first-order and first-degree differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{4}$$

is known as Bernoulli's equation which is non-linear for any value of the real number n (except for $n = 0$ and $n = 1$).

Method of finding Solution to Bernoulli Equation

Step 1: First of all, rewrite the given differential equation in the standard Bernoulli equation.

Step 2: Divide both sides by y^n.

Step 3: Put $z = y^{1-n}$ and obtain the first-order linear differential equation in z.

Step 4: Solve linear differential equation in z by the method discussed in Section (10.1.3).

(e) Exact Differential Equations

To solve a differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

Step 1: Find $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$. If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then the given equation is an exact differential equation.

Step 2: The solution of the given equation is

$$\int_{\text{Treting y as a constant}} M(x,y)dx + \int_{\substack{\text{Take only those terms in } N\\ \text{which do not contain } x.}} N(x,y)dy$$

(f) To Convert Non-exact Differential Equations into Exact Differential Equations Using Integrating Factors

Consider a differential equation

$$M\,(x,\,y)\,dx + N(x,\;y)\,dy = 0 \tag{5}$$

which is not exact.

Suppose there exists a function $F(x,\,y)$ such that

$$F(x,y)\,[M(x,y)dx + N(x,y)dy] = 0 \tag{2}$$

is an exact equation. Then $F(x\,,y)$ is called an integrating factor (IF) of the differential equation (5).

Methods to find an integrating factor to a non-exact differential equation

$Mdx + Ndy = 0$ are given here.

Case 1: If $\dfrac{1}{N}[M.y - N.x] = f(x)$ then IF $= e^{\int f(x)dx}$

Case 2: If $\dfrac{1}{M}[N.x - M.y] = g(y)$ then IF $= e^{\int g(y)dy}$

Case 3: If the differential equation of the form $y\,f_1(xy)\,dx + x\,f_2(x.y)\,dy = 0$

Then IF $= \dfrac{1}{M.x - N.y}$ provided $M\;\;x - N\;\;y \neq 0$

Case 4: If the given differential equation $M\,dx + N\,dy = 0$ is homogeneous equation and $M\;\;x + N\;\;y\;\; \neq 0$ then

$$\text{IF} = \frac{1}{M.x + N.y}$$

Case 5: Let the differential equation be of the form

$$x^a\,y^b\,(m.y\,dx + n.x\,dy) + x^c\,y^d\,(m_2\,y\,dx + n^2 x\,dy) = 0$$

where a, b, c, d, m_1, n_1, m_2, n_2 are all constants and $m_1m_2 - n_1n_2 \neq 0$

Then the integrating factor is $x^h\,y^k$

where $\dfrac{a+h+1}{m_1} = \dfrac{b+k+1}{n_1}$ and $\dfrac{c+h+1}{m_2} = \dfrac{d+k+1}{n_2}$.

OBJECTIVE-TYPE QUESTIONS

1. For the differential equation $f(x,y)\dfrac{dy}{dx} + g(x,y) = 0$ to be exact

(a) $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ (b) $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial y}$

(c) $f = g$ (d) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 g}{\partial y^2}$

[GATE (CE) 1997]

2. If c is a constant, solution of the differential equation $\dfrac{dy}{dx} = 1 + y^2$ is

(a) $y = \sin\,(x + c)$ (b) $y = \cos\,(x + c)$

(c) $y = \tan\,(x + c)$ (d) $y = e^x + c$

[GATE (CE) 1999]

3. The solution of the differential equation $\dfrac{dy}{dx} + y^2 = 0$, is

(a) $y = \frac{1}{x+C}$ (b) $y = -\frac{x^3}{3} + C$

(c) $y = Ce^x$ (d) None of these

[GATE (ME) 2003]

4. The solution of differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x+C}$ (b) $y = -\frac{x^2}{3} + C$

(c) $y = Ce^x$ (d) $y = x + C$

[GATE (EC) 2003]

5. Transformation to linear form by substitution $v = y^{1-n}$ of the equation $\frac{dy}{dt} + p(t)y = q(t)y^n$; $n > 0$ will be

(a) $\frac{dv}{dt} + (1-n)pv = (1-n)q$

(b) $\frac{dy}{dt} + (1-n)pv = (1+n)q$

(c) $\frac{dv}{dt} + (1+n)pv = (1-n)q$

(d) $\frac{dv}{dt} + (1+n)v = (1+n)q$

[GATE (Civil Engg) 2005]

6. If $x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$, and $y(1) = 0$ then what is $y(e)$?

(a) e (b) 1

(c) $1/e$ (d) $1/e^2$

[GATE (ME) 2005]

7. The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{x^2}$ (b) $(1+x)e^{-x^2}$

(c) $(1-x)e^{x^2}$ (d) $(1-x)e^{-x^2}$

[GATE (ME) 2006]

8. The solution of the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$, with initial condition $x(0) = 0$ is

(a) $e^{-2t}\,u(t)$ (b) $e^{2t}\,u(t)$

(c) $e^{-t}\,u(t)$ (d) $e^{t}\,u(t)$

[GATE (ECE) 2006]

9. Which of the following is a solution of the differential equation $\frac{dx(t)}{dt} + 3x(t) = 0$:

(a) $x(t) = 3e^{-t}$ (b) $x(t) = 2e^{-3t}$

(c) $x(t) = -\frac{3}{2}t^2$ (d) $x(t) = 3t^2$

[GATE (ECE) 2007]

10. The solution of differential equation $\frac{dy}{dx} = x^2 y$ with condition that $y = 1$ at $x = 0$ is

(a) $y = e^{1/2x}$ (b) $\log y = \frac{x^3}{3} + 4$

(c) $\log y = \frac{x^2}{2}$ (d) $y = e^{x^3/3}$

[GATE (Civil Engg) 2007]

11. Solution of $\frac{dy}{dx} = -\frac{x}{y}$ at $x = 1$ and $y = \sqrt{3}$ is

(a) $xy^2 = 2$ (b) $x + y^2 = 4$

(c) $x^2 + y^2 = 4$ (d) $x^2 - y^2 = -2$

[GATE (Civil Engg) 2008]

12. A function $y(t)$ satisfies the following differential equation $\frac{dy(t)}{dt} + y(t) = \delta(t)$, where $\delta(t)$ is the delta function. Assuming zero initial condition and denoting the unit step function by $u(t)$, $y(t)$ can be of the form

(a) e^t (b) e^{-t}

(c) $e^t u(t)$ (d) $e^{-t}u(t)$

[GATE (EE) 2008]

13. Consider the differential equation $\frac{dy}{dx} = 1 + y^2$, which one of the following can be a particular solution of this differential equation

(a) $y = \tan(x+3)$ (b) $y = \tan x + 3$

(c) $x = \tan(y+3)$ (d) $x = \tan y + 3$

[GATE (IN) 2008]

14. The solution of $x\frac{dy}{dx} + y = x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y = \frac{x^4}{5} + \frac{1}{x}$ (b) $y = \frac{4x^4}{5} + \frac{4}{5x}$

(c) $y=\dfrac{x^4}{5}+1$ (d) $y=\dfrac{x^5}{5}+1$

[GATE (ME) 2009]

15. Solution of the differential equation $3y\dfrac{dy}{dx}+2x=0$ represents a family of

(a) ellipses (b) circles
(c) parabolas (d) hyperbolas

[GATE (Civil Engg) 2009]

16. The solution of the differential equation $\dfrac{dy}{dx}=ky,\ y(0)=c$ is

(a) $x=ce^{-ky}$ (b) $x=ke^{cy}$
(c) $y=ce^{kx}$ (d) $y=ce^{-kx}$

[GATE (EC) 2011]

17. With initial condition $x(1) = 0.5$, the solution of the differential equation $t\dfrac{dx}{dt}+x=t$ is

(a) $x=t-\dfrac{1}{2}$ (b) $x=t^2-\dfrac{1}{2}$

(c) $x=\dfrac{t^2}{2}$ (d) $x=\dfrac{t}{2}$

[GATE (EE) 2012]

18. The solution of the differential equation $\dfrac{dy}{dx}-y^2=0$, given $y = 1$ at $x = 0$ is

(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1-x}$

(c) $\dfrac{1}{(1-x)^2}$ (d) $\dfrac{x^3}{3}+1$

[GATE (CH) 2013]

19. The solution to $\dfrac{dy}{dx}+y\cot x=\operatorname{cosec} x$ is

(a) $y = (c + x) \cot x$
(b) $y = (c + x) \operatorname{cosec} x$
(c) $y = (c + x) \operatorname{cosec} x \cot x$
(d) $y=(c+x)\dfrac{\operatorname{cosec} x}{\cot x}$

[GATE (BT) 2013]

20. Which ONE of the following is a linear non-homogeneous differential equation, where x and y are the independent and dependent variables respectively?

(a) $\dfrac{dy}{dx}+xy=e^{-x}$

(b) $\dfrac{dy}{dx}+xy=0$

(c) $\dfrac{dy}{dx}+xy=e^{-y}$

(d) $\dfrac{dy}{dx}+e^{-y}=e^{-y}=0$ **[GATE (EC) 2014]**

21. The solution of the initial-value problem $\dfrac{dy}{dx}=-2xy;\ y(0)=2$ is

(a) $1+e^{-x^2}$ (b) $2e^{-x^2}$

(c) $1+e^{x^2}$ (d) $2e^{x^2}$

[GATE (ME) 2014]

22. The integrating factor for the differential equation $\dfrac{dy}{dx}-\dfrac{y}{1+x}=(1+x)$ is

(a) $\dfrac{1}{1+x}$ (b) $(1 + x)$

(c) $x(1 + x)$ (d) $\dfrac{x}{1+x}$

[GATE (CH) 2014]

23. The general solution of the differential equation $\dfrac{dy}{dx}=\cos(x+y)$, with c as a constant, is

(a) $y + \sin (x + y) = x + c$

(b) $\tan\left(\dfrac{x+y}{2}\right)=y+c$

(c) $\cos\left(\dfrac{x+y}{2}\right)=x+c$

(d) $\tan\left(\dfrac{x+y}{2}\right)=x+c$

ANSWERS

1. (a)	2. (c)	3. (a)	4. (a)	5. (a)	6. (d)	7. (a)	8. (a)	9. (b)	10. (d)
11. (c)	12. (d)	13. (a)	14. (a)	15. (a)	16. (c)	17. (d)	18. (b)	19. (b)	20. (a)
21. (b)	22. (a)	23. (d)							

11 Linear Differential Equations of Higher Order with Constant Coefficients

11.1 INTRODUCTION

Linear differential equations play an important role in the study of many practical problems in science and engineering. Constant coefficient equations arise in the theory of vibrations, electric circuits, etc. Variable coefficient equations arises in many areas of electric circuits, physics and mathematical modeling of physical problems, etc.

The solution of constant coefficient equations can be obtained in terms of known standard functions.

A linear ordinary differential equation of order n, with constant coefficient is written as

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = R(x) \tag{1}$$

where $a_0, a_1, a_2, \ldots, a_n$ are all constants and $a_0 \neq 0$, R is a function of x only.

If $R(x) = 0$ then it is called a ***homogeneous equation*** otherwise it is said to be a ***non-homogeneous equations***.

11.2 THE DIFFERENTIAL OPERATOR *D*

Sometimes, it is convenient to write the given linear differential equation in a simple form using the differential operator D.

The part $\frac{d}{dx}$ of $\frac{dy}{dx}$ may be regarded as an operator D stands for $\frac{d}{dx}$, similarly $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \frac{d^4}{dx^4}, \cdots \frac{d^n}{dx^n}$ may be regarded as operators. D^2 stands for $\frac{d^2}{dx^2}$, D^3 stands for $\frac{d^3}{dx^3}, \ldots, D^n$ stands for $\frac{d^n}{dx^n}$.

Thus, we can write the given differential equation (1) is in the form of

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = R(x)$$

or $$[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]\, y = R(x)$$

or $$F(D)\, y = R(x) \tag{2}$$

where $F(D) \equiv a_0\, D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$.

11.3 SOLUTION OF HIGHER ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENT

Consider the n^{th} order homogeneous linear equation with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = R(x) \tag{3}$$

Let $y = e^{mx}$ be a solution of (3). Substituting $y = e^{mx}$,

$\dfrac{d^k y}{dx^k} = y^{(k)} = m^k e^{mx}$, $k = 1, 2, 3, \ldots, n$ (3) and canceling e^{mx}, we get the auxiliary (or characteristic) equations

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0 \tag{4}$$

The degree of the equation (4) is same as the order of the differential equation.

The auxiliary equation (4) has n roots. All the roots may be real and distinct, all or some of the roots may be complex, all or some of the roots may be equal. Consider the following cases.

Case I

When the auxiliary equation have all real and distinct roots say $m_1, m_2, m_3, \ldots, m_n$. Then the n linearly independent solutions $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, y_3 = e^{m_3 x}, \ldots y_n = e^{m_n x}$ of equation (3).

Hence, $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$ is a general solution of (3), where C_1, C_2, C_3, …, C_n are all constants.

Case II

When the auxiliary equation may have some multiple roots. Let r be the multiplicity of the root m_1, i.e., $m = m_1$ is repeated r times, and the remaining $(n - r)$ roots are real and distinct. Putting $m = m_1$, we obtain $y_1(x) = e^{m_1 x}$ as one of the solutions.

Now, the remaining $(r - 1)$ linearly independent solutions corresponding to the multiple root $m = m_1$ are given by $y_1, xy_1, x^2 y, x^3 y_1, \ldots, x^{r-1} y_1$.

Hence, $y(x) = (C_1 + xC_2 + x^2 C_3 + \ldots + x^{r-1} C_r)\, e^{m_1 x} + \cdots + C_n e^{m_1 x}$ is a general solution of (3), where $C_1, C_2, \ldots C_r, C_n$ are all constants.

Case III

When the auxiliary equation may have some imaginary roots. If $p + iq$ is a root then $p - iq$ is also a root of auxiliary equation. In this case, the linearly independent solutions are given by $e^{px} \cos qx$ and $e^{px} \sin qx$. If the auxiliary equation (4) has r complex conjugate pairs of roots $p_k \pm iq_k$, $k = 1, 2, 3, \ldots, r$, the corresponding linearly independent solutions are

$$e^{p_1 x} \cos q_1 x, e^{p_1 x} \sin q_1 x, e^{p_2 x} \cos q_2 x, e^{p_2 x} \sin q_2 x, \ldots, e^{p_r x} \cos q_r x .$$

Therefore, the general solution corresponding to linearly independent solutions is

$$y(x) = e^{p_1 x} (C_1 \cos q_1 x + C_2 \sin q_1 x) + e^{p_2 x} (C_3 \cos q_2 x + C_4 \sin q_2\, x)$$
$$+ \cdots + e^{p_r x} (C_{r_1} \cos q_{r_2} x + C_{r_2} \sin q_r x) + \cdots + C_n e^{m_n x}$$

Note: **Multiple Complex Roots**

If $p + iq$ is a multiple root of order m then $p - iq$ is also a multiple root of order m.

Then the corresponding linearly independent solutions are

$$e^{p_1 x}\cos q_1 x,\ e^{p_1 x}\sin q_1 x + xe^{p_1 x}\cos q_1 x,\ xe^{p_1 x}\sin q_1 x, \cdots x^{m-1}e^{p_2 x}\cos q_1 x,\ x^{m-1}\sin q_1 x$$

Therefore, the general solution corresponding to linearly independent solutions is

$$y(x) = e^{p_1 x}\,[(C_1 + xC_2 + x^2C_3 + \cdots x^{m-1}C_m)\cos q_1 x$$
$$+ (C_1' + xC_2' + x^2C_3' + x^{m-1}C_m')\sin q_1 x] + \cdots + C_n \cdot e^{m_n x}$$

where $C_1, C_2, \ldots, C_m, C_1', C_2', \ldots C_m', C_n$ are arbitrary constants.

Example 1 Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0.$

Solution Putting $y = e^{mx}$ in the given differential equation and we obtain the auxiliary equation

$$m^3 - m^2 - 4m + 4 = 0 \tag{1}$$

or $\quad (m - 1)(m^2 - 4) = 0$

or $\quad m = 1, 2, -2$

The roots of auxiliary equation (1) are $m = 1, 2, -2$, which are real and distinct.

The solution of the equation is given by

$$y = C_1e^x + C_2e^{2x} + C_3e^{-2x}, \text{ where } C_1, C_2, C_3 \text{ are arbitrary constants.}$$

Example 2 Solve $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 4y = 0$

Solution We have $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 4y = 0 \qquad (1)$

Putting $y = e^{mx}$ in (1), we obtain the auxiliary equation as

$$m^4 - 5m^2 + 4 = 0 \tag{2}$$

or $\quad (m^2 - 1)(m^2 - 4) = 0$

or $\quad m = 1, -1, 2, -2$

The rods of the auxiliary equation (2) are 1, –1, 2, –2.

The general solution of (1) is given by

$$y = C_1e^x + C_2e^{-x} + C_3e^{2x} + C_4e^{-2x}, \text{ where } C_1, C_2, C_3 \text{ and } C_4 \text{ are arbitrary constants.}$$

Example 3 Solve $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 0.$

Solution We have $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 0 \qquad (1)$

Substituting $y = e^{mx}$ in (1), we obtain the auxiliary equation as

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0 \tag{2}$$

or $\quad (m^2 - 2m + 2)^2 = 0$

or $\quad m^2 - 2m + 2 = 0 \qquad$ (two times)

or $\quad m = \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i \qquad$ (two times)

The general solution of (1) is given by

$$y = e^x [(C_1 + xC_2)\cos x + (C_3 + xC_4)\sin x]$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Example 4 Solve $(D^2 - 1)^2 (D^2 + 1)^2 y = 0;\ D \equiv \dfrac{d}{dx}$.

Solution We have $(D^2 - 1)^2 (D^2 + 1)^2 y = 0$ (1)

Substituting $y = e^{mx}$ in (1), we obtain the auxiliary equation as

$$(m^2 - 1)^2 (m^2 + 1)^2 = 0 \quad (2)$$

or $m^2 - 1 = 0$ (two times) and $(m^2 + 1)^2 = 0$ (two times)

or $m = \pm 1$ (two times) and $m = \pm i$ (two times)

The general solution is given by

$$y = (C_1 + xC_2)e^x + (C_3 + xC_4)e^{-x} + [(C_5 + xC_6)\cos x + (C_7 + xC_8)\sin x]$$

where C_1, C_2, C_3, C_4, C_5, C_6, C_7 and C_8 are arbitrary constants.

Example 5 Solve $\dfrac{d^4 y}{dx^4} - k^4 y = 0$.

Solution We have $\dfrac{d^4 y}{dx^4} - k^4 y = 0$ (1)

Substituting $y = e^{mx}$ in (1), we obtain the auxiliary equation as

$$m^4 - k^4 = 0 \quad (2)$$

or $(m^2 - k^2)(m^2 + k^2) = 0$

or $m = \pm k, +ik$

The general solution of (1) is given by

$$y = C_1 e^{kx} + C_2 e^{-kx} + C_3 \cos kx + C_4 \sin kx$$

where C_1, C_2, C_3 and C_4 are all constants.

Example 6 Solve $\dfrac{d^5 y}{dx^5} - \dfrac{d^3 y}{dx^3} = 0$.

Solution We have $\dfrac{d^5 y}{dx^5} - \dfrac{d^3 y}{dx^3} = 0$ (1)

Substituting $y = e^{mx}$ in (1), we obtain the auxiliary equation as

$$m^5 - m^3 = 0 \quad (2)$$

or $m^3(m^2 - 1) = 0$

or $m = 0, 0, 0, \pm 1$

The roots of (2) are $m = 0$ (repeated three times) and $m = \pm 1$ real and distinct.

The general solution of (1) is given by

$$y = (C_1 + xC_2 + x^2 C_3) + C_4 e^x + C_5 e^{-x}$$

where C_1, C_2, C_3, C_4, and C_5 are arbitrary constants.

EXERCISE 11.1

Solve the following differential equations.

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$
2. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$
3. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 0$
4. $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} - 9\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 4y = 0$
5. $\dfrac{d^6y}{dx^6} + y = 0$
6. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$
7. $4\dfrac{d^4y}{dx^4} - 12\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 27\dfrac{dy}{dx} - 18y = 0$
8. $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} - 2y = 0$
9. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 4y = 0$; given that $y(0) = 1$, $y'(0) = 0$, $y''(0) = \dfrac{1}{2}$.
10. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} + 11\dfrac{d^2y}{dx^2} + 18\dfrac{dy}{dx} + 18y = 0$; given that $y(0) = 0$, $y'(0) = 3$, $y''(0) = -11$, $y'''(0) = -23$.

Answers

1. $y = C_1e^{-x} + C_2e^{4x}$
2. $y = (C_1 + xC_2)e^{2x}$
3. $y = C_1e^{x} + C_2e^{-2x} + C_3e^{-5x}$
4. $y = (C_1 + xC_2 + x^2C_3)e^{-x} + C_4e^{4x}$
5. $y = C_1e^{x} + C_2e^{-x} + e^{x/2}\left(C_3 \cos\dfrac{\sqrt{3}}{2}x + C_4 \sin\dfrac{\sqrt{3}}{2}x\right) + e^{-x/2}\left(C_5 \cos\dfrac{\sqrt{3}}{2}x + C_6 \sin\dfrac{\sqrt{3}}{2}x\right)$
6. $y = C_1e^{x} + C_2e^{-2x} + C_3e^{3x}$
7. $y = C_1e^{x} + C_2e^{2x} + C_3e^{-\frac{3x}{2}} + C_4e^{\frac{3x}{2}}$
8. $y = C_1e^{2x} + (C_2x + C_3)e^{-x}$
9. $y = \left[e^{x} + (x+1)e^{-2x}\right]/2$
10. $y = \cos 3x + \sin 3x + e^{-x}(\cos x + \sin x)$

11.4 SOLUTION OF HIGHER ORDER NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

As already shown in Section 11.2, the complete solution of

$$[a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n]y = R(x) \tag{5}$$

or $\quad F(D)y = R(x)$, where $F(D) \equiv (a_0D^n + a_1D^{n-1} + \cdots + a_n)$ is

$$y = \text{Complementary function} + \text{Particular integral} = \text{C.F.} + \text{P.I.}$$

where the CF consists the general solution of the differential equation $F(D)y = 0$.

In Section 11.3, we have discussed different cases of finding the Complementary Function (CF) by taking the differential equation as $F(D)y = 0$.

Methods of finding the particular integral (PI) will be discussed now.

The particular integral of the differential equation $F(D)y = R(x)$ is $\frac{1}{F(D)} \cdot R(x)$

It is obviously a function of x which when operated by $F(D)$ gives $R(x)$.

Now, as $F(D)\frac{1}{F(D)}R = R$, therefore $\frac{1}{F(D)}$ can be regarded as the inverse operator of $F(D)$.

Hence, the particular integral of the equation

$$F(D)y = R \text{ will be } \frac{1}{F(D)} \cdot R$$

11.5 GENERAL METHODS OF FINDING PARTICULAR INTEGRALS (PI)

The operator $\frac{1}{D-\alpha}$, where α being a constant. If R is any function of x then

$$\frac{1}{(D-\alpha)}R = e^{\alpha x}\int e^{-\alpha x} \cdot R dx$$

Proof Let

$$y = \frac{1}{(D-\alpha)}R \tag{6}$$

Operating (6) both sides by $(D - \alpha)$, we get

$$(D-\alpha)y = (D-\alpha)\left(\frac{1}{D-\alpha}R\right)$$

$$(D-\alpha)y = R$$

or
$$\frac{dy}{dx} - \alpha y = R : D \equiv \frac{d}{dx} \tag{7}$$

Equation (7) is a linear differential equation,

whose integrating factor $= e^{-\int \alpha dx} = e^{-\alpha x}$

Thus, the complete solution of (7) is given by

$$y \cdot e^{-\alpha x} = \int R \cdot e^{-\alpha x} dx$$

$$y = e^{\alpha x}\int R \cdot e^{-\alpha x} dx$$

Hence,
$$\frac{1}{(D-\alpha)}R = e^{\alpha x}\int R \cdot e^{-\alpha x} dx$$

Similarly,
$$\frac{1}{D+\alpha}R = e^{-\alpha x}\int R \cdot e^{\alpha x} dx$$

Note: The above method can be used to evaluate the particular integral in which R is of the form tan αx, cot αx, cosec αx, sec αx or any other form not covered by shorter method (to be discussed later on).

11.6 SHORT METHODS OF FINDING THE PARTICULAR INTEGRAL WHEN 'R' IS OF A CERTAIN SPECIAL FORMS

Consider the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2}}{dx^{n-2}} + \cdots + a_n y = R(x) \tag{8}$$

or $\quad [D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]y = R(x);\ D \equiv \dfrac{d}{dx}$

or $\quad F(D)y = R(x)$, where $F(D) \equiv (a_0 D^n + a_1 D^{n-1} + \cdots + a_n)$

Therefore, the particular integral (PI) is

$$\text{PI} = \frac{1}{[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]} R(x) = \frac{1}{F(D)} \cdot R(x)$$

Case I

When $R(x) = e^{ax}$

We have $\quad F(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$

Therefore,

$$\begin{aligned} F(D)e^{ax} &= [D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]e^{ax} \\ &= D^n e^{ax} + a_1 D^{n-1} e^{ax} + a_2 D^{n-2} e^{ax} + \cdots + a_n e^{ax} \\ &= a^n e^{ax} + a_1 a^{n-1} e^{ax} + a_2 a^{n-2} e^{ax} + \cdots + a_n e^{ax} \\ &= (a^n + a_1 a^{n-1} + a_2 a^{n-2} + \cdots + a_n) e^{ax} \\ F(D)e^{ax} &= f(a)e^{ax} \end{aligned}$$

Operating both sides by $\dfrac{1}{F(D)}$, we get

$$\begin{aligned} e^{ax} &= \frac{1}{F(D)} [f(a) \cdot e^{ax}] \\ &= f(a) \cdot \frac{1}{F(D)} e^{ax} \end{aligned}$$

Hence, $\quad \dfrac{1}{F(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$, provided $f(a) \neq 0$. (9)

If $f(a) = 0$ then the above rule fails. We consider $F(D) = (D - a)\,\phi(D)$, where $(D - a)$ is a factor of $F(D)$. (10)

Then

$$\begin{aligned} \frac{1}{F(D)} e^{ax} &= \frac{1}{(D-a)\,\phi(D)} e^{ax} \\ &= \frac{1}{(D-a)} \left[\frac{1}{\phi(D)} e^{ax} \right] \end{aligned}$$

$$= \frac{1}{(D-a)}\left[\frac{1}{\phi(a)} e^{ax}\right]; \phi(a) \neq 0 \text{ [By Eq. (9)]}$$

$$= \frac{1}{\phi(a)}\left[\frac{1}{(D-a)} e^{ax}\right]$$

$$= \frac{1}{\phi(a)} e^{ax} \int e^{ax} \cdot e^{-ax}\, dx \qquad \left[\text{By } \frac{1}{D-\alpha} e^{\alpha x} = e^{\alpha x} \int e^{\alpha x} \cdot e^{-\alpha x}\, dx\right]$$

$$= x \frac{e^{ax}}{\phi(a)} \tag{11}$$

Differentiating (10) with respect to D, we get

$$F'(D) = \phi(D) + (D-a) \cdot \phi'(D)$$

Putting $D = a$, we get $F'(a) = \phi(a)$. (12)

Using (12), Eq. (11) reduces to

$$\frac{1}{F(D)} e^{ax} = x \frac{e^{ax}}{\phi(a)}$$

$$\frac{1}{F(D)} e^{ax} = x \frac{e^{ax}}{F'(a)}, \text{ provided } F'(a) \neq 0 \tag{13}$$

If $F'(a) = 0$ then the above rule can be repeated to give

$$\frac{1}{F(D)} e^{ax} = x^2 \frac{e^{ax}}{F''(a)}; \text{ provided } F''(a) \neq 0, \text{ and so on.}$$

Case II

When $R(x) = \sin(ax + b)$ or $\cos(ax + b)$, we know that

$$D \sin(ax+b) = a \cos(ax+b)$$

$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

$$D^3 \sin(ax+b) = -a^3 \cos(ax+b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

In general, $(D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$.

Hence,

$$F(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b)$$

Operating both sides $\frac{1}{F(D^2)}$, we get

$$\sin(ax+b) = f(-a^2) \cdot \frac{1}{F(D^2)} \cdot \sin(ax+b)$$

or
$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b); \text{ provided } f(-a^2) \neq 0. \tag{14}$$

Similarly,

$$\frac{1}{F(D^2)} \cos(ax+b) = \frac{1}{F(-a^2)} \cos(ax+b); \text{ provided } f(-a^2) \neq 0. \tag{15}$$

If $f(-a^2) = 0$, the above rule is fails.

In such a situation, we proceed as follows.

$$e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b) \qquad \text{(by Euler's formula)}$$

Thus,

$$\frac{1}{F(D^2)} e^{i(ax+b)} = \frac{1}{F(D^2)}[\cos(ax+b) + i\sin(ax+b)]$$

or
$$\frac{1}{F'(D^2)} e^{i(ax+b)} = \frac{1}{F(D^2)}[\cos(ax+b) + i\sin(ax+b)]$$

or
$$\frac{x}{F'(D^2)}[\cos(ax+b) + i\sin(ax+b)] = \frac{1}{F(D^2)}[\cos(ax+b) + i\sin(ax+b)]$$

Equating both sides real and imaginary parts, we have

$$\frac{1}{F(D^2)}\cos(ax+b) = x\cdot\frac{1}{[F'(D^2)]_{D^2=-a^2}}\cos(ax+b)$$

$$= \frac{x}{F'(-a^2)}\cos(ax+b)\text{ ; provided } F'(-a^2) \neq 0$$

and
$$\frac{1}{F(D^2)}\sin(ax+b) = \frac{x}{F'(-a^2)}\sin(ax+b)\text{ ; provided } F'(-a^2) \neq 0$$

If $F'(-a^2) = 0$ then the above rules fails and repeating the above process, we have

$$\frac{1}{F(D^2)}\cos(ax+b) = \frac{x^2}{F''(-a^2)}\cos(ax+b)\text{ ; provided } F''(-a^2) \neq 0$$

and
$$\frac{1}{F(D^2)}\sin(ax+b) = \frac{x^2}{F''(-a^2)}\sin(ax+b)\text{ ; provided } F''(-a^2) \neq 0$$

Case III

When $R(x) = x^n$; n being positive integer

$$\text{PI} = \frac{1}{F(D)}\cdot R(x) = \frac{1}{F(D)}\cdot x^n$$

Our aim to make the coefficient of the leading term of $F(D)$ unity, take the denominator in the numerator and then expand it by the Binomial theorem. Operate the resulting expansion on x^n.

Case IV

When $R(x) = e^{ax}\cdot P(x)$, where $P(x)$ is a function of x.

Let V is a function of x; then by successive differentiation of $e^{ax}\cdot V$, we have.

$$D[e^{ax}\cdot V] = e^{ax}(DV) + a\,e^{ax}\cdot V = e^{ax}(D+a)\,V$$

$$D^2[e^{ax}\cdot V] = e^{ax}(D+a)^2\cdot V$$

$$D^3[e^{ax}\cdot V] = e^{ax}(D+a)^3\,V$$

In general,

$$D^n[e^{ax} V] = e^{ax}(D + a)^n V$$

Hence,

$$F(D)\,[e^{ax} \cdot V] = e^{ax}\, F(D + a)\, V$$

Operating both sides by $\dfrac{1}{F(D)}$, we get

$$e^{ax}\, V = \frac{1}{F(D)}[e^{ax} \cdot F(D+a)]\,V$$

Putting $F(D + a)\, V = P(x)$, we have $V(x) = \dfrac{P(x)}{F(D+a)}$ and so we have

$$e^{ax} \frac{1}{F(D+a)} P = \frac{1}{F(D)}(e^{ax} \cdot P)$$

or

$$\frac{1}{F(D)}(e^{ax} \cdot P) = e^{ax} \frac{1}{F(D+a)} P$$

Case V

When $R(x) = x \cdot P(x)$ where P is a function of x.

$$\text{PI} = \frac{1}{F(D)}\, x\, P(x)$$

By successive differentiation of xV, we have

$$D(xV) = x\, DV + V$$

$$D^2(xV) = x\, D^2V + 2DV$$

In general,

$$D^n\,(xV) = x\, D^nV + n\, D^{n-1}\, V$$

$$= x\, D^nV + \left[\frac{d}{dD} D^n\right]V$$

Hence,

$$F(D)\,[xV] = x\, F(D)\, V + F'(D)\, V \tag{16}$$

Let $F(D)\, V = P$; then P will also be a function of x.

$$V = \frac{1}{F(D)} P$$

Putting, the value of V in Eq. (16), we get

$$F(D) \cdot x \frac{1}{F(D)} P = x \cdot F(D) \cdot \frac{1}{F(D)} P + F'(D) \cdot \frac{1}{F(D)} P \tag{17}$$

Operating (17) by $\dfrac{1}{F(D)}$ both sides, we get

$$\frac{1}{F(D)}\left[F(D) \cdot x \frac{1}{F(D)} P\right] = \frac{1}{F(D)}(xP) + \frac{1}{F(D)}\left[F'(D) \cdot \frac{1}{F(D)} P\right]$$

$$x \cdot \frac{1}{F(D)} P = \frac{1}{F(D)}(xP) + \frac{1}{F(D)} F'(D) \cdot \frac{1}{F(D)} P$$

Hence, $$\frac{1}{F(D)}(xP) = x\frac{1}{F(D)} P - \frac{F'(D)}{[F(D)]^2} P$$

Note: The above formula is not convenient to evaluate the PI of $x^n e^{ax}$, $n > 1$.

Example 7 Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{3x}$.

Solution The given DE can be written as

$$(D^2 - 5D + 6)\, y = e^{3x},\ D \equiv \frac{d}{dx} \tag{1}$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$
$$(m - 2)(m - 3) = 0$$
$$m = 2, 3$$

The CF $= C_1 e^{2x} + C_2 e^{3x}$; C_1, C_2 being arbitrary constants.

$$\text{PI} = \frac{1}{D^2 - 5D + 6} e^{3x}$$
$$= \frac{1}{(D-3)(D-2)} e^{3x}$$
$$= \frac{1}{(D-3)} e^{2x} \int e^{3x} \cdot e^{-2x}\, dx$$

$$\left[\text{By using general method } \frac{1}{D-\alpha} R = e^{\alpha x} \int R \cdot e^{-\alpha x}\, dx\right]$$

$$= \frac{1}{D-3} e^{3x}$$
$$= e^{3x} \int e^{3x}\, e^{-3x}\, dx$$
$$= e^{3x} \int 1\, dx = x\, e^{3x}$$

The complete solution of the equation (1), is

$$y = \text{CF} + \text{PI}$$
$$y = C_1 e^{2x} + C_2 e^{3x} + x\, e^{3x}$$

Example 8 Solve $\dfrac{d^2y}{dx^2} + 4y = e^{2x}$.

Solution The given equation can be rewritten as

$$(D^2 + 4)\, y = e^{2x};\ D \equiv \frac{d}{dx}$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$\text{CF} = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{PI} = \frac{1}{D^2 + 4} e^{2x}$$

$$= \frac{1}{2^2 + 4} e^{2x} \qquad \text{[using Case I]}$$

$$= \frac{1}{8} e^{2x}$$

The complete solution of given equation is

$$y = \text{CF} + \text{PI}$$

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} e^{2x}\text{; } C_1, C_2 \text{ are arbitrary constants.}$$

Example 9 Solve $\frac{d^3y}{dx^3} - y = (e^x + 1)^2$.

Solution The given equation can be rewritten as

$$(D^3 - 1)\, y = (e^x + 1)^2 \ \ldots (1);\ D \equiv \frac{d}{dx}$$

The auxiliary equation is

$$m^3 - 1 = 0$$

or $\quad (m - 1)(m^2 + m + 1) = 0$

or $\quad m = 1, \dfrac{-1 \pm \sqrt{3}\, i}{2}$

$$\text{CF} = C_1 e^x + e^{-\frac{x}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2} x\right) + C_3 \sin\left(\frac{\sqrt{3}}{2} x\right) \right]$$

Now,

$$\text{PI} = \frac{1}{D^3 - 1} (e^x + 1)^2$$

$$= \frac{1}{D^3 - 1} (e^{2x} + 2e^x + 1)$$

$$= \frac{1}{D^3 - 1} e^{2x} + \frac{2}{D^3 - 1} e^x + \frac{1}{D^3 - 1} \cdot 1$$

$$= \frac{1}{2^3 - 1} e^{2x} + \frac{2}{D^3 - 1} e^x + \frac{1}{D^3 - 1} e^{ox}$$

$$= \frac{1}{7} e^{2x} + \frac{2}{D^3 - 1} e^x + \frac{1}{O^3 - 1} e^{ox}$$

$$= \frac{1}{7} e^{2x} + \frac{2}{D^3 - 1} e^x - \frac{1}{D^3 - 1} e^{ox}$$

$$= \frac{1}{7} e^{2x} + 2 \cdot \frac{x}{3D^2} e^x - 1 \quad \left[\text{Using } \frac{1}{F(D)} e^{ax} = \frac{x}{F'(D)} e^{ax} \ \text{ if } f(a) = 0\right]$$

$$\text{PI} = \frac{1}{7} e^{2ax} + \frac{2x}{3} e^x - 1$$

The complete solution is

$$y = \text{CF} + \text{PI} = C_1 e^x + e^{-\frac{x}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2} x\right) + C_3 \sin\left(\frac{\sqrt{3}}{2} x\right)\right] + \frac{1}{7} e^{2x} + \frac{2}{3} x e^x - 1$$

where C_1, C_2 and C_3 are arbitrary constants.

Example 10 Solve $\dfrac{d^3 y}{dx^3} + \dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx} - y = \cos 2x$.

Solution The given equation can be rewritten as

$$(D^3 + D^2 - D - 1)y = \cos 2x \ \dots (1); \ D \equiv \frac{d}{dx}$$

The auxiliary equation is

$$m^3 + m^2 - m - 1 = 0$$

or $\quad m^2(m + 1) - (m + 1) = 0$

or $\quad (m + 1)(m^2 - 1) = 0$

or $\quad m = 1, -1, -1$

$\therefore \quad \text{CF} = C_1 e^x + (C_2 + C_3 x) e^{-x}$

Now,

$$\text{PI} = \frac{1}{D^3 + D^2 - D - 1} \cos 2x$$

$$= \frac{1}{D(-4) - 4 - D - 1} \cos 2x \qquad \text{[using Case II, replace } D^2 \text{ by } -2^2]$$

$$= \frac{1}{-5D - 5} \cos 2x$$

$$= \frac{-1}{5} \frac{1}{D + 1} \cos 2x$$

$$= \frac{-1}{5} \cdot \frac{D - 1}{D^2 - 1} \cos 2x$$

$$= -\frac{1}{5}(D-1)\cdot\frac{1}{-4-1}\cos 2x$$

$$= \frac{1}{25}(D-1)\cos 2x$$

$$= \frac{1}{25}[D\cos 2x - \cos 2x]$$

$$= \frac{1}{25}[-2\sin 2x - \cos 2x]$$

$$= -\frac{1}{25}[2\sin 2x + \cos 2x]$$

The complete solution is

$$y = \text{CF} + \text{PI} = C_1 e^{x} + (C_2 + x\, C_3)\, e^{-x} - \frac{1}{25}(2\sin 2x + \cos 2x)$$

where C_1, C_2, and C_3 are constants.

Example 11 Solve $(D^3 + a^2 D)\, y = \sin ax;\ D \equiv \frac{d}{dx}$

Solution The given equation is

$$(D^3 + a^2 D)y = \sin ax \ldots (1);\ D \equiv \frac{d}{dx}$$

The auxiliary equation is

$$m^3 + a^2 m = 0$$

or $$m(m^2 + a^2) = 0 \text{ or } m = 0, \pm ia$$

$$\text{CF} = C_1 + C_2 \cos ax + C_3 \sin ax$$

Now $$\text{PI} = \frac{1}{D^3 + a^2 D}\sin ax$$

$$= \frac{1}{D^3 + a^2 D}\sin ax$$

$$= \frac{1}{(D^2 + a^2)}\cdot\left(\frac{1}{D}\sin ax\right)$$

$$= \frac{1}{(D^2 + a^2)}\cdot\left(-\frac{\cos ax}{a}\right)$$

$$= -\frac{1}{a}\cdot\left[\frac{1}{D^2 + a^2}\cos ax\right]$$

$$= -\frac{1}{a}\frac{x}{2a}\sin ax \qquad \left[\because \frac{1}{D^2 + a^2}\cos ax = \frac{x}{2a}\sin ax, \text{ if } F(a) = 0\right]$$

$$= -\frac{x}{2a^2}\sin ax$$

The complete solution is

$$= C_1 + C_2 \cos ax + C_3 \sin ax - \frac{x}{2a^2}\sin ax$$

Example 12 Solve $(D^2 - 9)y = x^3;\ D \equiv \frac{d}{dx}$.

Solution The auxiliary equation of the given DE is

$$m^2 - 9 = 0$$

or $$m = \pm 3$$

$$\text{CF} = C_1 e^{3x} + C_2 e^{-3x}$$

Now, $$\text{PI} = \frac{1}{D^2 - 9} x^3$$

$$= \frac{1}{-9\left(1 - \frac{D^2}{9}\right)} x^3$$

$$= -\frac{1}{9}\left(1 - \frac{D^2}{9}\right)^{-1} \cdot x^3$$

$$= -\frac{1}{9}\left[1 + \frac{D^2}{9} + \frac{D^4}{81} + \cdots\right] \cdot x^3$$

$$= -\frac{1}{9}\left[x^3 + \frac{1}{9}(D^2 x^3) + \frac{1}{81}(D^4 x^3)\right]$$

$$= -\frac{1}{9}\left[x^3 + \frac{1}{9} \cdot 6x + 0\right]$$

$$= -\frac{x^3}{9} - \frac{2}{27}x$$

The complete solution is

$$y = C_1 e^{3x} + C_2 e^{-3x} - \frac{x^3}{9} - \frac{2}{27}x.$$

Example 13 Solve $(D^2 + 4D - 12)y = e^{2x} \cdot (x - 1);\ D \equiv \frac{d}{dx}$

Solution The auxiliary equation of the given DE is

$$m^2 + 4m - 12 = 0$$

$$(m - 2)(m + 6) = 0 \text{ or } m = 2, -6$$

$$\text{CF} = C_1 e^{2x} + C_2 e^{-6x}$$

Now,

$$\text{PI} = \frac{1}{(D^2+4D-12)} e^{2x}\cdot(x-1)$$

$$= \frac{1}{(D-2)(D+6)} e^{2x}\cdot(x-1)$$

$$= e^{2x}\cdot\frac{1}{(D+2-2)(D+2+6)}(x-1) \qquad \text{[Using Case IV]}$$

$$= e^{2x}\cdot\frac{1}{D(D+8)}(x-1)$$

$$= e^{2x}\cdot\frac{1}{8D\left(1+\frac{D}{8}\right)}(x-1)$$

$$= e^{2x}\cdot\frac{1}{8D}\left(1+\frac{D}{8}\right)^{-1}(x-1)$$

$$= e^{2x}\cdot\frac{1}{8D}\left(1-\frac{D}{8}\right)(x-1) \qquad \text{[Neglecting higher powers of } D\text{]}$$

$$= e^{2x}\cdot\frac{1}{8D}\left[(x-1)-\frac{1}{8}\right]$$

$$= e^{2x}\cdot\frac{1}{8D}\left[x-\frac{9}{8}\right]$$

$$= \frac{e^{2x}}{8}\left[\frac{x^2}{2}-\frac{9x}{8}\right] \qquad \left[\because\ \frac{1}{D}f(x)=\int f(x)\,dx\right]$$

The complete solution is

$$y = \text{CF}+\text{PI} = C_1e^{2x}+C_2e^{-6x}+\frac{e^{2x}}{8}\left[\frac{x^2}{2}-\frac{9x}{8}\right]$$

where C_1 and C_2 are arbitrary constants.

Example 14 Solve $(D^2-2D+2)\,y = x+e^x\,.\cos x;\ D\equiv\frac{d}{dx}$.

Solution The auxiliary equation of given equation is

$$m^2-2m+2=0$$

or

$$m = \frac{2\pm\sqrt{4-8}}{2} = 1\pm i$$

$\therefore$ $\text{CF} = e^x\,[C_1\cos x + C_2\sin x]$; C_1 and C_2 are arbitrary constants.

Now,

$$\text{PI} = \frac{1}{(D^2-2D+2)}\cdot(x+e^x\cos x)$$

$$= \frac{1}{(D^2 - 2D + 2)} x + \frac{1}{(D^2 - 2D + 2)} e^x \cdot \cos x$$

$$= \frac{1}{2}\left(1 + \frac{D^2 - 2D}{2}\right)^{-1} \cdot x + e^x \cdot \frac{1}{\{(D+1)^2 - 2(D+1) + 2\}} \cos x$$

$$= \frac{1}{2} \cdot \left[1 + D - \frac{D^2}{2} + \cdots\right] x + e^x \cdot \frac{1}{D^2 + 1} \cos x$$

$$= \frac{1}{2}(x+1) + e^x \frac{x}{2D} \cos x$$

$$= \frac{1}{2}(x+1) + \frac{xe^x}{2} \sin x$$

The complete solution is

$$y = e^x[C_1 \cos x + C_2 \sin x] + \frac{1}{2}(x+1) + \frac{xe^x}{2} \sin x$$

Example 15 Solve $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = x e^x \sin x$.

Solution The given DE can be rewritten in the form of

$$(D^2 - 2D + 1)y = x\, e^x \cdot \sin x; \; D \equiv \frac{d}{dx} \tag{1}$$

The auxiliary equation of (1) is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad (m-1)^2 = 0$$

or $m = 1, 1$

CF $= (C_1 + x\, C_2)\, e^x$; C_1 and C_2 are arbitrary constants.

Now,

$$\text{PI} = \frac{1}{(D-1)^2} x\, e^x \sin x$$

$$= \frac{1}{(D-1)^2} e^x \cdot (x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2} (x \sin x)$$

$$= e^x \cdot \frac{1}{D^2} x \sin x$$

$$= e^x \cdot \frac{1}{D} \int x \sin x \, dx$$

$$= e^x \cdot \frac{1}{D} [-x \cos x + \sin x]$$

$$= e^x \left[-\int x \cos x \, dx + \int \sin x \, dx \right]$$
$$= e_x \, [-x \sin x - \cos x - \cos x]$$
$$= -e^x \, (x \sin x + 2 \cos x)$$

Hence, the complete solution is

$$y = (C_1 + x\,C_2)\, e^x - e^x \,(x \sin x + 2 \cos x)$$

Example 16 Solve $(D^2 + 2D + 1)\, y = x \cdot \sin x;\ D \equiv \dfrac{d}{dx}$.

Solution The auxiliary equation of the given DE is

$$m^2 + 2m + 1 = 0 \text{ or } (m+1)^2 = 0$$

or $$m = -1, -1$$

$$\text{CF} = (C_1 + x\,C_2)\, e^{-x};\ C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

Now,

$$\text{PI} = \frac{1}{(D+1)^2}\, x \sin x$$

$$= x \cdot \frac{1}{(D+1)^2} \cdot \sin x - \frac{2\,(D+1)}{[(D+1)^2]^2} \cdot \sin x \qquad \text{[Using Case V]}$$

$$= x \cdot \frac{1}{D^2 + 2D + 1} \sin x - \frac{2\,(D+1)}{[D^2 + 2D + 1]^2} \sin x$$

$$= x \cdot \frac{1}{2D} \sin x - \frac{2\,(D+1)}{4D^2} \sin x$$

$$= \frac{x}{2} \int \sin x \, dx - \frac{1}{2} \frac{(D+1)}{-1^2} \sin x$$

$$= -\frac{x}{2} \cos x + \frac{1}{2} [\cos x + \sin x]$$

Hence, the complete solution is

$$y = (C_1 + x C_2) e^{-x} - \frac{x}{2} \cos x + \frac{1}{2} [\cos x + \sin x]$$

Example 17 Solve $(D^2 + 1) y = x^2 \cdot \sin 2x;\ D \equiv \dfrac{d}{dx}$.

Solution The auxiliary equation of the given DE is

$$m^2 + 1 = 0 \text{ or } m = \pm\, i$$

$$\text{CF} = C_1 \cos x + C_2 \sin x;\ C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

Now, $$\text{PI} = \frac{1}{D^2 + 1}\, x^2 \sin 2x$$

$$= \text{IP of } \frac{1}{D^2 + 1}\, x^2\, e^{2ix} \qquad \text{[IP = Imaginary Part]}$$

$$= \text{IP of } e^{2ix} \cdot \frac{1}{(D+2i)^2 + 1} \cdot x^2$$

$$= \text{IP of } e^{2ix} \cdot \frac{1}{D^2 + 4iD - 3} \cdot x^2$$

$$= \text{IP of } e^{2ix} \cdot \frac{1}{-3}\left[1 - \left(\frac{4iD}{3} + \frac{D^2}{3}\right)\right]^{-1} \cdot x^2$$

$$= \text{IP of } e^{2ix} \cdot \frac{1}{-3}\left[1 + \frac{4iD}{3} + \frac{D^2}{3} - \frac{16}{9}D^2 + \cdots\right] \cdot x^2$$

$$= \text{IP of } \left(\frac{e^{2ix}}{-3}\right)\left[x^2 + \frac{8ix}{3} - \frac{13}{9} \cdot 2\right]$$

$$= \text{IP of } \left(\frac{\cos 2x + i \sin 2x}{-3}\right) \cdot \left[x^2 + \frac{8ix}{3} - \frac{26}{9}\right]$$

$$= -\frac{1}{3}\left[\frac{8x}{3}\cos 2x + \frac{1}{9}(9x^2 - 26)\sin 2x\right]$$

The complete solution is

$$y = \text{CF} + \text{PI} = C_1 \cos x + C_2 \sin x - \frac{1}{3}\left[\frac{8x}{3}\cos 2x + \frac{1}{9}(9x^2 - 26)\sin 2x\right]$$

EXERCISE 11.2

Solve the following differential equations:

1. $(D^2 - 4D + 4)y = e^{2x} + \cos 2x;\ D \equiv \dfrac{d}{dx}$.

2. $(D^6 + 1)y = \sin \dfrac{3x}{2} \cdot \sin \dfrac{x}{2};\ D \equiv \dfrac{d}{dx}$.

3. $(D^4 - 2D^3 + D^2)y = x^3;\ D \equiv \dfrac{d}{dx}$.

4. $(D^2 - 4D - 5)y = e^{2x} + 3\cos(4x + 3);\ D \equiv \dfrac{d}{dx}$.

5. $(D^2 - 1)\, y = x\, e^x \cdot \sin x;\ D \equiv \dfrac{d}{dx}$.

6. $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x;\ D \equiv \dfrac{d}{dx}$.

7. $(D^2 - 2D + 2)y = x + e^x \cos x;\ D \equiv \dfrac{d}{dx}$.

8. $(D^3 - 6D + 13)y = 8e^{3x} \sin 4x + 2^x;\ D \equiv \dfrac{d}{dx}$.

9. $(D^2 + 4)y = \sin x,\ y(0) = 1,\ y'(0) = 1;\ D \equiv \dfrac{d}{dx}$.

10. $(D^2 - 1)y = \cos h\, x \cdot \cos x;\ D \equiv \dfrac{d}{dx}$.

11. $(D^2 - 6D + 13)y = 6\, e^{3x} \cdot \sin x \cdot \cos x;\ D \equiv \dfrac{d}{dx}$.

12. $(D^3 + 1)y = e^x + 5\, e^{2x} + 3;\ D \equiv \dfrac{d}{dx}$.

13. A body executes damped forced vibrations given by the equation.

$$\frac{d^2 x}{dt^2} + 2K\frac{dx}{dt} + b^2 x = e^{-Kt} \cdot \sin \omega t.$$

Solve the equation for both the cases, when $\omega^2 \neq b^2 - K^2$ and $\omega^2 = b^2 - K^2$.

Answers

1. $y = (C_1 + xC_2)e^{2x} + \dfrac{x^2 e^{2x}}{2} - \dfrac{1}{8}\sin 2x$

2. $y = (C_1 \cos x + C_2 \sin x) + e^{\left(\frac{\sqrt{3}}{2}x\right)}\left(C_3 \cos\dfrac{x}{2} + C_4 \sin\dfrac{x}{2}\right)$
$$+ e^{\frac{-\sqrt{3}}{2}x}\left(C_5 \cos\frac{x}{2} + C_6 \sin\frac{x}{2}\right) + \frac{1}{126}\cos 2x + \frac{x}{12}\sin x$$

3. $y = C_1 + xC_2 + (C_3 + xC_4)e^x + \dfrac{1}{20}x^5 + \dfrac{1}{2}x^4 + 3x^3 + 12\, x^2$

4. $y = C_1 e^{5x} + C_2 e^{-x} - \dfrac{1}{9}e^{2x} - \dfrac{3}{697}[16 \sin(4x + 3) + 21 \cos(4x + 3)]$

5. $y = C_1 e^x + C_2 e^{-x} - \dfrac{e^x}{25}[2(1 + 5x)\cos x + (5x - 14)\sin x]$

6. $y = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x) + xe^x + \dfrac{1}{10}(3 \sin x + \cos x)$

7. $y = e^x(C_1 \cos x + C_2 \sin x) + \dfrac{1}{2}(x + 1) + \dfrac{xe^x}{2}\sin x$

8. $y = e^{3x}(C_1 \cos 3x + C_2 \sin 3x) - \dfrac{2e^{3x}}{3}\sin 4x + \dfrac{2^x}{[(\log 2)^2 - 6\log 2 + 13]}$

9. $y = \cos 2x + \dfrac{1}{3}(\sin x + \sin 2x)$

10. $y = C_1 e^x + C_2 e^{-x} + \dfrac{2}{5}\sin x \cdot \sin h\, x - \dfrac{1}{5}\cos x \cdot \cos h\, x$

11. $y = e^{3x}\,[C_1 \cos 2x + C_2 \sin 2x] - \dfrac{3x\,e^{3x}}{4} \cos 2x$

12. $y = C_1 e^{-x} + e^{\frac{x}{2}} \left[C_2 \cos\left(\dfrac{\sqrt{3}}{2}\,x\right) + C_3 \sin\left(\dfrac{\sqrt{3}}{2}\,x\right)\right] + \dfrac{5}{9} e^{2x} + \dfrac{e^x}{2} + 3$

13. When $\omega^2 \neq b^2 - K^2$

$$y = e^{-Kt}\left[C_1 \cos \sqrt{(b^2 - K^2)}\; t + C_2 \sin \sqrt{(b^2 - K^2)}\; t\right] + \frac{e^{-Kt} \sin \omega t}{b^2 - K^2 - \omega^2}$$

when $\omega^2 = b^2 - K^2$

$$y = e^{-Kt}\,[C_1 \cos \omega t + C_2 \sin \omega t] - \frac{t}{2\omega}\, e^{-Kt} \cos \omega t$$

11.7 SOLUTIONS OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In several applied-mathematics problems, there are more than one dependent variable, each of which is a function of one independent variable, usually say time 't'. The formulation of such problems leads to a system of simultaneous linear differential equations. Such a system can be solved by the method of elimination, Laplace transform method. Here, only the method of elimination is considered.

11.7.1 Method of Elimination

Consider a system of two ordinary differential equations in two dependent variables x and y and one independent variable 't' given by

$$f_1(D)\,x + f_2(D)y = f(t) \tag{18}$$

$$g_1(D)\,x + g_2(D)\,y = g(t) \tag{19}$$

where $f_1(D), f_2(D), g_1(D), g_2(D)$, are all functions and $D \equiv \dfrac{d}{dt}$

Step 1 Elimination of y from the given system, results in a differential equation exclusively in x alone.

Step 2 Solve this differential equation obtained in Step 1 for 'x'.

Step 3 Substituting x (obtained in Step 2) in a similar manner, obtain a differential equation only in y.

Step 4 Solve the differential equation obtained in Step 3 for y.

Example 18 Solve $\dfrac{dx}{dt} + 4x + 3y = t,\ \dfrac{dy}{dx} + 2x + 5y = e^t$.

Solution Rearranging the given differential equations, we get

$$(D + 4)\,x + 3y = t \tag{1}$$

$$2x + (D + 5)\,y = e^t \tag{2}$$

where $\quad D \equiv \dfrac{d}{dt}$

Multiply (1) by $(D + 5)$, and multiply (2) by 3. Then subtracting, we get

$$[(D + 4)(D + 5) - 2 \times 3]x = (D + 5)t - 3e^t$$

$$[D^2 + 9D + 20 - 6]\, x = 1 + 5t - 3e^t$$

$$[D^2 + 9D + 14]\, x = 1 + 5t - 3e^t \qquad (3)$$

Equation (3) is a linear differential equation in x and t with constant coefficients.
Hence, the auxiliary equation is

$$m^2 + 9m + 14 = 0$$

$$m = -2, -7$$

$$\text{CF} = C_1 e^{-2t} + C_2 e^{-7t}$$

Its

$$\text{PI} = \frac{1}{(D^2 + 9D + 14)}(1 + 5t - 3e^t)$$

$$= \frac{1}{(D^2 + 9D + 14)} 1 + \frac{5}{14\left(1 + \frac{D^2 + 9D}{14}\right)} t - \frac{3}{(D^2 + 9D + 14)} e^t$$

$$= \frac{1}{(D^2 + 9D + 14)} e^{0t} + \frac{5}{14}\left[1 + \frac{D^2 + 9D}{14}\right]^{-1} t - \frac{3}{(D^2 + 9D + 14)} e^t$$

$$= \frac{1}{(0 + 0 + 14)} e^{0t} + \frac{5}{14}\left[1 - \frac{D^2 + 9D}{14} + \right] t - \frac{3}{1^2 + 9.1 + 14} e^t$$

$$= \frac{1}{14} + \frac{5}{14} \cdot \left(t - \frac{9}{14}\right) - \frac{3}{24} e^t$$

$$\text{PI} = \frac{1}{14} + \frac{5}{14} \cdot \left(t - \frac{9}{14}\right) - \frac{1}{8} e^t$$

Hence, the complete solution of (3) is

$$x(t) = \text{CF} + \text{PI}$$

$$x(t) = C_1 e^{-2t} + C_2 e^{-7t} + \frac{1}{14} + \frac{5}{14}\left(t - \frac{9}{14}\right) - \frac{e^t}{8} \qquad (4)$$

$$\therefore \qquad \frac{dx}{dt} = -2C_1 e^{-2t} - 7C_2 e^{-7t} + \frac{5}{14} - \frac{e^t}{8}$$

Putting the values of $x(t)$ and $\frac{dx}{dt}$ in (1), we get

$$-2C_1 e^{-2t} - 7C_2 e^{-7t} + \frac{5}{14} - \frac{1}{8} e^t + 4C_1 e^{-2t}$$

$$+ 4C_2 e^{-7t} + \frac{4}{14} + \frac{20}{14}\left(t - \frac{9}{14}\right) - \frac{4e^t}{8} + 3y = t$$

$$-2C_1 e^{-2t} - 3C_2 e^{-7t} - \frac{5}{8}e^t + \frac{10}{7}\left(t - \frac{9}{14}\right) + \frac{9}{14} + 3y = t$$

or $$y(t) = \frac{1}{3}\left(3C_2 e^{-7t} - 2C_1 e^{-2t} + \frac{5}{8}e^t - \frac{3}{7}t + \frac{27}{98}\right) \quad (5)$$

Equations (4) and (5) are the solutions of the given simultaneous equations.

Example 19 Solve $\frac{dx}{dt} - 7x + y = 0$ and $\frac{dy}{dt} - 2x - 5y = 0$

Solution The given differential equations are

$$\frac{dx}{dt} - 7x + y = 0 \quad (1)$$

$$\frac{dy}{dt} - 2x - 5y = 0 \quad (2)$$

Differentiating (1) and (2) w.r.t. 't' we get

$$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + \frac{dy}{dt} = 0 \quad (3)$$

$$\frac{d^2y}{dt^2} - 2\frac{dx}{dt} - 5\frac{dy}{dt} = 0 \quad (4)$$

Eliminating y and $\frac{dy}{dt}$ from (1), (2), and (3), we get

$$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + (2x + 5y) = 0 \quad (5) \quad \text{[from (2) and (3)]}$$

or $$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + 2x + 5\left(-\frac{dx}{dt} + 7x\right) = 0 \quad \text{[from (1) and (5)]}$$

$$\frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 37x = 0$$

$$(D^2 - 12D + 37)x = 0 \quad (6)$$

where $$D = \frac{d}{dt}$$

Equation (6) is a linear differential equation of second-order constant coefficients. We get

A.E. is $m^2 - 12m + 37 = 0$

$$m = 6 \pm i$$

Therefore, the solution of (6) is

$$x(t) = e^{6t}(C_1 \cos t + C_2 \sin t) \quad (7)$$

Differentiating (7)) w.r.t. 't', we get

$$\frac{dx}{dt} = 6e^{6t}(C_1 \cos t + C_2 \sin t) + e^{6t}(-C_1 \sin t + C_2 \cos t) \quad (8)$$

Putting the values of x and $\frac{dx}{dt}$ in (1), we get

$$y(t) = e^{6t}[(C_1 - C_2)\cos t + (C_2 + C_1)\sin t] \tag{9}$$

Equations (7) and (9) constitute the solution of the differential equation.

Example 20 Solve $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t$ and $\frac{dx}{dt} - x + y = \cos t$.

Solution Rearranging the given differential equation, we get

$$(D+3)x + Dy = \sin t \tag{1}$$

$$(D-1)x + y = \cos t \tag{2}$$

Multiplying (2) by D and subtracting from (1), we get

$$[(D+3) - D(D-1)]x = \sin t - D(\cos t)$$

$$[D + 3 - D^2 + D]x = \sin t + \sin t$$

or

$$(D^2 - 2D - 3)x = -2\sin t \tag{3}$$

Equation (3) is a second-order linear differential equation with constant coefficients. Its auxiliary equation is $m^2 - 2m - 3 = 0$

$$m = 3, -1$$

$\therefore$

$$\text{CF} = C_1 e^{3t} + C_2 e^{-t}$$

$$\text{PI} = \frac{1}{(D^2 - 2D - 3)}(-2\sin t)$$

$$= -\frac{2}{(D^2 - 2D - 3)}\sin t$$

$$\text{PI} = \frac{-2}{(-1) - 2D - 3}\sin t = \frac{-2}{-2D - 4}\sin t$$

$$= \frac{1}{D+2}\sin t$$

$$= \frac{D-2}{(D+2)(D-2)}\sin t = -\frac{1}{5}(D\sin t - 2\sin t)$$

$$= \frac{(D-2)}{-1-4}\sin t = -\frac{1}{5}(D\sin t - 2\sin t)$$

$$= -\frac{1}{5}(\cos t - 2\sin t)$$

$$= \frac{1}{5}(2\sin t - \cos t)$$

Complete solution is $x(t) = C_1 e^{3t} + C_2 e^{-t} + \frac{1}{5}(2\sin t - \cos t)$ (4)

Differentiating (4) w.r.t. 't', we get

$$\frac{dx}{dt} = 3\,C_1\,e^{3t} - C_2\,e^{-t} + \frac{1}{5}\,(2\cos t + \sin t)$$

Substituting the values of $x\,(t)$ and $\frac{dx}{dt}$ in (2), we get

$$3C_1\,e^{3t} - C_2\,e^{-t} + \frac{1}{5}(2\cos t + \sin t) - C_1\,e^{3t} - C_2\,e^{-t} - \frac{1}{5}(2\sin t - \cos t) = \cos t + y$$

$$2C_1\,e^{3t} - 2C_2\,e^{-t} + \frac{3}{5}\cos t - \frac{1}{5}\sin t + y = \cos t$$

$$y = \frac{1}{5}(\sin t + 2\cos t) - 2C_1 e^{3t} + 2C_2 e^{-t} \tag{5}$$

Equations (4) and (5) constitute the solutions of the given equations.

Example 21 Solve $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$$

Solution Rearranging the given equations, we get

$$Dx + (D-2)\,y = 2\cos t - 7\sin t \tag{1}$$

$$(D+2)\,x - Dy = 4\cos t - 3\sin t \tag{2}$$

Multiplying (1) by D and (2) by $(D-2)$ and adding, we get

$$[D^2 + (D+2)\,(D-2)]\,x = D\,[2\cos t - 7\sin t] + (D-2)\,[4\cos t - 3\sin t]$$

$$[D^2 + D^2 - 4]\,x = -2\sin t - 7\cos t - 4\sin t - 3\cos t - 8\cos t + 6\sin t$$

$$(2\,D^2 - 4)\,x = -18\cos t$$

or
$$(D^2 - 2)\,x = -9\cos t \tag{3}$$

Its AE is $m^2 - 2 = 0$

$$m = \pm\sqrt{2}$$

$\therefore$
$$\text{CF} = C_1\,e^{\sqrt{2}t} + C_2\,e^{-\sqrt{2}t}$$

$$\text{PI} = \frac{1}{D^2-2}(-9\cos t) = -9\frac{1}{D^2-2}\cos t = -9\frac{1}{-1-2}\cos t$$

$$= \frac{-9}{-3}\cos t = 3\cos t$$

$\therefore$
$$x(1) = \text{CF} + \text{PI} = 9\;e^{\sqrt{2}t} + C_2\;e^{-\sqrt{2}t} + 3\cos t \tag{4}$$

Putting the value of $x\,(t)$ in (2), we get

$$\frac{dy}{dt} = \sqrt{2}\,C_1\,e^{\sqrt{2}t} - \sqrt{2}\,C_2\,e^{-\sqrt{2}t} - 3\sin t + 2C_1\,e^{\sqrt{2}t} + 2C_2\,e^{-\sqrt{2}t} + 6\cos t - 4\cos t + 3\sin t$$

$$\frac{dy}{dt} = (2+\sqrt{2})C_1 e^{\sqrt{2}t} + (2-\sqrt{2})\,C_2\,e^{-\sqrt{2}t} + 2\cos t$$

On integrating, we get

$$y(t) = (\sqrt{2}+1)\,C_1\,e^{\sqrt{2}t} - (\sqrt{2}-1)C_2\,e^{-\sqrt{2}t} + 2\sin t + C_3 \tag{5}$$

Equations (4) and (5) constitute the solution of the given equation.

Example 22 Solve $t\dfrac{dx}{dt} + y = 0$, $t\dfrac{dy}{dt} + x = 0$.

Solution The given equations are

$$t\frac{dx}{dt} + y = 0 \tag{1}$$

$$t\frac{dy}{dt} + x = 0 \tag{2}$$

Differentiating (1) both sides w.r.t. 't', we get.

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$$

Multiplying both sides by 't', we get

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$$

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0 \qquad \text{(3) [From (2)]}$$

Equation (3) is a Cauchy's linear equation.

Now, our aim is to reduce Eq. (3) into a linear differential equation. With constant coefficient by putting $t = e^z$, i.e., $z = \log t$ and $t\dfrac{dx}{dt} = D_1$, $t^2\dfrac{d^2x}{dt^2} = D_1(D_1 - 1)$

where $D_1 \equiv \dfrac{d}{dz}$, we get

$$[D_1(D_1 - 1) + D_1 - 1]\,x = 0$$

$$(D_1^2 - 1)\,x = 0 \tag{4}$$

which is a linear equation.

Its AE is $m^2 - 1 = 0$, $m = \pm 1$

$$x(t) = \text{CF} = c_1 e^z + c_2\,e^{-z} = C_1\,t + C_2/t \tag{5}$$

Differentiating (5) both sides w.r.t. 't', we get

$$\frac{dx}{dt} = C_1 - \frac{C_2}{t^2}$$

Equation (1) becomes

$$t\left(C_1 - \frac{C_2}{t^2}\right) + y = 0$$

$$C_1 t - \frac{C_2}{t} + y = 0 \text{ or } y(t) = \frac{C_2}{t} - C_1 t \tag{6}$$

Equations (5) and (4) are the solution of given equations.

Example 23 Solve $\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x$.

Solution $\frac{dx}{dt} = 2y \Rightarrow Dx = 2y$ (1)

$$\frac{dy}{dt} = 2z \Rightarrow Dy = 2z \tag{2}$$

$$\frac{dz}{dt} = 2x \Rightarrow Dz = 2x \tag{3}$$

where $D = \frac{d}{dt}$

Equation (1), we have $\frac{dx}{dt} = 2y$

Differentiating both sides w.r.t. 't', we get

$$\frac{d^2x}{dt^2} = 2\frac{dy}{dx} = 2(2z) \qquad \text{[using (2)]}$$

$$\frac{d^2x}{dt^2} = 4z$$

Again differentiating both sides w.r.t. 't', we get

$$\frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 4(2x) \qquad \text{[using (3)]}$$

$$\frac{d^3x}{dt^3} = 8x \Rightarrow \frac{d^3x}{dt^3} - 8x = 0$$

$$\Rightarrow \quad (D^3 - 8)\,x = 0 \tag{4}$$

which is linear in x and t.

AE is $m^3 - 8 = 0$

$(m-2)\,(m^2 + 2m - 4) = 0$

$m = 2, -1 \pm i\sqrt{3}$

$$\therefore \quad x(t) = \text{CF} = C_1 e^{2t} + e^{-t}\,(C_2 \cos\sqrt{3}\,t + C_3 \sin\sqrt{3}\,t)$$

$$x(t) = C_1 e^{2t} + e^{-t}\, C_2 \cos(\sqrt{3}\,t - C_3)$$

From (3), we have $\frac{dz}{dt} = 2x$

$$\frac{dz}{dt} = 2C_1 e^{2t} + 2C_2 e^{-t} \cos(\sqrt{3}\,t - C_3)$$

on integrating, we get

$$z(t) = C_1 e^{2t} + C_2 e^{-t} \cos\left(\sqrt{3}\,t - C_3 + \frac{4\pi}{3}\right)$$

From (2), we have $\dfrac{dy}{dt} = 2z$

$$\frac{dy}{dt} = 2C_1 e^{2t} + C_2 e^{-t} \cos\left(\sqrt{3}\,t - C_3 + \frac{4\pi}{3}\right)$$

On integrating, we get

$$y(t) = C_1 e^{2t} + C_2 e^{-t} \cos\left(\sqrt{3}\,t - C_3 + \frac{4\pi}{3} - 2\frac{\pi}{3}\right)$$

$$= C_1 e^{2t} + C_2 e^{-t} \cos\left(\sqrt{3}\,t - C_3 + \frac{2\pi}{3}\right)$$

EXERCISE 11.3

Solve the following simultaneous equations:

1. $\dfrac{dx}{dt} + 2x + 3y = 0, \dfrac{dy}{dt} + 2y = 2e^{2t}$
2. $\dfrac{dx}{dt} - 3x - 2y = 0, \dfrac{dy}{dx} + 5x + 3y = 0$
3. $\dfrac{dx}{dt} + y = \sin t, \dfrac{dy}{dt} + x = \cos t$; given that $x = 2$, $y = 0$, when $t = 0$
4. $\dfrac{dy}{dx} = y, \dfrac{dz}{dx} = 2y + z$
5. $\dfrac{dx}{dt} = 2y - 1, \dfrac{dy}{dt} = 2x + 1$
6. $(D + 6)\,y - Dx = 0$, $(3 - D)\,x - 2Dy = 0$, where $D = \dfrac{d}{dt}$
   With $x = 2$, $y = 3$ at $t = 0$
7. $\dfrac{dx}{dt} - y = t, \dfrac{dy}{dt} + x = t^2$
8. $t^2 \dfrac{d^2x}{dt^2} + t\dfrac{dx}{dt} + 2y = 0, t^2 \dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} - 2x = 0$
9. $\dfrac{dx}{dt} = 3x + 8y, \dfrac{dy}{dt} = -x - 3y$, with $x(0) = 6$, $y(0) = -2$
10. $(D + 2)\,x + (D - 1)\,y = -\sin t$, $(D - 3)\,x + (D + 2)\,y = 4\cos t$, $D = \dfrac{d}{dt}$

Answers

1. $x(t) = C_1 e^t + C_2 e^{-5t} - \frac{6}{7}e^{2t};\ y(t) = -C_1 e^t + C_2 e^{-5t} + \frac{8}{7}e^{2t}$

2. $x(t) = C_1 \cos t + C_2 \sin t,\ y(t) = \frac{1}{2}\left[(C_2 - 3C_1)\cos t - (C_1 + 3C_2)\sin t\right]$

3. $x(t) = e^t + e^{-t},\ y(t) = \sin t - e^t + e^{-t}$

4. $y = \frac{1}{2}C_1 e^x,\ z = (C_1 + x\, C_2)e^x$

5. $x(t) = C_1 e^{2t} - C_2 e^{-2t} - \frac{1}{2},\ y(t) = C_1 e^{2t} - C_2 e^{-2t} + \frac{1}{2}$

6. $x(t) = 4e^{2t} - 2e^{-3t}$
   $y(t) = e^{2t} + 2e^{-3t}$

7. $x(t) = C_1 \cos t + C_2 \sin t + t^2 - 1$
   $y(t) = -C_1 \sin t + C_2 \cos t + t$

8. $x(t) = t\left[C_1 \cos(\log t) + C_2 \sin(\log t)\right] + t^{-1}\left[C_3 \cos(\log t) + C_4 \sin(\log t)\right]$
   $y(t) = t\left[C_1 \sin(\log t) - C_2 \cos(\log t)\right] + t^{-1}\left[C_4 \cos(\log t) + C_3 \sin(\log t)\right]$

9. $x(t) = 4e^t + 2e^{-t}$
   $y = -e^t + e^{-t}$

10. $x(t) = \frac{3}{5}C_1 e^{-\frac{t}{8}} + \frac{2}{5}\sin t - \frac{1}{5}\cos t,\ y(t) = C_1 e^{\frac{-t}{8}} + \sin t + \cos t$

SUMMARY

1. Solution of Higher Order Homogeneous Linear Differential Equations with Constant Coefficients

Consider the n^{th} order homogeneous linear equation with constant coefficients.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = R(x) \tag{1}$$

Case I When the auxiliary equation has all real and distinct roots say $m_1, m_2, m_3, \ldots, m_n$. Then the n linearly independent solutions $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, y_3 = e^{m_3 x}, \cdots y_n = e^{m_n x}$ of Eq. (1).

Hence, $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots C_n e^{m_n x}$ is a general solution of (1), where $C_1, C_2, C_3, \ldots, C_n$ are all constants.

Case II When the auxiliary equation may have some multiple roots. Let r be the multiplicity of the root m_1, i.e., $m = m_1$ is repeated r times, and the remaining $(n - r)$ roots are real and distinct. Putting $m = m_1$, we obtain $y_1(x) = e^{m_1 x}$ as one of the solutions.

Now, the remaining $(r - 1)$ linearly independent solutions corresponding to the multiple roots $m = m$, are given by $y_1, xy_1, x^2 y, x^3 y_1, \ldots, x^{r-1} y_1$.

Hence, $y(x) = (C_1 + xC_2 + x^2C_3 + \dots + x^{r-1}C_r)\ e^{m_1 x} + \cdots + C_n e^{m_n x}$ is a general solution of (1), where C_1, C_2, … C_r, C_n are all constants.

Case III When the auxiliary equation may have some imaginary roots.

If $p + iq$ is a root then $p - iq$ is also a root of the auxiliary equation. In this case, the linearly independent solutions are given by $e^{px} \cos qx$ and $e^{px} \sin qx$. The auxiliary equation (2) has r complex conjugate pairs of roots $p_k \pm iq_k$, $k = 1, 2, 3, \dots, r$, the corresponding linearly independent solutions are $e^{p_1 x} \cos q_1 x, e^{p_1 x} \sin q_1 x, e^{p_2 x} \cos q_2 x, e^{p_2 x} \sin q_2 x, \cdots, e^{p_r x} \cos q_r x$.

Therefore, the general solution corresponding to linearly independent solutions is

$$y(x) = e^{p_1 x}\,(C_1 \cos q_1 x + C_2 \sin q_1 x) + e^{p_2 x}(C_3 \cos q_2 x + C_4 \sin q_2\, x) + \cdots + e^{p_r x}(C_{r_1} \cos q_{r_2} x + C_{r_2} \sin q_r x)$$

Note: Multiple complex roots:

If $p + iq$ is a multiple root of order m then $p - iq$ is also a multiple root of order m. Then the corresponding linearly independent solutions are

$$e^{p_1 x} \cos q_1 x, e^{p_1 x} \sin q_1 x, + xe^{p_1 x} \cos q_1 x, xe^{p_1 x} \sin q_1 x, \cdots\ x^{m-1} e^{p_2 x} \cos q_1 x, x^{m-1x} \sin q_1 x$$

Therefore, the general solution corresponding to linearly independent solutions is

$$y(x) = e^{p_1 x}[(C_1 + xC_2 + x^2 C_3 + \cdots x^{m-1} C_m)\cos q_1 x + (C_1' + xC_2' + x^2 C_3' + x^{m-1)} C_m')\sin q_1 x] + \cdots + C_n e^{m_n x}$$

where $C_1, C_2, \dots, C_m, C_1', C'_2, \dots C'_m, C_n$ are arbitrary constants.

2. General Methods of Finding Particular Integrals (PI)

The operator $\dfrac{1}{D-\alpha}$, where a being a constant. If R is any function of x then

$$\frac{1}{(D-\alpha)} R \ = e^{\alpha x} \int e^{-\alpha x} \cdot R dx$$

3. Short Methods of Finding the Particular Integral when '*R*' is of a Certain Special Forms

Consider the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2}}{dx^{n-2}} + \cdots + a_n y \ = R(x) \tag{4}$$

or $\quad [D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]y = R(x);\ D \equiv \dfrac{d}{dx}$

or $\quad F(D)y = R(x)$, where $F(D) \equiv (a_0 D^n + a_1 D^{n-1} + \cdots + a_n)$.

Therefore, the particular integral (PI) is

$$\text{PI} = \frac{1}{[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]} R(x) = \frac{1}{F(D)} \cdot R(x)\,.$$

Case I When $R(x) = e^{ax}$

$$\text{PI} = \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}, \text{ provided } F(a) \neq 0.$$

If $F(a) = 0$ then the above rule fails, then. We consider $F(D) = (D - a)\phi\,(D)$; where $(D - a)$ is a factor of $F(D)$.

Then $\quad \dfrac{1}{F(D)} e^{ax} = \dfrac{1}{(D-a)\,\phi(D)} e^{ax}$

$$\frac{1}{F(D)} e^{ax} = x \frac{e^{ax}}{F'(a)}, \text{ provided } F'(a) \neq 0.$$

If $F'(a) = 0$ then the above rule can be repeated to give

$$\frac{1}{F(D)} e^{ax} = x^2 \frac{e^{ax}}{F''(a)}; \text{ provided } F''(a) \neq 0, \text{ and so on.}$$

Case II When $R(x) = \sin(ax + b)$ or $\cos(ax + b)$, we know that

$$\text{PI} = \frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b)\ ; \text{ provided } F(-a^2) \neq 0.$$

Similarly,

$$\frac{1}{F(D^2)} \cos(ax+b) = \frac{1}{F(-a^2)} \cos(ax+b)\ ; \text{ provided } F(-a^2) \neq 0.$$

Case III When $R(x) = x^n$; n being a positive integer.

$$\text{PI} = \frac{1}{F(D)} \cdot R(x) = \frac{1}{F(D)} \cdot x^n$$

Our aim to make the coefficient of the leading term of $F(D)$ unity, take the denominator in numerator and then expand it by the binomial theorem. Operate the resulting expansion on x^n.

Case IV When $R(x) = e^{ax} P(x)$, where $P(x)$ is a function of x.

$$\text{PI} = \frac{1}{F(D)} (e^{ax} \cdot P) = e^{ax} \frac{1}{F(D+a)} P$$

Case V When $R(x) = x\,P(x)$ where P is a function of x.

$$\text{PI} = \frac{1}{F(D)} x\,P(x)$$

$$= x \frac{1}{F(D)} P - \frac{F'(D)}{[F(D)]^2} P.$$

OBJECTIVE-TYPE QUESTIONS

1. The solution of differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$ is

(a) $Ae^x + Be^{-x}$

(b) $e^x (Ax + B)$

(c) $e^{-x}\left\{A\cos\left(\dfrac{\sqrt{3}}{2}x\right) + B\cos\left(\dfrac{\sqrt{3}}{3}x\right)\right\}$

(d) $e^{-x/2}\left\{A\cos\left(\dfrac{\sqrt{3}}{2}x\right) + B\sin\left(\dfrac{\sqrt{3}}{3}x\right)\right\}$

[GATE (ME) 2000]

2. The solution of the following differential equations with boundary condition $y(0) = -2$

and $y'(1) = -3, \dfrac{d^2y}{dx^2} = 3x - 2$

(a) $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} + 3x - 6$

(b) $y = 3x^3 - \dfrac{x^2}{2} - 5x + 2$

(c) $y = \dfrac{x^3}{3} - x^2 - \dfrac{5x}{2} - 2$

(d) $y = x^3 - \dfrac{x^2}{2} + 5x + \dfrac{3}{2}$

[GATE (CE) 2001]

3. The general solution of the differential equation $(D^2 - 4D + 4)y = a$, is of the form (given $D \equiv d/dx$), and C_1, C_2 are constants

(a) C_1e^{2x} (b) $C_1e^{2x} + C_2e^{-2x}$
(c) $C_1e^{2x} + C_2e^{2x}$ (d) $C_1e^{2x} + C_2xe^{-2x}$

[GATE (IN) 2005]

4. For the equation $\ddot{x}(t) + 3x(t) + 2x(t) = 5$, the solution $x(t)$ approaches the following values at $t \to \infty$

(a) 0 (b) 5/2
(c) 5 (d) 10

[GATE (EE) 2005]

5. The solution of the following differential equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ is given by

(a) $y = e^{2x} + e^{-3x}$ (b) $y = e^{2x} + e^{3x}$
(c) $y = e^{-2x} + e^{3x}$ (d) $y = e^{-2x+} e^{-3x}$

[GATE (ECE) 2005]

6. Statement for linked answer question (A) and (B). The complete solution for the ordinary differential equation

$$y'' + py' + qy = 0 \text{ is } y = C_1e^{-x} + C_2e^{-3x}$$

(A) Then p and q are

(a) 3, 3 (b) 3, 4
(c) 4, 3 (d) 4, 4

(B) Which of the following is a solution of the differential equation

$$y'' + py' + (q + 1)y = 0?$$

(a) e^{-3x} (b) xe^{-x}
(c) xe^{-2x} (d) x^2e^{-2x}

[GATE (ME) 2005]

7. The solution of $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 17y = 0$; $y(0) = 1$, $\dfrac{dy}{dx}(\pi/4) = 0$ in the range, $0 < x < \pi/4$ is given by

(a) $e^{-x}\left(\cos 4x + \dfrac{1}{4}\sin 4x\right)$

(b) $e^{x}\left(\cos 4x - \dfrac{1}{4}\sin x\right)$

(c) $e^{-4x}\left(\cos x + \dfrac{1}{4}\sin x\right)$

(d) $e^{-4x}\left(\cos 4x - \dfrac{1}{4}\sin 4x\right)$

[GATE (Civil Engg) 2005]

8. For $y'' + 4y' + 3y = 3e^{2x}$, the particular integral is

(a) $\dfrac{1}{15}e^{2x}$ (b) $\dfrac{1}{5}e^{2x}$

(c) $3e^{2x}$ (d) $C_1e^{-x} + C_2e^{-3x}$

[GATE (ME) 2006]

9. For the differential equation $\dfrac{d^2y}{dx^2} + k^2y = 0$, the boundary conditions are

(i) $y = 0$ for $x = 0$ and (ii) $y = 0$ for $x = a$. The form of non-zero solution of y (where m varies over all integers) are

(a) $y = \sum_m Am \sin\dfrac{m\pi x}{a}$

(b) $y = \sum_m Am \cos\dfrac{m\pi x}{a}$

(c) $y = \sum_m Ame^{m\pi/a}$

(d) $y = \sum_m Ame^{\frac{m\pi x}{a}}$

[GATE (ECE) 2006]

10. The solution of the differential equation $k^2\dfrac{d^2y}{dx^2} = y - y_2$ under the boundary conditions

(i) $y = y_1$ at $x = 0$, and (ii) $y = y_2$ at $x = \infty$ where k, y_1 and y_2 are constants; is

(a) $y = (y_1 - y_2) \exp(-x/k) + y_2$
(b) $y = (y_2 - y_1) \exp(-x/k) + y_1$
(c) $y = (y_1 - y_2) \sinh(x/k) + y_1$
(d) $y = (y_1 - y_2) \exp(-x/k) + y_2$

[GATE (ECE) 2007]

11. The degree of the differential equation $\frac{d^2x}{dt^2} + 2x^3 = 0$ is

(a) 0 (b) 1
(c) 2 (d) 3

[GATE (Civil Engg) 2007]

12. The general solution of $\frac{d^2y}{dx^2} - y = 0$ is

(a) $y = P\cos x + Q\sin x$
(b) $y = P\cos x$
(c) $y = P\sin x$
(d) $y = P\sin^2 x$

[GATE (Civil Engg) 2008]

13. Given that $\ddot{x} + 3x = 0$ and $x(0) = 1$, $\dot{x}(0) = 0$; what is $x(1)$?

(a) –0.99 (b) –0.16
(c) 0.16 (d) 0.99

[GATE (ME) 2008]

14. The homogeneous part of the differential equation $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r$, ($p$, q and r are constants) has real distinct roots if

(a) $p^2 - 4q > 0$ (b) $p^2 - 4q < 0$
(c) $p^2 - 4q = 0$ (d) $p^2 - 4q = r$

[GATE (IPE) 2009]

15. The solution of the differential equation $\frac{d^2y}{dx^2} = 0$ with boundary conditions: $\frac{dy}{dx} = 1$ at $x = 0$ and $\frac{dy}{dx} = 1$ at $x = 1$

(a) $y = 1$
(b) $y = x$
(c) $y = x + C$, where C is an arbitrary constant
(d) $y = C_1x + C_2$, where C_1 and C_2 are arbitrary constants

[GATE (IPE) 2009]

16. The order of the differential equation $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is

(a) 1 (b) 2
(c) 3 (d) 4

[GATE (ECE) 2009]

17. Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \delta(t) \text{ with}$$

$$y(t)\Big|_{t=0} = -2 \text{ and } \frac{dy}{dt}\Big|_{t=0} = 0$$

The numerical value of $\frac{dy}{dt}\Big|_{t=0}$ is

(a) -2 (b) -1
(c) 0 (d) 1

[GATE (EE) 2012]

18. The solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 0.25y = 0$, given $y = 0$ at $x = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$ is

(a) $xe^{0.5x} - xe^{-0.5x}$ (b) $0.5xe^{x} - 0.5xe^{-x}$
(c) $xe^{0.5x}$ (d) $-xe^{0.5x}$

[GATE (CH) 2013]

19. The maximum value of the solution $y(t)$ of the differential equation $y(t) + \ddot{y}(t) = 0$ with initial conditions $\dot{y}(0) = 1$ and $y(0) = 2$, for $t \geq 0$ is

(a) 1 (b) 2
(c) π (d) $\sqrt{2}$

[GATE (IN) 2013]

20. The solution for the differential equation $\frac{d^2x}{dt^2} = -9x$ with initial conditions $x(0) = 1$ and $\frac{dx}{dt}\Big|_{t=0} = 1$, is

(a) $t^2 + t + 1$

(b) $\sin 3t + \frac{1}{3}\cos 3t + \frac{2}{3}$

(c) $\frac{1}{3}\sin 3t + \cos 3t$

(d) $\cos 3t + t$

[GATE (EE) 2014]

21. With initial values $y(0) = y'(0) = 1$, the solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at $x = 1$ is _____.

[GATE (EC) 2014]

22. If a and b are constants, the most general solution of the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$ is

(a) ae^{-t} (b) $ae^{-t} + bte^{-t}$

(c) $ae^{t} + bte^{-t}$ (d) ae^{-2t}

[GATE (EC) 2014]

23. If the characteristic equation of the differential equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$$

has two equal roots then the values of α are

(a) ± 1 (b) 0, 0

(c) $\pm j$ (d) $\pm 1/2$

[GATE (EC) 2014]

24. Consider two solutions $x(t) = x_1(t)$ and $x(t)$ and $x(t) = x_2(t)$ of the differential equation $\frac{d^2x(t)}{dt^2} + x(t) = 0$, $t > 0$, such that $x_1(0) = 1$, $\left.\frac{dx_1(t)}{dt}\right|_{t=0} = 0, x_2(0) = 0, \left.\frac{dx_2(t)}{dt}\right|_{t=0} = 1.$

The Wronskian $W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix}$ at $t = \pi/2$ is

(a) 1 (b) –1

(c) 0 (d) $\pi/2$

[GATE (ME) 2014]

ANSWERS

1. (d)	3. (c)	3. (c)	4. (b)	5. (b)	6. (A) → (c); (B) → (c)	7. (a)	8. (b)
9. (a)	10. (d)	11. (b)	12. (a)	13. (d)	14. (a)	15. (c)	16. (b)
17. (d)	18. (c)	19. (wrong option, $\sqrt{5}$)	20. (c)	21. (0.54)	22. (b)	23. (a)	24. (a)

12 Solutions of Second-Order Linear Differential Equations with Variable Coefficients

12.1 INTRODUCTION

Linear differential equations play a role in the study of many practical problems in engineering and science. In this chapter, we consider the differential equations with variable coefficients involved in the area of physics, mechanics, electric circuits, and mathematical modeling of physical problems, etc. The important differential equations with variable coefficients are Bessel's equation, Legendre equation, Cauchy's equation, and Chebyshev equation, etc. We define the powerful general method (method of variation of parameters).

Standard Form of the Second-order LDE

Consider an LDE $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R$

where P, Q, and R are functions of x or constants.

12.2 COMPLETE SOLUTIONS OF $y''+Py'+Qy=R$ IN TERMS OF ONE KNOWN SOLUTION BELONGING TO THE CF

Given $$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+Qy=R \tag{1}$$

Let $y=u(x)$ be a known solution of the CF, so u is a solution of (1) when its right-hand side is taken to be zero. Thus, $y=u$ is a solution of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \tag{2}$$

so that $\dfrac{d^2u}{dx^2}+P\dfrac{du}{dx}+Qu=0$ (3)

Now, let the complete solution of (1) be $y = uv$ (4)
where v is a function of x, v will be determined.

From (4), $$\frac{dy}{dx} = v\frac{du}{dx}+u\frac{dv}{dx}$$

and $$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2}+2\frac{du}{dx}\cdot\frac{dv}{dx}+u\frac{d^2v}{dx^2}$$

Substituting the values of y, y' and y'' in (1), we get

$$\left(v\frac{d^2u}{dx^2}+2\frac{du}{dx}\cdot\frac{dv}{dx}+u\frac{d^2v}{dx^2}\right)+P\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+Quv=R$$

or $$v\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+u\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}\right)+2\frac{du}{dx}\cdot\frac{dv}{dx}=R$$

or $$v\cdot 0+u\frac{d^2v}{dx^2}+P\frac{dv}{dx}+2\frac{du}{dx}\cdot\frac{dv}{dx}=R \quad \text{[using (3)]}$$

or $$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\cdot\frac{dv}{dx}=\frac{R}{u} \tag{5}$$

Now, put $\dfrac{dv}{dx}=q$ so that $\dfrac{d^2v}{dx^2}=\dfrac{dq}{dx}$

Then (5) becomes

$$\frac{dq}{dx}+\left(P+\frac{2}{u}\frac{du}{dx}\right)q = \frac{R}{u} \tag{6}$$

which is a linear equation in q and x. Its integrating factor is

$$= e^{\int\left\{P+\frac{2}{u}\frac{du}{dx}\right\}dx}$$

$$= e^{\int Pdx+2\log u}$$

$$= e^{\log u^2}e^{\int Pdx}=u^2e^{\int Pdx}$$

and the solution of (6) is

$$q\cdot u^2e^{\int Pdx} = \int\frac{R}{u}u^2e^{\int Pdx}dx+c_1$$

or $$q = \frac{dv}{dx}=\frac{e^{-\int Pdx}}{u^2}\int R\cdot u\,e^{\int Pdx}+c_1\frac{e^{-\int Pdx}}{u^2}$$

Integrating both sides, we get

$$v = \int\left[\frac{e^{-\int Pdx}}{u^2}\int Rue^{\int Pdx}\right]dx+c_1\int\frac{e^{-\int Pdx}}{u^2}dx+c_2$$

Putting the value of v in (4), we get

$$y = u\int\left[\frac{e^{-\int P dx}}{u^2}\int Rue^{\int P dx}\right]dx + c_1 u\int\frac{e^{-\int P dx}}{u^2}dx + c_2 u$$

which includes the given solution $y = u$; and since it contains two arbitrary constants, it is the required complete solution.

12.3 RULES FOR FINDING AN INTEGRAL (SOLUTION) BELONGING TO COMPLEMENTARY FUNCTION (CF), i.e., SOLUTION OF $y'' + P(x)y + Q(x)y = 0$

Rule 1: $y = e^{ax}$ is a solution if $a^2 + Pa + Q = 0$
Rule 2: $y = e^x$ is a solution if $1 + P + Q = 0$
Rule 3: $y = e^{-x}$ is a solution if $1 - P + Q = 0$
Rule 4: $y = x^m$ is a solution if $m(m - 1) + Pmx + Qx^2 = 0$
Rule 5: $y = x$ is a solution if $P + Qx = 0$
Rule 6: $y = x^2$ is a solution if $2 + 2Px + Qx^2 = 0$

The working rule for finding the complete solution when one integral (solution) of CF is known or can be obtained by above rules.

Step 1 Put the given equation in the standard form $y'' + py' + Qy = R$ in which the coefficient of y'' is unity.

Step 2 Find an integral u of CF by using the following table:

Condition Satisfied	An integral of CF is
(i) $1 + P + Q = 0$	$u = e^x$
(ii) $1 - P + Q = 0$	$u = e^{-x}$
(iii) $a^2 + aP + Q = 0$	$u = e^{ax}$
(iv) $m(m - 1) + Pmx + Qx^2 = 0$	$u = x^m$
(v) $2 + 2px + Qx^2 = 0$	$u = x^2$

If a solution u is given in a problem then this step is omitted.

Step 3 Consider the complete solution of given equation is $y = u \cdot v$, where u has been obtained in Step 2; then find v using $\dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$ (7)

Step 4 Put $\dfrac{dv}{dx} = q$ so that $\dfrac{d^2v}{dx^2} = \dfrac{dq}{dx}$, put in (7). Then (7) will come out to be a linear equation in q and x if $R \neq 0$. Solve it as usual. If $R = 0$, then variables q and x will be separable.

Step 5 Now, replace q by $\dfrac{dv}{dx}$ and separate the variables v and x. Integrate and determine v. Put this value of v in $y = u \cdot v$. This will lead us to the desired complete solution of the given equation.

Example 1 Solve $x\dfrac{d^2v}{dx^2}-(2x-1)\dfrac{dy}{dx}+(x-1)y=0.$

Solution Putting the given equation in the standard form, we get

$$\frac{d^2y}{dx^2}-\left(2-\frac{1}{x}\right)\frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=0 \qquad (1)$$

Comparing (1) with $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+Qy=R,$ we have

$$P=-\left(2-\frac{1}{x}\right), Q=1-\frac{1}{x}, R=0$$

Here, $\quad 1+P+Q = 1+\dfrac{1}{x}-2+1-\dfrac{1}{x}=0$

i.e., $u=e^x$ is a part of CF of the solution of (1)

Let the complete solution of (1) is $y=uv$ (2)

Then v is given by $\dfrac{d^2v}{dx^2}+\left(P+\dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx}=\dfrac{R}{u}$

or $$\frac{d^2v}{dx^2}+\left(-2+\frac{1}{x}+\frac{2}{e^x}\cdot\frac{d}{dx}e^x\right)\cdot\frac{dv}{dx}=0$$

or $$\frac{d^2v}{dx^2}+\left(-2+\frac{1}{x}+\frac{2}{e^x}\cdot e^x\right)\cdot\frac{dv}{dx}=0$$

or $$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0 \qquad (3)$$

Let $\dfrac{dv}{dx}=q$ so that $\dfrac{d^2v}{dx^2}=\dfrac{dq}{dx}$

Equation (3) becomes

$$\frac{dq}{dx}+\frac{1}{x}q=0$$

which is linear in q and x, so

$$\text{IF}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

$\therefore$ $$q\cdot x=\int 0\cdot x\,dx+c_1$$

$$x\frac{dv}{dx}=c_1$$

$$\frac{dv}{dx}=\frac{c_1}{x} \text{ or } dv=c_1\frac{1}{x}dx$$

Integrating, $v=c_1\log x+c_2$

$\therefore$ $$y=e^x\cdot(c_1\log x+c_2)$$

$y=c_1e^x\log x+c_2e^x$, c_1, c_2 are arbitrary constant.

Example 2 Find the general solution of $(1-x^2)\,y'' - 2xy' + 2y = 0$, if $y = x$ is a solution of it.

Solution Rewriting the given equation in the standard form, we get

$$\frac{d^2y}{dx^2} - \left(\frac{2x}{1-x^2}\right)\frac{dy}{dx} + \left(\frac{2}{1-x^2}\right)y = 0 \tag{1}$$

Comparing (1) with $y'' + Py' + Qy = R$, we get

$$P = -\frac{2x}{1-x^2},\ Q = \frac{2}{1-x^2},\ R = 0$$

Here, $y = u = x$ is given to be a part of CF of the solution of (1).

Let the complete solution of (1) be $y = u\cdot v$ (2)

Then v is given by $\dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$

or $$\frac{d^2v}{dx^2} + \left(-\frac{2x}{1-x^2} + \frac{2}{x}\cdot 1\right)\frac{dv}{dx} = 0$$

or $$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{2x}{1-x^2}\right)\frac{dv}{dx} = 0 \tag{3}$$

Let $\dfrac{dv}{dx} = q$ so that $\dfrac{d^2v}{dx^2} = \dfrac{dq}{dx}$; then (5) becomes

$$\frac{dq}{dx} + \left(\frac{2}{x} - \frac{2x}{1-x^2}\right)q = 0$$

or $$\frac{dq}{q} + \left(\frac{2}{x} - \frac{2x}{1-x^2}\right)dx = 0$$

On integrating, $\log q + 2\log x + \log(1-x^2) = \log c_1$

$$qx^2(1-x^2) = c_1$$

or $$\frac{dv}{dx} = \frac{c_1}{x^2}\frac{1}{(1-x^2)}$$

or $$dv = c_1\cdot\frac{1}{x^2(1-x^2)}dx = c_1\left[\frac{1}{x^2} + \frac{1}{1-x^2}\right]dx$$

Integrating

$$v = c_1\left[-\frac{1}{x} + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right] + c_2$$

$\therefore$ the required solution is

$$y = u\cdot v = x\cdot\left[c_1\left(-\frac{1}{x} + \frac{1}{2}\log\frac{1+x}{1-x}\right) + c_2\right]$$

$$= xc_2 + c_1\left(-\frac{1}{x} + \frac{x}{2}\log\frac{1+x}{1-x}\right)$$

Example 3 Solve $x^2y'' + xy' - y = 0$; given that $x + \dfrac{1}{x}$ is one integral.

Solution Rewriting the given DE in the standard form,

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0 \tag{1}$$

Comparing (1) with $y'' + Py' + Qy = R$

$$P = \frac{1}{x}, Q = -\frac{1}{x^2}, R = 0$$

Here, given that $y = u = \left(x + \dfrac{1}{x}\right)$ is a part of CF of the solution of (1).

Let the complete solution of (1) is $y = u \cdot v$
Then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or
$$\frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2}{x + \frac{1}{x}} \cdot \frac{d}{dx}\left(x + \frac{1}{x}\right)\right] \cdot \frac{dv}{dx} = 0$$

or
$$\frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2x}{x + \frac{1}{x}} \cdot \left(1 - \frac{1}{x^2}\right)\right]\frac{dv}{dx} = 0$$

or
$$\frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2x}{x^2 + 1} \cdot \left(\frac{x^2 - 1}{x^2}\right)\right]\frac{dv}{dx} = 0$$

or
$$\frac{d^2v}{dx^2} + \frac{3x^2 - 1}{x(x^2 + 1)}\frac{dv}{dx} = 0 \tag{2}$$

Let $\dfrac{dv}{dx} = q$ so that $\dfrac{d^2v}{dx^2} = \dfrac{dq}{dx}$, (2) becomes

$$\frac{dq}{dx} + \frac{3x^2 - 1}{x(x^2 + 1)} \cdot q = 0$$

which is linear in q and x,

$$\frac{dq}{q} + \frac{3x^2 - 1}{x(x^2 + 1)}dx = 0$$

or
$$\frac{dq}{q} + \left(\frac{4x}{x^2 + 1} - \frac{1}{x}\right)dx = 0$$

Integrating, $\log q + 2\log(x^2+1) - \log x = \log c_1$

$$\frac{q(x^2+1)^2}{x} = c_1$$

$$\frac{dv}{dx} = c_1\frac{x}{(x^2+1)^2}$$

or $$dv = c_1\frac{x}{(x^2+1)^2}\mathrm{d}x$$

Integrating,

$$v = \frac{1}{2}c_1\int\frac{2x}{(x^2+1)}dx + c_2$$

Put $x^2 + 1 = t$
Then $2x\,dx = dt$

$$= \frac{1}{2}c_1\int\frac{dt}{t^2} + c_2 = \frac{-c_1}{2t} + c_2 = c_2 - \frac{c_1}{2(x^2+1)}$$

$\therefore$ the required solution is $y = u\cdot v$

$$y = \left(x + \frac{1}{x}\right)\cdot\left[c_2 - \frac{c_1}{(x^2+1)^2}\right]$$

$$y = c_2\left(\frac{x^2+1}{x}\right) - \frac{c_1}{2x}$$

Example 4 Solve $x^2y'' - 2x(1+x)\,y' + 2(1+x)\,y = x^3$.

Solution Rewriting the given DE in the standard form,

$$y'' - \frac{2(1+x)}{x}y' + \frac{2(1+x)}{x^2}y = x \tag{1}$$

Comparing (1) with $y'' + Py' + Qy = R$

$$P = -\frac{2(1+x)}{x},\quad Q = \frac{2(1+x)}{x^2},\quad R = x$$

Here, $P + Qx = 0$ so that $y = u = x$ is a part of CF of (1)
Let the general solution of (1) is $y = u\cdot v$
Then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left(\frac{-2(1+x)}{x} + \frac{2}{x}\frac{dx}{dv}\right)\frac{dv}{dx} = \frac{x}{x}$$

$$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \tag{2}$$

Let $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$, (2), becomes $\frac{dq}{dx} - 2q = 1$, which is linear in q and x

$$\text{IF} = e^{\int P dx} = e^{-\int 2dx} = e^{-2x}, \text{ and its solution is}$$

$$qe^{-2x} = \int 1 \cdot e^{-2x} dx + c_1$$

$$= \frac{-1}{2} e^{-2x} + c_1$$

$$q = \frac{-1}{2} + c_1 e^{2x}$$

$$\frac{dv}{dx} = \frac{-1}{2} + c_1 e^{2x}$$

$$dv = \frac{-1}{2} dx + c_1 e^{2x} dx$$

Integrating,

$$v = -\frac{1}{2} x + c_1 \frac{e^{2x}}{2} + c_2$$

$\therefore$ the required general solution is

$$y = u \cdot v$$

$$y = x \cdot \left[-\frac{x}{2} + c_1 \frac{e^{2x}}{2} + c_2 \right]$$

$$y = -\frac{1}{2} + c_1 \frac{x}{2} e^{2x} + xc_2$$

Example 5 Solve $y'' - (\cot x)y' - (1 - \cot x)y = e^x \sin x$

Solution

$$y'' - (\cot x)y' - (1 - \cot x)y = e^x \sin x \tag{1}$$

Comparing (1) with $y'' + Py' + Qy = R$

$$P = -\cot x,\ Q = -(1 - \cot x),\ R = e^x \sin x$$

Here, $1 + P + Q = 1 - \cot x - 1 + \cot x = 0$, so $y = u = e^x$ is a part of CF of (1)

Let the complete solution be $y = u \cdot v$; then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left(-\cot x + \frac{2}{e^x} \cdot e^x \right) \frac{dv}{dx} = \frac{e^x \sin x}{e^x}$$

or

$$\frac{d^2v}{dx^2} + (-\cot x + 2) \frac{dv}{dx} = \sin x \tag{2}$$

Let $\dfrac{dv}{dx} = q$, so that $\dfrac{d^2v}{dx^2} = \dfrac{dq}{dx}$

$$\frac{dq}{dx} + (2 - \cot x)q = \sin x, \text{ which is linear in } q \text{ and } x.$$

$$\text{IF} = e^{\int P dx} = e^{\int (2-\cos x)dx}$$

$$= e^{2x - \log \sin x} = e^{2x} \cdot e^{\log(\sin x)^{-1}}$$

$$= e^{2x} \cdot (\sin x)^{-1}$$

$$= \frac{e^{2x}}{\sin x}$$

$$\therefore \quad q \cdot \frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x}\, dx + c_1$$

$$q \cdot \frac{e^{2x}}{\sin x} = \int e^{2x} dx + c_1$$

$$q \cdot \frac{e^{2x}}{\sin x} = \frac{1e^{2x}}{2} + c_1$$

or $$q = \sin x + c_1 \frac{\sin x}{e^{2x}}$$

$$q = \frac{1}{2}\sin x + c_1 e^{-2x} \sin x$$

or $$\frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x} \sin x$$

$$dv = \left[\frac{1}{2}\sin x + c_1 e^{-2x} \sin x\right] dx$$

Integrating, $$v = -\frac{1}{2}\cos x + c_1 \cdot \frac{e^{-2x}}{1^2 + (-2)^2}[-2\sin x - \cos x] + c_2$$

$$u = -\frac{1}{2}\cos x - \frac{c_1 e^{-2x}}{5}(2\sin x + \cos x) + c_2$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - ba \cos bx)\right]$$

$\therefore$ the general solution is

$$y = u \cdot v = e^x \cdot \left[-\frac{1}{2}\cos x - \frac{c_1 e^{-2x}}{5}(2\sin x + \cos x) + c_2\right]$$

$$y = -\frac{e^x}{2}\cos x - \frac{c_1}{5}e^{-x}(2\sin x + \cos x) + c_2 e^{-x}$$

EXERCISE 12.1

Solve the following equations:

1. $x^2y'' - (x^2 + 2x)y' + (x + 2)y = x^3e^x$
2. $xy'' - 2(x + 1)y' + (x + 2)y = (x - 2)\, e^x$
3. $y'' - (1 - \cot x)y' - (y \cot x) = \sin^2 x$
4. $xy'' - (2x + 1)y' + (x + 1)y = x^3e^x$
5. $xy'' - (x + 2)y' + 2y = x^3$
6. $x^2y'' - y' + 4x^3y = -4x^5$, given that $y = e^{x^2}$ is a solution if the left-hand side is equated to zero.
7. $y'' + y = \sec x$, given that $\cos x$ is a part of CF.
8. $(x \sin x + \cos x)y'' - (x \cos x)y' + y \cos x = 0$
9. $\sin^2 x\left(\dfrac{d^2y}{dx^2}\right) = 2y$, given that $y = \cot x$ is a solution.

Answers

1. $y = c_1xe^x + c_2e^x + (x - 1)xe^x$
2. $y = \dfrac{1}{3}c_1x^3e^x + c_2e^x + \left(x - \dfrac{x^2}{2}\right)e^x$
3. $y = \dfrac{c_1}{2}(\sin x - \cos x) + c_2e^{-x} - \dfrac{1}{10}(\sin 2x - 2\cos 2x)$
4. $y = \left(\dfrac{c_1}{2}\right)x^2e^x + c_2e^x + \dfrac{1}{3}x^3e^x$
5. $y = c_1(x^2 + 2x + 2) + c_2e^x - x^3$
6. $y = \left(\dfrac{-c_1}{4}\right)e^{-x^2} + c_2e^{x^2} + x^2$
7. $y = c_1 \sin x + c_2 \cos x + x \sin x - \cos x \log \sec x$
8. $y = c_2\, x - c_1 \cos x$
9. $y = c_1(1 - x \cot x) + c_2 \cot x$

12.4 REMOVAL OF THE FIRST DERIVATIVE: REDUCTION TO NORMAL FORM

Reduce the differential equation $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, where P, Q, and R are functions of x, to the form $\dfrac{d^2v}{dx^2} + Iv = S$ which is known as the normal form of the given equation.

The given equation is $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ (8)

Let the complete solution of (8) be $y = u \cdot v$, where u and v are functions of x. Differentiating twice, $y = uv$ gives.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Putting the values of y, y' and y'' in (8), we get

$$v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + u\frac{d^2v}{dx^2} + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = R$$

or
$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R$$

or
$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)\cdot v = \frac{R}{u} \tag{9}$$

In order to remove the first derivative $\dfrac{dv}{dx}$ from (9), we take

$$P + \frac{2}{u}\frac{du}{dx} = 0 \text{ or } \frac{du}{u} = -\frac{1}{2}Pdx \tag{10}$$

Integrating, $\log u = -\dfrac{1}{2}\displaystyle\int Pdx$ or $u = e^{-\frac{1}{2}\int Pdx}$ (11)

Thus, the required suitable substitution for the dependent variable is $y = uv$, where u is given by (11).

Now, from (10), we have $\dfrac{du}{dx} = -\dfrac{1}{2}Pu$ so that

$$\frac{d^2u}{dx^2} = -\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}\frac{dP}{dx}\cdot u$$

or
$$\frac{d^2u}{dx^2} = -\frac{1}{2}P\left(-\frac{1}{2}Pu\right) - \frac{1}{2}\frac{dP}{dx}u, \quad \left[\text{putting the value of } \frac{du}{dx}\right]$$

$$\therefore \quad \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = \frac{1}{u}\cdot\left(\frac{1}{4}P^2u - \frac{1}{2}\frac{dP}{dx}\cdot u - \frac{1}{2}P^2u + Qu\right)$$

$$= Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = I \text{ (say)} \tag{12}$$

Also, take $S = R/u$ (13)

Using (10), (12), (13), (9) becomes

$$\frac{d^2 v}{dx^2} + Iv = S, \text{ where } I \text{ and } S \text{ are given by (12) and (13).}$$

12.5 WORKING RULE FOR SOLVING PROBLEMS BY USING NORMAL FORM

Step 1 Put the equation in the standard form $y'' + Py' + Qy = R$, in which the coefficient of $\frac{d^2 y}{dx^2}$ must be unity.

Step 2 To remove the first derivative, we choose $u = e^{-\frac{1}{2}\int P dx}$

Step 3 We now assume that the complete solution of the given equation is $y = u \cdot v$; then the given equation reduces to normal form

$$\frac{d^2 v}{dx^2} + Iv = S, \text{ where } I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \text{ and } S = R/u$$

Important Note The success in solving the given equation depends on the success in solving $\frac{d^2 v}{dx^2} + Iv = S$. Now this latter equation can be solved easily if I take two special forms.

(a) When I = constant then the resulting equation being with constant coefficients can be solved by the usual methods.

(b) When $I = \left(\frac{\text{constant}}{x^2}\right)$ then the resulting equation reduces to homogeneous form and, hence, it can be solved by using the usual methods.

Step 4 After getting v, the complete solution is given by $y = u \cdot v$

Example 6 Solve $y'' - (2\tan x)\, y' + 5y = 0$.

Solution

$$y'' - (2 \tan x)\, y' + 5y = 0 \tag{1}$$

Compare (1) with $y'' + Py' + Qy = R$

$$P = -2\tan x,\ Q = 5,\ R = 0$$

To remove the first derivative from (1),

Choose $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x dx}$

$$u = e^{\log \sec x} = \sec x$$

Calculate $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$$= 5 - \frac{1}{4} \cdot 4\tan^2 x - \frac{1}{2} \times -2\sec^2 x$$

$$= 5 - \tan^2 x + \sec^2 x$$

$$= 5 - \tan^2 x + \tan^2 x + 1 = 6 \text{ (constant)}$$

and $\quad S = R/u = 0/\sec x = 0$

Then the reduced equation is

$$\frac{d^2v}{dx^2} + Iv = S$$

$$\frac{d^2v}{dx^2} + 6v = 0 \text{ or } (D^2 + 6)v = 0, D \equiv \frac{d}{dx} \tag{2}$$

So, the solution of (2) is $v = \text{CF} = c_1 \cos(x\sqrt{6}) + c_2 \sin x\sqrt{6}$

$\therefore$ the complete solution is $y = u \cdot v$

$$y = \sec x(c_1 \cos x\sqrt{6} + c_2 \sin x\sqrt{6})$$

Example 7 Solve $y'' - (2 \tan x)\, y' + 5y = (\sec x) \cdot e^x$.

Solution Given $y'' - (2 \tan x)\, y' + 5y = (\sec x) \cdot e^x \quad (1)$

Compare (1) with $y'' + Py' + Qy = R$

$$P = -2 \tan x,\ Q = 5,\ R = (\sec x)e^x$$

Calculate
$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= 5 - \frac{1}{4} \cdot 4\tan^2 x - \frac{1}{2} \cdot (-2\sec^2 x)$$

$$= 5 - \tan^2 x + \sec^2 x = 6 \text{ (constant)}$$

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x\, dx}$$

$$= e^{\log \sec x} = \sec x$$

$$S = \frac{R}{u} = \frac{\sec x \cdot e^x}{\sec x} = e^x$$

$\therefore$ the reduced equation

$$\frac{d^2v}{dx^2} + Iv = S$$

$$\frac{d^2v}{dx^2} + 6v = e^x \text{ or } (D^2 + 6)v = e^x$$

Its auxiliary equation is $m^2 + 6 = 0$

$$m = \pm i\sqrt{6}$$

CF is $= c_1 \cos(x\sqrt{6}) + c_2 \sin(x\sqrt{6})$

and
$$\text{PI} = \frac{1}{D^2 + 6}e^x = \frac{1}{1^2 + 6}e^x = \frac{1}{7}e^x$$

$\therefore$
$$v = \text{CF} + \text{PI} = \left[c_1 \cos(x\sqrt{6}) + c_2 \sin x\sqrt{6}\right] + \frac{1}{7}e^x$$

$\therefore$ The complete solution is $y = u \cdot v$

$$y = \sec x \left[c_1 \cos(x\sqrt{6}) + c_2 \sin(x\sqrt{6}) + \frac{1}{7} e^x \right]$$

Example 8 Solve $y'' - \left(\frac{2}{x}\right) y' + \left(1 + \frac{2}{x^2}\right) y = xe^x$ by changing the dependent variable.

Solution Given $$y'' - \left(\frac{2}{x}\right) y' + \left(1 + \frac{2}{x^2}\right) y = xe^x \qquad (1)$$

Comparing (1) with $y'' + Py' + Qy = R$

$$P = -\frac{2}{x}, Q = \left(1 + \frac{2}{x^2}\right), R = xe^x$$

Calculate $$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= 1 + \frac{2}{x^2} - \frac{1}{4} \cdot \frac{4}{x^2} - \frac{1}{2} \cdot \frac{2}{x^2}$$

$$= 1 + \frac{2}{x^2} - \frac{1}{x^2} - \frac{1}{x^2} = 1$$

and $$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -\frac{2}{x} dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$S = \frac{R}{u} = \frac{xe^x}{x} = e^x$$

Then the equation (1) reduces to $\frac{d^2v}{dx^2} + Iv = S$

$$\frac{d^2v}{dx^2} + v = e^x \text{ or } (D^2 + 1)v = e^x \qquad (2)$$

Its AE is $m^2 + 1 = 0$

$$m = \pm i$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

and $$\text{PI} = \frac{1}{D^2 + 1} e^x = \frac{1}{1^2 + 1} e^x = \frac{1}{2} e^x$$

$\therefore$ $$V = \text{CF} + \text{PI}$$

$$V = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x$$

Then the required general solution is $y = u \cdot v$

$$y = x \cdot \left(c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x \right)$$

Example 9 Solve $y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2}\sin 2x$.

Solution Given $$y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2}\sin 2x \qquad (1)$$

Comparing (1) with $y'' + Py' + Qy = R$

$$P = -4x, Q = (4x^2 - 1),\ R = -3e^{x^2}\sin 2x$$

Calculate $$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= (4x^2 - 1) - \frac{1}{4}\cdot 16x^2 - \frac{1}{2}\cdot(-4)$$

$$= 4x^2 - 1 - 4x^2 + 2 = 1 \text{ (constant)}$$

and $$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -4x\,dx}$$

$$= e^{2\int x dx} = e^{x^2}$$

$$S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

Then (1) reduced to $\dfrac{d^2v}{dx^2} + Iv = S$

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

$$(D^2 + 1)v = -3\sin 2x$$

Its AE $m^2 + 1 = 0$

$$m = \pm i$$

The CF is $c_1\cos x + c_2\sin x$

$$\text{PI} = \frac{1}{D^2 + 1} - 3\sin 2x = \frac{-3}{-2^2 + 1}\sin 2x = \sin 2x$$

$\therefore$ $$V = c_1\cos x + c_2\sin x + \sin 2x$$

Then the complete solution is $y = u\cdot v$

$$y = e^{x^2}[c_1\cos x + c_2\sin x + \sin 2x]$$

EXERCISE 12.2

Solve the following differential equations:

1. $y'' + 4xy' + 4x^2y = 0$
2. $y'' - 4xy' + (4x^2 - 3)y = e^{x^2}$
3. $\dfrac{d}{dx}\left(\cos^2 x\cdot\dfrac{dy}{dx}\right) + y\cos^2 x = 0$
4. $x\dfrac{d}{dx}\left(x\dfrac{dy}{dx} - y\right) - 2x\dfrac{dy}{dx} + 2y + x^2y = 0$

5. $x^2\dfrac{d^2y}{dx^2} - 2x(3x-2)\dfrac{dy}{dx} + 3x(3x-4)y = e^{2x}$

6. $y'' + 2xy' + (x^2+5)y = xe^{-x^2/2}$

7. $(1-x^2)y'' - 4xy' - (1+x^2)\,y = x$

8. $y'' + \left(\dfrac{2}{x}\right)y' - y = 0$

9. $x^2y'' - 2xy' + (x^2+2)y = x^3e^x$

10. $y'' + \left(\dfrac{2}{x}\right)y' + y = \dfrac{\sin 2x}{x}$

Answers

1. $y = e^{-x^2}(c_1e^{x\sqrt{2}} + c_2e^{-x\sqrt{2}})$

2. $y = e^{x^2}[c_1e^x + c_2e^{-x} - 1]$

3. $y = \sec x[c_1\cos(x\sqrt{2}) + c_2\sin(x\sqrt{2})]$

4. $y = (c_1\cos x + c_2\sin x)\cdot x$

5. $y = x^{-2}e^{3x}\left[c_1x^2 + c_2x^{-1} + \dfrac{1}{3}x^2\cdot\log x\right]$

6. $y = e^{-x^2/2}\left[c_1\cos x + c_2\sin x + \dfrac{x}{4}\right]$

7. $y = (1-x^2)^{-1}\,[c_1\cos x + c_2\sin x + x]$

8. $y = x^{-1}[c_1e^x + c_2e^{-x}]$

9. $y = x\left[c_1\cos x + c_2\sin x + \dfrac{e^x}{2}\right]$

10. $y = x^{-1}\left[c_1\cos x + c_2\sin x - \dfrac{1}{3}\sin 2x\right]$

12.6 TRANSFORMATION OF THE EQUATION BY CHANGING THE INDEPENDENT VARIABLE

Consider a second order LDE with variable coefficients

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{14}$$

where P, Q, R are functions of x and let the independent variable be changed from x to z, where $z = f(x)$, (say).

Using the formula $\dfrac{df}{dx} = \dfrac{df}{dz}\cdot\dfrac{dz}{dx}$, we have $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx}$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} + \frac{dy}{dz}\cdot\frac{d}{dx}\left(\frac{dz}{dx}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}\cdot\frac{dz}{dx} + \frac{dy}{dx}\cdot\frac{d^2z}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\frac{d^2y}{dz^2}\cdot\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}+P\frac{dy}{dz}\cdot\frac{dz}{dx}+Qy=R$$

or $$\frac{d^2y}{dz^2}\cdot\left(\frac{dz}{dx}\right)^2+\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\frac{dy}{dz}+Qy=R$$

Dividing by $\left(\frac{dz}{dx}\right)^2$ on both sides, we get

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \tag{15}$$

where $P_1=\dfrac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1=\dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ and $R_1=\dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Here, P_1, Q_1, and R_1 are functions of x, but these can be converted to functions of z by using the relation $z=f(x)$. If the equating Q_1 to a constant quantity we see that P_1 also becomes constant then (15) can be solved (since it will be linear equation with constant coefficients) to obtain the required solution.

12.7 WORKING RULE FOR SOLVING EQUATIONS BY CHANGING THE INDEPENDENT VARIABLE

Step 1 Put the given DE in standard form by keeping the coefficient of y'' as unity, i.e.,

$$y''+Py'+Qy=R \tag{16}$$

Step 2 Suppose $Q=\pm kf(x)$; then we assume a relation between the new independent variable z and let the old independent variable x given by $\left(\frac{dz}{dx}\right)^2=kf(x)$.

Note Be careful that we omit the –ve sign of Q while writing this step. This is extremely important to find real values. Sometimes we assume that $\left(\frac{dz}{dx}\right)^2=f(x)$ whenever we anticipate complicated relation between z and x.

Step 3 We now solve $\left(\frac{dz}{dx}\right)^2=kf(x)$. Rejecting the –ve sign, we get

$$\frac{dz}{dx}=\pm\sqrt{kf(x)} \tag{17}$$

Now, separating the variables, (16) gives

$$dz = \sqrt{[kf(x)]}\,dx \quad \text{so that} \quad z = \int \sqrt{[kf(x)]}\,dx \tag{18}$$

where we have omitted the constant of integration since we are interested in finding just a relation between z and x.

Step 4 With the relationship (17) between z and x, we transform (16) to get an equation of the form

$$\frac{d^2y}{dx^2} + P_1\left(\frac{dy}{dz}\right) + Q_1 y = R_1 \tag{19}$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^z}$ and $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Now, by virtue of (17), $Q_1 = \dfrac{\pm kf(x)}{kf(x)} = \pm k$, a constant.

Then we calculate P_1

If P_1 is also constant. Then (19) can be solved because it will be a linear equation with constant coefficients. If, however, P_1 does not become constant then this rule will not be useful. The students must, therefore, be sure that P_1 comes out to be constant before proceeding further. The value of P_1, Q_1, and R_1 must be remembered for direct use in problems. R_1 can be converted to a function of z by using (17).

Step 5 After solving the equation (19) by usual methods, the variable z is replaced by x by using (18).

Example 10 Solve $y'' + (\cot x)y' + (4\,\mathrm{cosec}^2\,x)\,y = 0$.

Solution Given $y'' + (\cot x)y' + (4\,\mathrm{cosec}^2\,x)\,y = 0$ (1)

Comparing (1) with $y'' + Py' + Qy = R$

$$P = \cot x,\ Q = 4\,\mathrm{cosec}^2\,x,\ R = 0$$

We choose z such that $\left(\dfrac{dz}{dx}\right)^2 = 4\,\mathrm{cosec}^2\,x$

so that $\dfrac{dz}{dx} = 2\,\mathrm{cosec}\,x$

or $z = 2\log\tan(x/2)$

$$\frac{d^2z}{dx^2} = -2\,\mathrm{cosec}\,x\cdot\cot x$$

Calculate $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{-2\,\mathrm{cosec}\,x\cdot\cot x + 2\cot x\,\mathrm{cosec}\,x}{4\,\mathrm{cosec}^2\,x}$

$$P_1 = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4\,\text{cosec}^2 x}{4\,\text{cosec}^2 x} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

Then the reduced equation,

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\frac{d^2y}{dz^2} + y = 0 \tag{2}$$

or $\qquad (D_1^2 + 1)y = 0$ where $D_1 \equiv \dfrac{d}{dz}$

Its AE, is $m^2 + 1 = 0$ so that $m = \pm i$

Hence, the required solution is $y = \text{CF} = c_1 \cos z + c_2 \sin z$

$$y = c_1 \cos(2 \log \tan x/2) + c_2 \sin(2\log \tan x/2)$$

Example 11 Solve $(\cos x)y'' + (\sin x)y' - (2\cos^3 x)y = 2\cos^5 x$.

Solution Dividing by $\cos x$, the given equation in the standard form is

$$y'' + (\tan x)y' - (2\cos^2 x)\,y = 2\cos^4 x \tag{1}$$

Compare (1) with $y'' + Py' + Qy = R$

$$P = \tan x,\ Q = -2\cos^2 x,\ R = 2\cos^4 x$$

Choose z, such that $\left(\dfrac{dz}{dx}\right)^2 = 2\cos^2 x$

or $\qquad \dfrac{dz}{dx} = \sqrt{2}\cos x$ and $\dfrac{d^2z}{dx^2} = -\sqrt{2}\sin x$

or $\qquad dz = \sqrt{2}\cos x\,dx$ or $z = \sqrt{2}\sin x$

Calculate

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sqrt{2}\sin x + \tan x \cdot \sqrt{2}\cos x}{2\cos^2 x} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sqrt{2}\cos x}{\sqrt{2}\cos x} = -1$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cos^4 x}{2\cos^2 x} = \cos^2 x = 1 - \sin^2 x = \left(1 - \frac{z^2}{2}\right)$$

Then the reduced equation is

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\frac{d^2y}{dz^2} - y = 1 - \frac{z^2}{2} \text{ or } (D_1^2 - 1)y = \left(1 - \frac{z^2}{2}\right), \text{ where } D_1 \equiv \frac{d}{dz}$$

Its AE is $m^2 - 1 = 0$

$$m = \pm 1$$

$\text{CF} = c_1e^z + c_2e^{-z}$

$$\text{PI} = \frac{1}{(D_1^2 - 1)}\left(1 - \frac{z^2}{2}\right)$$

$$= \frac{1}{(D_1^2 - 1)}\cdot 1 + \frac{1}{2}\frac{1}{(1 - D_1^2)}z^2$$

$$= \frac{1}{(D_1^2 - 1)}e^{0z} + \frac{1}{2}(1 - D_1^2)^{-1}z^2$$

$$= \frac{1}{0 - 1}e^{0z} + \frac{1}{2}(1 + D_1^2 + \cdots)z^2$$

$$= -1 + \frac{1}{2}(z^2 + 2) = \frac{z^2}{2}$$

Hence, the required solution is $y = \text{CF} + \text{PI}$

$$y = c_1e^z + c_2e^{-z} + \frac{z^2}{2}$$

or $$y = c_1e^{\sqrt{2}\sin x} + c_2e^{-\sqrt{2}\sin x} + \sin^2 x \text{ as } z = \sqrt{2}\sin x$$

Example 12 Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3 \sin x^2$.

Solution Dividing by x, $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2 \sin x^2$

Compare (1) with $y'' + Py' + Qy = R$

$$P = -\frac{1}{x}, Q = -4x^2, R = 8x^2 \sin x^2$$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$$\frac{dz}{dx} = 2x \text{ so that } z = x^2 \text{ and } \frac{d^2z}{dx^2} = 2$$

Calculate $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}\cdot 2x}{4x^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{4x^2} = -1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{8x^2 \sin x^2}{4x^2} = 2\sin x^2 = 2\sin z$$

$\therefore$ the reduced equation

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

or $$\frac{d^2z}{dx^2} - y = 2\sin z$$

or $$(D_1^2 - 1)y = 2\sin z \quad (2), \text{ where } D_1 \equiv \frac{d}{dz}$$

Its AE is $$m^2 - 1 = 0$$

$$m = \pm 1$$

$$\text{CF} = c_1 e^z + c_2 e^{-z}$$

$$\text{PI} = \frac{1}{D_1^2 - 1} 2\sin z$$

$$= \frac{1}{-1^2 - 1} 2\sin z = -\sin z$$

Hence, the required solution is $y = \text{CF} + \text{PI}$

$$y = c_1 e^z + c_2 e^{-z} - \sin z$$

or $$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$$

Example 13 Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\sin\log(1+x)$.

Solution Dividing by $(1+x)^2$, the given equation in the standard form is

$$\frac{d^2y}{dx^2} + \frac{1}{1+x}\frac{dy}{dx} + \frac{1}{(1+x)^2} y = \frac{4}{(1+x)^2}\sin\log(1+x) \tag{1}$$

Comparing (1) with $y'' + Py' + Qy = R$

$$P = \frac{1}{1+x},\ Q = \frac{1}{(1+x)^2},\ R = \frac{4}{(1+x)^2}\sin\log(1+x)$$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = \frac{1}{(1+x)^2}$

or $\frac{dz}{dx} = \frac{1}{(1+x)}$ so that $z = \log(1+x)$

and $\frac{d^2z}{dx^2} = -\frac{1}{(1+x)^2}$

Calculate
$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\frac{1}{(1+x)^2} + \frac{1}{(1+x)} \times \frac{1}{(1+x)}}{\frac{1}{(1+x)^2}} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{1/(1+x)^2}{1/(1+x)^2} = 1 \text{ and}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{4(1+x)^{-2}\cos\log(1+x)}{(1+x)^{-2}}$$

$$= 4\cos\log(1+x) = 4\cos z$$

$\therefore$ the reduced equation is $\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

or $\frac{d^2y}{dz^2} + y = 4\cos z$

or $(D_1^2 + 1)y = 4\cos z$, where $D_1 \equiv \frac{d}{dz}$

Its AE is $m^2 + 1 = 0$

$$m = \pm i$$

$$\text{CF} = c_1\cos z + c_2\sin z$$

$$\text{PI} = \frac{1}{D_1^2 + 1}4\cos z$$

$$= 4\frac{z}{2.1}\sin z \qquad \left[\because \frac{1}{D^2+a^2}\cos az = \frac{z}{2a}\sin az\right]$$

$$= 2z\sin z$$

Hence, the required general solution is

$$y = \text{CF} + \text{PI} = c_1 \cos z + c_2 \sin z + 2z \sin z$$

or

$$y = c_1 \cos \{\log (1 + x)\} + c_2 \sin \{\log (1 + x)\} + 2\log(1 + x) \sin \{\log(1 + x)\}$$

EXERCISE 12.3

Solve the following differential equations:

1. $y'' - y' \cot x - (\sin^2 x)y = 0$
2. $y'' + (\tan x - 1)^2 y' - (n(n - 1) \sec^4 x)y = 0$
3. $xy'' + (4x^2 - 1) y' + 4x^3 y = 2x^3$
4. $y'' - (8e^{2x} + 2) y' + 4e^{2x} y = e^{6x}$
5. $xy'' - y' + 4x^3 y = x^5$
6. $x^6 y'' + 3x^5 y' + a^2 y = \dfrac{1}{x^2}$
7. $x^4 y'' + 2x^3 y' + n^2 y = 0$
8. $(1 + x)^2 y'' + (1 + x)y' + y = 4 \sin \{\log (1 + x)\}$

Answers

1. $y = c_1 e^{-\cos x} + c_2 e^{\cos x}$
2. $y = c_1 e^{-n\tan x} + c_2 e^{(n-1)\tan x}$ as $z = \tan x$
3. $y = e^{-x^2}\left[(c_1 + c_2 x^2) + \dfrac{1}{2}\right]$
4. $y = e^{2x}\left[c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3}) + \dfrac{1}{4}e^{2x} + 1\right]$
5. $y = c_1 \cos x^2 + c_2 \sin x^2 + \dfrac{x^2}{4}$
6. $y = c_1 \cos\left(\dfrac{a}{2x^2}\right) - c_2 \sin\left(\dfrac{a}{2x^2}\right) + \dfrac{1}{a^2 x^2}$
7. $y = c_1 \cos(n/x) - c_2 \sin(n/x)$
8. $y = c_1 \cos \log(1 + x) + c_2 \sin \log (1 + x) - 2 \log (1 + x)\cdot\cos (\log (1 + x)$

12.8 METHOD OF VARIATION OF PARAMETERS

This method is used for finding the complete solution of $\dfrac{d^2 y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ (20)

when the complementary function is known.

Let the CF be $y = au + bv$ (21)

where a and b are constants and u and v are functions of x. Then u and v must be solutions of

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{22}$$

Hence, $u'' + Pu' + Qu = 0$ and $v'' + Pv' + Qv = 0$ (23)
where $R \neq 0$, $au + bv$ will not represent the complete solution of (20)

Now, consider $y = Au + Bv$ (24)

The complete solution of (20), where A and B are not constants but functions of x so chosen that (20) shall be satisfied.

$$\therefore \quad \frac{dy}{dx} = Au' + Bv' + A'u + B'v \tag{25}$$

Choose $\quad A'u + B'v = 0$ (26)

Using (26), (25) becomes

$$\frac{dy}{dx} = Au' + Bv' \tag{27}$$

Differentiating (27) both sides w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = Au'' + Bv'' + A'u' + B'v' \tag{28}$$

Putting the values of y, y' and y'' from (24), (27), and (28) in (20), we get

$$(Au'' + Bv'' + A'u' + B'v') + P(Au' + Bv') + Q(Au + Bv) = R$$

$$A[u'' + Pu' + Qu] + B[v'' + Pv' + Qv] + A'u' + B'v' = R \tag{29}$$

Using (23), (29) becomes

$$A\cdot 0 + B\cdot 0 + A'u' + B'v' = R$$

or

$$A'u' + B'v' = R \tag{30}$$

Solving equations (26) and (30) for A' and B', we get

$$A' = \frac{-Rv}{\omega} \text{ and } B' = \frac{Ru}{\omega}$$

where W = Wronskian of u and $v = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

i.e., $\quad W = uv' - u'v$

On integrating find A and B, putting the values of A and B in (24), we get the complete solution of Eq. (20).

12.9 WORKING RULE FOR SOLVING SECOND ORDER LDE BY THE METHOD OF VARIATION OF PARAMETERS

Step 1 Put the given DE in standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{31}$$

Step 2 Put $R = 0$ in (31); then (31) becomes

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = O \tag{32}$$

Suppose the complementary function (CF) of (32) is

$y = au + bv$; a and b are constants.

Step 3 Replace the parameters a and b by $A(x)$ and $B(x)$; then the complete solution of (31) is

$$y = Au + Bv$$

where $A = \int \frac{-Rv}{w} dx$ and $B = \int \frac{Ru}{w} dx$

and w = Wronskian of u and $v = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

$$w = uv' - u'v$$

Now, find A and B, then put in $y = Au + Bv$.

12.10 SOLUTION OF THIRD-ORDER LDE BY METHOD OF VARIATION OF PARAMETERS

Consider a third order LDE such as

$$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = X \tag{33}$$

Put $X = 0$ in (33); then (34) becomes

$$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \tag{34}$$

Suppose $y = c_1u + c_2v + c_3w$ is the solution of (34), where $y = u$, $y = v$ and $y = w$ satisfy the given DE (33).

Then let the complete solution of (33) be

$$y = Au + Bv + Cw \tag{35}$$

where A, B, and C are functions of x. Then using the method of variation of parameters, as in the case of LDE of second order, we get

$$A' = \frac{XW(v, w)}{W(u, v, w)} \tag{36}$$

$$B' = -\frac{XW(u, w)}{W(u, v, w)} \tag{37}$$

$$C' = \frac{XW(u, v)}{W(u, v, w)} \tag{38}$$

where $W(u, v, w) = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$ and $w(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$, etc.

Now, integrating equations (36), (37), and (38), we get the values of A, B, and C respectively. Putting these values in (3), we get the solution of the given differential equation (33).

Example 14 Solve $\frac{d^2y}{dx^2} + n^2y = \sec nx.$

Solution We can write the given equation in the form of

$$y'' + n^2y = \sec nx \text{ or } (D^2 + n^2)y = \sec nx \tag{1}$$

Put RHS = 0,

$$(D^2 + n^2)y = 0 \tag{2}$$

Its AE is $m^2 + n^2 = 0$

$$m^2 = -n^2$$

$$m = \pm in$$

Solution of (2) is $y = \text{CF} = a \cos nx + b \sin nx$

Consider the complete solution of (1) as

$$y = A \cos nx + B \sin nx \tag{3}$$

where A and B are function of x.

Compare (3) with $y = Au + Bv$

$$u = \cos nx,\ v = \sin nx$$

$$W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix}$$

$$= n\cos^2 nx + n \sin^2 nx = n(\cos^2 nx + \sin^2 nx)$$

$$W = n$$

$$A = \int \frac{-R.v}{w} dx = -\int \frac{\sec nx \cdot \sin nx}{n} dx$$

$$= -\frac{1}{n}\int \tan nx\, dx = -\frac{1}{n^2} - \log \sec nx$$

$$= \frac{1}{n^2} \log \cos nx + c_1 \qquad [\because\ -\log x = \log 1/x]$$

and $$B = \int \frac{R.u.}{w} dx = \int \frac{\sec nx \cdot \cos nx}{n} dx = \frac{1}{n} + c_2$$

Putting the values of A and B in (3), we get the required solution of (1).

$$y = \left(\frac{1}{n^2} \log \cos nx + c_1\right) \cos nx + \left(\frac{1}{n} + c_2\right) \sin nx$$

$$y = c_1 \cos nx + c_2 \sin nx + \frac{1}{n^2} \cos nx \cdot \log \cos nx + \frac{x}{n} \sin nx$$

Example 15 Solve $(D^2 + 1)y = \text{cosec } x$.

Solution

$$(D^2 + 1)y = \text{cosec } x \tag{1}$$

Here, $R = \text{cosec } x$

Putting RHS = 0 = R in (1), we get

$$(D^2 + 1)y = 0 \tag{2}$$

AE $(m^2 + 1) = 0$

$$m = \pm i$$

Complete solution of (2) is $y = \text{CF} = a \cos x + b \sin x$

Consider the complete solution of (1) as

$$y = A \cos x + B \sin x \tag{3}$$

where A and B are functions of 'x'. Compare (3) with $y = Au + Bv$;

$$u = \cos x,\ v = \sin x$$

Wronskian of u, v = $W(u,v) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$= \cos^2 x + \sin^2 x = 1$$

Now, $$A = \int \frac{-R.v}{w} dx = -\int \frac{\text{cosec } x}{1} \cdot \sin x \cdot dx$$

$$A = -\int dx = -x + c_1$$

and $$B = \int \frac{R.u.}{w} dx = \int \frac{\text{cosec } x \cdot \cos x}{1} dx$$

$$= \int \cot x \, dx = \log \sin x + c_2$$

Putting these values of A and B in (3), we get

$$y = (-x + c_1) \cos x + (\log \sin x + c_2) \sin x$$
$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x$$

Example 16 Solve $(D^2 + 4)y = 4 \tan 2x$.

Solution

$$(D^2 + 4)y = 4\tan 2x \tag{1}$$

Here, $R = 4 \tan 2x$

Putting RHS = 0, Eq. (1) becomes

$$(D^2 + 4)y = 0 \tag{2}$$

Its AE is $m^2 + 4 = 0$

$$m = \pm 2i$$

Complete solution of (2) is $y = \text{CF} = a \cos 2x + b \sin 2x$

Consider the complete solution of (1) as

$$y = A \cos 2x + B \sin 2x \tag{3}$$

where A and B are functions of x.

Compare (3) with $y = Au + Bv$

$$u = \cos 2x,\ v = \sin 2x$$

Wronskian of u, v = $W(u,v) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$$= 2 \cos^2 2x + 2 \sin^2 2x = 2$$

Now, $$A = -\int \frac{R.v}{w} dx = -\int \frac{4 \tan 2x \cdot \sin 2x}{2} dx$$

$$= -2 \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -2 \int \frac{1 - \cos^2 2x}{\cos 2x} dx$$

$$= -2 \int \sec 2x \, dx + 2 \int \cos 2x \, dx$$

$$= -\log(\sec 2x + \tan 2x) + \sin 2x + c_1$$

and $$B = \int \frac{R \cdot u}{w} dx = \int \frac{4 \cdot \tan 2x \cdot \cos 2x}{2} dx$$

$$= 2 \int \sin 2x \, dx$$

$$= -\cos 2x + c_2$$

Putting these values of A and B in (3), we get

$$y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$$

Example 17 Solve $x^2y'' + xy' - y = x^2e^x$.

Solution The given equation in the standard form $y'' + Py' + Qy = R$ is

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = e^x \tag{1}$$

Here, $R = e^x$

Putting RHS = 0 in (1), we get

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \text{ or } x^2y'' + xy' - y = 0$$

or $$(x^2D^2 + xD - 1)y = 0; \text{ where } D \equiv \frac{d}{dx} \tag{2}$$

which is a homogeneous equation; putting $x = e^z$ and $D_1 \equiv \dfrac{d}{dz}$; then (2) becomes

$$[D_1(D_1 - 1) + D_1 - 1]y = 0$$

$$[D_1^2 - 1]y = 0 \tag{3}$$

Its AE is $m^2 - 1 = 0$ so that $m = \pm 1$

Hence, the solution of (3) is $y = \text{CF} = ae^z + be^{-z}$

$$y = ax + bx^{-1}$$

Consider the complete solution of (1) as

$$y = Ax + Bx^{-1} \tag{4}$$

where A and B are functions of 'x'.

Compare (4) with $y = Au + Bv$

$$u = x, \ v = \frac{1}{x}$$

$$\text{Wronskian of } u, v = w(u,v) = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

$$= -\frac{1}{x^2} - \frac{1}{x} = -\frac{2}{x}$$

Now, $$A = -\int \frac{R \cdot v}{w} dx = -\int \frac{e^x \cdot 1/x}{-2/x} dx$$

$$= \frac{1}{2}\int e^x dx = \frac{1}{2}e^x + c_1$$

and $$B = \int \frac{R \cdot u}{w} dx = \int \frac{e^x \cdot x}{-2/x} dx$$

$$= -\frac{1}{2}\int x^2 e^x dx$$

$$= -\frac{1}{2}x^2 e^x + \left[xe^x - \int 1 \cdot e^x dx\right] + c_2$$

$$B = -\frac{1}{2}x^2 e^x + xe^x - e^x + c_2$$

Putting these values of A and B in (4), we get

$$y = c_1 x + c_2 x^{-1} + e^x - x^{-1} e^x$$

Example 18 Solve $(D^2 + 2D + 1)y = e^{-x}\log x$.

Solution

$$(D^2 + 2D + 1)y = e^{-x}\log x \tag{1}$$

Here, $R = e^{-x}\log x$

Put RHS = 0 in (1), we get

$$(D^2 + 2D + 1)y = 0 \tag{2}$$

Its AE is $m^2 + 2m + 1 = 0$

$$m = -1, -1$$

The complete solution of (2) is $y = \text{CF} = (a + bx)e^{-x}$

Consider the complete solution of (1) as

$$y = (A + Bx)e^{-x} \tag{3}$$

where A and B are functions of 'x'.

Compare (3) with $y = Au + Bv$

$$u = e^{-x}, v = xe^{-x}$$

Wronskian of u, v = $$w(u, v) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -e^{-x} \end{vmatrix}$$

$$= -e^{-2x}(x - 1) + xe^{-2x}$$

$$w = e^{-2x}$$

Now, $$A = -\int \frac{R \cdot v}{w} dx = -\int \frac{e^{-x}\log x \times xe^{-x}}{e^{-2x}} dx$$

$$= -\int x \log x \, dx$$

$$A = \frac{x^2}{2}\left[\frac{1}{2} - \log x\right] + c_1$$

and $$B = \int \frac{R \cdot u}{w} dx = \int \frac{e^{-x} \log x \cdot e^{-x}}{e^{-2x}} dx = \int \log x \, dx = x \log x - x + c_2$$

Putting these values of A and B in (3), we get

$$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{x^2}{2}\left(\frac{1}{2} - \log x\right)e^{-x} + e^{-2x}(x \log x - x)$$

EXERCISE 12.4

Solve the following DE by the method of variation of parameters:

1. $(D^2 - 2D + 1)y = x^{3/2} \cdot e^x$
2. $(D^2 - 1)y = e^{-2x} \cdot \sin(e^{-x})$
3. $(D^2 - 6D + 9)y = x^{-2} e^{3x}$
4. $(D^2 - 2D + 2)y = e^x \cdot \tan x$
5. $(D^2 + 3D + 2)y = e^x + x^2$
6. $(D^2 + 1)y = \log(\cos x)$
7. $(D^2 + 1)y = x \cdot \cos 2x$
8. $(D^2 - 3D + 2)y = (xe^x + 2x)$
9. $x^2 y'' + xy' - y = x^2 e^x$
10. $x^2 y'' - xy' = x^3 e^x$
11. $(D^2 - 2D + 1)y = e^x$
12. $(D^2 - 3D + 2)y = e^{2x}/(e^x + 1)$
13. $(D^2 - 2D + 1)y = xe^x \log x;\ x > 0$

Answers

1. $y = \left[c_1 + c_2 x + \frac{4}{35} x^{7/2}\right] e^x$
2. $y = c_1 e^x + c_2 e^{-x} - e^x \cos(e^{-x}) - \sin(e^{-x})$
3. $y = c_1 e^{3x} + c_2 \cdot x e^{3x} - e^{3x} \log x$
4. $y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cdot \cos x \log(\sec x + \tan x)$
5. $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^x + \left(\frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4}\right)$
6. $y = c_1 \cos x + c_2 \sin x + \log \cos x - 1 + \sin x \cdot \log(\sec x + \tan x)$
7. $y = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos 2x + \frac{4}{9} \sin 2x$
8. $y = c_1 e^x + c_2 e^{2x} - \frac{x^2}{2} e^x - xe^{-x} + x + \frac{3}{2}$
9. $y = c_1 x + c_2 x^{-1} + e^x - \frac{1}{x} e^x$
10. $y = c_1 + x^2 c_2 + (x - 1)e^x$
11. $y = (c_1 + xc_2)e^x + \frac{1}{2} x^2 e^x$
12. $y = c_1 e^x + c_2 e^{2x} + e^x \log(1 + e^x) + e^{2x} \log(1 + e^{-x})$
13. $y = (c_1 + xe_2)e^x - \frac{5}{36} x^3 e^x - \frac{x^3}{6} e^x \cdot \log x$

SUMMARY

1. Complete Solutions of $y'' + Py' + Qy = R$ in Terms of One Known Solution Belonging to the CF

Given $$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = R \quad (1)$$

Let $y = u(x)$ be a known solution of the CF, so u is a solution of (1) when its right-hand side is taken to be zero. Thus, $y = u$ is a solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad (2)$$

Working Rule for Finding Complete Solution when One Integral Solution of CF is known or can be Obtained by the Above Rules

Step 1: Put the given equation in standard form $y'' + py' + Qy = R$ in which the coefficient of y'' is unity.

Step 2: Find an integral u of CF by using the following table:

	Condition Satisfied	An integral of CF is
(i)	$1 + P + Q = 0$	$u = e^x$
(ii)	$1 - P + Q = 0$	$u = e^{-x}$
(iii)	$a^2 + aP + Q = 0$	$u = e^{ax}$
(iv)	$m(m-1) + Pmx + Qx^2 = 0$	$u = x^m$
(v)	$2 + 2px + Qx^2 = 0$	$u = x^2$

If a solution u is given in a problem then this step is omitted.

Step 3: Consider the complete solution of the given equation is $y = u\cdot v$, where u has been obtained in Step 2. Then find v using $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \quad (3)$$

Step 4: Put $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$, put in (3). Then (3) will come out to be a linear equation in q and x if $R \neq 0$. Solve it as usual. It $R = 0$ then variables q and x will be separable.

Step 5: Now, replace q by $\frac{dv}{dx}$ and separate the variables v and x. Integrate and determine v. Put this value of v in $y = u\cdot v$. This will lead us to the desired complete solution of the given equation.

2. Removal of the First Derivative (Reduction to Normal Form)

Working Rule for Solving Problems by using Normal Form

Step 1: Put the equation in the standard form $y'' + Py' + Qy = R$, in which the coefficient of $\frac{d^2y}{dx^2}$ must be unity.

Step 2: To remove the first derivative, we choose $u = e^{-\frac{1}{2}\int P\,dx}$

Step 3: We now assume that the complete solution of the given equation is $y = u\cdot v$; then the given equation reduces to normal form

$$\frac{d^2v}{dx^2} + Iv = S, \text{ where } I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \text{ and } S = R/u$$

Important Note: Success in solving the given equation depends on the success in solving $\frac{d^2v}{dx^2} + Iv = S$.

Now this latter equation can be solved easily if I takes two special forms.

(i) When I = constant then the resulting equation being with constant coefficients can be solved by usual methods.

(ii) When $I = \left(\frac{\text{constant}}{x^2}\right)$ then the resulting equation reduces to homogeneous form, and hence, it can be solved by using usual methods.

Step 4: After getting v, the complete solution is given by $y = u \cdot v$

3. Transformation of the Equation by Changing the Independent Variable

Working Rule for Solving Equations by Changing the Independent Variable

Step 1: Put the give DE in standard form by keeping the coefficient of y'' as unity i.e.

$$y'' + Py' + Qy = R \tag{4}$$

Step 2: Suppose $Q = \pm kf(x)$; then we assume a relation between the new independent variable z and let the old independent variable x given by $\left(\frac{dz}{dx}\right)^2 = kf(x)$.

Note: Carefully omit –ve sign of Q while writing this step. This is extremely important to find real values. Sometimes we assume that $\left(\frac{dz}{dx}\right)^2 = f(x)$ whenever we anticipate complicated relation between z and x.

Step 3: We now solve $\left(\frac{dz}{dx}\right)^2 = kf(x)$. Rejecting –ve sign, we get

$$\frac{dz}{dx} = \pm\sqrt{kf(x)} \tag{5}$$

Now separating variables, (5) gives

$$dz = \sqrt{[kf(x)]}\,dx \quad \text{so that} \quad z = \int \sqrt{[kf(x)]}\,dx \tag{6}$$

where we have omitted the constant of integration since we are interested in finding just a relation between z and x.

Step 4: With the relationship (6) between z and x, we transform (4) to get an equation of the form

$$\frac{d^2y}{dx^2} + P_1\left(\frac{dy}{dz}\right) + Q_1 y = R_1 \tag{7}$$

where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^z}$ and $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Now, by virtue of (5), $Q_1 = \dfrac{\pm kf(x)}{kf(x)} = \pm k$, a constant.

Then we calculate P_1.
If P_1 is also constant, then (7) can be solved because it will be a linear equation with constant coefficients. If, however, P_1 does not become constant then this rule will not be useful. The students must, therefore, be sure that P_1 comes out to be constant before proceeding further. The value of P_1, Q_1 and R_1 must be remembered for direct use in problems. R_1 can be converted to a function of z by using (6).
Step 5: After solving the equation (7) by usual methods, the variable z is replaced by x by using (6).

4. Method of Variation of Parameters

Working Rule for Solving Second Order LDE by the Method of Variation of Parameters

Step 1: Put the given DE in standard form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \tag{8}$$

Step 2: Put $R = 0$ in (8); then (8) becomes

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=O \tag{9}$$

Suppose the complementary function (CF) of (9) is

$$y = au + bv;\ a \text{ and } b \text{ are constants}$$

Step 3: Replace the parameters a and b by $A(x)$ and $B(x)$; then the complete solution of (8) is

$$y = Au + Bv$$

where $A=\int\frac{-Rv}{w}dx$ and $B=\int\frac{Ru}{w}dx$

and $\qquad w = \text{Wronskian of } u \text{ and } v=\begin{vmatrix} u & v \\ u' & v'\end{vmatrix}$

$$w = uv' - u'v$$

Now, find A and B, then $y = Au + Bv$.

OBJECTIVE-TYPE QUESTIONS

1. The general solution of the differential equation $x^2\frac{dy}{dx^2}-x\frac{dy}{dx}+y=0$ is

(a) $Ax + Bx^2$ (b) $Ax - B\log x$
(c) $Ax + Bx^2\log x$ (d) $Ax + Bx\log x$

[GATE (ME) 1998]

2. The differential equation

$$\frac{d^2y}{dx^2}+(x^2+4x)\frac{dy}{dx}+y=x^8-8 \text{ is}$$

(a) partial differential equation
(b) non-linear differential equation
(c) non-homogenous differential equation
(d) ordinary differential equation

[GATE (ME) 1999]

3. Consider the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ with the boundary conditions of $y(0) = 0$ and $y(1) = 1$. The complete solution of the differential equation is

(a) x^2 (b) $\sin\left(\frac{\pi x}{2}\right)$

(c) $e^x\sin\left(\frac{\pi x}{2}\right)$ (d) $e^{-x}\sin\left(\frac{\pi x}{2}\right)$

[GATE (ME) 2012]

4. Consider the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$. Which of the following is a solution to this differential equation for $x > 0$?

(a) e^x (b) x^2

(c) $1/x$ (d) $\ln x$

[GATE (EE) 2014]

5. If $y = f(x)$ is solution of $\frac{d^2y}{dx^2} = 0$ with the boundary conditions $y = 5$ at $x = 0$, and $\frac{dy}{dx} = 2$ at $x = 10$, $f(15) =$ _______.

[GATE (ME) 2014]

6. The differential equation $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + x^3 y = e^x$ is a

(a) non-linear differential equation of first degree

(b) linear differential equation of first degree

(c) linear differential equation of second degree

(d) non-linear differential equation of second degree **[GATE (CH) 2014]**

ANSWERS

1. (d)	2. (d)	3. (a)	4. (c)	5. (34 to 36)	6. (b)

13 Series Solutions

13.1 INTRODUCTION

The solutions of differential equations with variable coefficients such as Legendre's equation, Bessel's equation, etc., cannot be expressed in terms of standard functions. However, in such cases, the solution can be obtained in the form of an infinite series in terms of independent variables. In this chapter, the series solution method can defined in two ways:

(i) Power-series method

(ii) General series-solution method (Frobenius method)

13.2 CLASSIFICATION OF SINGULARITIES

Consider a homogeneous linear second-order differential equation with variable coefficients:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad (1);\ a_0 \neq 0$$

We can write the equation (1) in the form of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \qquad (2)$$

where $P(x) = \dfrac{a_1(x)}{a_0(x)}$, $Q(x) = \dfrac{a_2(x)}{a_0(x)}$

13.2.1 Analytic Function

A function $f(x)$ is said to be analytic at x_0 if $f(x)$ has Taylor's series expansion about x_0 such that $\sum_{n=0}^{\infty} \dfrac{f^n(x_0)}{n!}(x - x_0)^n$ exist and converges to $f(x)$ for all x in the interval including x_0.

Hence, we find that all polynomial functions, e^x, $\sin x$, $\cos x$, $\sinh x$, and $\cosh x$ are analytic everywhere. A rational function is analytic except at those values of x at which its denominator is zero, for example, the rational function defined by $\dfrac{x}{(x^2 - 3x + 2)}$ is analytic everywhere except at $x = 1$ and $x = 2$.

13.3 ORDINARY AND SINGULAR POINTS

A point $x = x_0$ is called an ordinary point of the equation

$$y'' + P(x)\, y' + Q(x)\, y = 0 \tag{3}$$

if both the function $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

If the point $x = x_0$ is not an ordinary point of the differential equation (3) then it is called a singular point of the differential equation (3) there are two types of singular points.

(i) Regular singular points

(ii) Irregular singular points

A singular point $x = x_0$ of the Differential Equation (DE) (3) is called a *regular singular point of the* (DE) (3) if both $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at $x = x_0$.

A singular point which is not regular is called an *irregular point*.

Example 1 Show that $x = 0$ and $x = 1$ are singular. Points of $x^2(x + 1)^2y'' + (x^2 - 1)y' + 2y = 0$ where the first is irregular and the other is regular.

Solution Dividing by $x^2(x + 1)^2$, the given equation becomes

$$y'' + \frac{(x-1)}{x^2(x+1)}y' + \frac{2}{x^2(x+1)^2}y = 0 \tag{1}$$

Comparing (1) with standard equation $y'' + p(x)y' + Q(x)y = 0$, we get

$$P(x) = \frac{(x-1)}{x^2(x+1)} \text{ and } Q(x) = \frac{2}{x^2(x+1)^2}$$

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$ and $x = -1$.

Hence, $x = 0$ and $x = -1$ are both singular points.

Also, $$(x - 0)\, P(x) = \frac{(x-1)}{x(x+1)} \text{ and } (x - 0)^2 Q(x) = \frac{2}{(x+1)^2}$$

Showing that both $(x +1)\, P(x)$ and $(x + 1)^2\, Q(x)$ are analytic at $x = -1$ and hence $x = -1$ is a regular singular points.

Example 2 $x^3\,(x - 1)\, y'' + 2\,(x - 1)\, y' + 5xy = 0.$

Solution Dividing by $x^3\,(x - 1)$, the given equation becomes

$$y'' + \frac{2}{x^3}y' + \frac{5}{x^2(x-1)}y = 0 \tag{1}$$

Here, $$P(x) = \frac{2}{x^3} \text{ and } Q(x) = \frac{5}{x^2(x-1)}$$

$x = 0$ and $x = 1$ are singular points.

Since $(x - 0)\, P(x) = \dfrac{2}{x^2}$ is not analytic at $x = 0$, we have $x = 0$ is an irregular singular point.

However, $x = 1$ is a regular singular point since both $(x - 1)\, P(x) = \dfrac{2(x-1)}{x^3}$ and $(x - 1)^2$ $Q(x) = \dfrac{5(x-1)}{x^2}$ are analytic at $x = 1$.

Example 3 Determine whether $x = 0$ is an ordinary point or a regular singular point of the differential equation $2x^2y'' + 7x(x+1)y' - 3y = 0$.

Solution Dividing by $2x^2$, the given equation becomes

$$y'' + \frac{7(x+1)}{2x}y' - \frac{3}{2x^2}y = 0 \tag{1}$$

Here, $$P(x) = \frac{7(x+1)}{2x},\ Q(x) = -\frac{3}{2x^2}$$

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, we have $P(x)$ and $Q(x)$ are not analytic at $x = 0$.
Thus, $x = 0$ is not an ordinary point and so $x = 0$ is a singular point.

Also, $(x-0)P(x) = \dfrac{7(x+1)}{2}$, $(x-0)^2\,Q(x) = -\dfrac{3}{2}$

Both $(x-0)P(x)$ and $(x-0)^2Q(x)$ are analytic at $x = 0$.
Hence, $x = 0$ is regular singular point.

EXERCISE 13.1

1. Show that $x = 0$ is an ordinary point of $y'' - x\,y' + 2y = 0$.
2. Show that $x = 0$ is an ordinary point of $(x^2 + 1)\,y'' + x\,y' - xy = 0$.
3. Show that $x = 0$ is a regular singular point of $x^2\,y'' + x\,y' + \left(x^2 - \dfrac{1}{4}\right)y = 0$.
4. Show that $x = 0$ is regular singular point and $x = 1$ is an irregular singular point of $x(x-1)^3y'' + 2(x-1)^3\,y' + 3y = 0$
5. Verify that the origin is a regular singular point of the equation $2x^2y'' + xy' - (x+1)y = 0$.
6. Determine the nature of the point $x = 0$ for the equation $x\,y'' + y\sin x = 0$.
7. Show that $x = 0$ is an ordinary point of $(x^2 - 1)\,y'' + x\,y' - y = 0$ but $x = 1$ is a regular singular point.

13.4 POWER SERIES

An infinite series of the form

$$\sum_{n=0}^{\infty} C_n(x-x_0)^n = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \cdots \tag{4}$$

is called a power series in $(x - x_0)$. The constants $C_0, C_1, C_2, \ldots$ are known as the coefficients and x_0 is called the center of the power series (4). Since n takes only positive integral values, the power series (4) does not contain negative or fractional powers. So power series (4) contains only positive powers.

The power series (4) converges (absolutely) for $|x| < R$, where

$$R = \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|, \text{ provided the limit exists.} \tag{5}$$

R is said to be the radius of convergence of the power series (4). The interval $(-R, R)$ is said to be the interval of convergence.

Result 1 A power series represents a continuous function within its interval of convergence.

Result 2 A power series can be differentiated termwise in its interval of convergence.

13.5 POWER-SERIES SOLUTION ABOUT THE ORDINARY POINT $x = x_0$

Consider the equation $y'' + P(x)\, y' + Q(x)y = 0$ (6)

Then $x = x_0$ is an ordinary point of (6) and has two nontrivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n\, (x - x_0)^n \tag{7}$$

And these power series converge in some interval of convergence $|x - x_0| < R$, about x_0.

In order to get the coefficient $C_n^{\,s}$ is (2), we take

$$y = \sum_{n=0}^{\infty} C_n\, (x - x_0)^n \tag{8}$$

Differentiating (8) twice in succession w.r.t. x, we have

$$\therefore \qquad y' = \sum_{n=1}^{\infty} n C_n\, (x - x_0)^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1) C_n\, (x - x_0)^{n-2} \tag{9}$$

Putting the above values of y, y', and y'' in (6), we get an equation of the form

$$A_0 + A_1\, (x - x_0) + A_2\, (x - x_0)^2 + \dots + A_n\, (x - x_0)^n + \dots = 0 \tag{10}$$

where the coefficients $A_0, A_1, A_2, \dots$, etc., are now some functions of the coefficients $C_0, C_{1,}\, C_2, \dots$, etc. Since (10) is an identity, all the coefficients $A_0, A_1, A_{2,} \dots$ of (10) must be zero, i.e.,

$$A_0 = 0, A_1 = 0, A_2 = 0, \dots., A_n = 0 \tag{11}$$

Solving Eq. (11),we obtain the coefficients of (8) in terns of C_0 and $C_{1,}$ Substituting these coefficients in (8), we obtain the required series solution (6) in powers of $(x - x_0)$.

Example 4 Find the solution in series of $y'' + x\, y' + x^2 y = 0$ about $x = 0$.

Solution $y'' + x\, y' + x^2 y = 0$ (1)

Compare (1) with $y'' + P(x)\, y' + Q(x)\, y = 0$

Here, $P(x) = x$, $Q(x) = x^2$

Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, we have $x = 0$ is an ordinary point.

$$\text{Consider} \qquad y = C_0 + C_1 x + C_2\, x^2 + \dots = \sum_{n=0}^{\infty} C_n\, x^n \tag{2}$$

Differentiating (2) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} C_n \cdot n\, x^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} C_n\, n(n-1)\, x^{n-2} \tag{3}$$

Putting the above values of y, y' and y'' in (1), we get

$$\sum_{n=2}^{\infty} C_n \cdot n(n-1)\, x^{n-2} + x \sum_{n=1}^{\infty} C_n \cdot n x^{n-1} + x^2 \sum_{n=0}^{\infty} C_n\, x^n = 0$$

or $$\sum_{n=2}^{\infty} C_n \cdot n(n-1)x^{n-2} + \sum_{n=1}^{\infty} C_n \cdot nx^n + \sum_{n=0}^{\infty} C_n \cdot x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} C_{n+2} \cdot (n+2)(n+1)x^n + \sum_{n=1}^{\infty} C_n\, n x^n + \sum_{n=2}^{\infty} C_{n-2}\, x^n = 0$$

or $$2C_2 + (6C_3 + C_1)x + \sum_{n=2}^{\infty} [(n+1)(n+2)C_{n+2} + nC_n + C_{n-2}]x^n = 0 \tag{4}$$

Equating the constant term and the coefficients of various powers of x to zero, we get

$$2\,C_2 = 0 \Rightarrow C_2 = 0$$

$$6\,C_3 + C_1 = 0 \Rightarrow \; C_3 = -\frac{1}{6}C_1$$

$(n + 1)(n + 2)\, C_{n+2} + nC_n + C_{n-2} = 0 \quad \forall\, n > 2$

or $$C_{n+2} = -\frac{nC_n + C_{n-2}}{(n+1)(n+2)} \quad \forall\, n > 2 \tag{5}$$

Putting $n = 2$ in (5), $C_4 = -\dfrac{2C_2 + C_0}{(3 \cdot 4)} = -\dfrac{C_0}{12}$

Putting $n = 3$ in (5), $C_5 = -\dfrac{3C_3 + C_1}{(4 \cdot 5)} = -\dfrac{-\frac{1}{2}C_1 + C_1}{20}$

$$C_5 = -\frac{C_1}{40}$$

Putting $n = 4$ in (5), $C_6 = -\dfrac{4C_4 + C_2}{(5 \cdot 6)} = -\dfrac{-\frac{4C_0}{12} + 0}{30}$

$$= -\frac{-\frac{1}{3} + C_0}{30} = \frac{C_0}{90}$$, and so on. Putting these values in (2),

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y = C_0 + C_1 x - \frac{1}{6}C_1 x^3 - \frac{1}{12}C_0 x^4 - \frac{1}{40}C_1 x^5 + \frac{1}{90}C_0 x^5 + \cdots$$

or $$y = C_0\left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 \cdots\right) + C_1\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 \cdots\right)$$

Example 5 Find the power-series solution of the equation $(x^2 + 1)\, y'' + x\, y' - xy = 0$ in powers of x.

Solution Dividing by $(x^2 + 1)$ can be written as

$$y'' + \left(\frac{x}{x^2 + 1}\right)y' - \left(\frac{x}{x^2 + 1}\right)y = 0 \tag{1}$$

Compare (1) with $y'' + P(x)y' + Q(x)y = 0$

Here,

$$P(x) = \frac{x}{x^2+1}, \quad Q(x) = -\frac{x}{x^2+1}$$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$. So $x = 0$ is an ordinary point.

Consider the power-series solution of the form

$$y = c_0 + c_1 x + c_2 x^2 + \ldots = \sum_{n=0}^{\infty} c_n x^n \tag{2}$$

Differentiating (2) twice in succession w.r.t. 'x', we get

or $$y' = \sum_{n=1}^{\infty} c_n \cdot n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \tag{3}$$

Substituting the values of y, y', and y'' in (1), we get

$$(x^2+1)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} c_n \cdot n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}x^n + \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=1}^{\infty} C_{n-1}x^n = 0$$

or $$2C_2 + (6C_3 + C_1 - C_0)x + \sum_{n=2}^{\infty}[n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_{n-1}]x^n = 0 \tag{4}$$

Equating the constant term and the coefficients of various powers of x to zero, we get

$$2C_2 = 0 \Rightarrow C_2 = 0$$

$$6C_3 + C_1 - C_0 = 0 \Rightarrow \frac{(C_0 - C_1)}{6} = C_3$$

$$n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_{n-1} = 0 \quad \forall\, n \ge 2$$

$$C_{n+2} = \frac{C_{n-1} - n^2 C_n}{(n+1)(n+2)} \quad \forall\, n \ge 2 \tag{5}$$

The above relation (5) is known as *recurrence relation.*

Putting $n = 2$ in (5), $C_4 = \frac{1}{12} C_1$ as $C_2 = 0$

Putting $n = 3$ in (5), $C_5 = \frac{-9C_3}{(4 \cdot 5)} = -\frac{9}{20} \cdot \left(\frac{C_0 - C_1}{6}\right)$

$$C_5 = -\frac{3}{40}(C_0 - C_1)$$

Putting the above values of $C_2, C_3, C_4, C_5, \ldots$ in (2)

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$y = C_0 + C_1 x + \frac{1}{6}(C_0 - C_1)x^3 + \frac{1}{12}C_1 x^4 - \frac{3}{40}(C_0 - C_1)x^5 + \cdots$$

$$y = C_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \cdots\right) + C_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \cdots\right)$$

which is the required solution near $x = 0$, where C_0 and C_1 are arbitrary constants.

Example 6 Solve $y'' - 2x^2 y' + 4xy = x^2 + 2x + 4$ in powers of x.

Solution The given equation is $y'' - 2x^2 y' + 4xy = x^2 + 2x + 4$ (1)

Compare (1) with $y'' + P(x)y' + Q(x)y = R$

Here, $P(x) = -2x^2$, $Q(x) = 4x$.

So $P(x)$ and $Q(x)$ are analytic at $x = 0$,

$x = 0$ is an ordinary point of (1).

Consider
$$y = C_0 + C_1 x + C_2 x^2 + \ldots = \sum_{n=0}^{\infty} C_n x^n \tag{2}$$

Differentiating (2) twice in succession w.r.t. 'x', we have

$$y' = \sum_{n=1}^{\infty} n \cdot C_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} \tag{3}$$

Substituting these values of y, y', and y'' in (1),

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 2x^2 \sum_{n=1}^{\infty} n \cdot C_n x^{n-1} + 4x \sum_{n=0}^{\infty} C_n x^n = (x^2 + 2x + 4)$$

or
$$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}x^n - \sum_{n=1}^{\infty} 2nC_n x^{n+1} + \sum_{n=0}^{\infty} 4C_n x^{n+1} = (x^2 + 2x + 4)$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}x^n - \sum_{n=2}^{\infty} 2(n-1)C_{n-1}x^n + \sum_{n=1}^{\infty} 4C_{n-1}x^n = (x^2 + 2x + 4)$$

or
$$2C_2 + (6C_3 + 4C_0)x + (12C_4 + 2C_1)x^2 + \sum_{n=3}^{\infty} [(n+2)(n+1)C_{n+2} - 2(n-1)C_{n-1} + 4C_{n-1}]x^n = (x^2 + 2x + 4) \tag{4}$$

Equating to zero the coefficients of various powers of x in (4), we get

$$2C_2 = 4 \Rightarrow C_2 = 2$$

$$6C_3 + 4C_0 = 2 \Rightarrow C_3 = \frac{1}{3} - \frac{2}{3}C_0$$

$$12C_4 + 2C_1 = 1 \Rightarrow C_4 = \frac{1}{12} - \frac{1}{6}C_1 \text{ and}$$

$$(n+2)\,(n+1)\,C_{n+2} - 2\,(n-1)\,C_{n-1} + 4\,C_{n-1} = 0 \;\forall\; n \geq 3 \tag{5}$$

Putting $n = 3, 4, 5, \ldots$ in (5), we get

$$20\,C_5 - 4\,C_2 + 4\,C_2 = 0 \text{ so that } C_5 = 0$$

$$30\,C_6 = 2\,C_3 \text{ so that } C_6 = \frac{1}{15}\left(\frac{1}{3} - \frac{2}{3}C_0\right)$$

$$= \frac{1}{45} - \frac{2}{45}C_0$$

$$42C_7 = 4\,C_4 \text{ so that}$$

$$C_7 = \frac{2}{21}\left(\frac{1}{12} - \frac{1}{6}C_1\right) = \frac{1}{126} - \frac{1}{63}C_1\text{, and so on.}$$

Putting these values in (2), the required solution is

$$y = C_0 + C_1\,x + 2\,x^2 + \left(\frac{1}{3} - \frac{2\,C_0}{3}\right)x^3 + \left(\frac{1}{12} - \frac{C_1}{6}\right)x^4 + \left(\frac{1}{45} - \frac{2}{45}C_0\right)x^6 + \left(\frac{1}{126} - \frac{C_1}{63}\right)x^7 + \cdots$$

or $y = C_0\left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 - \ldots\right) + C_1\left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 - \ldots\right)$

$$+\,2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + \cdots$$

Example 7 Find the general solution of $y'' + (x-3)\,y' + y = 0$ near $x = 2$.

Solution The given equation is $y'' + (x-3)\,y' + y = 0$ (1)

Compare (1) with $y'' + P(x)\,y' + Q(x)\,y = 0$

Here, $P(x) = (x-3)$ and $Q(x) = 1$

Since both $P(x)$ and $Q(x)$ are analytic at $x = 2$, so $x = 2$ is an ordinary point of (1).
To find the solution near $x = 2$, we shall find a series solution in powers of $(x-2)$.

Consider $\quad y = C_0 + C_1(x-2) + C_2(x-2)^2 + C_3(x-2)^3 + \ldots$

$$y = \sum_{n=0}^{\infty} C_n\,(x-2)^n \tag{2}$$

Differentiating (2) twice in succession w.r.t. 'x', we get.

$$y' = \sum_{n=1}^{\infty} n \cdot C_n\,(x-2)^{n-1} \quad\text{and}\quad y'' = \sum_{n=2}^{\infty} n\,(n-1)\,C_n\,(x-2)^{n-2}$$

Putting the values of y, y', and y'' in (1), we get $n = 2$

$$\sum_{n=2}^{\infty} n\,(n-1)\,C_n\,(x-2)^{n-2} + (x-3)\sum_{n=1}^{\infty} n \cdot C_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n\,(x-2)^n = 0$$

or

$$\sum_{n=2}^{\infty} n\,(n-1)\,C_n\,(x-2)^{n-2} + [(x-2)-1]\sum_{n=1}^{\infty} n \cdot C_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n\,(x-2)^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-2)^{n-1} - \sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or $$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}(x-2)^n + \sum_{n=1}^{\infty} nC_n(x-2)^n - \sum_{n=0}^{\infty}(n+1)C_{n+1}(x-2)^n + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or $$(2C_2 - C_1 + C_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)C_{n+2} + nC_n - (n+1)C_{n+1} + C_n](x-2)^n = 0 \tag{3}$$

Equating to zero the coefficients of various powers of $(x - 2)$, we get.

$$2\,C_2 - C_1 + C_0 = 0 \text{ so that } C_2 = (C_1 - C_0)/2$$

$$(n + 2)\,(n + 1)\,C_{n+2} + (n + 1)\,C_n - (n + 1)\,C_{n+1} = 0\ \forall\ n \geq 1$$

or $$C_{n+2} = \frac{(C_{n+1} - C_n)}{(n+2)}\ \forall\ n \geq 1 \tag{1}$$

Putting $n = 1, 2, 3, \ldots$ in (4), we get

$$C_3 = \frac{C_2 - C_1}{3} = \frac{\frac{(C_1 - C_0)}{2} - C_1}{3}$$

$$C_3 = -\left(\frac{C_0 + C_1}{6}\right)$$

$$C_4 = \frac{C_3 - C_2}{4} = \frac{1}{4}\left[-\frac{C_0 + C_1}{6} - \frac{C_1 - C_0}{2}\right]$$

$$= \frac{1}{12}C_0 - \frac{1}{16}C_1, \text{ and so on.}$$

Putting these values in (2), the required solution near $x = 2$ is

$$y = C_0 + C_1(x-2) + \left(\frac{C_1 - C_0}{2}\right)(x-2)^2 - \left(\frac{C_0 + C_1}{6}\right)(x-2)^3$$
$$+ \left(\frac{1}{12}C_0 - \frac{1}{6}C_1\right)(x-2)^4 + \cdots$$

or $$y = C_0\left[1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 + \cdots\right]$$
$$+ C_1\left[(x-2)^2 + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 + \cdots\right]$$

Example 8 Find the power-series solution in powers of $(x - 1)$ of the initial-value problem $xy'' + y' + 2y = 0$, $y(1) = 1$, $y'(1) = 2$.

Solution The given equation is

$$x\,y'' + y' + 2y = 0 \tag{1}$$

Dividing by x on (1), we get

$$y'' + \frac{1}{x}y' + \frac{2}{x}y = 0 \tag{2}$$

Comparing (2) with $y'' + P(x)\,y' + Q(x)\,y = 0$

Here, $P(x) = \frac{1}{x},\ Q(x) = \frac{2}{x}$

Since $P(x)$ and $Q(x)$ are analytic at $x = 1$, so $x = 1$ is an ordinary point of (1)

Consider $$y = C_0 + C_1\,(x-1) + C_2(x-1)^2 + C_3(x-1)^3 + \cdots = \sum_{n=0}^{\infty} C_n\,(x-1)^n \tag{3}$$

Differentiating (3) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} n\cdot C_n\,(n-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n\cdot(n-1)\,C_n\,(x-1)^{n-2} \tag{4}$$

Putting the values of y, y' and y'' in (1), we get

$$x\sum_{n=2}^{\infty} n(n-1)C_n\,(x-1)^{n-2} + \sum_{n=1}^{\infty} n\cdot C_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n\,(x-1)^n = 0$$

or $$[(x-1)+1]\sum_{n=2}^{\infty} n(n-1)C_n\,(x-1)^{n-2} + \sum_{n=1}^{\infty} n\cdot C_n\,(x-1)^{n-1} + \sum_{n=0}^{\infty} 2C_n\,(x-1)^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n\,(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)C_n\,(x-1)^{n-2} + \sum_{n=1}^{\infty} n\cdot C_n\,(x-1)^{n-1} + \sum_{n=0}^{\infty} 2C_n\,(x-1)^n = 0$$

or $$\sum_{n=1}^{\infty}(n+1)nC_{n+1}\,(x-1)^n + \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}\,(x-1)^n$$

$$+ \sum_{n=0}^{\infty}(n+1)C_{n+1}\,(n-1)^n + \sum_{n=0}^{\infty} 2C_n\,(x-1)^n = 0$$

Equating to zero the coefficients of various powers of $(x-1)$, we get

$(2\,C_2 + C_1 + 2\,C_0) = 0$ so that $C_2 = -(C_1 + 2\,C_0)/2$ and $(n+1)\,n\,C_{n+1} + (n+2)\,(n+1)\,C_{n+2}$

$+ (n+1) + 2\,C_n = 0 \quad \forall\, n \geq 1$

or $(n+1)\,(n+2)\,C_{n+2} + (n+1)^2\,C_{n+1} + 2\,C_n = 0\ \forall\, n \geq 1$

or $$C_{n+2} = -\frac{(n+1)^2\,C_{n+1} + 2C_n}{(n+1)(n+2)} \quad \forall\, n \geq 1 \tag{5}$$

Given that $y(1) = 1$ and $y'(1) = 2$, putting $x = 1$ in (3), we get,

$C_0 = 1$, and $C_1 = 2$

$\therefore$ $C_2 = -(2+2)/2 = -2$

Putting $x = 1, 2, 3, \ldots$in (5), we get

$$C_3 = -\frac{2^2 C_2 + 2C_1}{2\cdot 3} = -\frac{4x-2+(2\times 2)}{2\cdot 3} = \frac{2}{3}$$

$$C_4 = -\frac{3^2 C_3 + 2C_2}{3\cdot 4} = -\frac{9\times\left(\frac{2}{3}\right)+2(-2)}{3\cdot 4} = -\frac{1}{6}$$

$$C_5 = -\frac{4^2 C_4 + 2C_3}{4\cdot 5} = -\frac{16\cdot\left(\frac{-1}{6}\right)+2\left(\frac{2}{3}\right)}{4\cdot 5} = -\frac{1}{15} \text{ and so on.}$$

Putting these values in (3), we obtain the required solution.

$$y = 1 + 2(x-1) - 2(x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots$$

EXERCISE 13.2

Find the series solution of the following equations:

1. $(1 - x^2)\, y'' + 2x\, y' - y = 0$ about $x = 0$
2. $(1 + x^2)\, y'' + x\, y' - y = 0$ near $x = 0$
3. $(x^2 - 1)\, y'' + x\, y' - y = 0$ near $x = 0$
4. $y'' + x^2\, y = 2 + x + x^2$ about $x = 0$
5. $y'' - x\, y' - py = 0$, where p is any constant.
6. $y'' - x\, y' + 2y = 0$ near $x = 1$
7. $(x^2 - 1)\, y'' + 3\, x\, y' + xy = 0$, $y(0) = 2$, $y'(0) = 3$
8. $(x^2 + 2x)\, y'' + (x + 1)\, y' - y = 0$, near $x = -1$
9. $y'' - x\, y' = e^{-x}$, $y(0) = 2$, $y'(0) = -3$

   $$\left[\text{Hint: Use } e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$
10. $(1 - x^2)\, y'' + 2\, y = 0$, $y(0) = 4$, $y'(0) = 5$

Answers

1. $y = C_0\left(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 - \ldots\right) + C_1\left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \ldots\right)$

2. $y = C_0\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{15}x^6 - \ldots\right) + C_1 \cdot x$

3. $y = C_0\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + \ldots\right) + C_1 \cdot x$

4. $y = C_0\left(1-\frac{x^4}{12}+\frac{x^8}{672}-...\right)+C_1\left(x-\frac{x^5}{20}+\frac{x^9}{1440}...\right)+x^2+\frac{x^3}{6}+\frac{x^4}{12}-\frac{x^6}{30}-\frac{x^7}{252}...$

5. $y = C_0\left[1+\frac{p}{2!}x^3+\frac{p(p+2)}{4!}x^4+\cdots\right]+C_1\left[x+\frac{(p+1)}{3!}x^3+\frac{(p+1)(p+3)}{5!}x^5+\cdots\right]$

6. $y = C_0\left[1-(x-1)^2-\frac{1}{3}(x-1)^3-\cdots\right]+C_1\left[(x-1)+\frac{1}{2}(x-1)^2-\cdots\right]$

7. $y = 2+3x+\left(\frac{11}{6}\right)x^3+\frac{1}{4}x^4-\cdots$

8. $y = C_0\left[1-\frac{1}{2}(x+1)^2-\frac{1}{8}(x+1)^4-\frac{1}{16}+(x+1)^6+...\right]+C_1(x+1)$

9. $y = 2-3x+\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{8}x^4-\cdots$

10. $y = 4+5x-4x^2-\frac{5}{3}x^3-\frac{1}{3}x^5+\cdots$

13.6 FROBENIUS METHOD

The series solution when $x = 0$ is a regular singular point of the differential equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0\cdot \tag{12}$$

Step 1 Suppose that a trial solution of (12) be of the form.

$$y = x^k\,(C_0+C_1x+C_2\,x^2+\ldots+C_m\,x^m+\ldots)$$

i.e.,
$$y = \sum_{m=0}^{\infty} C_m\cdot x^{m+k},\ C_0\neq 0 \tag{13}$$

Step 2 Differentiate (13) and obtain

$$y' = \sum_{m=0}^{\infty} C_m\,(m+k)x^{m+k-1} \text{ and } y'' = \sum_{m=0}^{\infty} C_m\,(m+k)(m+k-1)x^{m+k-2} \tag{14}$$

Using (13) and (14), (12) reduces to an identity.

Step 3 Equating to zero the coefficient of the smallest power of x in the identity obtained in Step 2 above, we obtain a quadratic equation in k. The quadratic equation so obtained is called an *indicial equation*.

Step 4 **Solving the indicial equation, the following cases arise:**

(i) The roots of the indicial equation are distinct and do not differ by an integer.

(ii) The roots of the indicial equation are equal.

(iii) The roots of the indicial equation are distinct and differ by an integer.

(iv) The roots of the indicial equation are unequal, differing by an integer and making *a* coefficient of y indeterminate.

Step 5 We equate to zero the coefficient of the general power in the identity obtained in Step 2. The equation so obtained will be called the ***recurrence relation***, because it connects the coefficients C_m, C_{m-2} or C_m, C_{m-1}, etc.

Step 6 If the recurrence relation connects C_m and C_{m-2} then we, in general, determine C_1 by equating to zero the coefficient of the next higher power. On the other hand, if the recurrence relation connects C_m and C_{m-1}, this step may be omitted.

Step 7 After getting various coefficients with the help of steps 5 and 6 above, the solution is obtained by substituting these in (13) above.

13.6.1 Case 1

***The roots of the indicial equation are distinct and do not differ by an integer*.**

If k_1 and k_2 do not differ by an integer then, in general, two independent solutions u and v are obtained by putting $k = k_1$ and $k = k_2$ in the series of y. Then the general solution is $y = au + bv$, where a and b are arbitrary constants.

Example 9 Solve in series the differential equation

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)y = 0$$

Solution The given equation is

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)y = 0 \tag{1}$$

Dividing by $2x^2$ both sides on (1), we get

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \left(\frac{1-x^2}{2x^2}\right)y = 0 \tag{2}$$

Comparing (2) with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we get

$$P(x) = -\frac{1}{2x},\ Q(x) = \frac{1-x^2}{2x^2}$$

so that at $x = 0$, $xP(x) = -\frac{1}{2}$ and $x^2Q(x) = \frac{1-x^2}{2}$. Thus, both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$ and so $x = 0$ is a regular singular point.

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{m+k},\ C_0 \neq 0 \tag{3}$$

$$\therefore \qquad y' = \sum_{m=0}^{\infty} C_m (m+k) x^{m+k-1}, \; y'' = \sum_{m=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k-2}$$

Substituting y, y', and y'' in (1), we have

$$2x^2 \sum_{m=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k-2} - x \sum_{m=0}^{\infty} C_m (m+k) x^{m+k-1} + (1-x^2) \sum_{m=0}^{\infty} C_m x^{m+k} = 0$$

or $$2\sum_{m=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k} - \sum_{m=0}^{\infty} C_m (m+k) x^{m+k} + \sum_{m=0}^{\infty} C_m x^{m+k} - \sum_{m=0}^{\infty} C_m x^{m+k+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \cdot [2(m+k)(m+k-1) - (m+k) + 1] \cdot x^{m+k} - \sum_{m=0}^{\infty} C_m x^{m+k+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \cdot \left[2(m+k)^2 - 3(m+k) + 1\right] x^{m+k} - \sum_{m=0}^{\infty} C_m \cdot x^{m+k+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \cdot (2m+2k-1) \cdot (m+k-1) x^{m+k} - \sum_{m=0}^{\infty} C_m x^{m+k+2} = 0 \qquad (4)$$

[$\because$ $2(m+k)^2 - 2(m+k) + 1 = 2(m+k)^2 - 2(m+k) - (m+k) + 1$
$= 2(m+k)(m+k-1) - (m+k-1) = (m+k-1)(2m+2k-1)$]

which is an identity. Equating to zero the coefficient of the smallest power of x, (4) gives the indicial equation

$$C_0 (2k-1)(k-1) = 0 \text{ or } (2k-1)(k-1) = 0 \qquad [\because C_0 \neq 0]$$

so that $k = 1, \dfrac{1}{2}$

The roots of the indicial equation are unequal, not differing by an integer.

To obtain the recurrence relation, we equate to zero the coefficient of x^{m+k} and obtain

$$C_m = (2k + 2m - 1)(k + m - 1) - C_{m-2} = 0$$

or $$C_m = \frac{1}{(2m+2k-1)\cdot(m+k-1)} C_{m-2} \qquad (5)$$

To obtain C_1, we now equate to zero the coefficient of x^{k+1} and we get $C_1(2k+1)k = 0$ so that $C_1 = 0$

Now, from (5) and $C_1 = 0$, we have

$$C_1 = C_3 = C_5 = \cdots = 0 \qquad (6)$$

Putting $m = 2$ in (5), $C_2 = \dfrac{1}{(2k+3)(k+1)} C_0 \qquad (7)$

Putting $m = 4$ in (5) and using (7) gives

$$C_4 = \frac{1}{(2k+5)(k+3)} C_2 = \frac{1}{(k+1)(k+3)(2k+3)(2k+5)} C_0$$

And so on putting these values in (3), i.e.,

$$y = x^k (C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots)$$

$$y = C_0 x^k \left[1 + \frac{x^2}{(k+1)(2k+3)} + \frac{x^4}{(k+1)(k+3)(2k+3)(2k+5)} + \cdots\right] \tag{8}$$

Putting $k = 1$ and replacing C_0 by a in (8) gives

$$y = ax\left[1 + \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} + \cdots\right] = au \text{ (say)}$$

Next, putting $k = \frac{1}{2}$ in (8) and replacing C_0 by b, gives

$$y = bx^{\frac{1}{2}}\left[1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} + \cdots\right] = bv \text{ (say)}$$

Hence, the required general series solution is given by

$y = au + bv$, where a and b are arbitrary constants.

Example 10 Verify that $x = 0$ is a regular singular point of $2x^2 y'' + xy' - (x + 1)y = 0$ and find two independent Frobenius series solutions of it.

Solution The given differential equation

$$2x^2y'' + xy' - (x + 1)y = 0 \tag{1}$$

Dividing by $2x^2$ both sides of (1), we get

$$y'' + \frac{1}{2x}y' - \left(\frac{x+1}{2x^2}\right)y = 0 \tag{2}$$

Comparing (2) with $y'' + P(x)\, y' + Q(x)y = 0$,

$$P(x) = \frac{1}{2x}, Q(x) = -\left(\frac{x+1}{2x^2}\right)$$

so that $x(p)x = \frac{1}{2}$ and $x^2Q(x) = -\left(\frac{x+1}{2}\right)$. Since $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$, so $x = 0$ is a regular singular point.

Suppose the series solution of (1) given by

$$y = \sum_{m=0}^{\infty} C_m\, x^{m+k},\ C_0 \neq 0 \tag{3}$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m\,(m+k)\,x^{m+k-1},\ y'' = \sum_{m=0}^{\infty} C_m\,(m+k)(m+k-1)x^{m+k-2}$$

Putting the values of y, y', and y'' in (1), we get

$$2x^2 \sum_{m=0}^{\infty}(m+k)(m+k-1)C_m\, x^{m+k-2} + x\sum_{m=0}^{\infty}(m+k)C_m\, x^{m+k-1} - (x+1)\sum_{m=0}^{\infty}C_m\, x^{m+k} = 0$$

$$\text{or } \sum_{m=0}^{\infty}2(m+k)(m+k-1)C_m\, x^{m+k} + \sum_{m=0}^{\infty}(m+k)C_m\, x^{m+k} - \sum_{m=0}^{\infty}C_m\, x^{m+k-1} - \sum_{m=0}^{\infty}C_m\, x^{m+k} = 0$$

or $$\sum_{m=0}^{\infty} [2(m+k)(m+k-1)+(m+k)-1]C_m\, x^{m+k} - \sum_{m=0}^{\infty} C_m\, x^{m+k+1} = 0$$

or $$\sum_{m=0}^{\infty} [2(m+k)^2 - (m+k)-1]C_m\, x^{m+k} - \sum_{m=0}^{\infty} C_m\, x^{m+k+1} = 0$$

or $$\sum_{m=0}^{\infty} [\{2(m+k)+1\}(m+k-1)]C_m\, x^{m+k} - \sum_{m=0}^{\infty} C_m\, x^{m+k+1} = 0 \quad (4)$$

Equating to zero the coefficient of the smallest power of x, namely, x^k, the above equation (4) in x gives the indicial equation

$$C_0\,(2k+1)\,(k-1) = 0 \text{ so that } k = 1,\ -\frac{1}{2},\ C_0 \neq 0$$

Here, the difference of the roots $= k_1 - k_2 = 1 - \frac{1}{2}, = \frac{3}{2}$ (which is not an integer).

Next, we equate to zero the coefficient of x^{m+k} in (4) and obtain the recurrence relation

$$\{2\,(m+k)+1)\}\,(m+k-1)\,C_m - C_{m-1} = 0$$

Or $$C_m = \frac{1}{\{2(m+k)+1\}(m+k-1)} C_{m-1} \quad (5)$$

Putting $m = 1, 2, 3, 4, \ldots$ in (5), we have

$$C_1 = \frac{1}{(2k+3)} C_0 \quad (6)$$

$$C_2 = \frac{1}{(2k+5)(k+1)} C_1 = \frac{1}{(2k+3)(2k+5)\,k\,(k+1)} C_0, \text{ (7) by (6) and so on}$$

Putting these values in (3), we get

$$y = C_0\, x^k \left[1 + \frac{x}{(2K+3)k} + \frac{x^2}{(2k+3)(2k+5)\,k\,(k+1)} + \ldots\right] \quad (8)$$

Putting $k = 1$, and replacing C_0 by a in (8), we get

$$y = ax\left[1 + \frac{x}{5} + \frac{x^2}{70} + \ldots\right] = au \text{ (say)}$$

Next, putting $k = -\frac{1}{2}$ and replacing C_0 by b in (8), we get

$$y = bx^{-\frac{1}{2}}\left[1 - x - \frac{x^2}{2} + \cdots\right] = bv \text{ (say)}$$

Hence, the general solution of (1) is given by

$y = au + bv$, where a and b are arbitrary constants.

Example 11 Show that $x = 0$ is a regular singular point of

$(2x + x^3)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 6xy = 0$ and find its solution about $x = 0$ using the series method.

Solution The given differential equation is

$$(2x + x^3)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6xy = 0 \tag{1}$$

Dividing by $(2x + x^3)$, we have

$$\frac{d^2y}{dx^2} - \frac{1}{(2x+x^3)}\frac{dy}{dx} + \frac{6}{(2+x^2)}y = 0 \tag{2}$$

Comparing (2) with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = -\frac{1}{(x^3+2x)}, \quad Q(x) = \frac{6}{x^2+2}$$

At $x = 0$, $P(x)$ and $Q(x)$ are not analytic so that $x = 0$ is a singular point.

Now, $\quad (x-0)\,p(x) = x\left\{\dfrac{-1}{x(2+x^2)}\right\} = -\dfrac{1}{x^2+2}$ and $(x-0)^2\,Q(x) = -\dfrac{x^2\cdot 6}{x^2+2} = -\dfrac{6x^2}{x^2+2}$

Since $x\,P(x)$ and $x^2Q(x)$ are both analytic at $x = 0$, so $x = 0$ is a regular solution point of (1).
Let the series solution of (1) be given by

$$y = \sum_{m=0}^{\infty} C_m\, x^{m+k} = x^k\,(C_0 + C_1 x + C_2 x^2 + \cdots) \tag{3}$$

Differentiating (3) w.r.t. 'x', both sides, we have

$$\frac{dy}{dx} = \sum_{m=0}^{\infty}(m+k)C_m\, x^{m+k-1}, \quad \frac{d^2y}{dx^2} = \sum_{m=0}^{\infty}(m+k)(m+k-1)C_m\, x^{m+k-2}$$

Putting the values of y, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$(2x+x^3)\sum_{m=0}^{\infty}(m+k)(m+k-1)C_m\, x^{m+k-2} - \sum_{m=0}^{\infty}(m+k)C_m\, x^{m+k-1} - 6x\sum_{m=0}^{\infty}C_m\, x^{m+k} = 0$$

or $$\sum_{m=0}^{\infty}2(m+k)(m+k-1)C_m\, x^{m+k-1} + \sum_{m=0}^{\infty}(m+k)(m+k-1)\,x^{m+k-1}$$

$$-\sum_{m=0}^{\infty}(m+k)C_m\, x^{m+k-1} - 6\sum_{m=0}^{\infty}C_m\, x^{m+k+1} = 0$$

or $$\sum_{m=0}^{\infty}[2(m+k)(m+k-1) - (m+k)]C_m\, x^{m+k-1} + \sum_{m=0}^{\infty}[(m+k)(m+k-1) - 6]C_m\, x^{m+k+1} = 0$$

or $$\sum_{m=0}^{\infty}[2(m+k)^2-3(m+k)]C_m\, x^{m+k-1}+\sum_{m=0}^{\infty}[(m+k)^2-(m+k)-6]C_m\, x^{m+k+1} = 0$$

or $$\sum_{m=0}^{\infty}(m+k)(2m+2k-3)C_m\, x^{m+k-1}+\sum_{m=0}^{\infty}(m+k-3)(m+k+2)C_m\, x^{m+k+1} = 0 \tag{4}$$

Equating to zero the coefficients of the smallest power of x, i.e., x^{k-1}, in (4) gives the indicial equation

$$C_0\, k\,(2k-3)=0 \text{ so that } C_1 \neq 0,\ k=0, \text{ and } \frac{3}{2}$$

Roots of the indicial equation are distinct and do not differ by an integer and the difference of the roots $= \frac{3}{2}-0=\frac{3}{2} \neq$ not an integer.

Next, equating to zero the coefficient of x^{m+k-1} in (4), we get

$$(m+k)\,(2m+2k-3)C_m+(m+k-5)\,(m+k)C_{m-2}=0$$

or $$C_m = -\frac{(m+k-5)(m+k)}{(m+k)(2m+2k-3)}C_{m-2}$$

or $$C_m = -\frac{(m+k-5)}{(2m+2k-3)}C_{m-2} \tag{5}$$

Putting $m = 3, 5, 7, 9, \ldots$ in (5) and noting that $C_1 = 0$, we get

$$C_1 = C_3 = C_5 = C_7 = C_9 = \ldots = 0 \tag{6}$$

Next, putting $m = 2, 4, 6, 8, \ldots$ in (5), we get

$$C_2 = -\frac{k-3}{2k+1}C_0,\ C_4 = -\frac{k-1}{2k+5}C_2 = \frac{(k-1)(k-3)}{(2k+1)(2k+5)}C_0 \cdots \text{ and so on}$$

Putting these values in (3), we have

$$y = C_0\, x^k\left[1-\frac{k-3}{2k+1}x^2+\frac{(k-1)(k-3)}{(2k+1)(2k+5)}x^4 \cdots\right] \tag{7}$$

Putting $k = 0$ and replacing C_0 by a in (7), we have

$$y = a\left[1+3x^2+\left(\frac{3}{5}\right)x^4-\cdots\right] = au \text{ (say)}$$

Next, putting $k=\frac{3}{2}$ and replacing C_0 by b in (7), we have

$$y = bx^{\frac{3}{2}}\left[1+\frac{3}{8}x^2-\frac{3}{128}x^4+\cdots\right] = bv \text{ (say)}$$

Hence, the required solution is $y = au + bv$, where a and b are arbitrary constants.

EXERCISE 13.3

Find the series solutions of the following equations about $x = 0$:

1. $2x(1-x)\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}+3y=0$
2. $8x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+y=0$
3. $2x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-(x+1)y=0$
4. $2x^2\dfrac{d^2y}{dx^2}+(2x^2-x)\dfrac{dy}{dx}+y=0$
5. $2x(1-x)\dfrac{d^2y}{dx^2}+(5-7x)\dfrac{dy}{dx}-3y=0$

Answers

1. $y=a\left[1-3x+\dfrac{3}{1\cdot 3}x^2-\ldots\right]+b\,x^{1/2}[1-x]$
2. $y=ax^{1/2}+bx^{1/4}$
3. $y=a\,x\left[1+\dfrac{1}{5}x+\dfrac{1}{70}x^2+\cdots\right]+b\,x^{-\frac{1}{2}}\left[1-x-\dfrac{x^2}{2}+\cdots\right]$
4. $y=a\,x\left[1-\dfrac{2x}{3}+\dfrac{4x^2}{3\cdot 5}-\dfrac{8x^3}{3\cdot 5\cdot 7}+\cdots\right]+b\,x^{\frac{1}{2}}\left[1-x+\dfrac{x^2}{2}-\dfrac{x^3}{6}+\cdots\right]$
5. $y=a\left[1+\dfrac{3x}{5}+\dfrac{3x^2}{7}+\dfrac{3x^3}{9}+\cdots\right]+b\,x^{-3/2}$

13.6.2 Case II

The roots of the indicial equation are equal.

In this case, we have $k_1 = k_2$. One of the linearly independent solutions is obtained by substituting $k = k_1$ in the series solution. The second solution can be obtained by setting $v(x) = u(x)$. $w(x)$ and determining $u(x)$ by the method of variation of parameters. We shall illustrate an alternate simple procedure using the following example.

Example 12 Find the series solution about $x = 0$, of the equation $xy'' + y' + xy = 0$, by the Frobenius method.

Solution The given differential equation is

$$x\,y'' + y' + xy = 0 \tag{1}$$

Dividing by x,

$$y'' + \frac{1}{x}y' + y = 0 \tag{2}$$

Comparing (2) with $y'' + P(x)\, y' + Q(x) = 0$, we have

$$P(x) = \frac{1}{x},\ Q(x) = 1$$

So that $xP(x) = 1$ and $x^2 Q(x) = x^2$, which are analytic at $x = 0$.

So $x = 0$ is a regular singular point.

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m\, x^{m+k},\ C_0 \neq 0 \tag{3}$$

$$\therefore \qquad y' = \sum_{m=0}^{\infty} C_m\, (m+k)\, x^{m+k-1},\ y'' = \sum_{m=0}^{\infty} C_m\, (m+k)(m+k-1)\, x^{m+k-2}$$

Putting these values of y, y' and y'' in (1), we get

$$x \sum_{m=0}^{\infty} C_m\, (m+k)(m+k-1)\, x^{m+k-2} + \sum_{m=0}^{\infty} C_m\, (m+k)\, x^{m+k-1} + x \sum_{m=0}^{\infty} C_m\, x^{m+k} = 0$$

or
$$\sum_{m=0}^{\infty} C_m\, (m+k)(m+k-1)\, x^{m+k-1} + \sum_{m=0}^{\infty} C_m\, (m+k)\, x^{m+k-1} + \sum_{m=0}^{\infty} C_m\, x^{m+k+1} = 0$$

or
$$\sum_{m=0}^{\infty} C_m\, [(m+k)(m+k-1) + (m+k)]\, x^{m+k-1} + \sum_{m=0}^{\infty} C_m \cdot x^{m+k+1} = 0$$

or
$$\sum_{m=0}^{\infty} C_m\, (m+k)^2\, x^{m+k-1} + \sum_{m=0}^{\infty} C_m \cdot x^{m+k+1} = 0 \tag{4}$$

Equating to zero the coefficient of the smallest power of x, i.e., x^{k-1}, (4) gives the indicial equation

$$C_0\, k^2 = 0 \quad \text{or} \quad k^2 = 0 \quad (\because \quad C_0 \neq 0)$$

so that $k = 0, 0$ (equal roots).

For recurrence relation, equating to zero the coefficient of x^{m+k-1} in (4) gives

$$C_m\, (m + k)^2 + C_{m-2} = 0 \ \text{ or } \ C_m = -\frac{C_{m-2}}{(m+k)^2} \tag{5}$$

For finding C_1, equating to zero the coefficient of x^k gives

$C_1\, (k+1)^2 = 0$ so that $C_1 = 0$ (for $k = 0$ is a root of the indicial equation)

Using $C_1 = 0$ and (5), we get

$$C_1 = C_3 = C_5 = C_7 = C_9 = \ldots = 0$$

Putting $m = 2, 4, 6, 8, \ldots$ in (5), we get

$$C_2 = -\frac{C_0}{(k+2)^2},\ C_4 = -\frac{C_2}{(k+4)^2} = \frac{C_0}{(k+2)(k+4)^2},$$

$$C_6 = -\frac{C_4}{(k+6)^2} = -\frac{C_0}{(k+2)^2\,(k+4)^2\,(k+6)^2}, \text{ and so on.}$$

Putting these values in (3), we get

$$y = C_0\, x^k \left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2\,(k+4)^2} - \frac{x^6}{(k+2)^2\,(k+4)^2\,(k+6)^2} + \cdots\right] \tag{6}$$

Putting $k = 0$ replacing C_0 by a in (6), gives

$$y = a\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right] = au \ \text{(say)} \tag{7}$$

To find another independent solution, substituting (6) into the LHS of (1) and simplifying, we find

$$x^2\, y'' + y' + xy = C_0\, k^2.\, x^{k-1} \tag{8}$$

Differentiating both sides partially w.r.t. 'k', gives

$$\frac{\partial}{\partial k}\cdot\left[x^2\,\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy\right] = \frac{C_0}{x}\,\frac{\partial}{\partial k}\cdot(k^2\cdot x^k)$$

or

$$\left[x^2\,\frac{d^2}{dx^2} + \frac{d}{dx} + x\right]\frac{dy}{dk} = \frac{C_0}{x}\left[2k\,x^k + k^2\,x^k\log x\right] \tag{9}$$

The presence of the factor k in each term of RHS of (9) shows that a second solution is $\left(\dfrac{\partial y}{\partial k}\right)_{k=0}$, where y is given by (6).

Differentiating (6) partially w.r.t. 'k', we get

$$\begin{aligned}\frac{dy}{dk} &= C_0\, x^k\log x\left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2\,(k+4)^2} - \frac{x^6}{(K+2)^2\,(k+4)^2\,(k+6)^2} + \cdots\right] \\ &\quad + C_0 x^k\left[-\frac{x^2}{(k+2)^2}\times\frac{-2}{(k+2)} + \frac{x^4}{(k+2)^2\,(k+4)^2}\times\left\{\frac{-2}{k+2} - \frac{2}{k+4}\right\}\right. \\ &\quad \left. - \frac{x^6}{(k+2)^2\,(k+4)^2\,(k+6)^2}\left\{-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right\} + \cdots\right]\end{aligned} \tag{10}$$

Putting $k = 0$ and replacing C_0 by b in (10), we get

$$\begin{aligned}\left(\frac{\partial y}{\partial K}\right)_{K=0} &= b\log x\cdot\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) \\ &\quad + b\left[\frac{x^2}{2^2} + \frac{x^2}{2^2\cdot 4^2}\cdot\left(1+\frac{1}{2}\right) + \frac{x^2}{2^2\cdot 4^2\cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\right] \\ &= b\left[u\log x + \left\{\frac{x^2}{2^2} - \frac{x^2}{2^2\cdot 4^2}\left(1 + \frac{1}{2}\right) + \cdots\right\}\right] \\ &= bv \text{ say [using (7)]}\end{aligned}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Note [To find $\dfrac{d}{dk}\left[\dfrac{1}{(k+2)^2(k+4)^2(k+6)^2}\right]$

Put $$u = \frac{1}{(k+2)^2(k+4)^2(k+6)^2}$$

$\therefore$ $\log u = -2\log(k+2) - 2\log(k+4) - 2\log(k+6)$

Differentiating w.r.t. 'k', we get

$$\frac{1}{u}\frac{du}{dk} = -\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}$$

or $$\frac{du}{dk} = u\cdot\left[-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right] \text{ etc.}$$

Other terms can be similarly differentiated easily.

Example 13 Find the series solution of $(x - x^2)y'' + (1 - 5x)y' - 4y = 0$ about $x = 0$.

Solution The given differential equation is

$$(x - x^2)y'' + (1 - 5x)y' - 4y = 0 \tag{1}$$

Dividing by $(x - x^2)$, $$y'' + \frac{(1-5x)}{(x-x^2)}y' - \frac{-4}{(x-x^2)}y = 0 \tag{2}$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, we have

$$P(x) = \frac{1-5x}{x-x^2},\quad Q(x) = \frac{-4}{(x-x^2)}$$

So that $$xP(x) = \frac{1-5x}{1-x} \text{ and } x^2Q(x) = -\frac{4x}{1-x}$$

Since $xP(x)$ and $x^2 Q(x)$ are analytic at $x = 0$, hence, $x = 0$ is a regular singular point.

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{m+k},\ C_0 \neq 0 \tag{3}$$

$\therefore$ $$y' = \sum_{m=0}^{\infty} C_m\cdot(m+k)x^{m+k-1},\ y'' = \sum_{m=0}^{\infty} C_m(m+k)(m+k-1)x^{m+k-2}$$

Putting the values of y, y', and y'' in (1), we get

$$(x-x^2)\sum_{m=0}^{\infty} C_m(m+k)(m+k-1)x^{m+k-2} + (1-5x)\sum_{m=0}^{\infty} C_m(m+k)x^{m+k-1} - 4\sum_{m=0}^{\infty} C_m x^{m+k} = 0$$

or $$\sum_{m=0}^{\infty} C_m(m+k)(m+k-1)x^{m+k-1} - \sum_{m=0}^{\infty} C_m(m+k)(m+k-1)x^{m+k} + \sum_{m=0}^{\infty} C_m(m+k)x^{m+k-1}$$

$$-5\sum_{m=0}^{\infty} C_m(m+k)x^{m+k} - 4\sum_{m=0}^{\infty} C_m x^{m+k} = 0$$

or $\sum_{m=0}^{\infty} C_m[(m+k)(m+k-1)+(m+k)]x^{m+k-1} - \sum_{m=0}^{\infty} C_m[(m+k)(m+k-1)+5(m+k)+4]x^{m+k} = 0$

or $$\sum_{m=0}^{\infty} C_m (m+k)^2 x^{m+k-1} - \sum_{m=0}^{\infty} C_m \cdot[(m+k)^2 + 4(m+k)+4]x^{m+k} = 0$$

or $$\sum_{m=0}^{\infty} C_m (m+k)^2 x^{m+k-1} - \sum_{m=0}^{\infty} C_m (m+k-2)^2 x^{m+k} = 0 \tag{4}$$

Equating to zero the coefficient of the smallest power of x, i.e., x^{k-1} in (4), gives the indicial equation

$$C_0 k^2 = 0 \text{ or } k^2 = 0 \qquad (\therefore\ C_0 \neq 0)$$

so that $k = 0, 0$ (equal roots).

For recurrence relation, equating to zero the coefficient of x^{m+k-1} in (4), we get

$$C_m (m+k)^2 - C_{m-1} (m+k+1)^2 = 0$$

or $$C_m = \frac{(m+k+1)^2}{(m+k)^2} C_{m-1} \tag{5}$$

Putting $m = 1, 2, 3, 4, \ldots$ in (5) and simplifying

$$C_1 = \frac{(k+2)^2}{(k+1)^2} C_0$$

$$C_2 = \frac{(k+3)^2}{(k+2)^2} C_1 = \frac{(k+3)^2}{(k+2)^2}\frac{(k+2)^2}{(k+1)^2} C_0 = \frac{(k+3)^2}{(k+1)^2} C_0$$

$$C_3 = \frac{(k+4)^2}{(k+3)^2} C_2 = \frac{(k+4)^2}{(k+3)^2}\frac{(k+3)^2}{(k+1)^2} C_0 = \frac{(k+4)^2}{(k+1)^2} C_0 \text{ and so on}$$

Putting these values in (3), i.e.,

$$y = x^k [C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots], \text{ we get.}$$

$$y = x^k C_0 \left[1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + \cdots\right] \tag{6}$$

Putting $k = 0$ and replacing C_0 by a in (6), we get

$$y = a[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots] = au \text{ (say)} \tag{7}$$

Differentiating partially w.r.t. 'k', (6) gives

$$\frac{\partial y}{\partial k} = C_0 x^k \log x \left[1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + \cdots\right]$$

$$+ C_0 x^k \left[\frac{(k+2)^2}{(k+1)^2} x \cdot \left\{\frac{2}{k+2} - \frac{2}{k+1}\right\} + \frac{(k+3)^2}{(k+1)^2} x^2 \left\{\frac{2}{(k+3)} - \frac{2}{(k+1)}\right\} + \cdots\right]$$

Putting $k = 0$ and replacing C_0 by b

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b\log x\cdot[1+22\,x+32\,x^2+42\,x^3+\cdots]+b\left[22\,x(1-2)+32\,x^2\cdot 2\left(\frac{2}{3}-2\right)+\cdots\right]$$

$$= b[u\log x - 2(1.2x + 2\cdot 3x^2 + \cdots)] = bv \text{ (say) [(using (7)]}$$

Hence, the required solution of (1) is $y = au + bv$, where a and b are arbitrary constants.

EXERCISE 13.4

Find the series solution of the following equations about $x = 0$:

1. $x\,y'' + y' + x^2\,y = 0$
2. $x^2\,y'' - x\,(1 + x)\,y' + y = 0$
3. $x\,y'' + (1 + x)\,y' + 2y = 0$
4. $x\,y'' + y' - y = 0$

Answers

1. $$y = a\left[1-\frac{x^3}{3^2}+\frac{x^6}{3^4\,(2!)^2}-\frac{x^9}{3^6\,(3!)^2}+\ldots\right]+b\left[u\log x+2\left\{\frac{x^3}{3^3}-\frac{1}{3^5\cdot(2!)^2}\left(1+\frac{1}{2}\right)x^6+\ldots\right\}\right]$$

2. $$y = ax\left[1+x+\frac{x^2}{2}+\frac{x^3}{2\cdot 3}+\ldots\right]+b\left[\log x\left\{1+x+\frac{x^2}{2}+\frac{x^3}{2\cdot 3}+\ldots\right\}+x^2\left\{-1-\frac{3}{4}x+\ldots\right\}\right]$$

3. $$y = a\cdot\left[1-2x+\frac{3}{2!}x^2-\frac{4}{3!}x^3+\cdots\right]$$
$$+b\left[\log x\left\{1-2x+\frac{3}{2!}x^2-\frac{4}{3!}x^3+\ldots\right\}+2\left(2-\frac{1}{2}\right)x-\frac{3}{2!}\left(-\frac{1}{3}+2+\frac{1}{2}\right)x^2+\cdots\right]$$

4. $$y = (a+b\log x)\left[1+x+\frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}+\cdots\right]$$
$$-2b\left[x+\frac{1}{(2!)^2}\left(1+\frac{1}{2}\right)x^2+\frac{1}{(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3+\cdots\right]$$

13.6.3 Case III

When the roots of the indicial equation are distinct and differ by an integer.

Let the roots k_1 and k_2 of the indicial equation differ by an integer (say $k_1 < k_2$) and if one of the coefficients of y becomes indeterminate when $k = k_2$, we modify the form y by replacing a by $b(k - k_2)$.

Therefore, the complete solution is

$$y = au + bv$$

Example 14 Find the series solution of $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + x^2 y = 0.$

Solution The given differential equation is

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + x^2 y = 0 \tag{1}$$

Dividing by x^2, $$\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} + y = 0 \tag{2}$$

Compare (2) with $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Here, $P(x) = \dfrac{5}{x}, Q(x) = 1$

At $x = 0$, $P(x)$ is not analytic so that $x = 0$ is a singular point. Also, $xP(x) = 5$ and $x^2Q(x) = x^2$. Since both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$, so $x = 0$ is a regular singular point.

Let $y = \sum_{m=0}^{\infty} C_m x^{m+k}$ be the series solution of (1). (3)

$$\therefore \quad y' = \sum_{m=0}^{\infty} (m+k)\cdot C_m x^{m+k-1}, \; y'' = \sum_{m=0}^{\infty} (m+k)(m+k-1)C_m x^{m+k-2}$$

Substituting the values of y, y', and y'' in (1), we have

$$x^2 \sum_{m=0}^{\infty} (m+k)(m+k-1)C_m x^{m+k-2} + 5x\sum_{m=0}^{\infty} (m+k)C_m \cdot x^{m+k-1} + x^2 \sum_{m=0}^{\infty} C_m x^{m+k} = 0$$

or $$\sum_{m=0}^{\infty} (m+k)(m+k-1)C_m \cdot x^{m+k} + 5\sum_{m=0}^{\infty} (m+k)C_m \cdot x^{m+k} + \sum_{m=0}^{\infty} C_m \cdot x^{m+k+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m [(m+k)(m+k-1) + 5(m+k)]x^{m+k} + \sum_{m=0}^{\infty} C_m \cdot x^{m+k+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m [(m+k)^2 + 4(m+k)]x^{m+k} + \sum_{m=0}^{\infty} C_m \cdot x^{m+k+2} = 0 \tag{4}$$

Equating to zero the coefficient of the smallest power of x, i.e., x^k in (4), we get the indicial equation

$$C_0 (m^2 + 4m) = 0 \text{ or } m(m+4) = 0 \qquad (\because \;\; C_0 \neq 0)$$

so that $m = 0, -4$.

Hence, the roots are distinct and differ by an integer. Next, equating to zero the coefficient of x^{m+k} in (4), we get

$$C_m[(m+k)^2 + 4(m+k)] + C_{m-2} = 0$$

Or $$C_m = -\frac{C_{m-2}}{(m+k)(m+k+4)} \tag{5}$$

Coefficient of $x^{k+1} = 0$

$$(m+1)(m+5)\,C_1 = 0 \Rightarrow C_1 = 0 \quad (\therefore\ m \neq -1, -5)$$

Putting $m = 2, 3, 4, 5, \ldots$ in (5), we get

$$C_2 = -\frac{C_0}{(k+2)(k+6)}$$

$$C_3 = -\frac{C_1}{(k+3)(k+7)} = 0 \qquad (\therefore\ C_1 = 0)$$

$$C_4 = -\frac{C_2}{(k+4)(k+8)} = \frac{C_0}{(k+2)(k+4)(k+6)(k+8)}$$

$$C_5 = -\frac{C_3}{(k+5)(k+9)} = 0 \qquad (\therefore\ C_3 = 0)$$

$$C_6 = -\frac{C_4}{(k+6)(k+10)} = \frac{-C_0}{(k+2)(k+4)(k+6)(k+8)(k+10)}, \text{ and so on.}$$

Putting these values in (3), we get

$$y = C_0\, x^k \left[1 - \frac{x^2}{(k+2)(k+6)} + \frac{x^4}{(k+2)(k+4)(k+6)(k+8)} - \cdots\right] \tag{6}$$

Putting $k = 0$ in (6), we get

$$y = a\left[1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8)} - \cdots\right] = au \ \text{(say)} \tag{7}$$

If we put $k = -4$ in (6), the coefficients become infinite. To avoid this difficulty, we put $a = b\,(k + 4)$, so that

$$y = b\,x^k\left[(k+4) - \frac{(k+4)\,x^2}{(k+2)(k+6)} + \frac{x^4}{(k+2)(k+6)(k+8)} - \cdots\right] \tag{8}$$

Now, $$\frac{\partial y}{\partial k} = y \log x + b\,x^k\left[1 + \frac{(k^2+8k+20)}{(k^2+8k+12)^2}x^2 - \frac{(3k^2+32k+76)}{(k^3+16k^2+76k+96)^2} + \cdots\right]$$

$$\therefore \quad \left(\frac{\partial y}{\partial k}\right)_{k=-4} = (y)_{k=-4} \log x + bx^{-k}\left[1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right]$$

$$= bx^{-4} \log x\left[0 - 0 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{16} + \cdots\right] + bx^{-4}\left[1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right]$$

$$= bx^{-4} \log x\left[-\frac{x^4}{16} - \frac{x^6}{16} - \cdots\right] + bx^{-4}\left[1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right]$$

$= bv$ (say)

Hence, the general solution if given by

$$y = au + bv$$

$$= a\left[1-\frac{x^2}{12}-\frac{x^4}{384}-...\right]+bx^{-4}\log x\left[-\frac{x^4}{16}-\frac{x^6}{16}-\cdots\right]+bx^{-4}\left[1+\frac{x^2}{4}-\frac{x^4}{4}+\cdots\right]$$

EXERCISE 13.5

Find the series solution of the following equations about $x = 0$:

1. $(1 - x^2)y'' - xy' + 4y = 0$
2. $x(1 - x)\, y'' - 3\, x\, y' - y = 0$
3. $x^2y'' + xy' + (x^2 - 1)y = 0$
4. $x^2y'' + xy' + (x^2 - 4)y = 0$

Answers

1. $y = C_0\,(1-2x^2)+C_1\left(x-\frac{x^3}{2}-\frac{x^5}{8}+\frac{x^7}{16}-\cdots\right)$
2. $y = (a + b\log x)\,(x + 2x^2 + 3x^2 + 4x^4 + \ldots) + b(1 + x + x^2 + x^3 + \ldots)$
3. $y = ax\left[1-\frac{x^2}{2\cdot 4}+\frac{x^4}{2\cdot 4^2\cdot 6}-\cdots\right]$

$$+bx^{-1}\log x\cdot\left[-\frac{x^2}{2}+\frac{x^4}{2^2\cdot 4}-\cdots\right]+bx^{-1}\left[1+\frac{x^2}{2^2}-\frac{3}{2^2\cdot 3^3}x^4+\cdots\right]$$

4. $y = a\,x^2\left[1-\frac{x^2}{2\cdot 6}+\frac{x^4}{2\cdot 4\cdot 6\cdot 8}-\cdots\right]+b\,x^{-2}\log x\left[\frac{-1}{2^2\cdot 4}+\frac{x^4}{2^3\cdot 4\cdot 6}-\cdots\right]$

$$+\ x^{-2}\left[1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\cdots\right]$$

13.6.4 Case IV

When the roots of the indicial equation are distinct, differ by an integer and making one or more coefficients indeterminate.

Let k_1 and k_2 are roots of indicial equation and let $k_1 < k_2$, and let a coefficient of y series become indeterminate when $k = k_1$. The complete solution is given by putting $k = k_1$ in y which contains two arbitrary constants. If we put $k = k_1$ in y then we obtain a series which is a constant multiple of one of the two series contained in the first solution.

Example 15 Find the series solution of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y\ = 0$$

Solution The given differential equation is

$$(1-x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=0 \tag{1}$$

Dividing $(1-x^2)$, $$\frac{d^2y}{dx^2}+\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{1}{1-x^2}y=0 \tag{2}$$

Compare (2) with $$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

Here, $P(x)=\dfrac{2x}{1-x^2},\ Q(x)=\dfrac{1}{1-x^2}$

At $x = 1$, $P(x)$ and $Q(x)$ are not analytic, so $x = 1$ is a singular point. Also,

$(x-1)\,P(x)=(x-1)\dfrac{2x}{(1-x)(1+x)}=-\dfrac{2x}{1+x}$ and $(x-1)^2\,Q(x)=(x-1)^2\dfrac{1}{(1-x)(1+x)}=-\left(\dfrac{x-1}{x+1}\right)$

are analytic at $x = 1$ so that $x = 1$ is a regular singular point.

Let the series solution of the given differential equation is

$$y=\sum_{m=0}^{\infty}C_m\,x^{m+k} \tag{1}$$

$$\therefore \quad y'=\sum_{m=0}^{\infty}C_m\cdot(m+k)\,x^{m+k-1}$$

$$y''=\sum_{m=0}^{\infty}C_m\cdot(m+k)(m+k-1)x^{m+k-2}$$

Putting the values of y, y', and y'' in (1) , we get

$$(1-x^2)\sum_{m=0}^{\infty}C_m\,(m+k)(m+k-1)\,x^{m+k-2}+2x\sum_{m=0}^{\infty}C_m\,(m+k)\,x^{m+k-1}+\sum_{m=0}^{\infty}C_m\,x^{m+k}=0$$

or $$\sum_{m=0}^{\infty}C_m\,(m+k)(m+k-1)\,x^{m+k-2}-\sum_{m=0}^{\infty}C_m\,(m+k)(m+k-1)x^{m+k}+2\sum_{m=0}^{\infty}C_m\,(m+k)x^{m+k}+\sum_{m=0}^{\infty}C_m\cdot x^{m+k}=0$$

or $$\sum_{m=0}^{\infty}C_m\cdot(m+k)(m+k-1)\,x^{m+k-2}-\sum_{m=0}^{\infty}C_m\,[(m+k)(m+k-1)-2(m+k)-1]\,x^{m+k}=0$$

or $$\sum_{m=0}^{\infty}C_m\,(m+k)(m+k-1)\,x^{m+k-2}-\sum_{m=0}^{\infty}C_m\,[(m+k)(m+k-3)-1]\,x^{m+k}=0 \tag{4}$$

Equating to zero the coefficient of smallest power of x, i.e., x^{k-2},

$$C_0\,(k-1)\cdot k = 0 \text{ so that } k = 0, 1 \qquad (\because\ C_0 \neq 0)$$

Next, equating to zero the coefficient of x^{k-1}, we get $C_1\, k\,(k+1) = 0$

Clearly, the coefficient C_1 becomes indeterminate when $k = 0$

Now, equating to zero the coefficient of x^{m+k}, we get

$$C_{m+2}\,(m+k+1)\,(m+k+2) - C_m\,[(m+k)\,(m+k-3-1] = 0$$

$$\therefore \qquad C_{m+2} = \frac{(m+k)(m+k-3)-1}{(m+k+1)(m+k+2)} C_m \tag{5}$$

Putting $k = 0, 1, 2, 3, \ldots$in (5), we get

$$C_2 = \frac{m(m-3)-1}{(m+1)(m+2)} C_0$$

$$C_3 = \frac{(m+1)(m-2)-1}{(m+2)(m+3)} C_1$$

$$C_4 = \frac{(m+2)(m-1)-1}{(m+3)(m+4)} C_2$$

$$= \frac{[(m+2)(m-1)-1]\cdot[m(m-3)-1]}{(m+1)(m+2)(m+3)(m+4)} C_0, \text{ and so on}$$

Thus, $\qquad y = \sum_{m=0}^{\infty} C_m\, x^{m+k}$ gives

$$y = C_0\, x^k + C_1\, x^{k+1} + C_2\, x^{k+2} + C_3\, x^{k+3} + C_4\, x^{k+4} + \cdots$$

$$= C_0\, x^k \left[1 + \frac{m(m-3)-1}{(m+1)(m+2)} x^2 + \frac{[(m+2)(m-1)-1]\cdot[m(m-3)-1]}{(m+1)(m+2)(m+3)(m+4)} x^4 + \cdots\right]$$

$$+ C_1\, x^k \left[x + \frac{(m+1)(m-2)-1}{(m+2)(m+3)} x^3 + \right] \tag{6}$$

Substituting $k = 0$ and choosing $C_0 = a$ and $C_1 = b$ in (6), we get

$$y = a\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + b\left(x - \frac{x^3}{3} + \cdots\right)$$

EXERCISE 13.6

Find the series solution of the following differential equations:

1. $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$ about $x = 0$

2. $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0$

3. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + xy = 0$

4. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + x^2y = 0$

Answers

1. $y = C_0\left(1 - x^2 - \dfrac{x^4}{3} - \dfrac{x^6}{5} + \ldots\right) + C_1 x$

2. $y = a\,x^{1/2}\,(1-x) + b\left(1 - 3x + \dfrac{3x^2}{1\cdot 3} + \dfrac{3x^3}{3\cdot 5} + \ldots\right)$

3. $y = \dfrac{1}{x}(C_0 \cos x + C_1 \sin x)$

4. $y = a\left(1 - \dfrac{x^4}{12} + \dfrac{x^6}{90} - \cdots\right) + b\left(x - \dfrac{x^2}{6} - = \dfrac{x^5}{40} - \cdots\right)$

SUMMARY

1. Analytic Function

A function $f(x)$ is said to be analytic at x_0 if $f(x)$ has Taylor's series expansion about x_0 such that

$\sum_{n=0}^{\infty} \dfrac{f^n(x_0)}{n}(x-x_0)^n$ exists and converges to $f(x)$ for all x in the interval including x_0.

Hence, we find that all polynomial functions, e^x, $\sin x$, $\cos x$, $\sinh x$, and $\cosh x$ are analytic everywhere. A rational function is analytic except at those values of x at which its denominator is zero, for example, the rational functional defined by $\dfrac{x}{(x^2 - 3x + 2)}$ is analytic everywhere except at $x = 1$ and $x = 2$.

2. Ordinary and Singular Points

A point $x = x_0$ is called an ordinary point of the equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{1}$$

If both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

If the point $x = x_0$ is not an ordinary point of the differential equation (1) then it is called a singular point of the differential equation (1). There are two types of singular points.

(i) Regular singular points, and

(ii) Irregular singular points.

A singular point $x = x_0$ of the differential equation (DE) (1) is called a regular singular point of the (DE) (1) if both $(x - x_0)\,P(x)$ and $(x - x_0)2\,Q(x)$ are analytic at $x = x_0$

A singular point which is not regular is called an irregular point.

3. Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} C_n (x-x_0)^n = C_0 + C_1 (x-x_0) + C_2 (x-x_0)^2 + \cdots \quad (2)$$

is called a power series $(x - x_0)$. The constants C_0, C_1, C_2 … are known as the coefficients and x_0 is called the center of the power series (2). Since n takes only positive integral values, the power series (2) does not contain negative or fractional powers. So the power series (1) contains only positive powers.
The power series (2) converges (absolutely) for $|x| < R$, where

$$R = \lim_{x\to\infty}\left|\frac{C_n}{C_{n+1}}\right|, \text{ provided the limit exists.} \quad (3)$$

R is said to be the radius of convergence of the power series (2). The interval $(-R_1\ R)$ is said to be the interval of convergence.

Result I A power series represents a continuous function within its interval of convergence.

Result II A power series can be differentiated termwise in its interval of convergence.

4. Power Series Solution about the Ordinary Point $x = x_0$

Let the equation $y'' + P(x)\, y' + Q(x)y = 0$ (4)
Then $x = x_0$ is an ordinary point of (4) and has two non-trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n\ (x-x_0)^n \quad (5)$$

And this power series converges in some interval of convergence $|x - x_0| < \mathrm{R}$ about x_0.
In order to get the coefficient C_n's in (5), we take

$$y' = \sum_{n=1}^{\infty} nC_n\ (x-x_0)^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1)C_n\ (x-x_0)^{n-2} \quad (6)$$

Putting the above values of y, y' and y'' in (4), we get the equation of the form

$$A_0 + A_1\,(x - x_0) + A_2\,(x - x_0) + \cdots + A_n\,(x - x_0)^n + \cdots \quad (7)$$

where the coefficients A_0, A_1, A_2, ….etc., are now some functions of the coefficients C_0, C_1, C_2, …etc. Since (7) is an identity, all the coefficients. A_0, A_1, A_2, … of (7) must be zero, i.e.,

$$A_0 = 0, A_1 = 0, A_2 = 0, \ldots., A_n = 0 \quad (8)$$

Solving the equation (7), we obtain the coefficients of (5) in terns of C_0 and C_1. Substituting these coefficients in (5), we obtain the required series solution (4) in powers of $(x - x_0)$.

5. Frobenius Method

Series solution when $x = 0$ is a regular singular point of the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad (9)$$

Step 1 Suppose that a trial solution of (9) be of the form

$$y = x^k\,(C_0 + C_1 x + C_2\, x^2 + \ldots + C_{\mathrm{m}}\, x^m + \ldots)$$

i.e., $$y = \sum_{m=0}^{\infty} C_m \cdot x^{m+k}, \; C_0 \neq 0 \tag{10}$$

Step 2 Differentiate (10) and obtain.

$$y' = \sum_{m=0}^{\infty} C_m (m+k) x^{m+k-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k-2} \tag{11}$$

Using (10) and (11), (9) reduces to an identity.

Step 3 Equating to zero the coefficient of the smallest power of x in the identity obtained in Step 2 above, we obtain a quadratic equation in k. The quadratic equation so obtained is called the indicial equation.

Step 4 Solve the indicial equation if the following cases arise:

(i) The roots of the indicial equation are distinct and do not differ by an integer.
(ii) The roots of the indicial equation are equal.
(iii) The roots of the indicial equation are distinct and differing by an integer.
(iv) The roots of the indicial equation are unequal, differing by an integer and making a coefficient of y indeterminate.

Step 5 We equate to zero the coefficient of the general power in the identity obtained in Step 2. The equation so obtained will be called the *recurrence relation*, because it connects the coefficients C_m, C_{m-2} or C_m, C_{m-1}, etc.

Step 6 If the recurrence relation connects C_m and C_{m-2}, then in general, determine C_1 by equating to zero the coefficient of the next higher power on the other hand, if the recurrence relation connects C_m and C_{m-1}. This step may be omitted.

Step 7 After getting various coefficients with the help of Steps 5 and 6 above, the solution is obtained by substituting these in (2) above.

14 Partial Differential Equations (PDE's)

14.1 INTRODUCTION

A partial differential equation is an equation involving two (or more) independent variables x, y, and a dependent variable z and its partial derivatives such as

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y} \text{ etc.}$$

We shall use the following standard notations:

$$p = \frac{\partial z}{\partial x} = z_x, \; q = \frac{\partial z}{\partial y} = z_y, r = \frac{\partial^2 z}{\partial x^2} = z_{xx}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy}, \; t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

14.1.1 Order of Partial Differential Equation

The order of a PDE is the order of the highest partial derivative appearing in the equation.

14.1.2 Degree of Partial Differential Equation (PDE)

The degree of a PDE is the degree of the highest order partial derivative occurring in it after the equation is made free from radicals and fractions so far as partial derivatives are concerned (i.e., the differential equation is rationalized).

Example: (i) $p + q = 1$ or $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ is of order 1 and degree 1.

(ii) $p + r + s = 1$ or $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 1$ is of order 2 and degree 1.

14.2 LINEAR PARTIAL DIFFERENTIAL EQUATION

A differential equation is said to be a *linear partial differential equation* if the dependent variable and all its derivatives appear in it in first degree.

A PDE which is not linear is called a *nonlinear partial differential equation*.

14.3 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE

Let z be the dependent variable and x, y be independent variables. The most general form of PDE of a first-order is given by

$$f(x, y, p, q) = 0 \tag{1}$$

Now, we shall classify PDE of first order into linear, semi-linear, quasi-linear and nonlinear.

(i) Linear PDE of Order One

A differential equation contains p and q and no higher derivative is called of first order. If the degree of p, q, and z are one then it is said to be a linear partial differential equation of first order.

Thus, it is of the form $P(x, y)p + Q(x, y)q + f(x, y)z = R(x, y)$ which are functions of x and y and do not contain any term of z.

(ii) Semi-Linear Partial Differential Equation of First Order

A partial differential equation is said to be semi-linear if it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

where $P(x, y)$ and $Q(x, y)$ are functions of x and y only and do not contain any term of z, and $R(x, y, z)$ contains some terms that are not of first degree in z.

(iii) Quasi-Linear Partial Differential Equation of First Order

A partial differential equation is said to be quasi-linear if it is of the form

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

where $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are functions depending on z also.

(iv) Nonlinear Partial Differential Equation of First Order

A differential equation is said to be a nonlinear partial differential equation of first order if p and/or q. occur in more than one degree.

(v) Homogeneous and Nonhomogeneous Partial Differential Equations

If each term of a linear PDE contains either the dependent variable or one of its partial derivatives then the equation is said to be homogeneous otherwise it is said to be nonhomogeneous.

14.4 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

(i) By Elimination of Arbitrary Constants

Consider an equation $F(x, y, z, a, b) = 0$ (2a)

where a and b denote arbitrary constants. Let z be a function of two independent variables x and y. Differentiating (2a) w.r.t. x and y partially, we get

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \tag{2b}$$

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \tag{2c}$$

By eliminating a, b from (2a), (2b), and (2c), we get an equation of the form

$$f(x, y, z, p, q) = 0 \tag{2d}$$

which is a partial differential equation of first order.

Note 1 If the number of arbitrary constants equals the number of independent variables in (2a), then the PDE obtained by elimination is of first order.

Note 2 If the number of arbitrary constants is more than the number of independent variables then the PDE obtained is of second or higher orders.

Example 1 Construct the partial differential equation (PDE) by eliminating the arbitrary constants.

$$z = a(x + y) + b(x - y) + abt + C \tag{1}$$

Solution Differentiating (1) partially both sides w.r.t. to x, y, and t, we get.

$$\frac{\partial z}{\partial x} = a + b \tag{2}$$

$$\frac{\partial z}{\partial y} = a - b \tag{3}$$

$$\frac{\partial z}{\partial t} = ab \tag{4}$$

Since $(a + b)^2 - (a - b)^2 = 4ab$ from (2), (3), and (4), we get

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\frac{\partial z}{\partial t}$$

Example 2 Find the partial differential equation of $z = ax^2 + by^2$ (1), where a, b are arbitrary constants.

Solution Differentiating (1) partially w.r.t. 'x' and y, we get

$$\frac{\partial z}{\partial x} = 2ax$$

$$\frac{\partial z}{\partial y} = 2by$$

or $$a = \frac{1}{2x}\frac{\partial z}{\partial x}, \quad b = \frac{1}{2y}\frac{\partial z}{\partial y}$$

Eliminating the two arbitrary constants a and b, we get

$$z = \frac{1}{2}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] \text{ or } 2z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

Example 3 Find a partial differential equation by eliminating a, b, c from.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Solution Differentiating (1) w.r.t. x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad c^2x+a^2z\frac{\partial z}{\partial x}=0 \tag{2}$$

and
$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \quad \text{or} \quad c^2y+b^2z\frac{\partial z}{\partial y}=0 \tag{3}$$

Now, differentiating (2) with respect to x and (3) w.r.t. y, we get

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\cdot\frac{\partial^2 z}{\partial x^2}=0 \tag{4}$$

$$c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2z\cdot\frac{\partial^2 z}{\partial y^2}=0 \tag{5}$$

From (2),
$$c^2=-\left(\frac{a^2z}{x}\right)\frac{\partial z}{\partial x}$$

Putting the value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0$$

or
$$zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\left(\frac{\partial z}{\partial x}\right)=0 \tag{6}$$

Similarly, (3) and (5) gives rise to

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\left(\frac{\partial z}{\partial y}\right)=0 \tag{7}$$

Equations (6) and (7) are the required solutions.

Example 4 Find the PDE of all spheres whose centers lie on the z-axis.

Solution Equation $x^2+y^2+(z-a)^2=b^2$ (1) where a, b are arbitrary constants.

Differentiating (1) partially both sides w.r.t. x and y, we get

$$2x+2(z-a)\frac{\partial z}{\partial x}=0 \tag{1}$$

$$2y+2(z-a)\frac{\partial z}{\partial y}=0 \tag{2}$$

From (2), $(z-a)=\dfrac{y}{\partial z/\partial y}$ (3)

Substituting (3) in (1), we get

$$2x-2\left(\frac{y}{\dfrac{\partial z}{\partial y}}\right)\cdot\frac{\partial z}{\partial x}=0$$

or $$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$$

or $$x\,q - y\,p = 0$$

Example 5 Find the PDE from $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$ where α is a parameter.

Solution $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$ (1)

Differentiating (1) partially w.r.t. x and y, we get

$$2(x-a) = 2z\left(\frac{\partial z}{\partial x}\right)\cdot\cot^2\alpha \text{ or } (x-a) = z\left(\frac{\partial z}{\partial x}\right)\cot^2\alpha \tag{2}$$

$$2(y-b) = 2z\left(\frac{\partial z}{\partial y}\right)\cot^2\alpha \text{ or } (y-b) = z\left(\frac{\partial z}{vy}\right)\cot^2\alpha \tag{3}$$

From (2) and (3), (1) becomes

$$z^2\left(\frac{\partial z}{\partial x}\right)^2\cot^4\alpha + z^2\left(\frac{\partial z}{\partial y}\right)^2\cot^4\alpha = z^2\cot^2\alpha$$

or $$p^2 + q^2 = \tan^2\alpha$$

Example 6 Obtain a PDE by eliminating a and b from
$z = (a+x)(b+y)$

Solution $z = (a+x)(b+y)$ (1)

Differentiating (1) partially both sides w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = (b+y) \tag{2}$$

$$\frac{\partial z}{\partial y} = (a+x) \tag{3}$$

From (2) and (3), (1) becomes

$$z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

or $$z = p.q$$

(ii) By Elimination of Arbitrary Function ϕ from the Equation $\phi(u, v) = 0$ (3)

where u and v are functions of x, y, and z.

Taking z as dependent variable and x and y as independent variables so that

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial y}{\partial x} = 0,\ \frac{\partial x}{\partial y} = 0 \tag{4}$$

Differentiating (3) w.r.t. 'x' we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

or
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

or
$$\frac{\partial \phi}{\partial u}\bigg/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\bigg/\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right) \tag{5}$$

Similarly, differentiating (1) w.r.t. y, we get
$$\frac{\partial \phi}{\partial u}\bigg/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)\bigg/\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right) \tag{6}$$

Eliminating ϕ with the help of (5) and (6), we get
$$\left(\frac{\partial v}{\partial x}+q\frac{\partial v}{\partial x}\right)\bigg/\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)\bigg/\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)$$

or
$$\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\cdot\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)\cdot\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)$$

or
$$Pq + Qq = R \tag{7}$$

where
$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$$
$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$$
$$R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

Thus, we obtain a linear PDE of first order and of first degree in p and q.

Note If the given equation between x, y, z contains two arbitrary functions then, in general, their elimination gives rise to equations of higher order.

Example 7 Obtain the partial differential equation by eliminating the arbitrary function $z = f\left(\frac{y}{x}\right)$.

Solution The given relation is $z = f\left(\frac{y}{x}\right)$ (1)

Differentiating (1) partially w.r.t. x and y, we get
$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) \tag{2}$$

and
$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \tag{3}$$

Dividing (2) by (3), we get
$$\frac{p}{q} = -\frac{y}{x} \text{ or } px + qy = 0$$

which is the required PDE.

Example 8 Obtain a partial differential equation by eliminating the arbitrary function f from $(x + y + z) = f(x^2 + y^2 + z^2)$.

Solution Given $(x + y + z) = f(x^2 + y^2 + z^2)$ (1)

Differentiating (1) partially w.r.t. 'x' and y, we get

$$1 + p = f'(x^2 + y^2 + z^2) \cdot (2x + 2zp) \quad (2)$$

$$1 + q = f'(x^2 + y^2 + z^2) \cdot (2y + 2zq) \quad (3)$$

Dividing (2) by (3), we get

$$\frac{1+p}{1+q} = \left(\frac{x+zp}{y+zq}\right) \quad \text{or} \quad (y - z)p + (z - x)q = x + y$$

which is the required PDE.

Example 9 Obtain a partial differential equation by eliminating the arbitrary functions f and g from $y = f(x - at) + g(x + at)$.

Solution Given $y = f(x - at) + g(x + at)$ (1)

Differentiating (1) partially w.r.t. x and t, we get

$$\frac{\partial y}{\partial x} = f'(x - at) + g'(x + at) \quad (2)$$

$$\frac{\partial y}{\partial t} = -af'(x - at) + ag'(x + at) \quad (3)$$

Again differentiating partially (2) and (3) w.r.t. x and t, we get

$$\frac{\partial^2 y}{\partial t^2} = f''(x - at) + g''(x + at) \quad (4)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2[f''(x - at) + g''(x + at)] \quad (5)$$

From (4) and (5), we get

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

which is the required PDE.

Example 10 Obtain a PDE by eliminating the arbitrary functions f and g from

$$z = f(x^2 - y) + g(x^2 + y).$$

Solution Given $z = f(x^2 - y) + g(x^2 + y)$ (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2xg'(x^2 + y) \quad (2)$$

$$\frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y) \quad (3)$$

Again differentiating (2) and (3) w.r.t. 'x' and y, we get

$$\frac{\partial^2 z}{\partial y^2} = 2[f'(x^2 - y) + g'(x^2 + y)] + 4x^2[f''(x^2 - y) + g''(x^2 + y)] \tag{4}$$

and $$\frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x + y) \tag{5}$$

Again, (2) $\Rightarrow f'(x^2 - y) + g'(x^2 + y) = \dfrac{1}{2x}\dfrac{\partial z}{\partial x}$ (6)

Using equations (5) and (6) in (4), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x}\frac{\partial z}{\partial x} + 4x^2 \frac{\partial^2 z}{\partial y^2}$$

or $$r = \frac{p}{x} + 4x^2 t$$

or $$p + 4x^3 t = r.x$$

Example 11 Obtain a PDE from $F(x y + z^2, x + y + z) = 0$ by eliminating the arbitrary function F.

Solution Given $F(xy + z^2, x + y + z) = 0$ (1)

Let $$u(x, y, z) = xy + z^2 \tag{2}$$

$$v(x, y, z) = x + y + z \tag{3}$$

Then Eq. (1) may be written as.

$$F(u, v) = 0 \tag{4}$$

Differentiating (4) partially w.r.t. x, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \tag{5}$$

From (2) and (3), we get $\dfrac{\partial u}{\partial x} = y, \dfrac{\partial v}{\partial z} = 2z, \dfrac{\partial u}{\partial y} = x, \dfrac{\partial v}{\partial x} = 1, \dfrac{\partial v}{\partial z} = 1$ (6)

Using (6) in (5), we get

$$\frac{\partial F}{\partial u}\cdot(y + 2pz) + \frac{\partial F}{\partial v}(1 + p) = 0 \tag{7}$$

Again differentiating (4) w.r.t. y, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

or $$\frac{\partial F}{\partial x}(x + 2qz) + \frac{\partial F}{\partial v}(1 + q) = 0 \tag{8}$$

From (7) and (8), we get

$$(2z - x)p + (y - 2z)q = (x - y)$$

EXERCISE 14.1

Obtain partial differential equations by eliminating the arbitrary constants/functions.

I. ***Elimination of arbitrary constants***

1. $z = ax + by + ab$; a, b arbitrary constants.
2. $z = ax + a^2y^2 + b$; a, b arbitrary constants.
3. $az + b = a^2x + y$; a, b arbitrary constants.
4. $z = axy + b$; a, b arbitrary constants.
5. $z = axe^y + \frac{1}{2}a^2 e^{2y} + b$; a, b arbitrary constants.
6. $z = a(x + y) + b(x - y) + abt + C$; a, b, c are arbitrary constants.
7. $z = A\, e^{pt} \sin px$; p, A are arbitrary constants.
8. $z = a\, e^{-b^2 x} \cos by$; a, b, are arbitrary constants.
9. $z = xy + y\sqrt{x^2 + a^2} + b$; a, b arbitrary constants.

II. ***Elimination of arbitrary functions***

10. $z = f(x^2 - y^2)$
11. $z = f(\sin x + \cos y)$
12. $z = e^{ax + by} f(ax - by)$
13. $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$
14. $z = x + y + f(x\, y)$
15. $z = f(x + iy) + g(x - iy)$
16. $lx + my + nz = f(x^2 + y^2 + z^2)$
17. $\phi(x^2 + y^2, x^2 - z^2) = 0$
18. $\phi(ax + by + cz, x^2 + y^2 + z^2) = 0$
19. $\phi(x^2 + y^2, z - xy) = 0$
20. $z = x^2 f(x - y)$

Answers

I. ***Elimination of arbitrary constants***

1. $z = px + qy + pq$
2. $q = 2yp^2$
3. $pq = 1$
4. $px = qy$
5. $q = px + p^2$
6. $p^2 - q^2 = 4\left(\frac{\partial z}{\partial t}\right)$
7. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$
8. $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$
9. $pq = py + qx$

II ***Elimination of arbitrary functions***

10. $py + qx = 0$
11. $p \sin y + q \cos x = 0$
12. $bp + aq = 2\, abz$
13. $p - q = (y - x)/z$
14. $px - qy = x - y$
15. $r + t = 0$
16. $(l + np)y + (lq - mp)\, z = (m + nq)x$
17. $yp - xq = \frac{xy}{z}$

18. $(bz - cy)p + (cx - az)q = ay - bx$

19. $xq - yp = x^2 - y^2$

20. $2z = p\,x + qx$

14.5 LAGRANGE'S METHOD OF SOLVING THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

The general form of a quasi-linear partial differential equation of the first order is

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z) \tag{8a}$$

is known as *Lagrange's linear equation*.

The general solution of the Lagrange's linear partial differential equation

$$Pp + Qq = R \tag{8a}$$

is given by $\phi\,(u, v) = 0$ (8b)

where ϕ is arbitrary function and $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$, form a solution of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Working Rule for Solving *Pp* + *Qq* = *R* by Lagrange's Method

Step 1 Put the given linear PDE of the first order in the standard form $Pp + Qq = R$ (9a)

Step 2 Write down Lagrange's auxiliary equation for (8a), namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{9b}$$

Step 3 Solve (9b) by using the well-known methods. Let $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$ be the two independent solution of (8c).

Step 4 The general solution (or integral) of (9a) is written in one of the following three forms:

$\phi\,(u, v) = 0$ $u = \phi(v)$ or $v = \phi(u)$

Example 12 Solve $xzp + yzq = xy$

Solution The given equation is $xzp + yzq = xy$ (1)

Lagrange's auxiliary equations for (1) are

$$\underset{\text{(i)}}{\frac{dx}{xz}} = \underset{\text{(ii)}}{\frac{dy}{yz}} = \underset{\text{(iii)}}{\frac{dz}{xy}} \tag{2}$$

From (i) and (ii), $\dfrac{dx}{xz} = \dfrac{dy}{yz}$ or $\dfrac{dx}{x} = \dfrac{dy}{y}$

On integrating $\log x = \log y + \log C_1$

$$\log \frac{x}{y} = \log C_1$$

$$\frac{x}{y} = C_1$$

From (ii) and (iii), $\dfrac{dy}{yz} = \dfrac{dz}{xy}$ or $\dfrac{dy}{z} = \dfrac{dz}{x}$

or $$xdy = zdz$$

On integrating, $$xy = \frac{z^2}{2} + C_2$$

or $$xy - \frac{z^2}{2} = C_2$$

The general solution is $\phi\left(\dfrac{x}{y}, xy - \dfrac{z^2}{2}\right) = 0$

where ϕ is an arbitrary function.

Example 13 Solve $yzp + zxq = xy$.

Solution The given equation $yzp + zxq = xy$ (1)

Lagrange's auxiliary equation for (1) are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \tag{2}$$

From (1) and (2), $$\frac{dx}{yz} = \frac{dy}{zx}$$

or $$\frac{dx}{y} = \frac{dy}{x} \quad \text{or} \quad xdx - ydy = 0 \tag{3}$$

On integrating, $x^2 - y^2 = C_1$

From (2) and (3), $$\frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dy}{z} = \frac{dz}{y} \quad \text{or} \quad ydy - zdz = 0$$

On integrating, $y^2 - z^2 = C_2$

The required general solution is $\phi\,(x^2 - y^2, y^2 - z^2) = 0$ where ϕ is an arbitrary function.

Example 14 The given equation is $y^2p - xyq = x(z - 2y)$.

Solution The given equation is $y^2p - xyz = x(z - 2y)$ (1)

Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \tag{2}$$

Taking first two fractions of (2), we get

$$\frac{dx}{y^2} = \frac{dy}{-xy} \quad \text{or} \quad \frac{dx}{y} = \frac{dy}{-x}$$

or $$xdx + ydy = 0$$

On integrating, we get

$$x^2 + y^2 = C_1 \tag{3}$$

Now, taking the last two fractions of (2), we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \text{or} \quad \frac{dz}{dy} = \frac{x(z-2y)}{-xy}$$

$$\frac{dz}{dy} = -\frac{z}{y} + 2$$

or
$$\frac{dz}{dy} + \left(\frac{1}{y}\right)z = 2 \tag{4}$$

which is a linear equation in z and y, its IF = $e^{\int \frac{1}{y}dy} = e^{\log y} = y$. Hence, the solution of (4) is

$$z.y = \int 2y\, dy + C_2 \quad \text{or} \quad zy - y^2 = C_2 \tag{5}$$

$\therefore$ the required general solution is

$$\phi(x^2 + y^2, zy - y^2) = 0$$

where ϕ is an arbitrary function.

Example 15 Solve $xp + yq = 3z$.

Solution The given equation is $xp + yq = 3z$. (1)

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z} \tag{2}$$

Taking first two fractions, $\frac{dx}{x} = \frac{dy}{y}$

On integrating, we get $\frac{x}{y} = C_1$ (3)

Taking the last two fractions,

$$\frac{dy}{y} = \frac{dz}{3z} \quad \text{or} \quad 3\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $3 \log y - \log z = \log C_2$

$$\log \frac{y^3}{z} = \log C_2$$

or
$$\frac{y^3}{z} = C_2 \tag{4}$$

The required solution is

$$\phi\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0 \tag{1}$$

where ϕ is an arbitrary function.

Example 16 Solve $xz\,(z^2 + xy)p - zy\,(z^2 + xy)q = x^4$.

Solution The given equation is $xz\,(z^2 + xy)p - yz\,(z^2 + xy)q = x^4$ (1)

The auxiliary equations are

$$\frac{dx}{zx(z^2+xy)} = \frac{dy}{-zy(z^2+xy)} = \frac{dz}{x^4} \tag{2}$$

Taking the first two fractions, we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

On integrating, $xy = C_1$ (3)

Taking the first and last fractions,

$$x^3\,dx = (z^3 + xyz)\,dz$$

Integrating, $$\frac{x^4}{4} = \frac{z^4}{4} + \frac{xyz^2}{2} + C_2$$

or $$x^4 - z^4 - 2xyz^2 = C_2 \tag{4}$$

The general solution is

$$\phi\,(xy,\, x^4 - z^4 - 2xyz^2) = 0$$

Example 17 Solve $p - q = \log\,(x + y)$

Solution The given equation is

$$p - q = \log\,(x + y) \tag{1}$$

The auxiliary equations $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{\log(x+y)}$ (2)

Taking the first two fractions, $\dfrac{dx}{1} = \dfrac{dy}{-1}$

Integrating, $x + y = C_1$ (3)

Taking the first and last fractions,

$$dx = \frac{dz}{\log(x + y)}$$

$$dx = \frac{dz}{\log C_1} \qquad \text{[from Eq. (3)]}$$

or $$\log C_1\,dx = dz$$

Integrating, $$x\,.\,\log C_1 = z + C_2$$

$$x \log\,(x + y) = z + C_2$$

or $$x \log\,(x + y) - z = C_2 \tag{4}$$

The general solution is

$$\phi\,[(x \log\,(x + y) - z,\, x + y)] = 0$$

Example 18 Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - xz)q = (x - y)z$.

Solution The given equations is

$$(x^2 - y^2 - yz)p + (x^2 - y^2 - xz)q = (x - y)z \tag{1}$$

Auxiliary equations are

$$\frac{dx}{(x^2 - y^2 - yz)} = \frac{dy}{(x^2 - y^2 - xz)} = \frac{dz}{(x - y)z} \tag{2}$$

$$dx - dy - dz = x^2 - y^2 - yz - x^2 + y^2 + zx - xz + yz = 0$$

Integrating $x - y - z = C_1$ (3)

Now, taking first and second and equate to third.

$$\frac{xdx - ydy}{x^3 - xy^2 - x^2y - y^3} = \frac{dz}{(x - y)z}$$

or $$\frac{xdx - ydy}{(x - y)(x^2 - y^2)} = \frac{dz}{(x - y)z}$$

or $$\frac{xdx - ydy}{(x^2 - y^2)} = \frac{dz}{z}$$

or $$\frac{1}{2}\, d\log(x^2 - y^2) = d(\log z)$$

On integrating, we get

$$\frac{(x^2 - y^2)}{z^2} = C_2 \tag{4}$$

The general solution is

$$\phi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$$

EXERCISE 14.2

Solve the following equations:

1. $p + q = 1$
2. $xp + yq = z$
3. $zp = -x$
4. $p(\tan x) + q(\tan y) = \tan z$
5. $\left(\frac{y^2 z}{x}\right)p + (zx)q = y^2$
6. $yp + x\,q = xyz^2(x^2 - y^2)$
7. $p - 2q = 3x^2 \sin(y + 2x)$
8. $x^2p + y^2q = z^2$
9. $p + 3q = 5z + \tan(y - 3x)$
10. $x(y - z)p + y(z - x)q = z(x - y)$
11. $(yz)p + (x\,z)q = (xy)$
12. $(y^2z)p + (x^2z)q = xy^2$
13. $zp - zq = z^2 + (x + y)^2$
14. $(y + z)p + (z + x)q = (x + y)$
15. $(y - z)p + (x - y)q = (z - x)$

Answers

1. $\phi(x-y, x-z) = 0$
2. $\phi\left(\frac{x}{z}, \frac{y}{z}\right) = 0$
3. $(x^2 + z^2, y) = 0$
4. $\phi\left(\frac{\sin z}{\sin y}, \frac{\sin x}{\sin y}\right) = 0$
5. $\phi(x^3 - y^3, x^2 - y^2) = 0$
6. $\phi\left[(x^2 - y^2), \frac{(x^2 - y^2)}{x^2} + \frac{2}{z}\right] = 0$
7. $\phi\ [2x + y, x^3 \sin(y + 2x) - z] = 0$
8. $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$
9. $\phi[y - 3x, e^{-5x}\{5z + \tan(y - 3x)\}] = 0$
10. $\phi\ [x + y + z, xyz] = 0$
11. $(x^2 - y^2, y^2 - z^2) = 0$
12. $\phi\ [(x^3 - y^3), (x^2 - z^2)] = 0$
13. $\phi[(x + y)\ e^{2y}\{z^2 + (x + y)^2\}] = 0$
14. $\phi\left[\left(\frac{x-y}{y-z}\right), \frac{y-z}{\sqrt{x+y+z}}\right] = 0$
15. $\phi\left[x+y+z, \frac{x^2}{2} + yz\right] = 0$

14.6 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Nonlinear partial differential equations of first order contain p and q of degree other than one and/or product terms of p and q.

Its complete solution is given by $f(x, y, z, a, b) = 0$ where a and b are any two arbitrary constants. Some standard forms of nonlinear first-order partial differential equations are given by the following:

(i) Standard Form I

Only p and q present (or x, y, z are absent).

Consider equations of the form

$$f(p.\ q) = 0 \tag{10}$$

Suppose $p = a$; then $f(a, q) = 0$

Solving $q = g\ (a)$

Consider $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + q\,dy$$

or $$dz = adx + g(a)dy$$

On integrating both sides, we get

$$z = ax + g(a)y + C$$

where a and C are arbitrary constants; thus, the complete solution is $z = ax + by + C$ (11)

where a, b satisfy the equation $f(a, b) = 0$, i.e., $b = g(a)$

(ii) Standard Form II

Clairaut's Equation

$$z = px + qy + f(p, q) \tag{12a}$$

The compete solution of (12a) is

$$z = ax + by + f(a, b) \tag{12b}$$

which is obtained by replacing p by a and q by b in the given equation (12a).

(iii) Standard Form III

Only p, q, and z present; Consider equation of the form

$$f(p, q, z) = 0 \tag{13a}$$

Putting $q = a\,p$ in (13a), we get $f(p, ap, z) = 0$ (13b)

In Eq. (13b) solving for p, we get $p = g(z)$.

Now, $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + q\,dy$$

$$dz = pdx + apdy$$

$$dz = p(dx + ady)$$

$$dz = g(z)\,(dx + ady)$$

On integrating,

$$x + ay = \int \frac{dz}{g(z)} + b\text{, where } a \text{ and } b \text{ are two arbitrary constants.}$$

(iv) Standard Form IV

The equation is of the form

$$f_1(x, p) = f_2(y, q) \tag{14}$$

Suppose $f_1(x, p) = f_2(y, q) = a$ = constant.

Solving each equation for p and q, we get

$$p = g_1(x, a) \text{ and } q = g_2(y, a)$$

Now, putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = g_1(x, a)dx + g_2(y, a)dy.$$

Integrating, $$z = \int g_1(x,a)\,dx + \int g_2(y,a)\,dy + b$$

which is the required complete solution.

Example 19 Solve $p + q = pq$.

Solution The given equation is of the form $f(p, q) = 0$.

Hence, its solution is given by $z = ax + by + C$ (1)

where $$a + b = ab \text{ or } b = \frac{a}{a-1}$$

$\therefore$ the complete solution is

$$z = ax + \left(\frac{a}{a-1}\right)y + C$$

Example 20 Solve $p^2 + q^2 = 1$.

Solution The given equation is in the form of $f(p, q) = 0$.

Let the solution be $z = ax + by + C$ (1)

where $a^2 + b^2 = 1$ or $b^2 = (1 - a^2)$

$$b = \pm\sqrt{(1-a^2)}$$

$$\therefore \quad z = ax \pm \sqrt{(1-a^2)}\, y + C$$

This is the required solution.

Example 21 Solve $x^2 p^2 + y^2 q^2 = z^2$.

Solution We have $\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$

$$\Rightarrow \quad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\text{or} \quad \left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2 + \left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2 = 1 \qquad (1)$$

Putting $\frac{\partial z}{z} = \partial Z, \frac{\partial x}{x} = \partial X, \frac{\partial y}{y} = \partial Y,$ we get

$$\log z = Z, \log x = X, \log y = Y$$

and $\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$

$$P^2 + Q^2 = 1, \qquad (2) \text{ where } P = \frac{\partial Z}{\partial X},\ Q = \frac{\partial Z}{\partial Y}$$

Let the required solution be $Z = AX + BY + C$ (3)

where $P = \frac{\partial Z}{\partial X} = A, Q = \frac{\partial Z}{\partial Y} = B$

$$A^2 + B^2 = 1 \text{ or } B = \pm\sqrt{(1-A^2)}$$

$$\therefore \quad Z = AX \pm \sqrt{(1-A^2)}\, Y + C$$

$$\text{or} \quad \log z = A \log x \pm \sqrt{(1-A^2)} \log y + C$$

Example 22 Solve $z = px + qy + (p^2 + q^2)$.

Solution The complete solution of the given PDE is

$$Z = a\,x + b\,y + (a^2 + b^2),\ \text{where } a = p \text{ and } b = q$$

Example 23 Solve $z = px + qy + 2\sqrt{pq}$.

Solution The complete solution of the given PDE is

$$z = ax + by + 2\sqrt{a\,b}\ ,\ \text{where } a = p,\ b = q$$

Example 24 Solve $p\,(1 + q) = qz$.

Solution The given PDE is in the form of $f\,(p, q, z) = 0$

Let $u = x + a\,y$, where $z = f\,(u)$

Then putting $p = \dfrac{dz}{du}, q = a\dfrac{dz}{du}$ in the given equation, we get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\,z\frac{dz}{du}$$

or
$$\left(1 + a\frac{dz}{du}\right) = az$$

or
$$a\frac{dz}{du} = (az - 1)$$

$$\frac{dz}{du} = \frac{(az-1)}{a} \quad \text{or} \quad \frac{du}{dz} = \frac{a}{(az-1)}$$

$$du = \frac{adz}{(az-1)}$$

On integrating, we get

$$u = \log\,(az - 1) + \log C$$
$$u = \log C\,(az - 1)$$
$$x + ay = \log C\,(az - 1)$$

Example 25 Solve $z^2\,(p^2 + q^2 + 1) = a^2$.

Solution The given equation is of the form $f\,(p, q, z) = 0$

Let $u = x + a\,y, z = f\,(u)$

Putting $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right] = a^2$$

or $$z^2(1+a^2)\left(\frac{dz}{du}\right)^2 = a^2 - z^2$$

$$z(1+a^2)\frac{dz}{du} = \pm\sqrt{(a^2 - z^2)}$$

or $$\pm\sqrt{(1+a^2)}\,\frac{z\,dz}{\sqrt{(a^2 - z^2)}} = du$$

On integrating, we get

$$u + c = \pm\sqrt{(1+a^2)}\,\sqrt{(a^2 - z^2)}$$

or $$(1 + a^2)(a^2 - z^2) = (x + ay + c)^2$$

Example 26 Solve $p(1 + q^2) = q(z - a)$.

Solution The given equation is of the form $f(p, q, z) = 0$.

Let $u = x + a\,y$; then putting $p = \dfrac{dz}{du}$ and $q = b\,\dfrac{dz}{du}$ in the given equation, we get

$$\frac{dz}{du}\left[1 + \left(b\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z - a)$$

$$1 + b^2\left(\frac{dz}{du}\right)^2 = b(z - a)$$

or $$b^2\left(\frac{dz}{du}\right)^2 = bz - ab - 1$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{(bz - ab - 1)}$$

or $$\frac{bdz}{\sqrt{(bz - ab - 1)}} = du$$

On integrating, we get

$$2\sqrt{(bz - ab - 1)} = u + C$$

$$4(bz - ab - 1) = (u + C)^2$$

$$4(bz - ab - 1) = (x + ay + C)^2$$

Example 27 Solve $p^2 - q^2 = x - y$.

Solution The given equation is $p^2 - x = q^2 - y$ which is of the form $f_1(x, p) = f_2(y, q)$.

Suppose $$p^2 - x = q^2 - y = a,$$

Then $$p^2 = a + x \text{ and } q^2 = a + y$$

or $$p = \sqrt{a + x} \quad \text{and} \quad q = \sqrt{a + y}$$

Putting the values of p and q in $d.z = pdx + qdy$, we get

$$dz = \sqrt{a + x}\,dx + \sqrt{a + y}\,dy$$

On integrating, we get

$$z = \frac{2}{3}(a + x)^{3/2} + \frac{2}{3}(a + y)^{3/2} + C$$

which is the required solution.

Example 28 Solve $p - x^2 = q + y^2$.

Solution The given equation is of the form

$$f_1(x, p) = f_2(y, q)$$

Suppose $\quad p - x^2 = q + y^2 = a$

Then $\quad p - x^2 = a$ and $q + y^2 = a$

or $\quad p = a + x^2,\ q = a - y^2$

Putting the values of p and q in $dz = pdx + qdy$, we get

$$dz = (a + x^2)dx + (a - y^2)dy$$

On integrating, we obtain

$$z = \left(ax + \frac{x^3}{3}\right) + \left(ay - \frac{y^3}{3}\right) + b$$

which is the required solution.

Example 29 Solve $yp = 2yx + \log q$.

Solution The given equation is $p - 2x = \frac{1}{y} \log q$, which is of the form $f_1(x, p) = f_2(y, q)$

Suppose $\quad p - 2x = \frac{1}{y}\log q = a$,

We get

$$p - 2x = a \text{ and } \frac{1}{y} \log q = a$$

or $\quad p = a + 2x,\ q = e^{ay}$

Putting the values of p and q in $dz = pdx + qdy$, we get

$$dz = (a + 2x)dx + e^{ay}\,dy$$

On integrating, we get

$$z = (ax + x^2) + \frac{1}{a}e^{ay} + b$$

which is the required solution.

EXERCISE 14.3

Solve the following equation:

1. $p^2 + q^2 = 2$
2. $pq = 1$
3. $\sqrt{p} + \sqrt{q} = 1$
4. $pq + p + q = 0$

5. $p^3 - q^3 = 0$
6. $p = e^q$
7. $p(1 + q^2) = q\,(z - a)$
8. $p^2 = qz$
9. $z = p^2 + q^2$
10. $z = px + qy + \sin(p + q)$
11. $p + q = z$
12. $p - x^2 = q + y^2$
13. $\sqrt{p} + \sqrt{q} = x + y$
14. $p + q = \sin x + \sin y$
15. $q(p - \cos x) = \cos y$
16. $(p^2 + q^2)\,y = qz$

Answers

1. $z = ax + \sqrt{(2-a^2)}\,y + C$
2. $z = ax + \dfrac{1}{a}y + C$
3. $z = ax + (1-\sqrt{a})^2 y + C$
4. $z = ax - \left(\dfrac{a}{a+1}\right)y + C$
5. $z = a\,(x + y) + C$
6. $z = a\,x + y\,(\log a) + C$
7. [**Hint:** Put $u = x + by$] $4(bz - ab - 1) = (x + by + c)^2$
8. $z = be^{(ax + a^2 y)}$
9. $4z\,(1 + a^2) = (x + ay + b)^2$
10. $z = a\,x + b\,y + \sin(a + b)$
11. $(1 + a)\log z = (x + a\,y + b)$
12. $z = \dfrac{1}{3}(x^3 - y^3) + a(x+y) + b$
13. $z = \dfrac{1}{3}(x+a)^3 + \dfrac{1}{3}(y-a)^3 + b$
14. $z = a\,(x - y) - (\cos x + \cos y) + C$
15. $z = ax + \sin x + \dfrac{1}{a}\sin y + b$
16. $z^2 = (c\,x + a)^2 + c^2\,y^2$

14.7 CLAIRAUT'S EQUATION

A nonlinear differential equation of the form

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \tag{15}$$

is called *Clairaut's equation.*

Putting $\dfrac{dy}{dx} = p$ in (15), we get

$$y = x.p + f(p)$$

or

$$x = \frac{1}{p}[y - f(p)] \tag{16}$$

Differentiating (16) both sides with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p}\left[1 - f'(p)\cdot\frac{dp}{dy}\right] + [y - f(p)]\cdot\left[-\frac{1}{p^2}\frac{dp}{dy}\right]$$

or

$$\frac{1}{p} = \frac{1}{p} - \frac{1}{p^2}[y - f(p) + pf'(p)]\frac{dp}{dy}$$

or $$\frac{1}{p^2}[y - f(p) + pf'(p)]\frac{dp}{dy} = 0 \tag{17a}$$

If $\dfrac{dp}{dy} = 0 \Rightarrow p = y' = c$

$\therefore$ the general solution of (15) is

$$y = cx + f(c) \tag{17b}$$

Note: The general solution of Clairaut's equation is obtained by replacing p by c, where c is an arbitrary constant.

Example 30 Solve $\sin px \cos y = \cos px \sin y + p$.

Solution Rewriting the given equation as

$$\sin px \cos y - \cos px \cdot \sin y = p$$

or $$\sin(px - y) = p$$

or $$px - y = \sin^{-1} p$$

or $$y = px - \sin^{-1} p \tag{1}$$

$\therefore$ The general solution of (1) is

$y = cx - \sin^{-1} c$; where c is an arbitrary constant.

Example 31 Solve $(y - px)(p - 1) = p$.

Solution Rewriting the given equation as

$$y - px = \frac{p}{p-1}$$

or $$y = px + \frac{p}{p-1} \tag{1}$$

The general solution of (1) is

$y = cx + \dfrac{c}{c-1}$; where c is an arbitrary constant.

Example 32 Solve $p = \log(px - y)$.

Solution The given equation

$$p = \log(px - y)$$

or $$px - y = e^p$$

or $$y = px - e^p \tag{1}$$

Equation (1) is a Clairaut's equation

Hence, its general solution is $y = cx - e^c$, where c is an arbitrary constant.

Example 33 Solve $p = \cos(y - px)$

Solution The given equation can be written as

$$\cos^{-1} p = y - px$$

or $$y = px + \cos^{-1} p \tag{1}$$

Equation (1) is a Clairaut's equation.

Hence, its general solution

$$y = cx + \cos^{-1} c \tag{2}$$

Now, for a singular solution, differentiating (2) w.r.t. 'c', we get

$$0 = x - \frac{1}{\sqrt{1-c^2}} \tag{3}$$

Eliminate c from (2) and (3), we rewrite (3) as $c = \dfrac{\sqrt{x^2-1}}{x}$

Putting the value 'c' in (2), we get

$$y = \sqrt{x^2-1} + \cos^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$$

which is the singular solution.

EXERCISE 14.4

Solve the following differential equations:

1. $(y - px)(2p + 3) = p$
2. $p = \tan(px - y)$
3. $y = px + ap(1 - p)$
4. $y = px + a/p$
5. $y = px + p - p^2$
6. $y = px + (1 + p^2)^{1/3}$
7. $\cos(y - px) = p^2$

Answers

1. $y = cx + \left(\dfrac{c}{2c+3}\right)$
2. $y = cx - \tan^{-1} c$
3. $y = cx + ac(1 - c)$
4. $y = cx + a/c$
5. $y = cx + c - c^2$
6. $y = cx + (1 + c^2)^{1/3}$
7. $y = cx + \cos^{-1} c^2$

where c is an arbitrary constant.

14.8 LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

14.8.1 Homogeneous Linear Partial Differential Equation with Constant Coefficients

A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants, is known as a linear PDE.

Consider a PDE of the form

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\cdot\partial y} + \cdots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right) +$$

$$\left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right) + N_0 \ z = f(x, y) \tag{18}$$

where the coefficients $A_0, A_1, \ldots A_n, B_0, \ldots B_{n-1}, M_0, M_1, N_0$ are constants. Then (18) is called an LPDE with constant coefficients.

Introducing the notation

$D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$. Then (18) can be rewritten as

$$\Big[(A_0 D_x^n + A_1 D_x^{n-1} \cdot D_y + \cdots A_n D_y^n) + (B_0 D_x^{n-1} + B_1 D_x^{n-2} D_y + \cdots$$

$$B_{n-1} D_y^{n-1}) + (M_0 D_x + M_1 D_y) + N_0\Big]_z = f(x, y)$$

or $\quad F(D_x, D_y)z = f(x, y) \quad (19)$

When all the derivatives appearing in (18) are of the same order then the resulting equation is called a linear homogeneous partial differential equation with constant coefficients and it is then of the form

$$(A_0 D_x^n + A_1 D_x^{n-1} D_y + \cdots A_n D_y^n) z = f(x, y) \tag{20}$$

On the other hand, when all the derivatives in (18) are not of the same order then it is called a nonhomogeneous LPDE with constant coefficients.

The general solution of (20) is the sum of the Complementary Function (CF or Z_c) and the Particular Integral (PI or Z_p). Thus, the General Solution (GS) = CF + PI.

14.8.2 Method of Finding the Complementary Function (CF) of the Linear Homogeneous PDE with Constant Coefficients

Let $$F(D_x, D_y)z = 0$$

i.e., $$(A_0 D_x^n + A_1 D_x^{n-1} D_y + \cdots A_n D_y^n) z = 0 \tag{21}$$

where $A_0, A_1, \ldots A_n$ are all constants.

The complementary function of (21) is the general solution of

$$(A_0 D_x^n + A_1 D_x^{n-1} D_y + \cdots A_n D_y^x) z = 0 \tag{22a}$$

or $$[(D_x - m_1 D_y)(D_x - m_2 D_y)\cdots(D_x - m_n D_y)] z = 0 \tag{22b}$$

where $m_1 \, m_2, \ldots m_n$ are some constants.

The solution of any one of the equations

$$(D_x - m_1 D_y)z = 0, (D_x - m_2 D_y)z = 0, \ldots (D_x - m_n D_y)z = 0 \tag{23}$$

is also a solution of (22b).

Now, we show that the general solution of $(D_x - mD_y)z = 0$ is $z = \phi(y + mx)$ where ϕ is an arbitrary function.

Now, $(D_x - mD_y)z = 0$ or $\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = 0$ or $p - mq = 0 \quad (24)$

which is in Lagrange's form $Pq + Qq = R$. Here, Lagrange's auxiliary equations for (24) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}$$
$$\text{(i)} \quad \text{(ii)} = \text{(iii)}$$

Taking (i) and (ii),

$$dy + mdx = 0$$

On integrating $y + m\,x = C_1$ (25)

Taking (iii), $dz = 0 \Rightarrow z = C_2$ (26)

Hence, from (25) and (26), the general solution of (24) is $z = \phi(mx + y)$, where ϕ is an arbitrary function. so, we assume that a solution of (22a) is of the form $z = \phi(y + mx)$ (27)

From (27), $$D_x z = \frac{\partial z}{\partial x} = m\phi'(y + mx)$$

$$D_x^2 z = \frac{\partial^2 z}{\partial x^2} = m^2\phi''(y + mx)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D_x^n z = \frac{\partial^n z}{\partial x^n} = m^n\phi^{(n)}(y + mx)$$

Again, $$D_y z = \frac{\partial z}{\partial y} = \phi'(y + mx)$$

$$D_y^2 z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y + mx)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D_y^n z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y + m\,x)$$

Also, in general $D_x^{\,r} D_y^{\,s} z = m^r\phi^{(r+s)}(y + mx)$

Substituting these values in (22a) and simplifying, we get

$$(A_0\, m^n + A_1\, m^{n-1} + A_2\, m^{n-2} + \cdots + A_n)\, \phi^{(n)}(y + m\,x) = 0$$

which is true if m is a root of the equation

$$A_0\, m^n + A_1\, m^{n-1} + A_2\, m^{n-2} + \cdots + A_n = 0 \tag{28}$$

Equation (28) is known as the Auxiliary Equation (AE) and is obtained by putting $D_x = m$ and $D_y = 1$ in $F(D_x, D_y) = 0$

Let $m_1, m_2, \ldots m_n$ be n roots of AE (28),Two cases arise.

Case 1

When $m_1, m_2, \ldots m_n$ are distinct. Then the part of CF corresponding to $m = m_r$ is $z = \phi_r(y + m_r x)$ for $r = 1, 2, 3, \ldots n$. Since the equation is linear, the sum of the solution is also a solution.

$\therefore$ CF of (22a) $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \cdots + \phi_n(y + m_nx)$

where $\phi_1, \phi_2, \ldots \phi_n$ are arbitrary functions.

Case 2

Repeated roots. If m is repeated 'r' times, the corresponding part of CF is

$$\phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_1(y+mx) + \cdots + x^{r-1}\,\phi_r(y+mx)$$

Working Rule for Finding CF

Step 1 Put the given equation in standard form

$$(A_0\,D_x^n + A_1\,D_x^{n-1}\,D_y + \cdots + A_n\,D_y^n)z = f(x, y) \tag{29}$$

Step 2 Replace D_x by m and D_y by 1 in the coefficient of z, we obtain auxiliary equation (AE) for (29) as

$$A_0\,m^n + A_1\,m^{n-1} + \cdots + A_n = 0 \tag{30}$$

Step 3 Solve (30) for m. Two cases will arise.

Case 1

Let $m = m_1, m_2, \ldots, m_n$ (distinct roots)

Then $\quad \text{CF} = f_1\,(y + m_1\,x) + f_2\,(y + m_2\,x) + \cdots + f_n\,(y + m_n\,x)$

Case 2

Let $m = m$, (r times). Then corresponding to these roots,

$$\text{CF} = f_1(y + m_1x) + xf_1(y + m_1x) + \ldots\, x^{r-1}\,f_r(y + m_1x).$$

Example 34 Solve $\dfrac{\partial^2 z}{\partial x^2} = a^2\,\dfrac{\partial^2 z}{\partial y^2}$.

Solution Auxiliary equation (AE) is obtained by replacing D_x by m and D_y by 1

So the AE is $m^2 - a^2 = 0$ or $m = \pm\, a$.

$\therefore$ the complementary function (CF) is $z = \phi_1(y + ax) + \phi_2(y - ax)$

Example 35 Solve $(2\,D_x^2 + 5\,D_x \cdot D_y + 2\,D_y^2)\,z = 0$.

Solution Given $(2\,D_x^2 + 5\,D_x\,D_y + 2\,D_y^2)\,z = 0$ (1)

Auxiliary equation of (1) is $2m^2 + 5m + 2 = 0$

$$(2m + 1)(m + 2) = 0$$

$$m = -\frac{1}{2}, -2$$

CF is $z = \phi_1\left(y - \dfrac{x}{2}\right) + \phi_2\,(y - 2x)$ (2) where ϕ_1 and ϕ_2 being arbitrary functions.

Let $\quad \phi_1\left(y - \dfrac{x}{2}\right) = \phi_1\left\{\dfrac{1}{2}(2y - x)\right\} = \psi_1\,(2y - x)$

Then (2) becomes

$$z = \psi_1(2y - x) + \phi_2(y - 2x),\ \psi_1 \text{ and } \phi_2 \text{ being arbitrary functions.}$$

Example 36 Solve $25\,r - 40\,s + 16\,t = 0$ or $(25D_x^2 - 40D_{xy} + 16D_y^2)z = 0$

Solution Given $(25D_x^2 - 40D_{xy} + 16D_y^2)z = 0$ (1)

AE is $25\,m^2 - 40\,m + 16 = 0$ or $(5m - 4)^2 = 0$

$m = \frac{4}{5}, \frac{4}{5}$. Then CF is $z = \phi_1\left(y + \frac{4}{5}x\right) + x\phi_2\left(y + \frac{4}{5}x\right)$

or $z = f_1\,(5y + 4x) + x\,f_2\,(5y + 4x)$, where f_1, f_2 being arbitrary functions.

Example 37 Solve $r + t + 2s = 0$ or $(D_x^2 + D_y^2 + 2D_{xy})z = 0$.

Solution Given $(D_x^2 + D_y^2 + 2D_{xy})z = 0$ (1)

AE is $m^2 + 2m + 1 = 0$ or $(m + 1)^2 = 0$

$$m = -1, -1$$

CF is $z = \phi_1(y - x) + x\phi_2(y - x)$

Example 38 Solve $(D_x^4 - D_y^4)z = 0$.

Solution Given $(D_x^4 - D_y^4)z = 0$ (1)

AE is $m^4 - 1 = 0$ or $(m^2 - 1)\,(m^2 + 1) = 0$

$m = \pm 1$, $m = \pm i$

Hence, CF is $z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix)$

Example 39 Solve $(D_x^3 D_y^2 + D_x^2 D_y^3)z = 0$.

Solution Given $(D_x^3 D_y^2 + D_x^2 D_y^3)z = 0$ (1)

Equation (1) can be rewritten as

$$D_x^2 D_y^2\,(D_x + D_y)z = 0$$

The CF is $z = \phi_1(y) + x\phi_2(y) + \phi_3(x) + y\phi_4(x) + \phi_5(y - x)$

Example 40 Solve $(D_x^3 - 4D_x^2 D_y + 4D_x D_y^2)z = 0$.

Solution Given $(D_x^3 - 4D_x^2 D_y + 4D_x D_y^2)z = 0$ (1)

The AE is $m^3 - 4\,m^2 + 4m = 0$

$m\,(m^2 - 4m + 4) = 0$

or $m\,(m - 2)^2 = 0$

$$m = 0, 2, 2$$

The CF is $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$

Example 41 Solve $(D_x^3 - D_y^3)z = 0$.

Solution Given $(D_x^3 - D_y^3)z = 0$ (1)

The AE is $m^3 - 1 = 0$

$m = 1, \omega, \omega^2$, where ω and ω^2 are complex cube roots of unity.

The CF is $z = \phi_1(y + x) + \phi_2(y + w\,x) + \phi_3(y + w^2\,x)$

Example 42 Solve $(D_x^2 + 3D_xD_y + 2D_y^2)z = 0$.

Solution Given $(D_x^2 + 3D_xD_y + 2D_y^{\,2})z = 0$ (1)

The AE is $m^2 + 3m + 2 = 0$

$$(m + 2)(m + 1) = 0$$

$$m = -2, -1$$

CF is $z = \phi_1(y - 2x) + \phi_2(y - x)$

Example 43 Solve $(D_x^2 - D_y^2)z = 0$.

Solution Given $(D_x^2 - D_y^2)z = 0$ (1)

The AE is $m^2 - 1 = 0$

$$m = \pm 1$$

CF is $z = \phi_1(y + x) + \phi_2(y - x)$

EXERCISE 14.5

Solve the following partial differential equations:

1. $(D_x^2 - D_xD_y - 6D_y^2)z = 0$, where $D_x = \dfrac{\partial z}{\partial x}, D_y = \dfrac{\partial z}{\partial y}$
2. $(D_x^3 - D_x^2 D_y - 8D_x D_y^2 + 12D_y^3)z = 0$
3. $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^3 z}{\partial x^3}\cdot\dfrac{\partial z}{\partial y} + 2\dfrac{\partial^2 z}{\partial x^2}\cdot\dfrac{\partial^2 z}{\partial y^2} - 5\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial^3 z}{\partial y^3} + 3\dfrac{\partial^4 z}{\partial y^4} = 0$
4. $(D_x^2 + 6D_x D_y + 9D_y^2)z = 0$
5. $(D_x^2 - D_y^2)z = 0$
6. $(9D_x^2 + 24D_x D_y + 16D_y^2)z = 0$
7. $(2D_x + D_y + 1)(D_x^2 + 3D_xD_y - 3D_x)z = 0$
8. $(D_x^3 - 6D_x^2 D_y + 11D_x D_y^2 - 6D_y^3)z = 0$
9. $(D_x^3 - 3D_x^2D_y + 2D_xD_y^2)z = 0$
10. $(D_x^4 - 2D_x^2 D_y^2 + D_y^4)z = 0$
11. $(2D_x + D_y + 5)(D_x - 2D_y + 1)^2 z = 0$

Answers

1. $z = f(y - 2x) + g(y + 3x)$
2. $z = g_1(y + 2x) + xg_2(y + 2x) + g_3(y - 3x)$

3. $z = f_1(y+x) + x f_2(y+x) + f_3\left[y - \frac{1}{2}(1 + i\sqrt{11}\,x)\right]$

$$+ f_3\left[y - \frac{1}{2}(1 + i\sqrt{11}\,x)\right] + i\left[f_4\left\{y - \frac{1}{2}(1 + i\sqrt{11}\,x)\right\} - f_5\left\{y - \frac{1}{2}(1 - i\sqrt{11}\,x)\right\}\right]$$

4. $z = f_1(y - 3x) + xf_2(y - 3x)$
5. $z = f_1(y + x) + f_2(y - x)$
6. $z = f_1\left(y - \frac{4}{3}x\right) + x f_2\left(y - \frac{4}{3}x\right).$
7. $z = f_1(y) + e^{-x/2} f_2(2y - x) + e^{3x} f_3(y - 3x)$

   or $z = f_1(y) + e^{-y} f_2(2y - x) + f_3(y - 3x)$
8. $z = f_1(y + x) + f_2(y + 2x) + xf_3(y + 2x)$
9. $z = f_1(y) + f_2(y + x) + f_3(y + 2x).$
10. $z = f_1(y - x) + x f_2(y - x) + f_3(y + x) + x f_4(y + x)$
11. $z = e^{-5y} f_1(2y - x) + e^{-x}[f_2(y + 2x) + xf_3(y + 2x)]$

14.9 METHOD OF FINDING THE PARTICULAR INTEGRAL (PI OR Z_P) OF THE LINEAR HOMOGENEOUS PDE WITH CONSTANT COEFFICIENTS

A PDE of the form $F(D_x, D_y)z = f(x, y)$ (31a)

$$\text{PI or } Z_P = \frac{1}{F(D_x, D_y)} f(x, y) \tag{31b}$$

14.9.1 Short Methods of Finding the Particular Integrals in Certain Cases

Before taking up the general method for finding PI of $F(D_x, D_y)z = f(x, y)$, we begin with cases when $f(x, y)$ is in two special forms. The methods corresponding to these forms are much shorter than the general methods to be discussed in the next section.

Short Method I

When $f(x, y)$ is of the form $\phi(ax + by)$. Two cases will arise.

Case I Replacing D_x by a and D_y by b in $F(D_x, D_y)$, suppose that $F(a, b) \neq 0$. If $F(D_x, D_y)$ is a homogeneous function of D_x and D_y of degree n then we have

$$\text{PI} = \frac{1}{F(D_x, D_y)} \phi(ax, by)$$

$$= \frac{1}{F(a, b)} \iint \cdots \int \phi(v)\, dv^n, \quad \text{where } v = ax + by.$$

After integrating $\phi(v)$ n times w.r.t. 'v', v must be replaced by $(ax + by)$.

Case II Replacing D_x by a and D_y by b in $F(D_x, D_y)$, suppose that $F(a, b) = 0$. Then $F(D_x, D_y)$ must be factorized.

In general, two types of factors will arise.

Let $F(D_x, D_y) = (bD_x - aD_y)^m \cdot G(D_x, D_y)$, where $G(a, b) \neq 0$

Then $$\text{PI} = \frac{1}{(bD_x - aD_y)^m}\varphi(ax+by) = \frac{x^m}{b^m\, m!}\varphi(ax+by)$$

Short Method II

When $f(x, y)$ is of the form $x^m y^n$, m and n being non-negative integers.

If $n < m$, $\dfrac{1}{F(D_x, D_y)}$ is expanded in powers of $\dfrac{D_y}{D_x}$. On the other hand, if $m < n$, $\dfrac{1}{F(D_x, D_y)}$ is expanded in powers of $\dfrac{D_x}{D_y}$. It will be noted that if $\dfrac{1}{F(D_x, D_y)}$ is expanded in two ways in any problem, we arrive at different answers.

However, the difference is not material as it can be incorporated in the arbitrary functions occurring in the CF of that problem.

14.9.2 General Method of Finding PI

Let $$F(D_x, D_y)\, z = f(x, y) \tag{32}$$

When $F(D_x, D_y)$ is a homogeneous function of D_x and D_y, we use the following results.

$$(R_1)\ \frac{1}{(D_x - mD_y)} f(x, y) = \int f(x, c - mx)\, dx, \text{ where } c = y + mx$$

$$(R_2)\ \frac{1}{(D_x + mD_y)} f(x, y) = \int f(x, c_1 + mx)\, dx, \text{ where } c_1 = y - mx$$

After performing integrations c and c_1 must be replaced by $y + mx$ and $y - mx$ respectively.

Example 44 Solve $(D_x^2 - 2D_x D_y + D_y^2)z = \sin x$.

Solution The auxiliary equation of the given equation is obtained by putting

$D_x = m$ and $D_y = 1$ in $(D_x^2 - 2D_x D_y + D_y^2) = 0$ is

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$

$\therefore$ $\quad$ CF $(Z_c) = f_1(y + x) + x f_2(y + x)$

Now, PI is $$Z_p = \frac{1}{(D_x - D_y)^2}\sin x$$

$$= \frac{1}{D_x^2 - 2D_x D_y + D_y^2}\sin x$$

[Replace D_x^2 by -1^2

D_y^2 by 0 and $D_x D_y = 0$]

$$= \frac{1}{-1-2\cdot 0+0}\sin x$$
$$= -\sin x$$

Thus, the general solution is

$Z = \text{CF} + \text{PI} = f_1(y + x) + x f_2(y + x) - \sin x$ where f_1 and f_2 are arbitrary functions.

Example 45 Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \cos 2y.$

Solution The auxiliary equation is

$$m^2 - 2m = 0$$
$$m(m-2) = 0 \text{ or } m = 0, 2$$

CF is $Z_c = f_1(y) + f_2(y + 2x)$

PI is
$$Z_p = \frac{1}{2}\frac{1}{D_x^2 - 2D_x D_y} 2\sin\cdot\cos 2y$$
$$= \frac{1}{2}\frac{1}{D_x^2 - 2D_x D_y}[\sin(x+2y) + \sin(x-2y)]$$
$$= \frac{1}{2}\frac{1}{D_x^2 - 2D_x D_y}\sin(x+2y) + \frac{1}{2}\frac{1}{D_x^2 - 2D_x D_y}\sin(x-2y)$$

[Replace D_x^2 by -1 and $D_x D_y$ by -2 in first z_p and D_x^2 by -1 and $D_x D_y$ by 2 in second z_p]

$$= \frac{1}{2}\frac{1}{-1+4}\sin(x+2y) + \frac{1}{2}\frac{1}{-1-4}\sin(x-2y)$$
$$= \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$
$$= \frac{1}{15}\sin x\cos 2y + \frac{4}{15}\sin 2y\cdot\cos x$$

$\therefore$ the general solution is

$$Z = Z_c + Z_p = f_1(y) + f_2(y+2x) + \frac{1}{15}(\sin x\cos 2y + 4\sin 2y\cdot\cos x),$$

where f_1 and f_2 are arbitrary functions.

Example 46 Solve $(D_x^2 + D_y^2)z = x^2y^2$.

Solution The auxiliary equation is

$$m^2 + 1 = 0 \text{ or } m = \pm i$$

CF is $Z_c = f_1(y + ix) + f_2(y - ix)$

PI is
$$Z_p = \frac{1}{(D_x^2 + D_y^2)}x^2y^2$$

$$= \frac{1}{D_x^2}\left[1+\frac{D_y^2}{D_x^2}\right]^{-1} x^2y^2$$

$$= \frac{1}{D_x^2}\left[1-\frac{D_y^2}{D_x^2}+\frac{D_y^4}{D_x^4}+\cdots\right]x^2y^2$$

$$= \frac{1}{D_x^2}(x^2\,y^2) - \frac{1}{D_x^4}D_y^2\,(x^2\,y^2) + 0 + \cdots$$

$$= \frac{x^4}{12}y^2 - 2.\frac{x^6}{3.4.5.6}$$

$$= \frac{1}{180}(15x^4\,y^2 - x^6)$$

$\therefore$ the general solution is

$$Z = \text{CF} + \text{PI} = f_1(y+ix) + f_2(y-ix) + \frac{1}{180}(15x^4y^2 - x^6)$$

Example 47 Solve $(D_x^2 + 3D_xD_y + 2D_y^2)Z = x + y.$

Solution The Auxiliary Equation (AE) of the given equation is

$$m^2 + 3m + 2 = 0 \text{ or } m = -1, -2$$

So CF is $Z_c = \phi_1(y-x) + \phi_2(y-2x)$

Now, PI is $Z_p = \dfrac{1}{D_x^2 + 3D_x\,D_y + 2D_y^2}(x+y)$ [Using Case I of Section 14.9.1]

$$= \frac{1}{1^2+3\cdot 1+2\cdot 1^2}\iint v\,dv\,dv, \text{ where } v = x+y$$

$$= \frac{1}{6}\int\frac{v^2}{2}d\,v$$

$$= \frac{1}{6}\cdot\frac{v^3}{6}$$

$$= \frac{1}{36}\cdot v^3$$

$$= \frac{1}{36}(x+y)^3$$

Then the general solution is

$$Z = \text{CF} + \text{PI} = \varphi_1\,(y-x) + \varphi_2\,(y-2x) + \frac{1}{36}(x+y)^3,$$

where ϕ and ϕ_2 are arbitrary functions.

Example 48 Solve $4r - 4s + t = 16 \log (x + 2y)$.

Solution The given equation can be rewritten as

$$4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log(x + 2y)$$

or $$(4D_x^2 - 4D_x D_y + D_y^2)z = 16 \log(x + 2y)$$

The AE is $4m^2 - 4m + 1 = 0$ so that $m = \frac{1}{2}, \frac{1}{2}$

CF is $$Z_c = \varphi_1\left(y + \frac{x}{2}\right) + x\varphi_2\left(y + \frac{x}{2}\right)$$

$$= \psi_1(2y + x) + x\psi_2(2y + x)$$

Now, PI is $$Z_p = \frac{1}{(2D_x - D_y)^2} \cdot 16 \log(x + 2y)$$

$$= 16 \frac{x^2}{2^2 \cdot 2!} \log(x + 2y)$$ [Using Case II of Section 14.9.1]

$$= 2x^2 \log(x + 2y)$$

Then the general solution is

$$Z = \text{CF} + \text{PI} = Z_c + Z_p$$

$$Z = \psi_1(2y + x) + x\psi_2(2y + x) + 2x^2 \log(x + 2y)$$

where ψ_1 and ψ_2 are arbitrary functions.

Example 49 Solve $(D_x^3 - 4D_x^2 D_y + 4D_x D_y^2) Z = 4 \sin(2x + y)$.

Solution The AE is $m^3 - 4m^2 + 4m = 0$

$$m(m^2 - 4m + 4) = 0$$

$$m(m - 2)^2 = 0$$

$$m = 0, 2, 2$$

CF is $$Z_c = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

Now, PI is $$Z_p = \frac{1}{D_x^3 - 4D_x^2 D_y + 4D_x D_y^2} 4\sin(2x + y).$$

$$= \frac{4}{D_x(D_x - 2D_y)^2} \sin(2x + y)$$

$$= \frac{4}{(D_x - 2D_y)^2}\left\{\frac{1}{D_x}\sin(2x + y)\right\}$$

$$= \frac{4}{(D_x - 2D_y)^2}\left\{-\frac{1}{2}\cos(2x + y)\right\},$$

since $\frac{1}{D_x}$ stands for integrating w.r.t. x treating y as constant.

$$= -2\cdot\frac{1}{(D_x - 2D_y)^2}\cos(2x + y)$$

$$= -2\cdot\frac{x^2}{1^2\cdot 2!}\cos(2x+y) = -x^2\cos(2x+y)$$

The general solution is

$$Z = \text{CF} + \text{PI} = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - x^2\cos(2x+y)$$

Example 50 Solve $(D_x^3 - 7D_x D_y^2 - 6D_y^3)Z = \sin(x+2y) + e^{3x+y}$.

Solution The AE is $m^3 - 7m - 6 = 0$

$(m+1)(m+2)(m-3) = 0$

$$m = -1, -2, 3$$

CF is $\quad Z_c = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

Now, PI is $\quad Z_p = \frac{1}{D_x^3 - 7D_x D_y^2 - 6D_y^3}\left[\sin(x+2y) + e^{3x+y}\right]$

PI corresponding to $\sin(x+2y)$

$$= \frac{1}{D_x^3 - 7D_x D_y^2 - 6D_y^3}\sin(x+2y)$$

$$= \frac{1}{1^3 - 7\cdot 1\cdot 2^3 - 6\cdot 2^3}\iiint \sin v\,dv\,dv\,dv, \text{ where } v = x+2y$$

$$= -\frac{1}{75}\iint(-\cos v)\,dv\,dv$$

$$= -\frac{1}{75}\int -\sin v\,dv = -\frac{1}{75}\cos v$$

$$= -\frac{1}{75}\cos(x+2y)$$

and PI corresponding to e^{3x+y}

$$= \frac{1}{D_x^3 - 7D_x D_y^2 - 6D_y^3}e^{3x+y} = \frac{1}{(D_x - 3D_y)}\left[\frac{1}{(D_x + D_y)(D_x + 2D_y)}\cdot e^{3x+y}\right]$$

$$= \frac{1}{(D_x - 3D_y)}\cdot\frac{1}{(3+1)(3+2)}\cdot\iint e^v\,dv\,dv, \text{ where } v = 3x+y.$$

$$= \frac{1}{20}\frac{1}{(D_x - 3D_y)}\int e^v\,dv = \frac{1}{20}\frac{1}{D_x - 3D_y}e^v$$

$$= \frac{1}{20}\frac{1}{D_x - 3D_y}e^{3x+y}$$

$$= \frac{1}{20}\frac{x}{1\cdot 1!}e^{3x+y} = \frac{x}{20}e^{3x+y}$$

Hence, the general solution is

$$Z = \text{CF} + \text{PI} = Z_c + Z_p$$

$$Z = \phi_1(y - x) + \phi_2\ (y - 2x) + (y + 3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20}e^{3x+y}$$

Example 51 Solve $r + (a + b)\, s + a\, b\, t = xy.$

Solution Given equation can be written as

$$(D_x^2 + (a+b)D_x\, D_y + ab\, D_y^2)z = xy$$

The AE is $\quad m^2 + (a + b)m + ab = 0$

or $\quad (m + a)\,(m + b) = 0$

$$m = -a, -b$$

CF is $\quad Z_c = \phi_1\,(y - ax) + \phi_2(y - bx)$

Now, PI is $\quad Z_p = \dfrac{1}{D_x^2 + (a+b)D_x\, D_y + ab\, D_y^2}\, xy$

$$= \frac{1}{D_x^2\left[1 + (a+b)\dfrac{D_y}{D_x} + ab\cdot\dfrac{D_y^2}{D_x^2}\right]}\, xy$$

$$= \frac{1}{D_x^2}\left[1 + (a+b)\frac{D_y}{D_x} + ab\,\frac{D_y^2}{D_x^2}\right]^{-1}\cdot xy$$

$$= \frac{1}{D_x^2}\left[1 - (a+b)\frac{D_y}{D_x} + \cdots\right]xy$$

$$= \frac{1}{D_x^2}\left[xy - \frac{(a+b)}{D_x}D_y\ xy\right] = \frac{1}{D_x^2}\left[xy - \frac{a+b}{D_x}\cdot x\right]$$

$$= \frac{1}{D_x^2}\left[xy - \frac{(a+b)}{2}x^2\right]$$

[since $\dfrac{1}{D_x}$ stands for integration w.r.t. 'x' and D_x and D_y stands for differentiation w.r.t. 'x' and y respectively]

$$= y\cdot\frac{x^3}{2\cdot 3} - \frac{(a+b)}{2}\cdot\frac{x^4}{3\cdot 4}$$

$$= \frac{x^3 y}{6} - \frac{(a+b)}{24}x^4$$

Hence, the general solution $Z = \text{CF} + \text{PI}$.

$$Z = Z_c + Z_p$$

$$Z = \phi_1(y-ax) + \phi_2(y-bx) + \frac{x^3y}{6} - \frac{(a+b)}{24}x^4$$

Example 52 Solve $(D_x^2 + D_x \cdot D_y - 6D_y^2)z = y \cdot \cos x$.

Solution The AE is $m^2 + m - 6 = 0$

$$m = 2, -3$$

CF is $Z_c = \phi_1(y+2x) + \phi_2(y-3x)$

Now, PI is
$$Z_p = \frac{1}{D_x^2 + D_x D_y - 6D_y^2} y . \cos x$$

$$Z_p = \frac{1}{(D_x - 2D_y)(D_x + 3D_y)} \cdot y \cos x$$

$$= \frac{1}{(D_x - 2D_y)} \int (3x + c)\cos x \, dx \quad \text{where } c = y - 3x$$

$$= \frac{1}{(D_x - 2D_y)}\left[(3x+c)\sin x - \int 3\sin x \, dx\right]$$

$$= \frac{1}{(D_x - 2D_y)}[y\sin x + 3\cos x]$$

$$= \int \left[(c' - 2x)\sin x + 3\cos x\right]dx, \quad \text{where } c' = y + 2x$$

$$= (c' - 2x)(-\cos x) - \int (-2)(-\cos x)\,dx + 3\sin x$$

$$= -y\cos x - 2\sin x + 3\sin x \qquad \text{as } c' = y + 2x$$

$$= \sin x - y\cos x$$

Hence, the general solution is

$$Z = Z_c + Z_p$$

$$Z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x$$

Example 53 Solve $(D_x^2 - D_y^2)z = \tan^3 x \ \tan y - \tan x \tan^3 y$.

Solution $(D_x^2 - D_y^2)z = \tan^3 x \ \tan y - \tan x \tan^3 y$

$$(D_x + D_y)(D_x - D_y)z = \tan^3 x \tan y - \tan .x \tan^3 y \qquad (1)$$

Its auxiliary equation (AE) is $(m+1)(m-1) = 0$

$$m = -1, 1$$

CF is $Z_c = \phi_1(y-x) + \phi_2(y+x)$

Now, PI is
$$z_p = \frac{1}{(D_x + D_y)(D_x - D_y)}(\tan^3 x \tan y - \tan x \tan^3 y)$$

$$= \frac{1}{(D_x + D_y)} \int \left[\tan^3 \cdot \tan(c-x) - \tan x \tan^3(c-x) \right] dx \quad \text{where } c = y + x$$

$$= \frac{1}{(D_x + D_y)} \int \Big[\tan x \tan(c-x)(\sec^2 x - 1)$$
$$- \tan x \tan(c-x)\{\sec^2(c-x) - 1\} \Big] dx$$

$$= \frac{1}{(D_x + D_y)} \int \left[\tan x \cdot \sec^2 x \tan(c-x) - \tan(c-x) \sec^2(c-x) \tan x \right] dx$$

$$= \frac{1}{(D_x + D_y)} \Bigg[\frac{\tan^2 x}{2} \tan(c-x) - \int \frac{\tan^2 x}{2} \sec^2(c-x)(-1)\, dx$$
$$\left\{ \frac{\tan^2(c-x)}{2 \times (-1)} \tan x - \int \frac{\tan^2(c-x)}{2 \times (-1)} \sec^2 x\, dx \right\} \Bigg]$$

$$= \frac{1}{2(D_x + D_y)} \left[\tan^2 x \tan(c-x) + \tan x \cdot \tan^2(c-x) \right];$$

$$\left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$+ \int (\sec^2 x - 1) \sec^2(c-x)\, dx - \int \{\sec^2(c-x) - 1\} \sec^2 x\, dx$$

$$= \frac{1}{2(D_x + D_y)} \Big[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x)$$
$$- \int \sec^2(c-x)\, dx + \int \sec^2 x\, dx \Big]$$

$$= \frac{1}{2(D_x + D_y)} \left[\tan^2 x \cdot \tan(c-x) + \tan x \tan^2(c-x) + \tan(c-x) + \tan x \right]$$

$$= \frac{1}{2(D_x + D_y)} \left[\tan^2 x \tan y + \tan x \tan^2 y + \tan y + \tan x \right] \text{ as } c = y + x.$$

$$= \frac{1}{2(D_x + D_y)} \left[\tan y(\tan^2 x + 1) + \tan x(\tan^2 y + 1) \right]$$

$$= \frac{1}{2(D_x + D_y)} [\tan y \cdot \sec^2 x + \tan x \cdot \sec^2 y]$$

$$= \frac{1}{2} \left[\int \tan(c' + x) \sec^2 dx + \int \tan x \cdot \sec^2(c' + x)\, dx \right], \text{ where } c' = y - x$$

$$= \frac{1}{2} \left[\tan(c' + x) \cdot \tan x - \int \sec^2(c' + x) \cdot \tan x\, dx + \int \tan x \cdot \sec^2(c' + x)\, dx \right]$$

[On integrating the first integral by parts keeping the second integral unchanged]

$$= \frac{1}{2}\tan(c' + x)\cdot\tan x$$

$$= \frac{1}{2}\tan y\cdot\tan x, \text{ as } c' = y - x$$

$\therefore$ The general solution is

$$Z = Z_c + Z_p$$

$$Z = \phi_1(y - x) + \phi_2(y + x) + \frac{1}{2}\tan x\cdot\tan y.$$

14.10 NONHOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A linear partial differential equation which is not homogeneous is called a *nonhomogeneous linear PDE.*

Consider a linear equation with constant coefficients:

$$F(D_x, D_y)z = f(x, y) \tag{33}$$

If (33) is a homogeneous linear PDE with constant coefficient then $F(D_x, D_y)$ can always be resolved into linear factors.

On the other hand, if (33) is not homogeneous then $F(D_x, D_y)$ cannot be resolved into linear factors.

Hence, we classify nonhomogeneous linear partial differential equations into the following two types:

1. When $F(D_x, D_y)$ can be resolved into linear factors, each of which is of first degree in D_x and D_y. For example, $(D_x^2 - D_y^2 + 2D_x + 1)z = x^3y^3$, i.e., $[(D_x + D_y + 1)(D_x - D_y + 1)]z = x^2y^3$.
2. When $F(D_x, D_y)$ cannot be resolved into linear form $(aD_x - bD_y - c)$

   For Example, $(2D_x^4 - 3D_x^2D_y + D_y^2)z = 2xy^2$, i.e., $(2D_x^2 - D_y)(D_x^2 - D_y)z = 2xy^2$.

14.10.1 Method of Finding CF when Equations are of the Type (1)

Working rule for finding the CF of nonhomogenous equations with constant coefficients, i.e.,

$$F(D_x, D_y)z = 0$$

When $F(D_x, D_y)$ can be resolved in to product of linear factors in D_x and D_y then

- Corresponding to each non-repeated factor $(aD_x - bD_y - c)$, the part of CF is $e^{\frac{cx}{a}}\phi(ay + bx)$ if $a \neq 0$.
- Corresponding to repeated factors $(aD_x - bD_y - c)^r$ then the part of CF is $e^{\frac{cx}{a}}[\phi_1(ay + bx) + x\phi_2(ay + bx) + x^2\phi_3(ay + bx) + \ldots + x^{r-1}\phi_r(ay + bx)]$ if $a \neq 0$

Particular Cases

(i) If $c = 0$ then corresponding to each nonrepeated factor $(aD_x - bD_y)$, the part of CF is $\phi(ay + bx)$.

(ii) If $c = 0$ then corresponding to each repeated factor $(aD_x - bD_y)^r$, the part of CF is

$$\phi_1(ay + bx) + x\phi_2(ay + bx) + x^2\, \phi_3(ay + bx) + \cdots + x^{r-1}\, \phi_r\,(ay + bx)$$

(iii) If $a = 1, b = 0, c = 0$ then corresponding to each nonrepeated factor D_x, the part of CF is $\phi(y)$.

(iv) If $a = 1, b = 0, c = 0$ then corresponding to each repeated factor $D_x^{\,r}$, the part of CF is

$$\phi_1(y) + x\, \phi_2(y) + \cdots + x^{r-1}\, \phi_r(y)$$

(v) If $a = 0, b = 1, c = 0$ then corresponding to each nonrepeated factor D_y, the part of CF is $\phi(x)$.

(vi) If $a = 0, b = 1, c = 0$ then corresponding to repeated factors $D_y^{\,r}$, the part of CF is

$$\phi_1(x) + \phi_2(x) + x^2\, \phi_3(x) + \cdots + x^{r-1}\, \phi_r(x)$$

(vii) If $a = 0, b = 1, c = 1$ then corresponding to each nonrepeated factor $(D_y + 1)$, the part of CF is $e^x\, \phi(x)$

(viii) If $a = 0, b = 1, c = 1$ then corresponding to repeated factor's $(D_y + 1)^r$, the part of CF is

$$e^x[\phi_1(x) + x\, \phi_2(x) + x^2\, \phi_3(x) + \cdots + x^{r-1}\, \phi_r(x)]$$

(ix) If $a = 1, b = 1, c = 0$ then corresponding to each nonrepeated factor $(D_x - D_y)$, the part of CF is $\phi(y + x)$.

(x) If $a = 1, b = 1, c = 0$ then corresponding to each repeated factors $(D_x - D_y)^r$, the point of CF is

$$\phi_1(y + x) + x\phi_2(y + x) + x^2\, \phi_3(y + x) + \cdots + x^{r-1}\, \phi_r(y + x)$$

Example 54 Solve $(D_x^2 - D_y^2 + D_x - D_y)z = 0$.

Solution Given $(D_x^2 - D_y^2 + D_x - D_y)z = 0$ (1)

Equation (1) can be written as

$$(D_x - D_y)\,(D_x + D_y + 1)z = 0$$

Hence, CF is $Z_c = \phi_1(y + x) + e^{-x}\, \phi_2(y - x)$

Example 55 Solve $D_x\, D_y\, (D_x - 2\, D_y - 3)z = 0.$ (1)

Solution Hence, the required solution is

$$z = \phi_1(y) + \phi_2(x) + e^{3x}\, \phi_3(y + 2x)$$

Example 56 Solve $(D_x^2 - a^2\, D_y^2 + 2\,ab\,D_x + 2\,a^2 b\,D_y)z = 0$.

Solution The given equation can be written as

$$[(D_x + aD_y)\,(D_x - aD_y) + 2ab(D_x + aD_y)]z = 0$$

$$(D_x + a\, D_y)\,(D_x - aD_y + 2ab)z = 0$$

The CF is $z = \phi_1(y - ax) + e^{-2abx}\, \phi_2(y + ax)$

Example 57 Solve $r + 2s + t + 2p + 2q + z = 0.$

Solution The given equation can be rewritten as

$$(D_x^2 + 2\,D_x\,D_y + D_y^2 + 2\,D_x + 2\,D_y + 1)z = 0$$

or $$(D_x + D_y)^2 + 2\,(D_x + D_y) + 1]z = 0$$

or $$(D_x + D_y + 1)^2 z = 0$$

So the required general solution is

$$z = e^{-x}\,[\phi_1\,(y-x) + x\,\phi_2\,(y-x)]$$

Example 58 Solve $(D_x^2 - D_x\,D_y + D_y - 1)\,z = 0$.

Solution $(D_x^2 - D_x\,D_y + D_y - 1)\,z = 0$

$$(D_x - 1)\,(D_x - D_y + 1)z = 0$$

The general solution is

$$z = e^{x}\,\phi_1(y) + e^{-x}\phi_2(y + x)$$

14.10.2 Method of Finding CF when Equations are of the Type(2)

$$F\,(D_x, D_y)z = 0 \tag{34}$$

where the symbolic operator $F\,(D_x, D_y)$ cannot be resolved into factors linear in D_x and D_y. In such a case, as a trial solution, consider $Z = Ae^{hx + ky}$ (35)

where A, h, k are constants. From (35), we have

$$D_x^r\,z \;=\; Ah^r e^{hx+ky},\; D_x^r D_y^s z = Ah^r k^s e^{hx+ky} \text{ and } D_y^s\,z = Ak^s e^{hx+ky}$$

Hence, substituting (35) in (34), we get

$$Af(h, k)\,e^{hx + ky} = 0$$

which will hold if $f(h, k) = 0$ (36)

and A is an arbitrary constant.

Now, for any chosen values of h or k, (36) gives one or more values of k or h. So there exist infinitely many pairs of numbers (h_i, k_i) satisfying (36)

Thus, $$Z = \sum_{i=1}^{\infty} A_i\,e^{h_i x + k_i y} \tag{37a}$$

where $$f(h_i, k_i) = 0 \tag{37b}$$

for each i is a solution of Eq. (34). So, if $f\,(D_x, D_y)$ has no linear factor, (37a) is taken as the general solution of (34).

See examples 62 and 64.

14.10.3 Method of Finding Particular Integrals (PI)

The methods of finding PI of nonhomogeneous linear partial differential equations are very similar to those used in solving homogeneous linear PDE with constant coefficients.

Here we are considering a few cases only.

Case I

When $f\,(x, y) = e^{ax + by}$

Then, $$\text{PI} = \frac{1}{f\,(D_x, D_y)}\,e^{ax+by} = \frac{1}{f\,(a,b)}\,e^{ax+by} \text{ provided } f(a, b) \neq 0$$

i.e., put $D_x = a$ and $D_y = b$

Case II

When $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

Then, PI = $\dfrac{1}{f(D_x, D_y)}\sin(ax+by)$ or $\cos(ax + by)$ is obtained by putting $D_x^2 = -a^2$,

$D_x \cdot D_y = -ab,\ D_y^2 = -b^2$

Case III

When $f(x, y) = x^m y^n$, where m and n are positive integers.

Then, $$\text{PI} = \frac{1}{f(D_x\, D_y)} x^m y^n = [f(D_x, D_y]^{-1} x^m v^n. \text{ [Using binomial theorem]}$$

Case IV

When $f(x, y) = e^{(ax + by)} V(x)$

Then, $$\text{PI} = \frac{1}{f(D_x\, D_y)} e^{(ax+by)}\ V(x)$$

$$= e^{(ax+by)} \frac{1}{f(D_x + a,\ D_y + b)} \cdot V(x)$$

Example 59 Solve $(D_x^2 - D_y^2 + D_x - D_y)z = 0.$

Solution The given equation is

$$(D_x^2 - D_y^2 + D_x + D_y)z = 0$$

$$[(D_x + D_y)(D_x - D_y) + (D_x - D_y)]\, z = 0$$

$$(D_x - D_y)(D_x + D_y + 1)\, z = 0$$

The general solution is $Z_c = \phi_1(y + x) + e^{-x}\, \phi_2(y - x)$

Example 60 Solve $(D_x^2 + 2D_x D_y + D_y^2 + 2D_x + 2D_y + 1)z = 0$

Solution The given equation is $(D_x^2 + 2D_x D_y + D_y^2 + 2D_x + 2D_y - 1)z = 0$

or $$(D_x + D_y + 1)^2\, z = 0$$

$\therefore$ the general solution is $Z_c = e^{-x}[\phi_1(y - x) + x\phi_2(y - x)]$

Example 61 Solve $(D_x - D_y - 1)(D_x - D_y - 2)z = e^{2x-y} + x$

Solution $(D_x - D_y - 1)(D_x - D_y - 2)z = e^{2x-y} + x$

To find the CF of $(D_x - D_y - 1)(D_x - D_y - 2)\, z = 0$

CF is $Z_c = e^x\, \phi_1(y + x) + e^{2x}\, \phi_2(y + x)$

PI corresponding to e^{2x-y} is

$$Z_p = \frac{1}{(D_x - D_y - 1)(D_x - D_y - 2)} \cdot e^{2x-y}$$

$$Z_p = \frac{1}{(2+1-1)(2+1-2)} e^{2x-y}$$

$$Z_p = \frac{1}{2} e^{2x-y}$$

Now, PI corresponding to x is

$$Z_p = \frac{1}{(D_x - D_y - 1)(D_x - D_y - 2)} x$$

$$= \frac{1}{2}(1 - D_x + D_y)^{-1}\left(1 - \frac{1}{2}D_x + \frac{1}{2}D_y\right)^{-1} \cdot x$$

$$= \frac{1}{2}(1 + D_x - D_y \cdots)\left(1 + \frac{D_x}{2} - \frac{D_y}{2} \cdots\right) \cdot x$$

$$= \frac{1}{2}\left(1 + D_x + \frac{D_x}{2} + \cdots\right) x$$

$$= \frac{1}{2}\left(1 + \frac{3}{2}D_x\right) \cdot x$$

$$= \frac{1}{2}x + \frac{3}{4}$$

$\therefore$ The complete solution is

$$Z = Z_c + Z_p$$

$$Z = e^x \phi_1(y + x) + e^{2x}\phi_2(y + x) + \frac{1}{2}e^{2x-y} + \frac{x}{2} + \frac{3}{2},$$

where ϕ_1, ϕ_2 are arbitrary functions.

Example 62 Solve $(D_x - D_y^2)z = 0$

Solution Here, $D_x - D_y^2$ is not a linear factor in D_x and D_y. So we use the method (2) as explained in Section 14.10.2.

Let $Z = Ae^{hx + ky}$ be a trial solution of the given equation.

$$D_x z = Ahe^{hx + ky} \text{ and } D_y^2 z = Ak^2 e^{hx + ky}$$

Putting these values in the given differential equation, we get

$$A(h - k^2)e^{hx + ky} = 0, \text{ so that } h - k^2 = 0 \text{ or } h = k^2$$

Replacing h by k^2, the most general solution of the given equation is $z = \sum A e^{k^2 x + ky}$, where A and k are arbitrary constants.

Example 63 Solve $(D_x - 2 D_y - 1)(D_x - 2 D_y^2 - 1)z = 0$.

Solution $(D_x - 2 D_y - 1)$ being linear in D_x and D_y, the part of CF corresponding to it is $e^x f(y + 2x)$. To find CF corresponding to the nonlinear factor $(D_x - 2 D_y^2 - 1)$, we proceed as follows:

Let a trial solution of $(D_x - 2D_y^2 - 1)z = 0$ (1)

be $\quad Z = Ae^{hx + ky}$ (2)

$\therefore \quad D_x z = Ahe^{hx + ky},\ D_y{}^2 z = Ak^2 e^{hx + ky}$

Hence, (1) becomes

$$A(h - 2k^2 - 1)e^{hx + ky} = 0 \text{ or } h - 2k^2 - 1 = 0 \text{ or } h = 2k^2 + 1$$

Replacing h by $2k^2 + 1$ in (2), the solution of (1), i.e., the part of CF corresponding to $(D_x - 2D_y^2 - 1)$ in the given equation is given by

$$Z_c = \sum A e^{(2k^2+1)x+ky}, \ A \text{ and } k \text{ being arbitrary constants.}$$

$\therefore$ the required solution is $Z = e^x \phi(y + 2x) + \sum Ae^{(2k^2+1)x + ky}$

Example 64 Solve $(D_x^2 + D_y^2 - n^2)z = 0.$

Solution $(D_x^2 + D_y^2 - n^2)z = 0$ (1)

Let a trial solution of (1) be $z = Ae^{hx + ky}$ (2)

$\therefore \quad D_x^2 z = Ah^2 e^{hx+ky},\ D_y^2 z = Ak^2 e^{hx+ky}$

Then (1) becomes

$$A(h^2 + k^2 - n^2)\, e^{hx + ky} = 0 \text{ or } h^2 + k^2 - n^2 = 0$$

or $\quad h^2 + k^2 = n^2$

Taking α as parameter, we see that (2) is satisfied if

$$h = n\cos\alpha \text{ and } k = n\sin\alpha$$

Putting these values in (2), the required general solution is

$$Z = \sum A e^{n(x\cos\alpha + y\sin\alpha)}, \ A \text{ and } \alpha \text{ being arbitrary constants.}$$

Example 65 Solve $(D_xD_y + aD_y + bD_y + ab)\, z = e^{mx + ky}$

Solution The given equation can be rewritten as

$$(D_x + b)\,(D_y + a)\, z = e^{mx + ky}$$

$\therefore$C.F is $\quad Z_c = e^{-bx}\, \phi_1(y) + e^{-ay}\, \phi_2(x)$ and

PI is $\quad Z_p = \dfrac{1}{(D_x + b)(D_y + a)}\, e^{mx+ny} = \dfrac{1}{(m+b)(n+a)}\, e^{mx+ny}$

Hence, the required general solution is

$$Z = Z_c + Z_p$$

$$Z = e^{-bx}\, \phi_1(y) + e^{-ay}\, \phi_2(x) + \frac{1}{(m+b)(n+a)}\, e^{mx+ny}$$

Example 66 Solve $(D_x - D_y - 1)\,(D_x - D_y - 2)\, Z = \sin(2x + 3y).$

Solution Given $(D_x - D_y - 1)\,(D_x - D_y - 2)\, Z = \sin(2x + 3y)$

The CF is $Z_c = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$ and

PI is
$$Z_p = \frac{1}{(D_x - D_y - 1)(D_x - D_y - 2)} \cdot \sin(2x + 3y)$$
$$= \frac{1}{D_x^2 - 2D_xD_y + D_y^2 - 3D_x + 3D_x + 2} \cdot \sin(2x + 3y)$$
$$= \frac{1}{-2^2 - 3\times -6 - (3)^2 - 3D_x + 3D_y + 2} \sin(2x + 3y)$$
$$= \frac{1}{-3D_x + 3D_y + 1} \sin(2x + 3y) = \frac{D_x}{-3D_x^2 + 3D_x D_y + D_x} \cdot \sin(2x + 3y)$$
$$= \frac{D_x}{-3\times -2^2 + 3\times -6 + D_x} \sin\,(2x + 3y) = \frac{D_x}{D_x - 6} \sin\,(2x + 3y)$$
$$= \frac{D_x\,(D_x + 6)}{D_x^2 - 36} \sin\,(2x + 3y) = \frac{D_x^2 + 6D_x}{-2^2 - 36} \sin\,(2x + 3y)$$
$$= -\frac{1}{40}\left[D_x^2 \sin(2x + 3y) + 6D_x \sin(2x + 3y)\right]$$
$$= -\frac{1}{40}[-4\sin(2x + 3y) + 12\cos(2x + 3y)]$$

$\therefore$ the required solution is
$$Z = e^x\,\varphi_1\,(y + x) + e^{2x}\,\varphi_2\,(y + x) - \frac{1}{40}[-4\sin(2x + 3y) + 12\cos(2x + 3y)]$$

or
$$Z = e^x\,\varphi_1\,(y + x) + e^{2x}\,\varphi_2\,(y + x) - \frac{1}{10}\sin(2x + 3y) - \frac{3}{10}\cos(2x + 3y)$$

14.11 EQUATION REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A partial differential equation of the form
$$a_0\,x^n \frac{\partial^n z}{\partial x^n} + a_1\,x^{n-1}\,y \frac{\partial^n z}{\partial x^{n-1}\,\partial y} + a_2\,x^{n-2}\,y^2 \frac{\partial^n z}{\partial x^{n-2}\,\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = f(x, y) \tag{38}$$

having variable coefficients in particular form (namely, the term $\frac{\partial^n z}{\partial x^n}$ is multiplied by the variable x^n, $\frac{\partial^n z}{\partial y^n}$ is multiplied y^n, $\frac{\partial^n z}{\partial x^r\,\partial y^{n-r}}$ is multiplied by $x^r y^{n-r}$ and so on) can be reduced to a linear partial differential equation with constant coefficients by the substitutions
$$x = e^u \text{ and } y = e^v \tag{39}$$

so that
$$u = \log x \text{ and } v = \log y \tag{40}$$

Equation (38) can be rewritten as

$$(a_0 x^n D_x^n + a_1 x^{n-1} y D_x^{n-1} D_y + a_2 x^{n-2} y^2 D_x^{n-2} D_y^2 + \cdots + a_n D_y^n)z = f(x, y) \tag{41}$$

where $a_0, a_1, a_2, \ldots a_n$ are constants.

Let $\quad D_x \equiv \dfrac{\partial}{\partial x}$ and $D_y \equiv \dfrac{\partial}{\partial y}$

Now, $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial x} = \dfrac{1}{x}\dfrac{\partial z}{\partial u}$

$\therefore \quad x\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}$ so that $x\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial u}$ or $xD_x = D'_u \equiv \dfrac{\partial}{\partial u}$

Similarly, $x^n D_x^n = D'_u(D'_u - 1)$

$$x^3 D_x^3 = D'_u(D'_u - 1)(D'_u - 2)$$

....

....

$$x^n D_x^n = D'_u(D'_u - 1)(D'_u - 2), \ldots (D'_u - n + 1)$$

Also, $\quad \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial y} = \dfrac{1}{y}\dfrac{\partial z}{\partial v}$

$\therefore \quad y\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial v}$ so that $y\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial v}$ or $y D_y = D'_v \equiv \dfrac{\partial}{\partial v}$

Similarly, $x^2 D_y^2 = D'_v(D'_v - 1)$, $x^3 D_x^3 = D'_v(D'_v - 1)(D'_v - 2)$, and so on and

$$x \cdot y D_x D_y = D'_u D'_v$$

Using the above values in (41) reduces to an equation having constant coefficients and now it can easily be solved by the methods already discussed for homogeneous and nonhomogeneous linear equations with constant coefficients. Finally, with the help of (39) and (40), the solution is obtained in terms of the old variables x and y.

Example 67 Solve $(x^2 D_x^2 + 2xy D_x D_y + y^2 D_y^2)z = 0$

Solution Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$ (1)

Also, let $\quad D_x = \dfrac{\partial}{\partial x}, D_y = \dfrac{\partial}{\partial y}, D'_u = \dfrac{\partial}{\partial u}, D'_v = \dfrac{\partial}{\partial v}$

Then the given equation $(x^2 D_x^2 + 2\, xy D_x D_y + y^2 D_y^2) = 0$ becomes

$$\left[D'_u(D'_u - 1) + 2\, D'_u\, D'_v + D'_v(D'_v - 1) \right] z = 0$$

or $\quad [(D'_u + D'_v)^2 - (D'_u + D'_v)]z = 0$ or $(D'_u + D'_v) - (D'_u + D'_v - 1)z = 0$

Hence, the required general solution is

$$Z = \phi_1(v - u) + e^u \phi_2(v - u)$$

$$Z = \phi_1(\log y - \log x) + x\, \phi_2(\log y - \log x)$$

$Z = \phi_1$ (log y/x) + x ϕ_2 (y/x)

$Z = f_1$ (y/x) + $x f_2$ (y/x), where f_1 and f_2 are arbitrary constants.

Example 68 Solve $x^2 \dfrac{\partial^2 z}{\partial x^2} - 4xy \dfrac{\partial^2 z}{\partial x \partial y} + 4y^2 \dfrac{\partial^2 z}{\partial y^2} + 6y \dfrac{\partial z}{\partial y} = x^3 y^4$.

Solution Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$.

Also, let $D_x = \dfrac{\partial}{\partial x}, D_y = \dfrac{\partial}{\partial y}, D'_u = \dfrac{\partial}{\partial u}, D'_v = \dfrac{\partial}{\partial v}$

Then the given equation $(x^2 D_x^2 - 4xy D_x D_y + 4x^2 D_y^2 + 6y D_y) = x^3 y^4$ becomes

$$\left[D'_u (D'_u - 1) - 4 D'_u D'_v + 4 D'_v (D'_v - 1) + 6 D'_v\right] Z = e^{3u+4v}$$

or
$$[(D'_u - 2D'_v)^2 - (D'_u - 2D'_v)] Z = e^{3u+4v}$$

$$(D'_u - 2D'_v)(D'_u - 2D'_v - 1) Z = e^{3u+4v}$$

The CF is
$$Z_c = \phi_1 (v + 2u) + e^u \phi_2 (v + 2u)$$
$$= \phi_1 (\log y + 2 \log x) + x \phi_2 (\log y + 2 \log x)$$
$$Z_c = f_1(yx^2) + x f_2(yx^2)$$

Now, PI is
$$Z_p = \frac{1}{(D'_u - 2D'_v)(D'_u - 2D'_v - 1)} e^{3u+4v}$$
$$= \frac{1}{(3 - 2 \times 4)(3 - 2 \times 4 - 1)} e^{3u+4v}$$
$$= \frac{1}{30} e^{3u+4v} = \frac{1}{30} x^3 y^4$$

Hence, the complete solution is $Z = Z_c + Z_p$

$$Z = f_1(yx^2) + xf^2(yx^2) + \frac{1}{30} x^3 y^4 \text{, where } f_1, f_2 \text{ are arbitrary constants.}$$

Example 69 Solve $[x^2 D_x^2 + 2xy D_x D_y - x D_x] = \dfrac{x^3}{y^2}$.

Solution Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$

Also, let $D_x = \dfrac{\partial}{\partial x}, D_y = \dfrac{\partial}{\partial y}, D'_u = \dfrac{\partial}{\partial u}, D'_v = \dfrac{\partial}{\partial v}$

Then the given equation $(x^2 D_x^2 + 2xy D_x D_y - x D_x) = \dfrac{x^3}{y^2}$ becomes

$$\left[D'_u (D'_u - 1) + 2 D'_u D'_v - D'_u\right] Z = e^{3u-2v}$$

or
$$(D'^2_u + 2 D'_u D'_v - D'_u) Z = e^{3u-2v}$$

or
$$D'_u (D'_u + 2D'_v - 2) z = e^{3u-2v}$$

The CF is $Z_c = \phi_1(v) + e^{2u}\phi_2(v-2u) = \phi_1(v) + (e^u)^2\,\phi_2(v-2u)$

or $Z_c = \phi_1(\log x) + x^2\,\phi_2(\log y - 2\log x)$

$$= f_1(y) + x^2 f_2(y/x^2)$$

Now, PI is $$Z_p = \frac{1}{D'_u(D'_u + 2D'_v - 2)}e^{3u-2u}$$

$$= \frac{1}{3\cdot(3-2\times-2-2)}e^{3u-2u} = -\frac{1}{9}(e^u)^3(e^u)^{-2}$$

$$= -\frac{1}{9}\frac{x^3}{y^2}$$

$\therefore$ The complete solution is

$$Z = f_1(y) + x^2 f_2(y/x^2) - \frac{1}{9}\frac{x^3}{y^2}$$

EXERCISE 14.6

Solve the following equations:

1. $(D_x + D_y - 1)(D_x + 2D_y - 2)z = 0$
2. $(D_x - D_y + 1)(D_x + 2D_y - 3)z = 0$
3. $(D_x^3 - 3D_x D_y + D_x + 1)z = e^{2x+3y}$
4. $(2D_x^2 + D_y^2 - 3D_y)z = 5\cos(3x - 2y)$
5. $(D_x^2 - D_y)z = \cos(3x - y)$
6. $r - s + p = 1$
7. $(D_x + D_y - 1)(D_x + 2D_y - 3)z = 4 + 3x + 6y$
8. $(D_x^2 - D_y)x = x e^{ax + a^2 y}$
9. $(D_x^2 - D_y^2 - 3D_x + 3D_y)z = xy + e^{(x+2y)}$
10. $[(D_x + D_y - 1)(D_x + D_y - 3)(D_x + D_{y)}]z = e^{x+y}\sin(2x + y)$
11. $x^2\dfrac{\partial^2 z}{\partial x^2} - y^2\dfrac{\partial^2 z}{\partial y^2} - y\dfrac{\partial z}{\partial y} + x\dfrac{\partial z}{\partial x} = 0$
12. $[x^2 D_x^2 - y^2 D_y^2]z = x^2 y$
13. $\left[x^2 D_x^2 - 2xy D_x D_y + y^2 D_y^2 - x D_x + 3y D_y\right]z = 8\left(\dfrac{y}{x}\right)$
14. $(x^2 D_x^2 - y^2 D_y^2)z = xy$
15. $(x^2 D_x^2 - y^2 D_y^2 + x D_x - y D_y)z = \log x$

Answers

1. $z = e^x \phi_1(y-x) + e^{2x} \phi_2(y-2x)$
2. $z = e^{-x} \phi_1(y+x) + e^{3x} \phi_2(y-2x)$
3. $z = \sum A e^{hx+ky} - \frac{1}{7} e^{2x+3y}$ where A, h, k are constants and h, k are related as $h^3 - 3h + k + 1 = 0$
4. $z = \phi_1(x) + e^{3x/2} \phi_2(2y-x) + \frac{1}{10}\left[4\cos(3x-2y) + 3\sin(3x-2y)\right]$
5. $z = \sum A e^{h(x+y)} - \frac{1}{82}\left[9\cos(3x-y) - \sin(3x-y)\right]$, where A and h are arbitrary constants.
6. $z = \phi_1(y) + e^{-x} \phi_2(y+x) + x$
7. $z = e^x \phi_1(y-x) + e^{3x} \phi_2(y-2x) + 6 + x + 2y$
8. $z = \sum A e^{hx+h^2 y} + e^{ax+a^2 y}\left[\frac{1}{4a}x^2 - \frac{1}{4a^2}x\right]$
9. $z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{6}x^2 y - \frac{1}{6}xy - \frac{1}{9}x^2 - \frac{1}{18}x^3 - \frac{2}{27}x - xe^{x+2y}$
10. $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x) + \frac{1}{30}\left[3\cos(2x+y) - 2\sin(2x+y)\right]$
11. $z = f_1(xy) + f_2(y/x)$
12. $z = f_1(xy) + xf_2(y/x) + \frac{1}{2}x^2 y$
13. $z = f_1(xy) + x^2 f_2(xy) + \left(\frac{y}{x}\right)$
14. $z = f_1(xy) + x f_2\left(\frac{y}{x}\right) + (xy)\log x$
15. $z = f_1(xy) + f_2\left(\frac{y}{x}\right) + \frac{1}{6}(\log x)^3$

14.12 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

The general form of a second-order PDE in the function z of two independent variables x, y is given by.

$$A(x,y)\frac{\partial^2 z}{\partial x^2} + B(x,y)\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\frac{\partial^2 z}{\partial y^2} + f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \tag{42}$$

This equation is linear in second-order terms PDE. Equation (42) is said to be linear or quasi-linear according to whether f is linear or nonlinear.

PDE (42) is classified as elliptic, parabolic, or hyperbolic accordingly as $B^2 - 4\,AC < 0, = 0$ or > 0.

Example 70 Classify the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$ (1)

Solution Compare (1) with $A\,Z_{xx} + B\,Z_{xy} + C\,Z_{yy} + f(x, y, z, p, q) = 0$

$$A = 1,\ C = 1$$

Now, $B^2 - 4\,AC = 0 - 4 \times 1 \times 1 = -4 < 0$

So the given equation is elliptic.

Example 71 Classify the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = f(x, y).$ (1)

Solution Compare (1) with the standard equation so $A = 1$, $C = 1$

Now, $B^2 - 4\,AC = 0 - 4 \times 1 \times 1 = -4 < 0$

The given equation is elliptic.

Example 72 Classify $a^2\,\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial z}{\partial t} = 0.$ (1)

Solution Here, $A = a^2$, $B = 0$, $C = 0$; so $B^2 - 4\,AC = 0 - 4a^2\cdot 0 = 0$

Hence, the equation (1) is parabolic.

Example 73 Classify $(1 - x^2)\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + (1 - y^2)\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + x^2 y\dfrac{\partial z}{\partial y} - 2z = 0.$ (1)

Solution Compare (1) with $A\dfrac{\partial^2 z}{\partial x^2} + B\dfrac{\partial^2 z}{\partial x\,\partial y} + C\dfrac{\partial^2 z}{\partial y^2} + f\left(x, y, z, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}\right) = 0$

Here, $A = (1 - x^2)$, $B = -2xy$, $C = (1 - y^2)$

$\therefore$ $B^2 - 4\,AC = (-2\,xy)^2 - 4\cdot(1 - x^2)(1 - y^2)$

$$= 4(x^2 + y^2 - 1)$$

The given differential equation is

Hyperbolic if $B^2 - 4\,AC < 0$, i.e., $x^2 + y^2 > 1$

- Parabolic if $B^2 - 4\,AC = 0$, i.e., $x^2 + y^2 = 1$, and
- Elliptic if $B^2 - 4\,AC < 0$, i.e., $x^2 + y^2 < 1$

Example 74 Classify $\dfrac{\partial^2 z}{\partial t^2} - a^2\,\dfrac{\partial^2 z}{\partial x^2} = 0.$ (1)

Solution Here, $A = -a^2$, $B = 0$, $C = 1$

$\therefore$ $B^2 - 4\,AC = 0 - 4 \times -a^2 \times 1 = 4\,a^2 > 0$

So the equation (1) is hyperbolic.

EXERCISE 14.7

Classify the following equations:

1. $\dfrac{\partial^2 z}{\partial t^2} + t^2 \dfrac{\partial^2 z}{\partial x \partial t} + x\dfrac{\partial^2 z}{\partial x^2} = 0$

2. $\dfrac{\partial^2 u}{\partial t^2} - 4\dfrac{\partial^2 u}{\partial x \partial t} + 4\dfrac{\partial^2 u}{\partial x^2}$

3. $\dfrac{\partial^2 u}{\partial t^2} + 4\dfrac{\partial^2 u}{\partial x \partial t} + 4\dfrac{\partial^2 u}{\partial x^2}$

4. $\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{C^2}\dfrac{\partial u}{\partial t}$

Answers

1. Hyperbolic if $t^2 > 4x$, Parabolic if $t^2 = 4x$ and elliptic if $t^2 < 4x$.
2. Hyperbolic.
3. Parabolic.
4. Parabolic

14.13 CHARPIT'S METHOD

Charpit's method is a general method for solving equations with two independent variables.

Consider the first-order partial differential equation in two variables x, y is

$$f(x, y, z, p, q) = 0 \tag{43}$$

Here, z depends on x and y, $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$

We have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy \tag{44}$$

We try to find another equation

$$g(x, y, z, p, q) = 0 \tag{45}$$

such that the solution of Eq. (43) is also a solution of Eq. (45). If these equations exists then the equations (43) and (45) are said to be *compatible*. We solve equations (43) and (45) for p and q and substitute in (44).

This will give the solution provided (44) is integrable.

Setting the derivatives of f and g with respect to x and y, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot p + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \tag{46}$$

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\cdot p + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \tag{47}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot q + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \tag{48}$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\cdot q + \frac{\partial q}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \tag{49}$$

Eliminating $\dfrac{\partial p}{\partial x}$ between equations (46) and (47), we get

$$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p}-\frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial p}-\frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}\cdot\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0 \tag{50}$$

Now, eliminating $\dfrac{\partial q}{\partial y}$ between equations (48) and (49), we get

$$\left(\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial q}-\frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial q}-\frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial q}\right)\cdot q+\left(\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q}-\frac{\partial g}{\partial p}\cdot\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0 \tag{51}$$

$$\because \qquad \frac{\partial p}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial q}{\partial x}$$

The last terms in equations (50) and (51) differ in sign only. Adding (50) and (51), we obtain

$$\left(\frac{\partial f}{\partial p}+p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)+\left(-\frac{\partial f}{\partial p}\right)\left(\frac{\partial g}{\partial x}\right)+\left(-\frac{\partial f}{\partial q}\right)\left(\frac{\partial g}{\partial y}\right)=0$$

which is a linear partial differential equation of the first order with x, y, z, p, q as independent variables and g as the dependent variable.

The auxiliary equations are

$$\frac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\frac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\frac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\frac{dx}{-\dfrac{\partial f}{\partial q}}=\frac{dy}{-\dfrac{\partial f}{\partial q}}=\frac{dg}{0} \tag{52}$$

Any integral of Eq. (52) which involves p or q or both can be taken in (44).

Example 75 Solve $px + qy = pq$.

Solution Here, the given equation is

$$f(x, y, z, p, q) \equiv px + qy - pq = 0 \tag{1}$$

The Charpit's auxiliary equations are

$$\frac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\frac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\frac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\frac{dx}{-\dfrac{\partial f}{\partial p}}=\frac{dy}{-\dfrac{\partial f}{\partial z}}$$

or

$$\frac{dx}{-(x-q)}=\frac{dy}{-(y-q)}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dp}{p+p.0}=\frac{dq}{q+q.0} \tag{2}$$

Taking the last two fractions of Eq. (2)

$$\frac{dp}{p}=\frac{dq}{q}$$

Integrating $\log p = \log q + \log a$ or $p = aq$ (3)

Putting $p = aq$ in (1), we get

$$aqx + qy - aq^2 = 0 \text{ or } aq = (ax + y);\ q \neq 0 \tag{4}$$

From (3) and (4), $p = ax + y$ and $q = \dfrac{ax+y}{a}$

Putting the values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy$$

or $$adz = (ax + y)\,(adx + dy)$$

$$= (ax + y)\,d(ax + y)$$

$$adz = u \cdot du \text{ where } u = ax + y$$

On integrating, we get

$$az = \frac{u^2}{2} + b$$

or $$az = \frac{(ax+y)^2}{2} + b$$

which is the complete integral of (1), a and b being arbitrary constants.

Example 76 Solve $(p^2 + q^2)\,x = pz$.

Solution Here, $f(x, y, z, p, q) \equiv (p^2 + q^2)\,x - pz = 0$ (1)

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{p^2 + q^2 - p^2} = \frac{dq}{-pq} = \frac{dz}{-p(2px - z) - q(2qx)} = \frac{dx}{-(2px - z)} = \frac{dy}{-(2qx)} \qquad (2)$$

Taking the first two fractions of (2), we have

$$\frac{dp}{q^2} = \frac{dq}{-pq} \quad \text{or} \quad pdp = -qdq$$

or $$pdp + qdq = 0$$

Integrating, $p^2 + q^2 = a^2$ (3)

From (1) and (3), we have

$$a^2x = pz \;\Rightarrow\; p = \frac{a^2x}{z}$$

Putting $p = \dfrac{a^2x}{z}$ in (3), we get

$$q = \frac{a}{z}\sqrt{(z^2 - a^2x^2}$$

Putting the values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{a^2x}{z}dx + \frac{a\sqrt{(z^2 - a^2x^2)}}{z}dy$$

or $$\frac{z\,dz - a^2 x\,dx}{\sqrt{(z^2 - a^2 x^2}} = ady$$

Putting $z^2 - a^2 x^2 = t$ so that $2(zdz - a^2 xdx) = dt$, we get

$$\frac{\frac{1}{2}dt}{\sqrt{t}} = ady \text{ or } \frac{1}{2}t^{-\frac{1}{2}}dt = ady$$

Integrating both sides, we have

$$t^{1/2} = ay + b \text{ or } \sqrt{(z^2 - a^2 x^2} = ay + b$$

or $$(z^2 - a^2 x) = (ay + b)^2$$

EXERCISE 14.8

Solve the following equations by Charpit's method:

1. $z^2 = pqxy$
2. $q + px = p^2$
3. $pxy + pq + qy = yz$
4. $z + px + qy = \dfrac{p^2 y}{2}$
5. $z = pq$
6. $q = 3p^2$

Answers

1. $z = ax^b y^{1/b}$
2. $z = axe^{-y} - \dfrac{a^2}{2}e^{-2y} + b$
3. $\log (z - ax) = y - a \log (a + y) + b$
4. $z = \dfrac{ax}{y^2} - \dfrac{a^2}{4y^3} + \dfrac{b}{y}$
5. $2\sqrt{z} = x\sqrt{a} + \left(\dfrac{1}{\sqrt{a}}\right)y + b$
6. $z = ax + 3a^2y + b$

14.14 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER: (MONGE'S METHOD)

Here, we study Monge's method of integrating

$$Rr + Ss + Tt = V \tag{53}$$

where r, s, t have their usual meaning and R, S, T, and V are functions of x, y, z, p, q.

We know that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y\,\partial x}dy$$

$$dp = rdx + sdy \tag{54}$$

and $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy$$

$$dq = sdx + tdy \tag{55}$$

From (54) and (55), we have

$$r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}$$

Substituting these values in (53), we have

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V$$

or $$Rdy(dp - sdy) + Ssdx\,dy + Tdx(dq - sdx) = Vdxdy$$

or $$(Rdpdy + Tdqdx - Vdxdy) - s(Rdy^2 - Sdxdy + Tdx^2) = 0 \tag{56}$$

Clearly, any relation between x, y, z, p, and q which satisfies (56), must also satisfy the following two relations

$$Rdpdy + Tdqdx - Vdxdy = 0 \tag{57a}$$

and $$Rdy^2 - Sdxdy + Tdx^2 = 0 \tag{57b}$$

The equations (57a) and (57b) are called ***Monge's subsidiary equations***.

Since Eq. (57b) is quadratic, it can be resolved into two linear equations in dx and dy, such that

$$dy - m_1 dx = 0 \tag{58a}$$

and $$dy - m_2 dx = 0 \tag{58b}$$

Now, from (57a) and (58a), combined if necessary with $dz = pdx + qdy$, obtain two integrals say $u_1 = \alpha$ and $v_1 = \beta$

Then the relation $u_1 = f(v_1)$ (59)

is the solution and is called an ***intermediate integral***.

Similarly, from (57a) and (58b), we get another integral

$$u_2 = g(v_2) \tag{60}$$

With the help of equations (59) and (60), we find the values of p and q in terms of x and y.

Substituting these values in $dz = pdx + qdy$ and integrating, we obtain the complete integral of the given equation.

Example 77 Solve $r = a^2 t$.

Solution We know that $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$

Substituting the values of r and t in $r = a^2 t$, we get

$$\frac{dp - sdy}{dx} = a^2\left(\frac{dq - sdx}{dy}\right) \tag{1}$$

or $$(dpdy - a^2 dqdx) - s(dy^2 - a^2\,dx^2) = 0$$

$\therefore$ The Monge's subsidiary equations are

$$dpdy - a^2 dqdx = 0 \tag{2}$$

and $$dy^2 - a^2 dx^2 = 0 \tag{3}$$

From (3), we have

$$dy - adx = 0 \tag{4}$$

and $$dy + adx = 0 \tag{5}$$

Now, from (4) and (2), we get

$$dp(adx) - a^2dq\,dx = 0 \text{ or } dp - adq = 0 \tag{6}$$

From (4) and (6), we get

$$u_1 \equiv y - ax = \alpha \text{ and } v_1 \equiv p - aq = \beta$$

$$\therefore \quad p - aq = f(y - ax) \tag{7}$$

is an intermediate integral.

Similarly, from (5) and (2), we get the second intermediate integral $p + aq = g(y + ax)$ (8)

Solving equations (7) and (8), we have

$$p = \frac{1}{2}[f(y - ax) + g(y + ax)]$$

and $$q = \frac{1}{2a}[g(y + ax) - f(y - ax)]$$

Putting the values of p and q in $dz = p\,dx + qdy$, we have

$$dz = \left[\frac{1}{2}f(y - ax) + g(y + ax)\right]dx + \left[\frac{1}{2a}\{g(y + ax) - f(y - ax)\}\right]dy$$

$$dz = \frac{1}{2a}g(y + ax)(dy + adx) - \frac{1}{2a}f(y - ax)(dy - adx)$$

Integrating, we get

$$z = \frac{1}{2a}\varphi_1(y + ax) - \frac{1}{2a}\varphi_2(y - ax)$$

or $$z = \psi_1(y + ax) + \psi_2(y - ax)$$

Example 78 Solve $pt - qs = q^3$.

Solution Given $pt - qs = q^3$ (1)

Putting $t = \dfrac{dq - sdx}{dy}$ in (1), we get

$$\frac{p(dq - sdx)}{dy} - qs = q^3$$

or $$(pdq - q^3dy) - s(pdx + qdy) = 0$$

The Monge's subsidiary equations are

$$pdq - q^3dy = 0 \tag{2}$$

and $$pdx + qdy = 0 \tag{3}$$

From (3), we have $dz = 0 \Rightarrow z = a$ (4)

From equations (2) and (3), we obtain

$$dq + q^2dx = 0 \qquad (\because \quad -pdx = +qdy)$$

or $$\frac{dq}{q^2} + dx = 0$$

$\therefore$ $$-\frac{1}{q} + x = b$$

or $$-\frac{1}{q} + x = f(z)$$

or $$-\frac{\partial y}{\partial z} + x = f(z) \text{ or } \frac{\partial y}{\partial z} - x = -f(z)$$

Integrating, we get

$$y - zx = +f_1(z) + c$$

or $$y = zx + f_1(z) + c$$

$$y = zx + f_1(z) + f(x)$$

Since c is a function of x which is regarded constant at the time of integration.

Example 79 Solve $y^2r - 2ys + t = p + 6y$. (1)

Solution Putting $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$ in (1), we get

$$y^2 = \left(\frac{dp - sdy}{dx}\right) - 2ys + \frac{dq - sdx}{dy} = p + 6y$$

or $[y^2dpdy - (p + 6y)\, dxdy + dqdx] - s[y^2dy^2 + 2y\, dx\, dy + dx^2] = 0$

$\therefore$ The Monge's subsidiary equations are

$$y^2dpdy - (p + 6y)\, dxdy + dqdx = 0 \tag{2}$$

and $$y^2dy^2 + 2y\, dxdy + dx^2 = 0 \tag{3}$$

From (3), $$(ydy + dx)^2 = 0$$

or $$ydy + dx = 0 \tag{4}$$

Integrating, we obtain

$$y^2 + 2x = \alpha \tag{5}$$

From (2) and (4), we get

$$ydp + (p + 6y)dy - dq = 0$$

or $$(ydp + pdy) + (6ydy - dq) = 0$$

$\therefore$ $$yp + 3y^2 - q = \beta \tag{6}$$

From (5) and (6), we obtain the intermediate integral is

$$(yp + 3y^2 - q) = f(y^2 + 2x)$$

or $$py - q = -3y^2 + f(y^2 + 2x)$$

Lagrange's subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{-3y^2 + f(y^2 + 2x)}$$

Taking the first two fractions, we have

$$ydy + dx = 0 \text{ or } y^2 + 2x = c$$

Again, taking the last two fractions, we have

$$dz = [3y^2 - f(y^2 + 2x)]\, dy$$

or $$dz = 3y^2dy - f(c)dy$$

Integrating, $$z = y^3 - y\,f(c) + d$$

or $$z = y^3 - y\,f(y^2 + 2x) + g(y^2 + 2x)$$

which is the required solution.

EXERCISE 14.9

Solve the following equations:

1. $r + (a + b)s + abt = xy$
2. $r - t\cos^2 x + p\tan x = 0$
3. $r = t$
4. $rx^2 - 2sx + t + q = 0$
5. $q^2r - 2pqs + p^2t = pq^2$
6. $x^2r + 2xys + y^2t = 0$
7. $r = 2y^2$

Answers

1. $z = \dfrac{x^3y}{6} - \dfrac{1}{24}(a+b)x^4 + \phi_1(y - ax) + \phi_2(y - bx)$
2. $z = \phi_1(y - \sin x) + \phi_2(y + \sin x)$
3. $z = \phi_1(x + y) + \phi_2(y - x)$
4. $z = \phi_1(y + \log x) + x\,\phi_2(y + \log x)$
5. $y = e^x f_1(z) + f_2(z)$
6. $z = y\,f_1(y/x) + f_2(y/x)$
7. $z = x^2y^2 + x\,f(y) + g(y)$

14.15 MONGE'S METHOD OF INTEGRATING

Here, we consider

$$Rr + Ss + Tt + U(rt - s^2) = V \tag{61}$$

where R, S, T, U, V are functions of x, y, z, p, q, and r, s, and t have their usual meanings.

We know that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy$$

and $$dq = \frac{\partial p}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy$$

Therefore, $$r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}$$

Putting the values of r and t in Eq. (61), we have

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) + U\left\{\left(\frac{dp - sdy}{dx}\right)\left(\frac{dq - sdx}{dy}\right) - s^2\right\} = V$$

or $(RdPdy + Tdqdx + UdPdq - Vdxdy) - s(Rdy^2 - Sdxdy + Tdx^2 + UdPdx + Udqdy) = 0$

Thus, the Monge's subsidiary equations are

$$L \equiv RdPdy + Tdqdx + UdPdq - Vdxdy = 0 \tag{62a}$$

and
$$M \equiv Rdy^2 - Sdxdy + Tdx^2 + UdPdx + Udqdy = 0 \tag{62b}$$

Equations (62a) and (62b) are called *Monge's subsidiary equations*. Now, Eq. (62a) cannot be factorized, because of the presence of the term $UdPdx + Udqdy$.

$\therefore$ we try to factorize $M + \lambda L = 0$

or $(Rdy^2 - Sdxdy + Tdx^2 + UdPdx + Udqdy) + \lambda(RdPdy + Tdqdx + UdPdq - Vdxdy) = 0$ (63)

where λ is some multiplier to be determined.

Consider the factors of (62b) to be

$$(Rdy + mTdx + KUdP)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0 \tag{64}$$

Comparing equations (63) and (64), we have

$$\frac{R}{m} + mT = -(S + \lambda V) \tag{65}$$

$$k = m \text{ and } \frac{R\lambda}{k} = U$$

Now, from the last two relations, we obtain $m = \dfrac{R\lambda}{U}$

$\therefore$ from Eq. (65), we get $\dfrac{U}{\lambda} + \dfrac{R\lambda T}{U} = -(S + \lambda V)$

or
$$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \tag{66}$$

(which is called the λ-*equation*)

Consider λ_1 and λ_2 as the roots of Eq. (66).

When $\lambda = \lambda_1,\ m = \dfrac{R\lambda_1}{U}$, then Eq. (64) becomes

$$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dP\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0 \tag{67}$$

or
$$(Udy + \lambda_1 Tdx + \lambda_1 UdP)\ (Udx + \lambda_1 Rdy + \lambda_1 Udq) = 0 \tag{68}$$

Similarly, when $\lambda = \lambda_1,\ m = \dfrac{R\lambda_2}{U}$ then Eq. (64) becomes

$$(Udy + \lambda_2 Tdx + \lambda_2\, UdP)\ (Udx + \lambda_2\, Rdy + \lambda_2\, Udq) = 0 \tag{69}$$

Now, solving equations (68) and (69), we obtain two intermediate integrals of the form

$$u_1 = f(v_1) \text{ and } u_2 = g(v_2) \tag{70}$$

Solving them, find the values of p and q and substitute in $dz = Pdx + qdy$. Integrating, we obtain the complete solution of the given equation.

Example 80 Solve $2s + (rt - s^2) = 1$.

Solution Given $2s + (rt - s^2) = 1$ (1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$$R = 0, S = 2, T = 0, U = 1, V = 1$$

Using the λ-equation $\lambda^2(UV + RT) + \lambda\, US + U^2 = 0$, we have

$$\lambda^2 + 2\lambda + 1 = 0 \text{ or } \lambda = -1, -1$$

$$\therefore \quad \lambda_1 = -1 \text{ and } \lambda_2 = -1$$

Putting $\lambda_1 = -1 = \lambda_2$ in one pair of equations

$$Udy + \lambda_1 Tdx + \lambda_1 UdP = 0$$

and

$$Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

$$\therefore \quad dy - dp = 0 \text{ and } dx - dq = 0$$

Integrating, we get

$$y - p = c_1 \text{ and } x - q = c_2$$

The intermediate integral is $(y - p) = f(x - q)$

Also, $\quad p = y - c_1$ and $q = x - c_2$

Substituting these values of p and q in $dz = pdx + qdy$, we have

$$dz = (y - c_1)\, dx + (x - c_2)dy$$

$$dz = (ydx + xdy) - c_1 dx - c_2 dy$$

$$\therefore \quad \boxed{z = xy - c_1 x - c_2 y + d}$$

which is the required complete integral.

Example 81 Solve $r + 3s + t + (rt - s^2) = 1$.

Solution Given $r + 3s + t + (rt - s^2) = 1$ (1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$$R = 1, S = 3, T = 1, U = 1, \text{ and } V = 1$$

$\therefore$ the λ-equation

$$\lambda^2(UV + RT) + US + V^2 = 0$$

$$\therefore \quad 2\lambda^2 + 3\lambda + 1 = 0 \text{ or } (2\lambda + 1)(\lambda + 1) = 0$$

or

$$\lambda = -1, -\frac{1}{2}$$

Let

$$\lambda_1 = -1 \text{ and } \lambda_2 = -\frac{1}{2}$$

The first intermediate integral is given by

$$Udy + \lambda_1 Tdx + \lambda_1 UdP = 0 \text{ and } Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or

$$dy - dx - dP = 0 \text{ and } \quad dx - \frac{1}{2}dy - \frac{1}{2}dq = 0$$

or

$$dy - dx - dP = 0 \text{ and } \quad -2dx + dy + dq = 0$$

Integrating, we get

$$y - x - p = c_1 \text{ and } -2x + y + q = c_2$$

$\therefore$ The first intermediate integral is

$$(y - x - p) = f(y - 2x + q) \tag{2}$$

$$= f(\alpha), \text{ where } \alpha = y - 2x + q$$

The second intermediate integral is given by

$$Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \text{ and } Udy + \lambda_1 Tdx + \lambda_2 UdP = 0$$

or

$$-dy + dx - dq = 0 \text{ and } 2dy - dx - dP = 0$$

Integrating, we get

$$-y + x - q = c_3 \text{ and } 2y - x - p = c_4$$

$\therefore$ the second intermediate integral is

$$2y - x - p = g(x - y - q) \tag{3}$$

$$= g(\beta), \text{ where } \beta = x - y - q$$

Now, $\alpha + \beta = -x$ or $d\alpha + d\beta = -dx$

and from (3) and (2),

$$y = g(\beta) - f(\alpha)$$

or

$$dy = g'(\beta)\, d\beta - f'(\alpha)\, d\alpha$$

and

$$P = y - x - f(\alpha),\ q = x - y - (\beta)$$

Putting the values of p and q in $dz = Pdx + qdy$, we get

$$dz = [y - x - f(\alpha)]dx + [x - y - \beta]dy$$

$$= (y - x)dx - f(\alpha)dx + (x - y)dy - \beta dy$$

$$= -(x - y)(dx - dy) - f(\alpha)(-d\alpha - d\beta) - \beta(g'(\beta)d\beta - f'(\alpha)d\alpha$$

$$dz = -(x - y)(dx - dy) + f(\alpha)d\alpha - \beta g'(\beta)d\beta + [f(\alpha)\, d\beta + \beta f'(\alpha)d$$

On integrating, we get

$$z = \frac{-(x-y)^2}{2} + \int f(\alpha)\, d\alpha - \int \beta g'(\beta) d\beta + \beta f(\alpha)$$

$$= \frac{-(x-y)^2}{2} + \phi(\alpha) - \beta g(\beta) + \int g(\beta)\, d\beta - \beta f(\alpha)$$

$$z = \frac{-(x-y)^2}{2} + \phi(\alpha) - \beta g(\beta) + \psi(\beta) + \beta f(\alpha)$$

which is the required integral.

Example 82 Solve $r + 4s + t + (rt - s^2) = 2$.

Solution Do same as Example 81; $\lambda_1 = -1$ and $\lambda_2 = -\dfrac{1}{3}$

and

$$2z = 2xy - x^2 - y^2 + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$$

EXERCISE 14.10

Solve the following by Monge's method:

1. $3r + 4s + t + (rt - s^2) = 1$
2. $r + t - (rt - s^2) = 1$
3. $5r + 6s + 3t + 2(rt - s^2) = -3$
4. $3s + (rt - s^2) = 2$
5. $2r + te^x - (rt - s^2) = 2e^x$

Answers

1. $z = c_1 x + c_2 y - \dfrac{x^2}{2} + 2xy - \dfrac{3y^2}{2} + c_3$
2. $z = \dfrac{x^2 + y^2}{2} + c_1 x + c_2 y + c_3$
3. $2z = 3xy - \dfrac{3}{2}x^2 - \dfrac{5}{2}y^2 - c_1 x - c_2 y + c_3$
4. $z = xy + \phi(\alpha) + \psi(\beta) + \beta y$, where $x = \beta - \alpha$; $y = \phi'(\alpha) - \psi'(\beta)$
5. $z = e^x + c_z x - c_1 y + y^2 + c_3$

SUMMARY

1. Lagrange's Method of Solving Linear Partial Differential Equations of First Order

Working Rule for Solving $P_p + Q_q = R$ by Lagrange's Method

Step I Put the given linear PDE of the first order in the standard form

$$Pp + Qq = R \tag{1}$$

Step II Write down Lagrange's auxiliary equation for (1), namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{2}$$

Step III Solve (2) by using the well-known methods, let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be the two independent solutions of (2).

Step IV The general solution (or integral) of (1) is written in one of the following three forms:

$$\phi\,(u, v) = 0,\ u = \phi\,(v) \text{ or } v = \phi\,(u)$$

2. Nonlinear Partial Differential Equations of First Order

A nonlinear partial differential equation of first order contains p and q of degree other than one and/or product terms of p and q.

Its complete solution is given by $f\,(x, y, z, a, b) = 0$ where a and b are any two arbitrary constants. Some standard forms of nonlinear first-order partial differential equations are given below:

(i) Standard Form I

Only p and q present or x, y, z, are absent.

Consider equations of the form

$$f(p.\ q) = 0 \tag{3}$$

Then the complete solution is

$$z = a\,x + b\,y + c \tag{4}$$

where a, b satisfy the equation $f\,(a, b) = 0$, i.e., $b = g\,(a)$

(ii) Standard Form II

Clairauts' equation

$$z = p\,x + q\,y + f\,(p, q) \tag{5}$$

The complete solution of (5) is

$$z = a\,x + b\,y + f(a, b) \tag{6}$$

which is obtained by replacing p by a and q by b in the given equation (5).

(iii) Standard Form III

Only p, q, and z present. Consider an equation of the form

$$f(p, q, z) = 0 \tag{6}$$

Putting $q = a\,p$ in (6), we get $f(p, a\,p, z) = 0$ (7)

Equating (7) and solving for p, we get $p = g(z)$.

Now, put the values of p and q in $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + q\,dy$

$$d\,z = pdx + apd\,yd$$
$$z = p(dx + ady)$$
$$d\,z = g(z)\,(dx + ady)$$

On integrating,

$$x + a\,y = \int \frac{dz}{g(z)} + b$$, where a and b are two arbitrary constants.

(iv) Standard Form IV: Equation of the form

$$f_1(x, p) = f_2(y, q) \tag{8}$$

Suppose $f_1(x, p) = f_2(y, q) = a$ = constant

Solving each equation for p and q, we get

$$p = g_1(x, a) \text{ and } q = g_2(y, a)$$

Now, putting these values of p and q in

$$dz = pdx + qdy, \text{ we get}$$
$$dz = g_1(x, a)\,dx + g_2(y, a)dy$$

Integrating, $z = \int g_1(x,a)dx + \int g_2(y,a)dy + b$

This is the required complete solution.

3. Method of Finding the Complementary function (CF) of the Linear Homogeneous PDE with Constant Coefficients

$$F(D_x, D_y)z = 0$$

i.e., $$\left(A_0 D_x^n + A_1 D_x^{n-1} D_y + \ldots A_n D_y^n\right)z = 0 \tag{9}$$

where $A_0, A_1, \ldots A_n$ are all constants.

The complementary function of (9) is the general solution of

$$\left(A_0 D_x^n + A_1 D_x^{n-1} D_y + \ldots A_n D_y^x\right)z = 0 \tag{10}$$

$$A_0\,m^n + A_1\,m^{\,n-1} + A_2\,m^{\,n-2} + \ldots + A_n = 0 \tag{11}$$

Equation (11) is known as the *auxiliary equation* (AE) and is obtained by putting $D_x = m$ and $D_y = 1$ in $F(D_x, D_y) = 0$

Let $m_1, m_2, \ldots m_n$ be n roots of AE (11). Then the following cases arise:

Case I When $m_1, m_2, \ldots m_n$ are distinct. Then the part of CF corresponding to $m = m_r$ is $z = \phi_r(y + m_n x)$ for $r = 1, 2, 3, \ldots n$. Since (10) is linear, the sum of the solution is also a solution.

$\therefore$ CF of (10) $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + m_nx)$

where $\phi_1, \phi_2, \ldots \phi_n$ are arbitrary functions.

Case II Repeated roots. If m is repeated 'r' times, the corresponding part of the CF is

$$\phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_1(y + mx) + \ldots + x^{r-1}\,\phi_r(y + mx).$$

4. Method of Finding the Particular Integral (PI or Z_P) of the Linear Homogeneous PDE with Constant Coefficients

The particular integral of the equation $F(D_x, D_y)z = f(x, y)$ is (12)

$$\text{PI or } Z_p = \frac{1}{F(D_x, D_y)} f(x,y) \quad (13)$$

(a) Short Methods of Finding the Particular Integrals in Certain Cases

Before taking up the general method for finding PI of $F(D_x, D_y)\, z = f(x, y)$

Short Method (I) When $f(x, y)$ is of the form $\phi(ax + by)$, two cases will arise.

Case I Replacing D_x by $a\, D_y$ by b in $F(D_x, D_y)$, suppose that $F(a, b) \neq 0$. If $F(D_x, D_y)$ is a homogeneous function of D_x and D_y of degree n, then we have

$$\text{PI} = \frac{1}{F(D_x, D_y)} \phi(ax+by)$$

$$= \frac{1}{F(a,b)} \iint \dots \int \phi(v)\, dv^n, \text{ where } v = ax + by.$$

After integrating $\phi(v)$ n times w.r.t. 'v', v must be replaced by $(ax + by)$.

Case II Replacing D_x by a, D_y by b in $F(D_x, D_y)$, suppose that $F(a, b) = 0$.

Then $F(D_x, D_y)$ must be factorized.

In general, two types of factors will arise.

Let $F(D_x, D_y) = (b\, D_x - a\, D_y)m.\ G(D_x, D_y)$, where $G(a, b) \neq 0$.

Then $$\text{PI} = \frac{1}{(bD_x - aD_y)^m} \phi(ax+by) = \frac{x^m}{b^m\, m!} \phi(ax+by)$$

Short Method II When $f(x, y)$ is of the form $x^m y^n$, m and n being non-negative integers.

If $n<m$, $\dfrac{1}{F(D_x, D_y)}$ is expanded in powers of $\dfrac{D_y}{D_x}$. On the other hand, if $m < n$, $\dfrac{1}{F(D_x,D_y)}$ is expanded in two ways. In many problems, we arrive at different answers.

However, the difference is not material as it can be incorporated in the arbitrary functions occurring in the CF of that problem.

(b) General Method of Finding PI of

$$F(D_x, D_y)\, z = f(x, y) \quad (14)$$

when $F(D_x, D_y)$ is a homogeneous function of D_x and D_y. We use the following results.

$$(R_1)\ \frac{1}{(D_x - mD_y)} f(x,y) = \int f(x, c-mx)\, dx, \text{ where } c = y + mx.$$

$$(R_2)\ \frac{1}{(D_x + mD_y)} f(x,y) = \int f(x, c_1-mx)\, dx, \text{ where } c_1 = y - mx.$$

After performing integrations, c and c_1 must be replaced by $y + mx$ and $y - mx$ respectively.

5. Nonhomogeneous Linear Partial Differential Equations with Constant Coefficients

A linear partial differential equation which is not homogeneous is called a nonhomogeneous linear PDE. Consider a linear equation with constant coefficients:

$$F(D_x, D_y)\, z = f(x, y) \quad (15)$$

If (15) is a homogeneous linear PDE with constant coefficients then $F(D_x, D_y)$ can always be resolved into factors. On the other hand, if (15) is not homogeneous then $F(D_x, D_y)$ cannot be resolved into linear factors.

Hence, we classify nonhomogeneous linear partial differential equations into the following two types:

(i) When $F(D_x, D_y)$ can be resolved into linear factors, each of which is of first degree in D_x and D_y.
For example, $\left(D_x^2 - D_y^2 + 2D_x + 1\right)z = x^3 y^3$, i.e., $[(D_x + D_y + 1)(D_x - D_y + 1)]\, z = x^2 y^3$.

(ii) When $F(D_x, D_y)$ cannot be resolved into linear form $(a D_x + b D_y + c)$
For example, $(2D_x^4 - 3D_x^2 D_y + D_y^2)\, z = 2xy^2$, i.e., $(2D_x^2 - D_y)(D_x^2 - D_y)\, z = 2xy^2$.

6. Method of Finding CF when Equations are of the Type (b)

$$F(D_x, D_y)\, Z = 0 \tag{16}$$

Where the symbolic operator $F(D_x, D_y)$ cannot be resolved into factors linear such as D_x and D_y. In such a case, as a trial solution, consider $Z = A\, e^{hx + ky}$ (17)

where A, h, k are constants from (17). We have

$$D_x^r z = A h^r e^{hx + Ky},\quad D_x^r D_y^s z = A h^r k^s e^{hx+ky} \text{ and } D_y^s z = A k^s e^{hx+ky}$$

Hence, substituting (17) in (16), we get

$$Af(h, k) e^{hx + ky} = 0$$

which will hold if $f(h, k) = 0$ (18)

and A is an arbitrary constant.

Now, for any chosen values of h or k, (18) gives one or more values of k or h. so there exist infinitely many pairs of numbers (h_i, k_i) satisfying (18).

Thus,
$$Z = \sum_{i=1}^{\infty} A_i e^{h_i x + k_i y} \tag{19}$$

where $f(h_i, k_i) = 0$ (20)

for each i is a solution of the equation (16). So, if $f(D_x, D_y)$ has no linear factor, (19) is taken as the general solution of (16).

7. Method of Finding Particular Integrals (PI)

The methods of finding PI of nonhomogeneous linear partial differential equations are very similar to those used in solving a homogeneous linear PDE with constant coefficients.

Here, we are considering a few cases only.

Case I When $f(x, y) = e^{ax + by}$ then

$$\text{PI} = \frac{1}{f(D_x, D_y)} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by} \text{ provided } f(a, b) \neq 0$$

i.e., put $D_x = a$ and $D_y = b$

Case II When $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$ then

$$\text{PI} = \frac{1}{f(D_x, D_y)} \sin(ax+by) \text{ or } \cos(ax + by)$$

is obtained by putting $D_x^2 = -a^2$, $D_x \cdot D_y = -ab$, $D_x^2 = -b^2$

Case III When $f(x, y) = x^m y^n$, where m and n are positive integers then

$$\text{PI} = \frac{1}{f(D_x D_y)} x^m y^n = [f(D_x, D_y)]^{-1} x^m y^n \text{ (expand by binomial theorem)}$$

Case IV When $f(x, y) = e^{ax+by} V(x)$ then

$$\text{PI} = \frac{1}{f(D_x D_y)} e^{ax+by} V(x) = e^{ax+by} \frac{1}{f(D_x + a, D_y + b)}$$

8. Equations Reducible to Linear Equations with Constant Coefficients

Consider a partial differential equation of the form

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 x^{n-2} y^2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \ldots + a_n y^n \frac{\partial^n z}{\partial y^n} = f(x, y) \qquad (21)$$

Putting $x = e^u$ or $u = \log x$ and $y = e^v$ or $v = \log y$

Therefore, $x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}$ so that $x\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$ or $D_x = D'_u$ and $D'_u \equiv \frac{\partial}{\partial u}$

Similarly,

$$x^2 D_x^2 = D_u (D_u - 1)$$

$$x^3 D_x^3 = D_u (D_u - 1)(D_u - 2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x^n D_x^n = D_u (D_u - 1)(D_u - 2), \ldots (D_u - n + 1)$$

Also,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial v}$$

$\therefore$

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v} \text{ so that } y\frac{\partial}{\partial y} = \frac{\partial}{\partial v} \text{ or } D_y = D'_y \equiv \frac{\partial}{\partial v}$$

Similarly, $x^2 D_y^2 = D_v (D_v - 1)$, $x^3 D_y^3 = D_v (D_v - 1)(D_v - 2)$, and so on.

Using the above values, (21) reduces to an equation having constant coefficients and now it can easily be solved by the methods already discussed for homogeneous and nonhomogeneous linear equations with constant coefficients.

9. Classification of Partial Differential Equation of Second Order

The general form of a second-order PDE is the function Z of two independent variables x, y given by

$$A(x,y)\frac{\partial^2 z}{\partial x^2} + B(x,y)\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\frac{\partial^2 z}{\partial y^2} + \left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \qquad (22)$$

This equation in linear in second-order terms PDE (22) is said to be linear or quasi-linear according as f is linear or non-linear.

PDE (22) is classified as elliptic, parabolic, or hyperbolic as $B^2 - 4AC < 0, = 0$ or > 0.

OBJECTIVE-TYPE QUESTIONS

1. The particular integral of $(4r - 4s + t) = 16 \log (x + 2y)$ is

(*a*) $2x \log(x + 2y)$ (b) $2x^2 \log(2x + y)$
(c) $2y^2 \log(x + 2y)$ (d) $2x^2 \log(x + 2y)$

2. The general solution of $\frac{\partial z}{\partial y} = a\frac{\partial z}{\partial x}$ is

(a) $z = f(y + ax)$ (b) $z = f(x^2 + y^2)$
(c) $z = f(x + ay)$ (d) $z = f(ax + by)$

3. The general solution of $(D^4 - D'^4)z = 0$ is

(a) $z = f_1(x^2 + y^2) + f_2(x^2 - y^2)$
(b) $z = f_1(x + y) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$
(c) $z = f_1(x^2 - y) + f_2(x - y^2) + f_3(y^2 + ix) + f_4(y - ix)$
(d) $z = f_1(x^2 - y^2) + f_2(x - y^2) + f_3(y^2 - ix) + f_4(y - ix)$

4. The PI of $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$ is

(a) $\frac{x}{3}\cos(y+4x)$ (b) $x\cos(y+4x)$
(c) $\cos(y+4x)$ (d) $-\frac{x}{3}\cos(y+4x)$

5. The PI of $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$ is

(a) $\cos(2x+y)$ (b) $\sin(2x+y)$
(c) $\frac{1}{25}\cos(2x+y)$ (d) $\frac{1}{25}\sin(2x+y)$

6. The general solution $(D^2 - 5DD' + 6D'^2)z = e^{3x-2y}$ is

(a) $z = f_1(y+2x) + f_2(y+3x) + \frac{1}{63}e^{3x-2y}$

(b) $z = f_1(3x+y) + yf_2(2x+y) + \frac{1}{63}e^{3x-2y}$

(c) $z = f_1(2y+x) + f_2(y+3x) + \frac{1}{63}e^{3x-2y}$

(d) $z = f_1(3y+2x) + f_2(2y+3x) + \frac{1}{63}e^{3x-2y}$

7. The general solution of $(D^2 + D'^2)z = \cos mx \cos ny$ is

(a) $z = f_1(y-ix) + f_2(y+ix) - \frac{1}{(m^2-n^2)}\cos mx \cos ny$

(b) $z = f_1(x+y) + f_2(y-x) - \frac{1}{(m^2-n^2)}\cos mx \cos ny$

(c) $z = f_1(y-ix) + f_2(y+ix) - \frac{1}{(m^2-n^2)}\cos mx \cos ny$

(d) $z = f_1(y+x) + f_2(y-x) - \frac{1}{(m^2+n^2)}\cos mx \cos ny$

8. The general solution of $\frac{\partial^2 z}{\partial x \partial y} = xy^2$ is

(a) $z = \frac{x^2y^3}{6} + f(y) + \phi(x)$

(b) $z = \frac{x^2y^2}{4} + f(y) + \phi(x)$

(c) $z = \frac{x^3y^2}{6} + f(y) + \phi(x)$

(d) $z = \frac{xy^3}{4} + f(y) + \phi(x)$

9. The general solution of $\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$ is

(a) $z = e^x \sin y + f(y) + \phi(x)$
(b) $z = e^y \sin x + f(y) + \phi(x)$
(c) $z = e^x \sin x + f(y) + \phi(x)$
(d) $z = e^y \cos y + f(x) + \phi(x)$

10. The PI of $(r + s - 6t) = y\cos x$ is

(a) $-y\cos x + \sin x$ (b) $\cos x + \sin x$
(c) $\cos x + y\sin x$ (d) $x\cos x + \sin x$

11. The general solution of $r = a^2t$ is

(a) $z = f_1(y + ax) + f_2(y - x)$
(b) $z = f_1(y + x) + f_2(y - ax)$
(c) $z = f_1(y + ax) + f_2(y - ax)$
(d) $z = f_1(y + x) + f_2(y^2 - ax)$

12. The general solution of $(D^2 + DD' + D' - 1)z = 0$ is

(a) $z = e^{-x}f_1(y) + e^x f_2(y - x)$
(b) $z = e^x f_1(y) + e^{-x} f_2(y + x)$
(c) $z = e^{-x}f_1(y) + e^{-x} f_2(y^2 - x)$
(d) $z = e^x f_1(y) + e^{-x} f_2(y + x^2)$

13. The type of partial differential equation $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is

(a) Parabolic (b) Elliptic
(c) Hyperbolic (d) Nonlinear

[GATE (IN) 3013]

14. The partial differential equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ is a

(a) Linear equation of order 2
(b) Non-linear equation of order 1
(c) Linear equation of order 1
(d) Non-linear equation of order 2

[GATE (ME) 3013]

ANSWERS

1. (d)	2. (c)	3. (b)	4. (d)	5. (c)	6. (a)	7. (a)	8. (a)	9. (b)	10. (a)
11. (c)	12. (a)	13. (a)	14. (d)						

15 Applications of Partial Differential Equations

15.1 INTRODUCTION

Many physical and engineering problems using mathematical modeling can be formulated as initial boundary-value problems consisting of partial differential equations with boundary conditions (Bc's) and initial conditions (Ic's). The method of separation of variables is a powerful technique to obtain the general solution of such boundary-value problems when the partial differential equation is linear. Assuming separability, this method reduces the boundary-value problem to two ordinary differential equations. The linear combination of the solutions of these ordinary differential equations, using the superposition principle, gives the general solution which is made to satisfy the given boundary conditions and initial conditions.

15.2 METHOD OF SEPARATION OF VARIABLES

Separation of variables is a powerful technique to solve partial differential equations. For a partial differential equation in the function u of two independent variables x and y, assume that the required solution is separable, i.e.,

$$u(x, y) = X(x) \cdot Y(y) \tag{1}$$

where X is a function of x alone only and Y is a function of y alone only. Then substitution of u from (1) and its derivatives reduces the partial differential equation to the form

$$F\,(X, X', X'' \ldots) = G\,(Y, Y', Y'' \ldots) \tag{2}$$

which is separable in X and Y. Since the LHS of Eq. (2) is a function of x alone and RHS of (2) is a function of y alone, Eq. (2) must be equal to a common constant k. Thus, (2) reduces to

$$F\,(X, X', X'' \ldots) = k \tag{3}$$

$$G\,(Y, Y', Y'' \ldots) = k \tag{4}$$

Thus, the determination of a solution to the partial differential equation reduces to the ordinary differential equation.

Example 1 Using the method of separation of variables, solve $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u$, where $u(x, 0) = 6\,e^{-3x}$.

Solution

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u \tag{1}$$

Consider (1) has the solution of the form

$$u(x, y) = X(x) \cdot Y(y) \tag{2}$$

where X is a function of x alone and Y is a function of y alone. Differentiating (2) both sides w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx}, \frac{\partial u}{\partial y} = X\frac{dY}{dy}$$

Substituting the values of u, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$ in Eq. (1), we get

$$Y\frac{dX}{dx} = 2x\frac{dY}{dy} + XY$$

or
$$\left(\frac{dX}{dx} - X\right)Y = 2X\frac{dY}{dy}$$

Separating the variables,

$$\frac{\frac{dX}{dx} - X}{2X} = \frac{\frac{dY}{dy}}{Y} = k \text{ (say), where } k \text{ is a constant.}$$

$\therefore$
$$\frac{1}{2X}\left(\frac{dX}{dx} - X\right) = k \text{ or } \frac{dX}{dx} = (1 + 2k)X$$

or
$$\frac{1}{X}\frac{dX}{dx} = 1 + 2k \tag{3}$$

and
$$\frac{1}{Y}\frac{dY}{dy} = k \tag{4}$$

Solving (3), we get $\log X = (1 + 2k)x + \log C_1$

or $\quad X(x) = C_1 e^{(1 + 2k)x}$

Solving (4), we get $Y(y) = C_2 e^{ky}$.

Substituting $X(x)$ and $Y(y)$ in (1), we get

$$u(x, y) = C_1 C_2 e^{(1 + 2k)x} \cdot e^{ky}$$

$$u(x, y) = Ae^{(1 + 2k)x + ky}, \text{ where } A = C_1 C_2 \tag{5}$$

Using $u(x, 0) = 6e^{-3x}$ in (5), we get

$$6e^{-3x} = Ae^{(1 + 2k)x}$$

$\therefore \quad A = 6,\ 1 + 2k = -3$ or $k = -2$

Hence, the required solution is

$$u(x, y) = 6e^{-(3x + 2y)}$$

Example 2 Using the method of separation of variables, solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given that $u = 0$ when $t \to \infty$, as well as $u(0, t) = 0 = u(l, t)$.

Solution The given equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

Consider (1) has the solution of the form

$$u(x, t) = X(x).\, T(t) \tag{2}$$

where X is a function of x only and T is a function of t only.

Differentiating (2) both sides w.r.t. x and t, we have

$$\frac{\partial u}{\partial x} = T\frac{d^2X}{dx^2}, \frac{\partial u}{\partial t} = X\frac{dT}{dt}, \text{ and } \frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Substituting these values in (1), we get

$$T\frac{d^2X}{dx^2} = X\frac{dT}{dt}$$

Separating the variables, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt} = -\lambda^2 \text{ (say) where } \lambda \text{ is any constant.}$$

$$\therefore \qquad \frac{d^2X}{dx^2} + \lambda^2 X = 0 \tag{3}$$

$$\frac{dT}{dt} + \lambda^2 T = 0 \tag{4}$$

Solutions of (3) and (4) are

$$X(x) = A\cos\lambda x + B\sin\lambda x \text{ and } T(t) = Ce^{-\lambda^2 t}$$

Substituting the values of X and T in (2), we get

$$u(x, t) = (A\cos\lambda x + B\sin\lambda x)\, Ce^{-\lambda^2 t}$$

$$= (A_1\cos\lambda x + B_1\sin\lambda x)\, e^{-\lambda^2 t}, \text{ where } A_1 = AC \text{ and } B_1 = BC \tag{5}$$

A_1 and B_1 are constants.

Using the BC $u(x, 0) = 0$ in (5), we get

$$A_1 e^{-\lambda^2 t} = 0 \Rightarrow A_1 = 0$$

$$\therefore \qquad u(x, t) = B_1 \sin\lambda x \cdot e^{-\lambda^2 t} \tag{6}$$

and $u(l, t) = 0$ in (5), we get

$$B_1 e^{-\lambda^2 t} \cdot \sin\lambda l = 0$$

$B_1 \neq 0$, $\therefore$ $\sin\lambda l = 0 = \sin m\pi$, m is any integer

$$\lambda l = m\pi \Rightarrow \lambda = \frac{m\pi}{l}$$

Therefore, Eq. (6) becomes

$$u(x, t) = B_1 e^{\frac{-\pi^2 t m^2}{l^2}} \sin \frac{m\pi x}{l}$$

Hence, $$u(x, t) = \sum_{m=1}^{\infty} B_m e^{\frac{-m^2 \pi^2 t}{l^2}} \sin \frac{m\pi x}{l}$$

Example 3 Use the method of separation of variables to solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ with $u(x, 0) = 0$ and $u_t(0, t) = 0$.

Solution The given equation is

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \tag{1}$$

Suppose (1) has the solution of the form

$$u(x, t) = X(x). T(t) \tag{2}$$

where X is a function of x only, T is a function of t only. Differentiating (2) w.r.t. t and x, we have

$$\frac{\partial u}{\partial t} = X \frac{dT}{dt} \text{ and } \frac{\partial^2 u}{\partial x \partial t} = \left(\frac{dX}{dx}\right)\left(\frac{dT}{dt}\right)$$

Substituting these values in (1), we have

$$\frac{dX}{dx} \cdot \frac{dT}{dt} = e^{-t} \cos x$$

Separating the variables, we get

$$e^t \frac{dT}{dt} = k \text{ or } \frac{dT}{dt} = k e^{-t} \tag{3}$$

$$k \frac{dX}{dx} = \cos x \tag{4}$$

Solving (3) and (4), we get $T(t) = -ke^{-t} + C_1$

$$X(x) = \frac{1}{k}(\sin x + C_2)$$

Substituting the values of $T(t)$ and $X(x)$ in (2), we get

$$u(x, t) = \frac{1}{k}(\sin x + C_2) \cdot (C_1 - ke^{-t}) \tag{5}$$

Differentiating (5) both sides w.r.t. 't', we get

$$u_t(x, t) = \frac{1}{k}(\sin x + C_2) \cdot k e^{-t} = e^{-t}(\sin x + C_2) \tag{6}$$

Using the given condition $u_t(0, t) = 0$ in (6) and $u(x, 0) = 0$ in (5), we get

$$u_t(0, t) = 0 \Rightarrow e^{-t} C_2 = 0 \text{ or } C_2 = 0$$

$$u(x, 0) = 0 \Rightarrow \frac{1}{k}(\sin x)[C_1 - k] = 0$$

$$C_1 - k = 0 \text{ or } C_1 = k$$

$$\therefore \quad u(x, t) = \sin x(1 - e^{-t})$$

which is the required solutions.

EXERCISE 15.1

Solve the following PDE by the method of separation of variables:

1. $4\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}$, with $u(0, y) = 8e^{-3y}$.
2. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ and $u(0, y) = e^{-5y}$.
3. $\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = u$, with $u(x, 0) = 3e^{-5x} + 2e^{-3x}$.
4. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$, with $u(0, y) = 0$, $\frac{\partial u(0, y)}{\partial x} = 1 + e^{-3y}$.
5. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Answers

1. $u(x, y) = 8\,e^{-3(4x + y)}$
2. $u(x, y) = e^{2x - 5y}$
3. $u(x, y) = 3e^{-5x - 5y} + 2e^{-3x - 2y}$
4. $u(x, y) = \frac{1}{\sqrt{2}} \sin h\,(\sqrt{2}\,x) + e^{-3y} \sin x$
5. $z(x, y) = \left[C_1 e^{ax} + C_2 e^{bx}\right] e^{-ky}$, where $a = (1 + \sqrt{1 + k})$, $b = (1 - \sqrt{1 + k})$

15.3 SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION

Consider an elastic string placed along the X-axis, stretched to the length l between two fixed points A at $x = 0$ and B at $x = l$.

Let $u(x, t)$ denote the deflection (displacement from equilibrium position). Then the vibrations of the string are governed by the one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{5}$$

where c is the phase velocity of the string with boundary conditions.

$$u(0, t) = 0, u(l, t) = 0 \tag{6}$$

The form of motion of the string will depend on the initial displacement or deflection at time $t = 0$

$$u(x, 0) = f(x) \tag{7}$$

and the initial velocity

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = u_t(x, 0) = g(x) \tag{8}$$

The initial boundary value problem (5) can be solved by the method of separation of variables. Suppose Eq. (5) has the solution of the form

$$u(x, t) = X(x)\, T(t) \tag{9}$$

where X is a function of x alone and T is a function of t alone. Differentiating (9) both sides w.r.t. 'x' and t, we get

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx}, \frac{\partial u}{\partial t} = X\frac{dT}{dt} \text{ and } \frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}, \frac{\partial^2 u}{\partial t^2} = X\frac{d^2T}{dt^2}$$

Substituting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$ in (5), we get

$$X\frac{d^2T}{dt^2} = c^2\, T\frac{d^2X}{dx^2}$$

Separating the variables, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2\, T}\frac{d^2T}{dt^2} \tag{10}$$

Both sides of (10) must be equal to the common constant k.

$$\therefore \qquad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2\, T}\frac{d^2T}{dt^2} = k$$

The two ordinary differential equations are

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2T}{dt^2} - kT = 0 \tag{11}$$

The boundary conditions (6) reduce to

$$\left.\begin{aligned} u(0, t) = 0 = X(0)\cdot T(t), &\quad \text{i.e., } X(0) = 0 \\ u(l, t) = 0 = X(l)\cdot T(t), &\quad \text{i.e., } X(l) = 0 \end{aligned}\right\} \tag{12}$$

There are three cases that arise.

Case I

If $k = 0$, Eq. (11) becomes

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2T}{dt^2} = 0$$

$$X(x) = A + Bx. \quad T(t) = C + Dt$$

Using the boundary condition $X(0) = 0 \Rightarrow A = 0$

$$X(l) = 0 \Rightarrow Bl = 0 \text{ or } B = 0$$

$$\therefore \quad X(x) = 0$$

Hence, the deflection $u(x, t) = 0$ is not possible.

Case II

If $k > 0$, i.e., $k = \lambda^2$ say, Eq. (11) becomes

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0 \text{ and } \frac{d^2T}{dt^2} + \lambda^2 T = 0$$

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x},\ T(t) = Ce^{\lambda t} + De^{-\lambda t}$$

Using the boundary condition (12), $X(0) = 0 \Rightarrow A + B = 0$ or $B = -A$

$$X(l) = 0 \Rightarrow Ae^{\lambda l} + Be^{-\lambda l} = 0$$

or $\quad Ae^{\lambda l} - Be^{-\lambda l} = 0$

or $\quad A[e^{\lambda l} - e^{-\lambda l}] = 0$

$$A = 0,\ e^{\lambda l} - e^{-\lambda l} \neq 0$$

$$\therefore \quad B = 0$$

$$X(x) = 0$$

Hence, the deflection $u(x, t) = 0$ is not possible.

Case III

If $k < 0$, i.e., $k = -\lambda^2$ say, Eq. (11) becomes

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \text{ and } \frac{d^2T}{dt^2} + \lambda^2 T = 0$$

Its solution is

$$X(x) = A \cos \lambda x + B \sin \lambda x \text{ and } T(t) = C \cos c\lambda t + D \sin c\lambda t$$

Using the boundary condition (12), $X(0) = 0 \Rightarrow A = 0$

$$X(l) = 0 \Rightarrow B \sin \lambda l = 0$$

$$B \neq 0,\ \sin \lambda l = 0 = \sin n\pi$$

or $\quad \lambda l = n\pi$

or $\quad \lambda = \dfrac{n\pi}{l}, \quad n = 1, 2, 3, \ldots$

$$\therefore \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, 3, \ldots$$

and $\quad T_n(t) = C_n \cos \dfrac{n\pi ct}{l} + D_n \sin \dfrac{n\pi ct}{l}$

The deflection of the vibrating string is

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c \cdot t}{l} + B_n \sin \frac{n\pi c \cdot t}{l} \right] \sin \frac{n\pi x}{l} \tag{13}$$

The unknown constants A_n and B_n are determined using initial conditions.

Note I When the initial displacement $u(x, 0) = f(x)$ is given.

Put $t = 0$ in (9), we get

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

Thus, A_n are the Fourier coefficients in the half-range Fourier sine series expansion of $f(x)$ in $(0, l)$,

Hence,
$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \quad n = 1, 2, 3, \ldots \tag{14}$$

Note II When the initial velocity $u_t(x, 0) = g(x)$ is given, then differentiating (13) w.r.t. 't'

$$u_t(x, t) = \sum_{n=1}^{\infty}\left[-\frac{A_n l}{n\pi c}\sin\frac{n\pi c t}{l} + B_n\frac{l}{n\pi c}\cos\frac{x\pi c t}{l}\right]\sin\frac{n\pi x}{l}$$

Put $t = 0$

$$u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{l}{n\pi c}\sin\frac{n\pi x}{l}$$

or
$$B_n = \frac{2}{n\pi c}\int_0^l g(x)\cdot\sin\frac{n\pi x}{l}\,dx \tag{15}$$

Hence, the general solution of the one-dimensional wave equation (5) is given by

$$u(x, t) = \sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi c t}{l} + B_n\sin\frac{n\pi c t}{l}\right]\sin\frac{n\pi x}{l}, \text{ where}$$

A_n and B_n are given by (14) and (15).

Result 1 When only displacement is given, $f(x) \neq 0$ and $g(x) = 0$, in which all $B_n's$ are zero.

Then the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi c t}{l}\right)\cdot\sin\frac{n\pi x}{l}$$

with A_n given by (14).

Result 2 When only initial velocity is given, i.e., $g(x) \neq 0$ and $f(x) = 0$ in which all $A_n's$ are zero.

Then the solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l}$$

with B_n given by (15).

Example 4 Solve the vibrating-string problem $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$ with the conditions $u(0, t) = 0 = u(l, t)$,

$$u(x, 0) = \begin{cases} x & \text{if } 0 < x < \dfrac{l}{2} \\ (l - x) & \text{if } \dfrac{l}{2} < x < l \end{cases}$$

and $\quad u_t(x, 0) = x(x - l);\ 0 < x < l.$

Solution The given vibrating-string problem $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (1)

Here, the initial displacement $u(x, 0) = f(x)$ and the initial velocity $u_t(x, 0) = g(x)$

$\therefore$ The solution of the given wave equation is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l} \right] \cdot \sin\frac{n\pi x}{l} \tag{2}$$

where $\quad A_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \cdot \sin\frac{n\pi x}{l}\, dx$

$$A_n = \frac{2}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x)\sin\frac{n\pi x}{l}\, dx \right]$$

$$= \frac{2}{l}\left[x \cdot \left(\frac{-l}{n\pi}\right) \cdot \cos\frac{n\pi x}{l} - 1 \cdot \left(\frac{-l^2}{n^2\pi^2}\right) \cdot \sin\frac{n\pi x}{l} \right]_0^{l/2}$$

$$+ \frac{2}{l}\left[(l - x)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l} - (-1)\left(\frac{-l^2}{n^2\pi^2}\right) \cdot \sin\frac{n\pi x}{l} \right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right]$$

$$A_n = \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Now, $\quad B_n = \dfrac{2}{n\pi c}\displaystyle\int_0^l g(x) \cdot \sin \cdot \frac{n\pi x}{l}\, dx$

$$= \frac{2}{n\pi c}\left[\int_0^l x(x - l) \cdot \sin \cdot \frac{n\pi x}{l}\, dx \right]$$

$$= \frac{2}{n\pi c}\left[x(x - l)\left(\frac{-l}{n\pi}\right) \cdot \cos\frac{n\pi x}{l} - (2x - l)\left(-\frac{l^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{l} + 2\left(\frac{-l^3}{n^3\pi^3}\right) \cdot \cos\frac{n\pi x}{l} \right]_0^l$$

$$= \frac{2}{n\pi c}\left[\frac{-2l^3}{n^3\pi^3}\left\{(-1)^n - 1\right\}\right]$$

$$= \frac{8l^3}{n^4\pi^4 c}, \text{ when } n \text{ is odd.}$$

Thus, the required solution of the given equation is

$$u(x, t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cdot\cos\frac{n\pi ct}{l}\cdot\sin\frac{n\pi x}{l}\right]$$

$$+ \frac{8l^3}{c\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi ct}{l}\cdot\sin\frac{(2n-1)\pi x}{l}\Bigg]$$

Example 5 Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, 0 \le x \le 1, t > 0$ with conditions $\left.\begin{aligned} u(0, t) = 0 \\ u_x(1, t) = 0\end{aligned}\right\} t > 0$ and

$$u(x, 0) = \begin{cases} x \text{ if } 0 < x < \frac{1}{4} \\ \frac{1}{2} - x \text{ if } \frac{1}{4} < x < \frac{1}{2} \\ 0 \text{ if } \frac{1}{2} < x < 1 \end{cases}$$

$$u_t(x, 0) = 0, \quad 0 < x < 1$$

Solution The given equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ (1)

with boundary condition $\left.\begin{aligned} u(0, t) = 0 \\ u_x(1, t) = 0\end{aligned}\right\}$ (2) and

initial conditions $u(x, 0) = f(x) = \begin{cases} x, & 0 < x < \frac{1}{4} \\ \left(\frac{1}{2} - x\right),, & \frac{1}{4} < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1. \end{cases}$

$$u_t(x, 0) = g(x) = 0, 0 < x < 1$$

Suppose (1) has the solution of the form

$$u(x, t) = X(x)\cdot T(t) \tag{3}$$

where X is a function of x only and T is a function of t only.

Differentiating (3) both sides w.r.t. x and t, we have

$$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2 X}{dx^2} \text{ and } \frac{\partial^2 u}{\partial t^2} = X\frac{d^2 T}{dt^2}$$

Substituting these values in Eq. (1), we have.

$$X\frac{d^2 T}{dt^2} = c^2 T\frac{d^2 X}{dx^2}$$

Separating the variables,

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2} = k \text{ (say)} \tag{4}$$

$$\therefore \quad \frac{d^2 X}{dx^2} - kX = 0 \tag{5}$$

and

$$\frac{d^2 T}{dt^2} - kc^2 T = 0 \tag{6}$$

using $u(0, t) = 0$, (3) gives $X(0)\cdot T(t) = 0$ so that $X(0) = 0$, $T(t) \neq 0$

Differentiating(3) partially w.r.t. 'x', we get

$$u_x(x, t) = X'(x)\cdot T(t) \tag{7}$$

Using $u_x(1, t) = 0$, (7) gives $X'(1)\cdot T(t) = 0$ so that $X'(1) = 0$, $T(t) \neq 0$

Thus, $$X(0) = 0 \text{ and } X'(1) = 0 \tag{8}$$

We now solve (5) under BC (8).

Then three cases arise.

Case I

When $k = 0$, using BC (8), $X(x) = 0$

$\therefore$ $u(x, t) = 0$. It is not possible.

Case II

When $k > 0$, i.e., $k = \lambda^2$ say using (8), $X(x) = 0$

$\therefore$ $u(x, t) = 0$. It is not possible.

Case III

When $k < 0$, say $k = -\lambda^2$

$$\frac{d^2 X}{dx^2} - \lambda^2 X = 0, \quad X(x) = A'\cos\lambda x + B'\sin\lambda x$$

$$X'(x) = -A'\lambda\sin\lambda x + B'\lambda\cos\lambda x$$

Using BC (8), we get $A' = 0$ and $B'\lambda\cos\lambda = 0$ so that $A' = 0$, $\cos\lambda = 0$ ($\therefore$ $\lambda \neq 0$) and $B' \neq 0$

Now, $$\cos\lambda = 0 = \cos\frac{(2n-1)\pi}{2} \text{ or } \lambda = \frac{(2n-1)\pi}{2}, n = 1, 2, 3, \ldots$$

Hence, the non-zero solution $X_n(x)$ of (5) is given by

$$X_n(x) = B_n' \sin \frac{(2n-1)}{2} \pi x \tag{9}$$

Now, $k = -\lambda^2 = -\frac{1}{4}(2n-1)^2 \pi^2$ so that (6) becomes

$\frac{d^2T}{dt^2} - \frac{1}{4}(2n-1)^2 \pi^2 c^2 T = 0$, whose general solution is

$$T_n(t) = C_n' \cos \frac{(2n-1)\pi c t}{2} + D_n' \sin \frac{(2n-1)\pi c t}{2} \tag{10}$$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{(2n-1)\pi c t}{2} + B_n \cdot \sin \frac{(2n-1)\pi c t}{2}\right] \sin \frac{(2n-1)\pi x}{2} \tag{11}$$

$$A_n = B_n' C_n' \quad B_n = B_n' D_n'$$

The initial condition $u_t(x, 0) = g(x) = 0$

$\therefore$ $B_n = 0$, Eq. (11), becomes

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{(2n-1)\pi c t}{2} \cdot \sin \frac{(2n-1)\pi x}{2} \tag{12}$$

where $A_n = \frac{2}{1}\int_0^1 f(x) \sin \frac{(2n-1)\pi x}{2}\, dx$

$$= 2\left[\int_0^{1/4} x \cdot \sin \frac{(2n-1)\pi x}{2}\, dx + \int_{1/4}^{1/2} \left(\frac{1}{2} - x\right) \cdot \sin \frac{(2n-1)\pi x}{2}\, dx + \int_{1/2}^{1} 0 \cdot \sin \frac{(2n-1)\pi x}{2}\, dx\right]$$

$$= 2\left[x \cdot \left\{\frac{-\cos \frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi}{2}}\right\} - (1)\left\{\frac{-\sin \frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2 \pi^2}{4}}\right\}\right]_0^{\frac{1}{4}}$$

$$+ 2\left[\left(\frac{1}{2} - x\right)\left\{\frac{-\cos \frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi}{2}}\right\} + \left\{\frac{-\sin \frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2 \pi^2}{4}}\right\}\right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{8 \sin \frac{(2n-1)\pi}{4}}{(2n-1)^2 \pi^2}$$

Hence, the required solution is

$$u(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\sin (2n-1)\pi x}{(2n-1)^2} \cos \frac{(2n-1)\pi c t}{2} \sin \frac{(2n-1)\pi x}{2}\right]$$

Example 6 Find the deflection $u(x, t)$ of the vibrating starting of length $l = \pi$, ends fixed, and $c^2 = 1$ corresponding to zero initial velocity and initial deflection $u(x, 0) = f(x) = k(\sin x - \sin 2x)$.

Solution The vibrating string governing the PDE is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (1)\ [\therefore c^2 = 1]$$

with conditions $u(0, t) = 0$, $u(\pi, t) = 0$ and

$$u(x, 0) = f(x) = k(\sin x - \sin 2x)$$

$$u_t(x, 0) = g(x) = 0$$

Since initial velocity $u_t(x, 0) = g(x) = 0$; then $B_n = 0$

The solution of the given equation (2) is given by

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi t}{\pi} \sin\frac{n\pi x}{\pi}$$

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos nt \cdot \sin nx \qquad (2)$$

where $$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} k(\sin x - \sin 2x)\sin nx\,dx$$

$$A_n = \frac{2k}{\pi}\left[\int_0^{\pi} \sin x \cdot \sin nx\,dx - \int_0^{\pi} \sin 2x \cdot \sin nx\,dx\right] \qquad (3)$$

But we know that $\int_0^{\pi} \sin mx \cdot \sin nx\,dx = 0$ if $m \neq n$

$$= \frac{\pi}{2} \text{ if } m = n$$

$\therefore$ Eq. (3) gives

$$A_1 = k,\ A_2 = -k \text{ and } A_n = 0\ \forall\ n \geq 3$$

Putting these values in Eq. (2), the required deflection is given by

$$u(x, t) = k(\cos t \sin x - \cos 2t \,.\, \sin 2x)$$

Example 7 A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $u = u_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $u(x, t)$.

Solution Let the equation to the vibrating string be

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

with the condition $u(0, t) = 0$, $u(l, t) = 0$

$$u_t(x, 0) = 0,\ u(x, 0) = u_0 \sin^3\left(\frac{\pi x}{l}\right)$$

Suppose (1) has the solution of the form

$$u(x, t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 \cos \lambda ct + C_4 \sin \lambda ct) \tag{2}$$

Using $u(0, t) = 0$, (2) gives

$$C_1 = 0$$

$$\therefore \quad u(x, t) = C_2 \sin \lambda x(C_3 \cos \lambda ct + C_4 \sin \lambda ct) \tag{3}$$

Using $u(l, t) = 0$, gives

$$0 = C_2 \sin \lambda l(C_3 \cos \lambda ct + C_4 \sin \lambda ct)$$

$$\therefore \quad \sin \lambda l = 0 = \sin n\pi \text{ or } \lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l},\quad n = 0, 1, 2, 3, \ldots$$

$$\therefore \quad u(x, t) = C_2 \sin\frac{n\pi x}{l}\left(C_3 \cos\frac{n\pi ct}{l} + C_4 \sin\frac{n\pi ct}{l}\right) \tag{4}$$

$$u_t(x, t) = \frac{\partial u}{\partial t} = C_2 \sin\frac{n\pi x}{l}\left(-\frac{n\pi c}{l}C_3 \sin\frac{n\pi ct}{l} + C_4\frac{n\pi c}{l}\cdot\cos\frac{n\pi ct}{l}\right)$$

Using $\quad u_t(x, 0) = 0 \Rightarrow C_4 = 0$, (4) reduces to

$$u(x, t) = C_2C_3 \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l}$$

$$= B_n \sin\frac{n\pi x}{l}\cdot\cos\left(\frac{n\pi ct}{l}\right) \qquad [\because\ B_n = C_2\, C_3]$$

or

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l} \tag{5}$$

But

$$u(x, 0) = u_0 \sin^3\left(\frac{\pi x}{l}\right) = \frac{u_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

$$\therefore \quad u = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{u_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Hence, $\quad B_1 = \dfrac{3u_0}{4},\ B_3 = \dfrac{u_0}{4}$ and all B's are zero.

$$\therefore \quad u(x, t) = \frac{u_0}{4}\left[3\sin\frac{\pi x}{l}\cdot\cos\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi ct}{l}\right]$$

Example 8 The vibrations of an elastic string are governed by the partial differential equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2$ (sin + sin $3x$). Find the deflection $u(x, t)$ of the vibrating string for $t > 0$.

Solution The given partial differential equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

With the condition $u(0, t) = 0$, $u(\pi, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = 2(\sin x + \sin 3x)$.

Let the solution of (1) be in the form of

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt).\sin nx \tag{2}$$

Then,
$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} 2(\sin x + \sin 3x)\cdot \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left[2\sin x\cdot \sin nx + 2\sin 3x \sin nx\right] dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left[\cos(n-1)x - \cos(n+1)x\right] dx + \frac{2}{\pi}\int_0^{\pi}\left[\cos(n-3)x - \cos(n+3)x\right] dx$$

$$= \frac{2}{\pi}\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\sin(n-3)x}{(n-3)} - \frac{\sin(n+3)x}{(n+3)}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(n+1)\sin(n-1)x - (n-1)\sin(n+1)x}{n^2-1}\right]_0^{\pi}$$

$$+ \frac{2}{\pi}\left[\frac{(n+3)\sin(n-3)x - (n-3)\sin(n+3)x}{n^2-9}\right]_0^{\pi}$$

$$= 0 \text{ (when } n \neq 1, 3)$$

Now,
$$A_1 = \frac{2}{\pi}\int_0^{\pi} 2(\sin x + \sin 3x)\sin x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[2\sin^2 x + 2\sin 3x\cdot \sin x\right] dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[(1 - \cos 2x) + \cos 2x - \cos 4x\right] dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[(1-\cos 4x)\, dx = \frac{2}{\pi}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi} = \frac{2}{\pi}(\pi - 0)\right.$$

$$A_1 = 2$$

and
$$A_3 = \frac{2}{\pi}\int_0^{\pi} 2(\sin x + \sin 3x)\cdot \sin 3x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}[2\sin 3x\cdot \sin x + 2\sin^2 3x]\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}[\cos 2x - \cos 4x + 1 - \cos 6x]\,dx$$

$$= \frac{2}{\pi}\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4} + x - \frac{\sin 6x}{6}\right]_0^{\pi} = 2$$

$$B_n = \frac{2}{n\pi}\int_0^{\pi} g(x)\cdot \sin nx\,dx = \frac{2}{n\pi}\int_0^{\pi} 0\cdot \sin nx dx = 0$$

∴ Eq. (2) becomes

$$u(x, t) = A_1 \cos t \sin x + A_3 \cos 3t \,.\, \sin 3x$$
$$u(x, t) = 2 \cos t \sin x + 2 \cos 3t \sin 3x$$
$$u(x, t) = 2\,(\cos t \sin x + \cos 3t \sin 3x)$$

EXERCISE 15.2

1. A string is stretched between two fixed points at a distance l apart, motion is started by displacing the string in the form $y = a \sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t = 0$. Show that the displacement of any point at a fixed distance x from one end at time t is given by

$$y\,(x, t) = a \sin\left(\frac{\pi x}{l}\right)\cdot\cos\left(\frac{\pi c t}{l}\right)$$

2. Solve completely the equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ representing the vibrations of a string of length l, fixed at both ends, given that $u(0, t) = 0$, $u(l, t) = 0$, $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$, $0 < x < l$.
3. If a string of length l is initially at rest in equilibrium position and each of its point is given the velocity $u_t(x, 0) = b\sin^3\left(\frac{\pi x}{l}\right)$, find the displacement $u(x, t)$.
4. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, under the conditions $u(0, t) = 0$, $u(\pi, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x$; $0 < x < \pi$.
5. A tant string of length $2l$ is fastened at both ends. The midpoint of the string is taken to a height b and then released from rest in that position. Find the displacement of the string.
6. A string is stretched between the fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by $f(x) = \begin{cases} \frac{2kx}{l}, & \text{when } 0 < x < l/2 \\ \frac{2k(l-x)}{l}, & \text{when } l/2 < x < l \end{cases}$

   Find the deflection of the string at any time t.
7. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $u = k(l\,k - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Answers

3. $u(x, t) = \dfrac{bl}{12\pi c}\left[9\sin\left(\dfrac{\pi x}{l}\right)\sin\left(\dfrac{\pi ct}{l}\right) - \sin\left(\dfrac{3\pi x}{l}\right)\cdot\sin\left(\dfrac{3\pi ct}{l}\right)\right]$

4. $u(x, t) = 2\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin nx.\cos nct$

6. $u(x, t) = \dfrac{8K}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\cos\dfrac{(2n-1)\pi ct}{l}\sin\dfrac{(2n-1)\pi x}{l}$

7. $u(x, t) = \dfrac{8Kl^2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{1}{n^3}\cos\left(\dfrac{n\pi ct}{l}\right)\cdot\sin\left(\dfrac{n\pi x}{l}\right)$, when n is odd.

15.4 ONE-DIMENSIONAL HEAT EQUATION

Consider the flow of heat by conduction in a bar OA. Let OA be taken as the x-axis. We suppose an element $PQRS$ of the bar at any point P is a function of x and time t. Suppose that the bar is raised to an assigned temperature distribution at time $t = 0$ and then heat is allowed to flow by conduction. We find $u(x, t)$ at any point x and at any point $t > 0$.

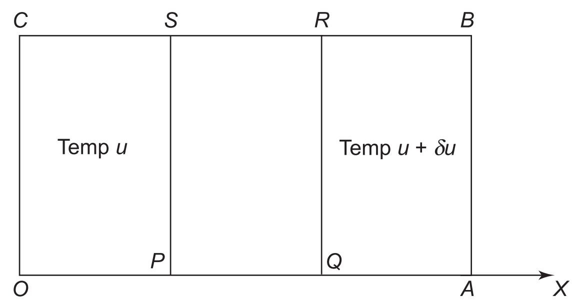


Fig. 15.1

We consider the following assumptions:

(i) The bar is homogeneous, i.e., the mass of the bar per unit volume is constant, say.

(ii) The sides of the bar are insulated and the loss of heat from the sides by conduction can be neglected.

(iii) The amount of heat crossing any section of the bar is given by $kA\left(\dfrac{\partial u}{\partial x}\right)\delta t$, where A is the area of cross section of the bar, $\dfrac{\partial u}{\partial x}$ is the temperature gradient at the section, and δt is the time of flow of heat, k is the thermal conductivity of the material of the bar.

Now, the quantity of heat flowing into the element across the section PS in time δt

$$= -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t$$

where the negative sign has been taken because heat flows in the decreasing direction.

Again, the quantity of heat flowing out of the element across the section QR in time δt

$$= -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

Hence, the quantity of heat retained by the element is

$$= -kA\left(\frac{\partial u}{\partial x}\right)_x \cdot \delta t + kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \cdot \delta t$$

or
$$= kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)x\right]\cdot \delta t \tag{16}$$

Suppose the heat above raises the temperature of the element by a small quantity δu. Then the same quantity of heat is again given by $= (\rho A\,\delta x)\sigma\delta u.$ (17)
where σ is the specific heat of the bar.

Since equations (16) and (17) are equal, we have

$$kA\,\delta t[u(x + \delta x, t) - u(x, t)] = (\rho A\,\delta x)\sigma\delta u.$$

or
$$k \cdot \frac{u(x + \delta x, t) - u(x, t)}{\delta x} = \rho\sigma \frac{\delta u}{\delta t}$$

Now, as $\delta x \to 0$ and $\delta t \to 0$

$\therefore \quad k\,\dfrac{\partial^2 u}{\partial x^2} = \rho\sigma\dfrac{\partial u}{\partial t}$ or $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ where $c^2 = \dfrac{k}{\rho\sigma}$ is called the *diffusivity* of the material of the bar.

15.5 SOLUTION OF ONE-DIMENSIONAL HEAT EQUATION

The one-dimensional heat-flow equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{18}$$

To solve Eq. (18) by the method of separation of variables, suppose Eq. (18) has the solution of the form

$$u(x, t) = X(x).\ T(t) \tag{19}$$

where X is a function of x alone and T is a function of t alone. Differentiating (19) both sides w.r.t. x and t, we get

$$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2 x}{dx^2} \quad \text{and} \quad \frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

Substituting in Eq. (18), we have

$$X\frac{dT}{dt} = c^2\,T\frac{d^2 X}{dx^2}$$

Separating the variables,

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{c^2 T}\frac{dT}{dt} \tag{20}$$

The LHS of (20) is a function of x alone and the RHS is a function of t alone. Since x and t are independent variables, (20) can hold good if each side is equal to a constant, say k. Then

$$\frac{d^2 X}{dx^2} = kX \text{ or } \frac{d^2 X}{dx^2} - kX = 0 \tag{21}$$

And $$\frac{dT}{dt} - kT = 0 \qquad (22)$$

Then three cases aries accordingly as k is zero, +ve or –ve.

Case I

When $k = 0$ then solutions (21) and (22) are

$$X(x) = A_1 x + B_1 \text{ and } T(t) = C_1$$

$$\therefore \quad u(x, t) = (A_1 x + B_1)C_1$$

Case II

When $k > 0$, i.e., $k = +\lambda^2$, (say) then solutions of (21) and (22) are

$$X(x) = A_2 e^{\lambda x} + B_2 e^{-\lambda x} \text{ and } T(t) = C_2\, e^{\lambda^2 k^2 t}$$

$$\therefore \quad u(x, t) = (A_2 e^{\lambda x} + B_2 e^{-\lambda x})\, C_3 e^{\lambda^2 k^2 t}$$

Case III

When $k < 0$, i.e., $k = -\lambda^2$, then solutions of (21) and (22) are

$$X(x) = A_3 \cos \lambda x + B_3 \sin \lambda x \text{ and } T(t) = C_3\, e^{-\lambda^2 k^2 t}$$

$$\therefore \quad u(x, t) = (A_3 \cos \lambda x + B_3 \sin \lambda x) C_3\, e^{-\lambda^2 k^2 t}$$

Now, we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with the problem of heat conduction, the temperature $u(x, t)$ must decrease with the increases of time. Accordingly, the solution in Case (III) is the only suitable solution.

Example 9 If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed as a function $f(x)$ in the bar then find the temperature at a subsequent time t.

Solution The one-dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Suppose $u(x, t)$ is the temperature.

Since the ends $x = 0$ and $x = l$ are kept at zero temperature, the boundary conditions is given by $u(0, t) = u(l, t) = 0$ for all t and the initial condition is given by

$$u(x, 0) = f(x)$$

Consider Eq. (1) has solution of the form

$$u(x, t) = X(x).T(t) \qquad (2)$$

where X is a function of x alone and T is a function of t alone.

Differentiating (2) both sides w.r.t. x and t, we have

$$\frac{\partial^2 u}{\partial x^2} = T \frac{d^2 X}{dx^2}, \quad \frac{\partial u}{\partial t} = X \frac{dT}{dt}$$

Substituting it in (1), we get

$$c^2 T \frac{d^2 X}{dx^2} = X \frac{dT}{dt}$$

Separating the variables,

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{c^2 T}\frac{dT}{dt} \tag{3}$$

Since x and t are independent variables, (3) can hold good if each side is equal to a constant, say k. Then (3) becomes

$$\frac{1}{X}\frac{d^2 X}{dx^2} = k \text{ or } \frac{d^2 X}{dx^2} - kX = 0 \tag{4}$$

$$\frac{1}{c^2 T}\frac{dT}{dt} = k \text{ or } \frac{dT}{dt} - kc^2 T = 0 \tag{5}$$

The boundary condition $\left.\begin{array}{l} u(0, t) = 0 = X(0)T(t) \Rightarrow X(0) = 0 \\ u(l, t) = 0 = X(l)T(t) \Rightarrow X(l) = 0 \end{array}\right\}, \; T(t) \neq 0$ (6)

Case I: When $k = 0$, the solution of (4) and (5) are

$$X(x) = Ax + B \quad \text{and} \quad T(t) = C$$

Using BC (6), $X(0) = 0 \Rightarrow B = 0$ and $X(l) = 0 \Rightarrow Al + 0 = 0$

$$A = 0$$

$\therefore$ $X(x) = 0$

Hence, $u(x, t) = 0$ which is not possible.

Case II: When $k > 0$, i.e., $k = \lambda^2$ (say) then the solutions of (4) and (5) are

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x},\; T(t) = Ce^{+\lambda^2 c^2 t}$$

Using BC (6), $X(0) = 0 \Rightarrow A + B = 0$ or $B = -A$

$$X(l) = 0 \Rightarrow A[e^{\lambda l} - e^{-l\lambda}] = 0$$

$$A = 0 \qquad (\therefore \; e^{\lambda l} - e^{-\lambda l} \neq 0)$$

$\therefore$ $B = 0$

$\therefore$ $X(x) = 0$

Hence, $u(x, t) = 0$, which is not possible.

Case III: When $k < 0$, i.e., $k = -\lambda^2$ say, then the solutions of (4) and (5) are

$$X(x) = A\cos\lambda x + B\sin\lambda x,\; T(t) = ce^{-\lambda^2 c^2 t}$$

Using BC (6), $X(0) = 0 \Rightarrow A = 0$

$$X(l) = 0 \Rightarrow B\sin\lambda l = 0$$

$$B \neq 0,\; \sin\lambda l = 0 = \sin n\pi$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}, \text{ where } n = 1, 2, 3, \ldots$$

Hence, the nonzero solution $X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$ and $T_n(t) = C_n e^{\frac{-n^2\pi^2c^2t}{l^2}}$

Therefore, $u_n(x, t) = X_n(x). T_n(t)$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{-n^2\pi^2c^2t}{l^2}} \tag{7}$$

Putting $t = 0$ in (7) and using $u(x, 0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} E_n \cdot \sin\left(\frac{n\pi x}{l}\right)$$

which is a Fourier sine series, so the constant E_n is given by

$$E_n = \frac{2}{l}\int_0^l f(x)\cdot \sin\left(\frac{n\pi x}{l}\right) dx, \quad n = 1, 2, 3, \ldots \tag{8}$$

Hence, (7) is required solution where E_n is given by (8).

Example 10 The ends A and B of a 20 cm long rod have the temperature at 30°C and 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.

Solution The initial temperature distribution in the rod is

$$u = 30 + \frac{(80-30)}{20}x = 30 + \frac{50}{20}x$$

i.e.,

$$u = 30 + \frac{5}{2}x$$

and the final distribution (i.e., in steady state) is

$$u = 40 + \frac{(60-40)}{20}x = 40 + x$$

To get u in the intermediate period, reckoning time from the instant when the end temperature changed, we consider

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady-state temperature distribution in the rod and $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

Thus, $u_2(x) = 40 + x$

Now, $u_1(x, t)$ satisfies the one-dimensional heat-flow equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Hence, the solution of (1) is

$$u(x, t) = (A\cos\lambda x + B\sin\lambda x).\ e^{-c^2\lambda^2 t}$$

Hence, u is in the form

$$u = 40 + x + \sum_{\lambda=1}^{\infty} (A_\lambda \cos \lambda x + B_\lambda \sin \lambda x) e^{-c^2 \lambda^2 t}$$

Since $u = 40°C$ when $x = 0$ and $u = 60°C$ when $x = 20$, we get

$$A_\lambda = 0 \text{ and } \lambda = \frac{n\pi}{20}$$

$$\therefore \qquad u = 40 + x \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20} \cdot e^{\frac{-c^2 n^2 \pi^2 t}{400}} \qquad (2)$$

Using the initial condition $u(x, 0) = 30 + \frac{5}{2}x$

$$30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20}$$

or
$$\frac{3}{2}x - 10 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20}$$

Hence,
$$B_n = \frac{2}{20} \int_0^{20} \left(\frac{3}{2}x - 10\right) \sin \frac{n\pi x}{20} dx$$

$$= \frac{1}{10}\left[\left(\frac{3x}{2} - 10\right)\left(\frac{-20}{n\pi}\cos\frac{n\pi x}{20}\right) - \frac{3}{2}\left(\frac{-400}{n^2\pi^2}\sin\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[-20\left(\frac{20}{n\pi}\right)(-1)^n - (-10)\left(\frac{20}{n\pi}\right)\right] = -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

Thus,
$$u(x, t) = 40 + x - \frac{20}{\pi} \sum_{n=1}^{\infty} \left(\frac{2(-1)^n}{n}\right) \sin\left(\frac{n\pi x}{20}\right) e^{\frac{-n^2 \pi^2 c^2 t}{400}}$$

Example 11 Obtained temperature distribution $u(x, t)$ in a uniform long bar whose one end is kept at 10°C and the other end is insulated. Further, it is given that $u(x, 0) = (1 - x)$; $0 < x < 1$.

Solution Suppose the bar is placed along the x-axis. Let its one end be at the origin and the other end be at $x = 1$.Then to solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (1)

With BC $\qquad u_x(1, t) = 0, u(0, t) = 10 \qquad (2)$

and IC $\qquad u(x, 0) = (1 - x), 0 < x < 1 \qquad (3)$

Consider $\qquad u(x, t) = y(x, t) + 10$

or $\qquad y(x, t) = u(x, t) - 10 \qquad (4)$

Using(4), (1), (2), and (3) reduce to

$$\frac{\partial y}{\partial t} = C^2 \frac{\partial^2 y}{\partial x^2} \qquad (5)$$

$$y_x(1, t) = 0, y(0, t) = 0 \tag{6}$$

$$y(x, 0) = u(x, 0) - 10 = -(x + 9) \tag{7}$$

Let the equation (5) has the solution of the form

$$y(x, t) = X(x) \cdot T(t) \tag{8}$$

Substituting this value of y in (5), we get

$$X \frac{dT}{dt} = c^2 T \frac{d^2 X}{dx^2} \text{ or } \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = k \text{ (say)} \tag{9}$$

$$\therefore \quad \frac{d^2 X}{dx^2} - k X = 0 \tag{10}$$

$$\frac{dT}{dt} - k c^2 T = 0 \tag{11}$$

Using(6) and (8) gives $X'(1)\ T(t) = 0$ and $X(0)\ T(t) = 0$

Since $T(t) \neq 0$, Then $X'(1) = 0$ and $X(0) = 0$ (12)

Cases I and II

When $k \geq 0$ then using BC (12) and equation (10), have a trivial solution.

$\therefore$ $u(x, t) = 0$, It is not possible.

Case III

When $k < 0$, i.e., $k = -\lambda^2$ say, then the solution of (10) is

$$X(x) = A \cos \lambda x + B \sin \lambda x. \tag{13}$$

So that $X'(x) = -A\lambda \sin \lambda x + B\lambda \cos \lambda x$ (14)

Using BC (12), equations (13) and (14) gives

$$A = 0 \text{ and } \cos \lambda = 0$$

where we have taken $B \neq 0$, since otherwise $X(x) = 0$, and, hence, $u(x, t) = 0$

Now, $\cos \lambda = 0 \Rightarrow \lambda = \frac{1}{2}(2n - 1)\pi, n = 1, 2, 3, ...$

$$\therefore \quad k = -\lambda^2 - \frac{1}{4}(2n - 1)^2 \pi^2$$

Hence, nonzero solution $X_n(x) = B_n \sin \frac{1}{2}(2n - 1)\pi x$

Now, Eq. (11) becomes

$$\frac{dT}{dt} = -\frac{(2n - 1)^2 \pi^2 k}{4} T \text{ or } \frac{dT}{dt} = -C_n^2\, dt \tag{15}$$

where $C_n^2 = \frac{1}{4}(2n - 1)^2 \pi^2 k$

Solving (15), $T_n(t) = D_n e^{-c_n^2 t}$

$$\therefore \quad y_n(x, t) = X_n(n)\ T_n(t) = E_n \sin \frac{(2n - 1)\pi x}{2} \cdot e^{-c_n^2 t}$$

are solutions of (5) satisfying (6), Here, E_n (= B_n D_n) in another arbitrary constant. In order to find a solution also satisfying(7), we consider

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-c_n^2 t} \tag{16}$$

Putting $t = 0$ in (16) and using(7), we have

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} \tag{17}$$

Multiplying both sides of (17) by sin $\frac{(2m-1)\pi x}{2}$ and then integrating w.r.t. x from 0 to 1, we get

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}dx \tag{18}$$

But $$\int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{19}$$

Using (19), (18) gives

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}dx = E_m$$

$$\therefore \quad E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}dx$$

$$= -2\left[(x+9)\left(\frac{-\cos\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi x}{2}}\right) - \left(1\frac{-\sin\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2\pi^2}{4}}\right)\right]_0^1 \quad \text{[using chain rule]}$$

$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi}$$

Using (16) and (4), the required solution is given by

$$u(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-c_n^2 t}$$

Example 12 A rod of length l with insulated sides, is initially at a uniform temperate u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature distribution $u(x, t)$.

Solution Reference Example 11. Here, $f(x) = u_0$. Then Eq. (8)

$$E_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx = \frac{2u_0}{l}\left[\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2u_0}{n\pi}\left[1-(-1)^n\right] \qquad [\therefore \cos n\pi = (-1)^n]$$

$$= \begin{cases} 0 \text{ if } n = 2m, & m, = 1, 2, 3, \ldots \\ \dfrac{4u_0}{n\pi} \text{ if } n = 2m - 1, & m = 1, 2, 3, \ldots \end{cases}$$

Hence, the solution (7) reduces to

$$u(x, t) = \sum_{m=1}^{\infty} E_{2m-1} \sin \frac{(2m-1)\pi x}{l} e^{-C_{2m-1}^2 \cdot t}$$

i.e. $$u(x, t) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \frac{l}{(2m-1)} \sin \frac{(2m-1)\pi x}{l} \cdot e^{-C_{2m-1}^2 \cdot t}$$

where $$C_{2m-1}^2 = (2m-1)^2 \frac{\pi^2 c^2}{l^2}$$

EXERCISE 15.3

1. Solve $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ for $0 < x < \pi$, $t = 0$, when $u_x(0, t) = u_x(\pi, t) = 0$ and $u(x, 0) = \sin x$.
2. Solve $\dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2}$, $t > 0$ by the method of separation of variables with conditions $u(0, t) = 0$, $u(2, t) = 0$, $t > 0$ and $u(x, 0) = x$, $0 < x < 2$.
3. Solve $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0 = u(\pi, t.)$; $t > 0$ and $u(x, 0) = x^2$, $0 < x < \pi$.
4. The temperature at one end of a 50 cm long bar with insulated sides, is kept at 0°C and the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter, find the temperature distribution.
5. Find the solution of $\dfrac{\partial^2 u}{\partial x^2} = h^2 \dfrac{\partial u}{\partial t}$ for which $u(0, t) = 0$, $u(l, t) = 0$ and $u(x, 0) = \sin\left(\dfrac{\pi x}{l}\right)$ by method of separation of variables.
6. Use the method of separation of variables to solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, given that $u = 0$ when $t \to \infty$ as well as $u = 0$ at $x = 0$ and $x = l$.
7. A homogeneous rod of conducting material of 100 cm length has its ends kept at 0°C and temperature initially is $u(x, 0) = x$, $0 \le x \le 50$, $u(x, 0) = 100 - x$, $50 \le x \le 100$. Find the temperature $u(x, t)$ at any time $t > 0$.
8. The temperature distribution in a bar of length π, which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by the partial differential equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$. Assuming the initial temperature as $u(x, 0) = \cos 2x$, find the temperature in the bar at any instant of time.

Answers

1. $u(x, t) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_{m=1}^{\infty} \dfrac{1}{4m^2 - 1} e^{-4m^2 c^2 t} \cos 2mx$

2. $u(x, t) = \sum_{n=1}^{\infty} \dfrac{\sin \frac{n\pi x}{2}}{\sin \frac{n\pi}{2}} e^{\frac{-3n^2 \pi^2 t}{4}}$

3. $u(x, t) = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx\, e^{-a^2 \pi^2 t}$

4. $u(x, t) = 50 - \dfrac{400}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \cos \dfrac{(2n-1)\pi x}{50}\, e^{-(2n-1)^2 \pi^2 c^2/2500}$

5. $u(x, t) = \sin \dfrac{\pi x}{l} e^{-\left(\frac{\pi^2}{h^2 l^2}\right)t}$

6. $u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\dfrac{n\pi x}{l}\right) \cdot e^{-\frac{n^2 \pi^2}{l^2}t}$

7. $u(x, t) = \dfrac{400}{\pi^2}\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)^2} e^{-\left(\frac{(2n+1)\pi c}{100}\right)^2 t} \cdot \sin \dfrac{(2n+1)\pi x}{100}$

8. $u(x, t) = e^{-4t} \cdot \cos 2x$

15.6 VIBRATING MEMBRANE—TWO-DIMENSIONAL WAVE EQUATION

Consider a tightly stretched membrane (like membrane of a drum). For this case, we have the following assumptions.

(i) The membrane is uniform.

(ii) The tension T per unit length is the same in all directions at every point.

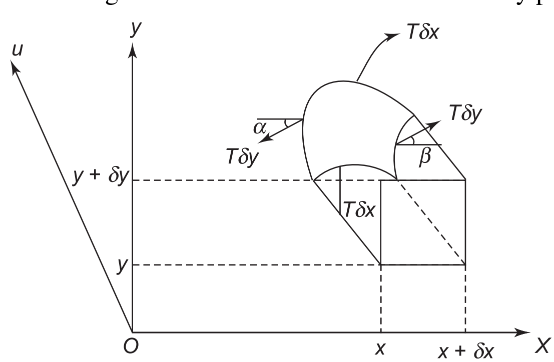


Fig. 15.2

Consider the forces on an element $\delta x\delta y$ of the membrane. Now, due to its displacement u, perpendicular to the xy-plane, forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Here, the forces $T\delta y$ (tangential to the membrane) on its opposite edge of length δy act at angles α and β to the horizontal.

Hence, their vertical component $= (T\delta y)\sin\beta - (T\,\delta y)\sin\alpha$

$$= T\delta y\,(\tan\beta - \tan\alpha) \qquad (\because\ \alpha, \beta \text{ are very small})$$

$$= T\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

$$= T\delta y\delta x\,\frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right]}{\delta x} = T\delta y\delta x\cdot\frac{\partial^2 u}{\partial x^2}$$

Similarly, the forces $T\delta x$ (vertical component of the force) acting on the edges of length δx have the vertical component

$$T\,\delta y\,\delta x\cdot\frac{\partial^2 u}{\partial y^2}$$

Hence, the equation of motion of the element is given by

$$(m\,\delta x\,\delta y)\,\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\delta x\,\delta y$$

or
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{ where } c^2 = \frac{T}{m}$$

This is the wave equation in two dimensions.

15.7 SOLUTION OF TWO-DIMENSIONAL WAVE EQUATION

Two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{23}$$

subject to the boundary conditions

$$u(0, y, t) = 0,\ u(a, y, t) = 0,\ u(x, 0, t) = 0,\ u(x, b, t) = 0$$

and initial conditions $u(x, y, 0) = f(x, y)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y)$

Suppose that Eq. (23) has solutions of the form

$$u(x, y, t) = X(x)\cdot Y(y)\cdot T(t) \tag{24}$$

where X is a function of x alone, Y is a function of y alone, and T is a function of t alone.

Substituting this in (23), we get

$$X''YT + XY''T = \frac{1}{c^2}\,XYT''$$

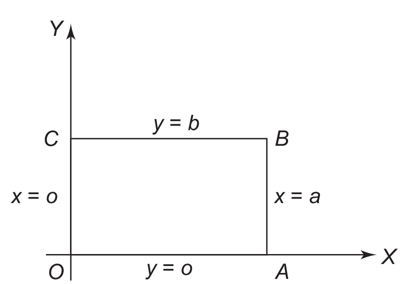


Fig. 15.3

or $$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T''}{c^2 T} \tag{25}$$

Since x, y, and t are independent variables, (25) can only be true if each term on each side is equal to a constant.

Let $$\frac{X''}{X} = k_1 \text{ so that } X'' - k_1 X = 0 \tag{26}$$

$$\frac{Y''}{Y} = k_2 \text{ so that } Y'' - k_2 Y = 0 \tag{27}$$

and $$\frac{T''}{c^2 T} = k_1 + k_2 \text{ so that } T'' - (k_1 + k_2)\, c^2 T = 0 \tag{28}$$

Using $u(0, y, t) = 0$ and $u(a, y, t) = 0$, (24) gives

$$X(0)\, Y(y)\, T(t) = 0 \text{ and } X(a)\, Y(y)\, T(t) = 0$$

We assume that $Y(y) \neq 0$, $T(t) \neq 0$, then $$\left.\begin{matrix} X(0) = 0 \\ X(a) = 0 \end{matrix}\right\} \tag{29}$$

Now, we solve (26) under BC (29).

Then three cases arise.

Case I

When $k_1 = 0$ then the solution of (26) is

$$X(x) = Ax + B$$

Using BC (29), $X(0) = 0 \Rightarrow B = 0$ and $X(a) = 0 \Rightarrow A.a = 0$

i.e., $A = 0$

$\therefore$ $X(x) = 0$

$u(x, y, t) = 0$, so we reject $k_1 = 0$

Case II

When $k_1 > 0$, i.e., $k_1 = \lambda_1^2$ then the solution of (26) is

$$X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x}$$

Using BC (29), $X(0) = 0 \Rightarrow A + B = 0$ or $B = -A$.

and $X(a) = 0 \Rightarrow A(e^{\lambda_1 a} - e^{-\lambda_1 a}) = 0$

$A = 0$ and $e^{\lambda_1 a} - e^{-\lambda_1 a} \neq 0$, then $B = 0$

$\therefore$ $X(x) = 0$

Hence, $u(x, y, t) = 0$, so we reject $k_1 = \lambda_1^2$.

Case III

When $k_1 < 0$, i.e., $k = -\lambda_1^2$. Then the solution of (26) is

$$X(x) = A \cos \lambda_1 x + B \sin \lambda_1 x$$

Using BC (29), $X(0) = 0 \Rightarrow A = 0$

and $$X(a) = 0 \Rightarrow B \sin \lambda_1 a = 0$$

$$B \neq 0 \text{ so that } \sin \lambda_1 a = 0 = \sin m\pi$$

$$\lambda_1 a = m\,\pi$$

$$\lambda_1 = \frac{m\pi}{a},\ m = 1, 2, 3, \ldots$$

Hence, the nonzero solution $X_m(x)$ of (26) is given by

$$X_m(x) = B_m \sin\left(\frac{m\pi x}{a}\right),\ m = 1, 2, 3, \ldots$$

Now next, when $k_2 = -\lambda_2^2$, so the solution of (27) is

$$Y(y) = C \cos \lambda_2\, y + D \sin \lambda_2\, y$$

Now, using the BC $u(x, 0, t) = 0$ and $u(x, b, t) = 0$, (27) gives

$$Y(0) = 0 \text{ and } Y(b) = 0$$

$$\therefore \quad Y(0) = 0 \Rightarrow C = 0 \text{ and } Y(b) = 0 \Rightarrow D \sin \lambda_2 b = 0$$

$$D \neq 0 \text{ so that } \sin \lambda_2 b = 0 = \sin n\pi$$

$$\lambda_2 b = n\pi$$

$$\lambda_2 = \frac{n\lambda}{b},\ n = 1, 2, 3, \ldots$$

Hence, the nonzero solution is

$$Y_n(y) = D_n \sin\left(\frac{n\pi y}{b}\right),\ n = 1, 2, 3, \ldots$$

Now, Eq. (28) becomes

$$\frac{T''}{c^2 T} = k_1 + k_2 = -(\lambda_1^2 + \lambda_2^2)$$

$$\frac{T''}{c^2 T} = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

or $$T'' + \lambda_{mn}^2\, T = 0 \tag{30}$$

where $$\lambda_{mn}^2 = c^2\,\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The solution of (30) is

$$T_{mn}(t) = E_{mn} \cdot \cos \lambda_{mn}\, t + F_{mn} \cdot \sin \lambda_{mn} t$$

$$\therefore \quad u_{mn}(x, y, t) = X_m(x) \cdot Y_n(y) \cdot T_{mn}(t)$$

$$u_{mn}(x, y, t) = (A_{mn} \cos \lambda_{mn}\, t + B_{mn} \sin \lambda_{mn}\, t) \cdot \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right) \tag{31}$$

where $A_{mn} = (B_m \cdot D_n \cdot E_{mn})$ and $B_{mn} = (B_m \cdot D_n \cdot F_{mn})$ are arbitrary constants.

Consider the more general solution of Eq. (23) is given by

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn} \cos \lambda_{mn}\, t + B_{mn} \cdot \sin \lambda_{mn}\, t\right] \cdot \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{32}$$

Differentiating (32) partially w.r.t. 't', we get

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-A_{mn}\cdot\lambda_{mn}\sin\lambda_{mn}t + B_{mn}\cdot\lambda_{mn}\cos\lambda_{mn}t\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (33)$$

Putting $t = 0$ in (32) and (33) using I.C. $u(x, y, 0) = f(x, y)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y)$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (34)$$

$$g(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (B_{mn}\,\lambda_{mn})\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (35)$$

which are double Fourier sine series. We get

$$A_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x, y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \quad (36)$$

and

$$B_{mn} = \frac{4}{ab\lambda_{mn}}\int_{x=0}^{a}\int_{y=0}^{b} g(x, y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \quad (37)$$

Thus, the desired deflection $u(x, y, t)$ of the given membrane is given by (32) where in A_{mn} and B_{mn} are given by (36) and (37) respectively.

Example 13 Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and initial deflection is $f(x, y) = A\sin\pi x.\sin 2\pi y$.

Solution The deflection of the square membrane is given by the two-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

The boundary conditions are $u(x, 0, t) = 0 = u(x, 1, t)$ and $u(0, y, t) = 0$, $u(1, y, t) = 0$

The initial conditions are $u(x, y, 0) = f(x, y) = A\sin\pi x\sin 2\pi y$,

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y) = 0$$

$\therefore$ The deflection of (1) is given by

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\lambda_{mn}t\sin(m\pi x)\cdot\sin(n\pi y) \quad (2)$$

where

$$A_{mn} = \frac{4}{1\cdot 1}\int_0^1\int_0^1 f(x, y)\sin(m\pi x)\sin(n\pi y)\,dxdy$$

$$= 4\int_0^1\int_0^1 A\sin\pi x\sin 2\pi y\cdot\sin(m\pi x)\cdot\sin(n\pi y)\,dx\,dy$$

$[\because\ a = b = 1,\ c = 1,\text{ and } \lambda^2_{mn} = \pi^2(m^2 + n^2)]$

$$= A\left[\left(\int_0^1 2\sin\pi x\cdot\sin m\pi x\,dx\right)\left(\int_0^1 2\sin 2\pi y\cdot\sin n\pi y\,dy\right)\right] \quad (3)$$

But
$$\left.\begin{aligned}\int_0^1 2\sin\pi x\cdot\sin m\pi x\,dx &= 0 \quad \text{if } m\neq 1\\ &= 1 \quad \text{if } m = 1\end{aligned}\right\} \quad (4)$$

and
$$\left.\begin{aligned}\int_0^1 2\sin 2\pi y\cdot\sin n\pi y\,dy &= 0 \quad \text{if } n\neq 2\\ &= 1 \quad \text{if } n = 2\end{aligned}\right\} \quad (5)$$

Using (4) and (5), (3) gives

$$A_{12} = A \text{ and } A_{mn} = 0 \text{ otherwise}$$

and $\lambda_{12}^2 = \pi^2(1^2+2^2) = \pi^2\cdot 5$ or $\lambda_{12} = \pi\sqrt{5}$

$\therefore$ Eq. (2) gives

$$\begin{aligned}u(x, y, t) &= A_{12}\cos\lambda_{12} t\sin\pi x\cdot\sin 2\pi y\\ &= A\cos(\pi\sqrt{5}\,t)\sin\pi x\cdot\sin 2\pi y\end{aligned}$$

Example 14 A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k\sin\pi y$. Show that the deflection at any instant is $k\sin 2\pi x.\ \sin\pi y.\cos(\sqrt{5}.\pi ct)$.

Solution Here $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)$ (1)

The BC's are $u(0, y, t) = 0 = u(1, y, t)$ and $u(x, 0, t) = 0 = u(x, 1, t)$

The IC's are $u(x, y, 0) = f(x, y) = k\sin 2\pi x\sin\pi y$ and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x, y) = 0$

The deflection of (1) is given by

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\lambda_{mn} t\sin m\pi x\cdot\sin n\pi y \quad (2)$$

$[\therefore\ a = b = 1,\ \lambda_{mn}^2 = c^2\pi^2(m^2+n^2)]$

where
$$\begin{aligned}A_{mn} &= \frac{4}{1\cdot 1}\int_0^1\int_0^1 f(x, y)\cdot\sin(m\pi x)\cdot\sin(n\pi y)\,dx\,dy\\ &= 4\int_0^1\int_0^1 (k\sin 2\pi x\cdot\sin\pi y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy\\ &= k\cdot\left[\left(\int_0^1 2\sin 2\pi x\cdot\sin m\pi x\,dx\right)\left(\int_0^1 2\sin\pi y\cdot\sin n\pi y\,dy\right)\right]\end{aligned}$$

But
$$\left.\begin{aligned}\int_0^1 2\sin 2\pi x\cdot\sin m\pi x\,dx &= 0 \quad \text{if } m\neq 2\\ &=1 \quad \text{if } m=2\end{aligned}\right\} \tag{4}$$

and
$$\left.\begin{aligned}\int_0^1 2\sin \pi y\cdot\sin n\pi y\,dy &= 0 \quad \text{if } n\neq 1\\ &=1 \quad \text{if } n=1\end{aligned}\right\} \tag{5}$$

Using(4) and (5), (3) gives

$$A_{21}=k \text{ and } \lambda_{21}^2 = c^2\lambda^2(2^2+1^2)=c^2\pi^2\,5$$

or
$$\lambda_{21} = c\pi\sqrt{5}$$

∴ The required solution is

$$u(x, y, t) = k\sin 2\pi x\cdot\sin\pi y\cdot\cos(\pi c\sqrt{5}\,t)$$ **Hence, proved.**

EXERCISE 15.4

1. Solve $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)$ with the conditions that $u = 0$, along the lines $x = 0$, $y = 0$, $x = a$, $y = 1$.
2. Proof show that for a rectangular membrane of sides a and b vibrating with its boundaries fixed, the eigenvalues and eigenfunctions are given by $\lambda_{mn} = \pi c\sqrt{\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)}$,

   $u_{mn} = \left[A_{mn}\cos(\lambda_{mn} t) - B_{mn}\sin(\lambda_{mn} t)\right]\cdot\sin\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)$,

   $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$
3. Find the deflection $u(x, y, t)$ of a rectangular membrane $0\le x\le a$, $0\le y\le b$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = xy\,(a - x)\,(b - y)$.

Answers

1. $u(x, y, t) = A\cos(c\pi t).\ \sin(\pi x)$.
3. $u(x, y, t) = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos(\lambda_{mn} t)$

   where $A_{mn} = \dfrac{64\,a^2 b^2}{\pi^6 (mn)^3}$ both m and n are odd, and $\lambda_{mn} = \pi c\sqrt{\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)}$

15.8 TWO-DIMENSIONAL HEAT FLOW

Consider the heat of flow in a metal plate in the directions of its length (say *x*-axis) and breadth (*y*-axis) respectively. Where, of course, there is no flow of heat along the direction of the normal to the plane of the rectangle, i.e., the temperature at any point is independent of the *z*-coordinate and depends on *x*, *y*, and time *t* only, the flow is called two-dimensional. Consider the flow of heat in a rectangular plate with sides δx and δy. Let the metal plate be of uniform thickness α, density ρ, specific heat σ, and thermal conductivity *k*.

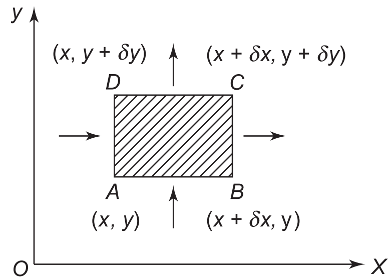


Fig. 15.4

Now, the quantity of heat that enters the plate per second from the sides *AB*.

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y$$

Similarly, the quantity of heat that enters the plate per second from the side *AD*

$$= -k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

The quantity of heat flowing out through the sides *CD* and *BC* per second is $-k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$ and $-k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ respectively.

Hence, total gain of heat by the rectangular element *ABCD* per second.

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{x+\delta x}$$

$$= k\alpha\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \tag{38}$$

But the rate of gain of heat is also $= \sigma\rho\delta x\,\delta y\dfrac{\partial u}{\partial t}$ (39)

Equating (38) and (39), we get

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = \sigma\rho\delta x\delta y\frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\,\delta x\,\delta y$ and applying limit $\delta x \to 0$ and $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \sigma\rho\frac{\partial u}{\partial t}$$

or $$\frac{\partial u}{\partial t} = \frac{k}{\sigma\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \text{ where } c^2 = \frac{k}{\sigma\rho} \tag{40}$$

Equation (3) gives the temperature distribution of the plate in the transient state. In the steady state, u is independent of t, so that $\frac{\partial u}{\partial t} = 0$. Hence the temperature distribution of the plate in the steady state is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, called *Laplace's equation* in two dimensional.

15.9 SOLUTION OF TWO-DIMENSIONAL HEAT EQUATION BY THE METHOD OF SEPARATION OF VARIABLES

The diffusion of heat in a rectangular metal plate of uniform, isotropic material, with both faces insulated and with the four edges kept at zero temperature is given by the transient temperature $u(x, y, t)$ which satisfies the following initial boundary-value problem consisting of the PDE

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{41}$$

in the region of the plate $0 \le x \le a$, $0 \le y \le b$, $t > 0$.

With boundary conditions $u(x, 0, t) = 0 = u(x, b, t)$ (42)

$0 < x < a$, $t > 0$

$$u(0, y, t) = 0 = u(a, y, t), \quad 0 < y < b, \quad t > 0 \tag{43}$$

and initial conditions $u(x, y, 0) = f(x, y)$, $0 < x < a$, $0 < y < b$. (44)

We can apply the method of separation of variables on (41), because the PDE (41) and the BC (42) and (43) are homogeneous.

Consider (41) has the solution of the form

$$u(x, y, t) = \phi(x, y).\, T(t) \tag{45}$$

Substituting u in (41), we have

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)\cdot T = \frac{1}{c^2}\phi\frac{dT}{dt}$$

or $$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)\frac{1}{\phi} = \frac{\dot{T}}{c^2 T} \qquad \left(\dot{T} \equiv \frac{dT}{dt}\right)$$

The mutual value of the LHS and RHS of the above equation must be a constant. Since the time-dependent part of the product solution exponentially decaying, a separation constant in the form of $-\lambda$ may be introduced.

Then the resulting equation are

$$\frac{dT}{dt} + \lambda^2 c^2 T = 0; \; t > 0 \tag{46}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda^2\phi; \; 0 < x < a, \; 0 < y < b \tag{47}$$

The eigenvalue λ relates to the decay of the time-dependent part. The boundary condition (42), (43) are reduced to

$$\phi(x, 0)\, T(t) = 0, \quad \phi(x, b)\, T(t) = 0$$
$$\phi(0, y)\, T(t) = 0, \quad \phi(a, y)\, T(t) = 0$$

If $T(t) = 0$, we get the trivial solution so that $u(x, y, t) = 0$ for all t.

Thus, the required boundary conditions are

$$\phi(x, 0) = 0, \quad \phi(x, b) = 0, \quad 0 < x < a \tag{48}$$
$$\phi(0, y) = 0, \quad \phi(a, y) = 0, \quad 0 < y < b \tag{49}$$

In the new two-dimensional eigenvalue problem consisting of equations (47), (48), (49), the PDE and BC are linear and homogeneous. So separation of variables can be applied again.

Suppose $\phi\,(x, y) = X(x).\ Y(y)$

Then Eq. (47) becomes

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2,\ 0 < x < a,\ 0 < y < b.$$

The ratios $\dfrac{X''}{X}$ and $\dfrac{Y''}{Y}$ should be a negative constant, denoted by $-\lambda_1^2$ and $-\lambda_2^2$ respectively because the sum of a function of x and a function of y can be constant only if each of these two functions are individually constants. The separate equations for x and y are

$$X'' + \lambda_1^2 X = 0, \quad 0 < x < a \tag{50}$$

$$Y'' + \lambda_2^2 Y = 0, \quad 0 < y < b \tag{51}$$

The three separations constants are connected by the relation $\lambda^2 = \lambda_1^2 + \lambda_2^2$ (52)

The boundary conditions (48) and (49) take the form

$$X(x)\, Y(0) = 0, \quad X(x)\, Y(b) = 0, \quad 0 < x < a$$
$$X(0)\, Y(y) = 0, \quad X(a)\, Y(y) = 0, \quad 0 < y < b$$

Again, $X(x) = 0$ for all x or $Y(y) = 0$ for all y, we get a trivial solution $u(x, y, t) = 0$

Thus, the appropriate boundary conditions are

$$X(0) = 0,\ X(a) = 0 \tag{53a}$$
$$Y(0) = 0,\ Y(b) = 0 \tag{53b}$$

Equations (50) and (53a), and (51) and (53b) form two independent eigenvalue problems, with the following solutions:

$$X_m(x) = \sin\left(\frac{m\pi x}{b}\right),\ \lambda_{1m}^2 = \left(\frac{m\pi}{a}\right)^2 \text{ and } Y_n(y) = \sin\left(\frac{n\pi y}{b}\right),\ \lambda_{2n}^2 = \left(\frac{n\pi}{b}\right)^2$$

Here, the indices m and n are independent. The solutions of the two-dimensional eigenvalue problems (47), (48), (49) are

$$\phi_{mn}(x, y) = X_m(x)\cdot Y_n(y) \text{ with } \lambda_{mn}^2 = \lambda_{1m}^2 + \lambda_{2n}^2$$

and the corresponding solution of (46) is

$$T_{mn} = e^{-\lambda_{mn}^2 \cdot c^2 t}$$

The PDE (41) and BC (42), (43) are satisfied by the functions

$$u_{mn}(x, y, t) = \phi_{mn}(x, y)\cdot T_{mn}(t)$$
$$= \sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{n\pi y}{b}\right)\cdot e^{-\lambda_{mn}^2 c^2 t}$$
$$(m, n = 1, 2, 3, \ldots)$$

Using the superposition principle, we obtain the double series

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\,\phi_{mn}(x, y)\cdot T_{mn}(t) \tag{54}$$

which satisfies equations (41), (42), (43).

The unknown coefficients A_{mn} are determined using the initial conditions (44) as follows.

In a problem involving a rectangle $0 < x < a$, $0 < y < b$, a double Fourier sine series is given by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Similarly, a double Fourier cosine series is given by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Clearly, other combinations of sines and cosines could be considered.

All these double series are of the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_m(x)\cdot\psi_n(y)$$

where $\phi_m(x)$, $\psi_n(y)$ are the eigenfunctions of a *Strum–Liouville* problem, satisfying the orthogonally relations.

The expansion formula in the case of a double Fourier sine series in the rectangle $0 < x < a$, $0 < y < b$ is

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Here, the unknown coefficients A_{mn} are given by

$$A_{mn} = \frac{4}{ab}\int_6^b\int_0^a f(x, y)\cdot\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)dx\,dy \tag{55}$$

with $m = 1, 2, 3, \ldots$and $n = 1, 2, 3, \ldots$

Now, consider the general solution is given by (54) as

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)e^{-\lambda_{mn}^2 c^2 t} \tag{56}$$

Using the initial condition (44) in (57) we have

$$f(x, y) = u(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

which is a double Fourier sine series. Here, A_{mn} are determined by (55), thus the complete solution to the BVP (41), (42), (43), and (44), is given by (56) with coefficient A_{mn} determined by (55).

15.10 SOLUTION OF TWO-DIMENSIONAL LAPLACE'S EQUATION BY THE METHOD OF SEPARATION OF VARIABLES

The two-dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{57}$$

is a rectangle in the xy-plane, $0 < x < a$ and $0 < y < b$ satisfying the following boundary conditions:

$$u(x, 0) = 0 \tag{58a}$$

$$u(x, b) = 0 \tag{58b}$$

$$u(0, y) = 0 \tag{58c}$$

$$u(a, y) = f(y) \tag{58d}$$

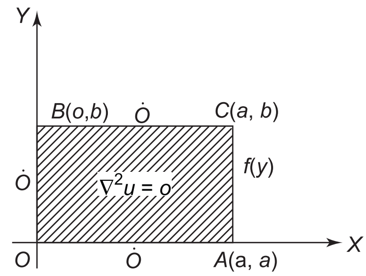


Fig. 15.5

i.e., u is zero on three sides OA, OB, BC and is prescribed by the given function $f(y)$ on the fourth side AC of the rectangle $OACB$.

To solve (57) by the separation of variables, suppose (57) has the solution of the form

$$u(x, y) = X(x).\ Y(y) \tag{59}$$

Substituting (59) in (57), we have

$X''Y + XY'' = 0$ where $'$ denotes differentiation w.r.t. x and y.

So $$\frac{Y''}{Y} = -\frac{X''}{X} \tag{60}$$

Both sides of (60) must be equal to a constant k since LHS of (60) is a functions of y only and the RHS of (60) is a function of x only .

Then $$\frac{Y''}{Y} = -\frac{X''}{X} = k$$

or $$Y'' - kY = 0 \tag{61}$$

or $$X'' + kX = 0 \tag{62}$$

The boundary conditions (58a), (58b), (58c), reduces to

$$u(x, 0) = 0 = X(x).\ Y(0), \text{ i.e., } Y(0) = 0 \tag{63a}$$

$$u(x, b) = 0 = X(x).\ Y(b), \text{ i.e., } Y(b) = 0 \tag{63b}$$

$$u(0, y) = 0 = X(0).\ Y(y), \text{ i.e., } X(0) = 0 \tag{63c}$$

If $k \geq 0$, (61) will have only trivial solutions. So we suppose $k < 0$, i.e., $k = -\lambda^2$. Then Eq. (61) becomes

$$Y'' + \lambda^2 Y = 0$$

having the general solution

$$Y(y) = C_1 \cos \lambda y + C_2 \sin \lambda y$$

Using (63a), $Y(0) = 0 = C_1 1 + C_2 0$, i.e., $C_1 = 0$

Using (63b), $Y(b) = 0 = C_1 \sin \lambda b$

Since $C_2 \neq 0$, $\sin \lambda b = 0 = \sin n\pi$ or $\lambda b = n\pi$

$$\lambda = \frac{n\pi}{b}, n = 1, 2, 3, \ldots \tag{64}$$

$$\therefore \quad Y(y) = \sin \frac{n\pi y}{b} \tag{65}$$

Now, the solution of (62) is

$$X(x) = C_3 e^{-\lambda x} + C_4 e^{\lambda x}$$

Using (63c), $X(0) = 0 = C_3 + C_4$, i.e., $C_3 = -C_4$

$$\text{or} \quad X(x) = 2C_4 \left(\frac{e^{\lambda x} - e^{-\lambda x}}{2} \right) = C_4 \sin h \lambda x$$

$$\therefore \quad X(x) = C_4 \sin \left(\frac{n\pi x}{b} \right) \tag{66}$$

Hence, the solution of (57) is given by

$$u_n(x, y) = A_n \sin h \left(\frac{n\pi x}{b} \right) \sin \left(\frac{n\pi y}{b} \right) \qquad (\text{where } C_4 = A_n)$$

By superposition principle,

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$$

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin h \left(\frac{n\pi x}{b} \right) \sin \left(\frac{n\pi y}{b} \right) \tag{67}$$

Using (58d), $f(y) = u(a, y)$

$$\therefore \quad f(y) = \sum_{n=1}^{\infty} \left[A_n \cdot \sin h \left(\frac{n\pi a}{b} \right) \sin \left(\frac{n\pi y}{b} \right) \right]$$

Thus, $A'_n s$ are Fourier coefficients of the Fourier half-range sine series of $f(y)$ in the interval $(0, b)$ and are given by

$$A_n \cdot \sin h \left(\frac{n\pi a}{b} \right) = \frac{2}{b} \int_0^b f(y) \cdot \sin \left(\frac{n\pi y}{b} \right) dy$$

$$\text{or} \quad A_n = \frac{2}{b \cdot \sin h \left(\frac{n\pi a}{b} \right)} \int_0^b f(y) \cdot \sin \left(\frac{n\pi y}{b} \right) dy \tag{68}$$

$$(n = 1, 2, 3, \ldots)$$

Hence, the harmonic function $u(x, y)$ satisfying the Laplace's equation (57) and four boundary conditions (58a), (58b), (58c), and (58d) is given by (67) with $A_n{}^{'s}$ determined by (68).

Example 15 Determine the transient temperature in a rectangular metal plate of uniform isotropic material with both faces insulated, with the four edges maintained at zero temperature and with initial temperature distribution given by $f(x, y) = xy$.

Solution The initial condition $u(x, y, 0) = f(x, y) = xy$.

Let the complete solution of the 2D heat equation be

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cdot e^{-\lambda_{mn}^2 c^2 t} \quad (1)$$

where
$$A_{mn} = \frac{4}{a\cdot b}\int_0^b\int_0^a f(x, y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$= \frac{4}{ab}\left[\int_0^b\int_0^a xy\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy\right]$$

$$= \frac{4}{ab}\left[\int_0^b y\sin\left(\frac{n\pi y}{b}\right)dy\right]\times\left[\int_0^a x\sin\left(\frac{m\pi x}{a}\right)dx\right] \quad \text{[Integrating by parts]}$$

$$= \frac{4}{ab}\left[-y\left(\frac{b}{n\pi}\right)\cos\left(\frac{n\pi y}{b}\right)+\frac{b^2}{n^2\pi^2}\sin\left(\frac{n\pi y}{b}\right)\right]_0^b \times$$
$$\left[(-x)\left(\frac{a}{m\pi}\right)\cos\left(\frac{m\pi x}{a}\right)+\frac{a^2}{m^2\pi^2}\sin\left(\frac{m\pi x}{a}\right)\right]_0^a$$

$$= \frac{4}{ab}\left[\left(-\frac{b^2}{n\pi}\cos n\pi\right)\cdot\left(-\frac{a^2}{m\pi}\cos m\pi\right)\right]$$

$$= \frac{4ab}{mn\pi^2}\cos n\pi\cdot\cos m\pi = \frac{4ab}{mn\pi^2}(-1)^{m+n}$$

Putting the values of A_{mn} in (1), the required solution is

$$u(x, y, t) = \frac{4ab}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{mn}\cdot\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{n\pi y}{b}\right)e^{-\lambda_{mn}^2 c^2 t}$$

Example 16 An infinite long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine temperature at any point of the plate in the steady state.

Solution In the steady state, the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

with the boundary conditions

$$u(0, y) = 0;\ \forall\ y \quad (2)$$
$$u(\pi, y) = 0;\ \forall\ y \quad (3)$$
$$u(x, \infty) = 0;\ \ 0 < x < \pi \quad (4)$$

and the initial condition

$$u(x, 0) = u_0 \quad (5)$$

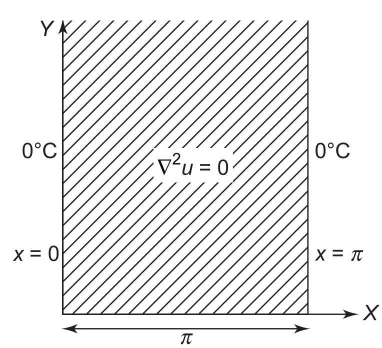


Fig. 15.6

Consider (1) has the solution of the form.

$$u(x, y) = X(x) \cdot Y(y) \tag{6}$$

Substituting the values of $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}$ in (1), we get

$$X''Y + XY'' = 0 \text{ or } \frac{X''}{X} = -\frac{Y''}{Y} \tag{7}$$

Now, the two sides can be equal only if both are equal to a constant, say k.

Then $$\frac{X''}{X} = k \Rightarrow X'' - kX = 0 \tag{8}$$

and $$-\frac{Y''}{Y} = k \Rightarrow Y'' + kY = 0 \tag{9}$$

Now, the boundary conditions (2) and (3) becomes

$$u(0, y) = 0 = X(0).\ Y(y), \text{ i.e., } X(0) = 0 \tag{10}$$

$$u(\pi, y) = 0 = X(\pi).\ Y(y), \text{ i.e., } X(\pi) = 0 \tag{11}$$

$$u(x, \infty) = 0 = X(x).\ Y(\infty), \text{ i.e., } Y(\infty) = 0 \tag{12}$$

If $k \geq 0$ then (1) has a trivial solution. Now, when $k < 0$, i.e., $k = -\lambda^2$ then (8) has the solution

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \tag{13}$$

Using BC (10) and (11),

$$X(0) = 0 \Rightarrow c_1 = 0$$

$$X(\pi) = 0 \Rightarrow c_2 \sin \lambda \pi = 0$$

Since $c_2 \neq 0$, $\sin \lambda\pi = 0$, i.e., $\sin \lambda\pi = \sin n\pi$

$$\lambda = n$$

$\therefore$ $$X(x) = c_2 \sin nx$$

Solution of (9) is

$$Y(y) = c_3 e^{\lambda y} + c_4 e^{-\lambda y}$$

Using $$Y(\infty) = 0 \Rightarrow c_3 = 0$$

$\therefore$ $$Y(y) = c_4 e^{-ny} \qquad [\because \quad \lambda = n]$$

Thus, $$u(x, y) = A \sin nx \cdot e^{-ny}$$

Hence, the most general solution satisfying the condition is of the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin nx \cdot e^{-ny} \tag{14}$$

Putting $y = 0$ in (14), we have

$$u(x, 0) = u_0 = \sum_{n=1}^{\infty} A_n \sin nx$$

Now, $$u_0 = \sum_{n=1}^{\infty} A_n \sin nx$$

$\Rightarrow$ $$A_n = \frac{2}{\pi}\int_0^{\pi} u(x,0) \cdot \sin nx dx = \frac{2}{\pi}\int_0^{\pi} u_0 \cdot \sin nx dx$$

$$= \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} \dfrac{4u_0}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Hence, $$u(x, y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nx \cdot e^{-ny}$$

Example 17 Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, which satisfies the conditions

$$u(0, y) = 0 = u(l, y),\ u(x, 0) = 0 \text{ and } u(x, a) = \sin\frac{n\pi x}{l}$$

Solution The Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

The boundary conditions are

$$u(0, y) = 0 = u(l, y) = u(x, 0) \tag{2}$$

Suppose (1) has the solution of the form

$$u(x, y) = X(x)\cdot Y(y)$$

Substituting the values of $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ in (1), we get

$$X''Y + XY'' = 0 \text{ or } \frac{X''}{X} = -\frac{Y''}{Y}$$

Now, the two sides can be equal only if both are equal to a constant, say K. Then

$$X'' - KX = 0 \tag{4}$$

$$Y'' + KY = 0 \tag{5}$$

The boundary condition (2) becomes

$$u(0, y) = 0 = X(0)\cdot Y(y), \quad \text{i.e., } X(0) = 0 \quad (\because\ Y(y) \neq 0)$$
$$u(l, y) = 0 = X(l)\cdot Y(y), \quad \text{i.e., } X(l) = 0 \quad (\because\ Y(y) \neq 0)$$
$$u(x, 0) = 0 = X(0)\cdot Y(y), \quad \text{i.e., } Y(0) = 0 \quad (\because\ X(x) \neq 0)$$

If $K \geq 0$, (4) has a trivial solution.

If $K < 0$, i.e., $K = -\lambda^2$, then equation (4) has the solution of the form

$$X(x) = C_1\cos\lambda x + C_2\sin\lambda x \tag{6}$$

Using $$X(0) = 0 \Rightarrow C_1 = 0 \text{ and } X(l) = 0 \Rightarrow C_2\sin\lambda l = 0$$

Since $$C_2 \neq 0,\ \sin\lambda l = 0 = \sin n\pi$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}$$

$\therefore \qquad X(x) = C_2 \sin\left(\frac{n\pi x}{l}\right)$

The solution of (5) is

$$Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

Using $\qquad Y(0) = 0 \Rightarrow C_3 + C_4 = 0$

$$C_4 = -C_3$$

$\therefore \qquad Y(y) = C_3 \left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]$

Hence, $\qquad u(x, y) = A \sin\left(\frac{n\pi x}{l}\right)\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right] \qquad$ [where $A = C_2 \cdot C_3$] $\qquad (7)$

Using $\qquad u(x, a) = \sin\left(\frac{n\pi x}{l}\right)$, (7) becomes

$$\sin\left(\frac{n\pi x}{l}\right) = A \sin \frac{n\pi x}{l} \cdot \left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right]$$

or $\qquad A = \dfrac{1}{e^{\frac{n\pi s}{l}} - e^{-\frac{n\pi a}{l}}} = \dfrac{1}{2 \cdot \sin h\left(\frac{n\pi a}{l}\right)}$

$\therefore \qquad u(x, y) = \dfrac{\sin\left(\frac{n\pi x}{l}\right)\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]}{2 \sin h\left(\frac{n\pi a}{l}\right)} = \sin\left(\frac{n\pi x}{l}\right)\dfrac{\sin h\left(\frac{n\pi y}{l}\right)}{\sin h\left(\frac{n\pi a}{l}\right)}$

Example 18 Find the steady-state temperature distribution in a thin sheet of metal plate which occupies the semi-infinite strip, $0 \le x \le L$ and $0 \le y < \infty$ when the edge $y = 0$ is kept at the temperature $u(x, 0) = kx(L - x)$, $0 < x < L$ while

1. the edges $x = 0$ and $x = L$ are kept at zero temperature
2. the edges $x = 0$ and $x = L$ are insulated

Assume that $u(x, \infty) = 0$.

Solution

Case 1: When edges are kept at zero temperature

The Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$

with the boundary conditions

$$u(0, y) = 0 = u(L, y) \qquad (2)$$

Suppose (1) has the solution of the form

$$u(x, y) = X(x) \cdot Y(y)$$

Substituting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in (1), we get

$$X''Y + XY'' = 0$$

Separating the variables

$$\frac{Y''}{Y} = -\frac{X''}{X} = K \text{ (say)}$$

Then,

$$X'' - KX = 0 \tag{3}$$

$$Y'' + KY = 0 \tag{4}$$

If $K \geq 0$, (3) has a trivial solution. If $K < 0$, i.e., $K = -\lambda^2$

The boundary conditions (2) is

$$u(0, y) = 0 = X(0) \cdot Y(y), \text{ i.e., } X(0) = 0 \qquad (\because \quad Y(y) \neq 0)$$

$$u(L, y) = 0 = X(L) \cdot Y(y), \text{ i.e., } X(L) = 0$$

Now, the solution of (3) with $X(0) = 0 = X(L)$, is

$$X_n(x) = \sin \frac{n\pi x}{L}, n = 1, 2, 3, \ldots$$

The solution of (4) with $Y(\infty) = 0$ is

$$Y(y) = \beta_n e^{-\lambda y} = \beta_n e^{-\frac{n\pi y}{L}}$$

Thus,

$$u(x, y) = \sum_{n=1}^{\infty} \beta_n e^{-\frac{n\pi y}{L}} \cdot \sin\left(\frac{n\pi x}{L}\right) \tag{5}$$

where $\beta_n = \frac{2}{L}\int_0^L kx\,(L - x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2K}{L}\left[x(L - x) \cdot \left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) - (L - 2x)\left(\frac{-L^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2K}{L} \cdot \frac{2L^3}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$= \frac{-8KL^2}{n^3\pi^3} \text{ when } n \text{ is odd}$$

$$u(x, t) = -\frac{8KL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n - 1)^3}\, e^{\frac{-(2n-1)\pi y}{L}} \cdot \sin\left(\frac{(2n - 1)\pi x}{L}\right)$$

Case 2: Both edges are insulated

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0, \quad \left(\frac{\partial u}{\partial x}\right)_{x=L} = 0$$

The boundary conditions are $X'(0) = 0 = X'(L)$.

The solution of (3) with $X'(0) = 0 = X'(L)$ is

$$X_n(x) = A\cos\left(\frac{n\pi x}{L}\right), n = 0, 1, 2, 3, \ldots$$

And the solution of (4) with $Y(\infty) = 0$ is

$$Y_n(y) = e^{\frac{-n\pi y}{L}}$$

Thus,
$$u(x, y) = \sum_{n=0}^{\infty} A_n e^{-\frac{n\pi y}{L}} \cdot \cos\left(\frac{n\pi x}{2}\right)$$

where
$$A_0 = \frac{1}{L}\int_0^L Kx(L-x)dx = \frac{K}{L}\left[\frac{Lx^2}{2} - \frac{x3}{3}\right]_0^L$$

$$= \frac{KL^2}{6} \text{ and}$$

$$A_n = \frac{2}{L}\int_0^L Kx(L-x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2K}{L}\left[x(L-x)\left(\frac{-L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) - (L-2x)\cdot\left(\frac{-L^2}{n^2\pi^2}\right)\cos\frac{n\pi x}{L} + (-2)\left(\frac{-L^3}{n^3\pi^3}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2L}{L}\cdot\frac{L^3}{n^2\pi^2}\left[(-1)^n + !\right]$$

$$= \frac{4KL^2}{n^2\pi^2} \text{ when } n \text{ is even.}$$

Hence, the required solution is

$$u(x, y) = \frac{KL^2}{6} + \frac{4KL^2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{4n^2}e^{-\frac{2ny}{L}}\cdot w\left(\frac{2n\pi x}{L}\right)\right]$$

Example 19 Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in a square of length π and $f(x) = \sin^2 x$, $0 < x < \pi$.

Solution Given

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The boundary conditions are

$$u(0, y) = 0,\ u(a, y) = 0,\ u(x, b) = 0 \text{ and } u(x, 0) = f(x) = \sin^2 x,\ 0 < x < \pi. \tag{2}$$

Suppose (1) has the solution of the form

$$u(x, y) = X(x) \cdot Y(y) \tag{3}$$

Substituting the values of $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ in (1), we get

$$\frac{X''}{X} = -\frac{Y''}{Y} = K = -\lambda^2 \qquad \text{(say)}$$

$\because$

$$X'' + \lambda^2 X = 0 \tag{4}$$

$$Y'' - \lambda^2 Y = 0 \tag{5}$$

Equation (2) then becomes

$$u(0, y) = 0 = X(0) \cdot Y(y), \text{ i.e., } X(0) = 0$$

$$u(a, y) = 0 = X(a) \cdot Y(y), \text{ i.e., } X(a) = 0 \tag{6}$$

$$u(x, b) = 0 = X(x) \cdot Y(b), \text{ i.e., } Y(b) = 0 \tag{7}$$

The solution of (4) with $X(0) = 0$, $X(a) = 0$, is

$$X_n(x) = \sin\frac{n\pi x}{a}, \qquad n = 1, 2, 3, \ldots.$$

The solution of (5) with $Y(b) = 0$, is

$$Y(y) = A_1 e^{\lambda y} + A_2 e^{-\lambda y}$$

$$0 = Y(b) = A_1 e^{\lambda b} + A_2 e^{-\lambda b}$$

or

$$A_1 = -\frac{A_2 e^{-\lambda b}}{e^{\lambda b}}$$

or

$$Y(y) = -\frac{A_2 e^{-\lambda b}}{e^{\lambda b}} \cdot e^{\lambda y} + A_2 e^{-\lambda y}$$

$$= \frac{A_2}{e^{\lambda b}}[e^{\lambda b} \cdot e^{-\lambda y} - e^{-\lambda b} \cdot e^{\lambda y}]$$

$$= A \sin h\{(b - y)\}$$

Thus,

$$Y_n(y) = A_n \cdot \sin h\left[\frac{n\pi}{a}(b - y)\right]$$

The required solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \cdot \sin h\left(\frac{n\pi(b - y)}{a}\right)$$

Using $u(x, 0) = \sin^2 x$, we have

$$A_n = \frac{2}{\pi \sin hn\pi} \int_0^{\pi} \sin^2 x \cdot xxdx$$

$$= \frac{2}{\pi \sin hn\pi} \left(\frac{-4}{n(x^2 - 4)} \right) = \frac{-8}{n\pi(x^2 - 4) \cdot \sin hn\pi}$$

Hence, $$u(x, y) = -\frac{8}{\pi} \sum \frac{\sin nx \cdot \sin h(n(\pi - y))}{\sin hn\pi \cdot n(n^2 - 4)}$$

Example 20 A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. The temperature along the short edge $y = 0$ is given by

$$u(x, 0) = 20x;\ 0 < x \le 5$$
$$= 20(10 - x);\ 5 < x < 10$$

While the two long edges $x = 0$ and $x = 10$ as well as the other short edges are kept at 0°C. Find the steady state temperature at any point (x, y) of the plate.

Solution In the steady state, the equation of the heat flow is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The boundary conditions are

$$\left.\begin{aligned} u(0, y) &= 0 \\ u(10, y) &= 0 \\ u(x, \infty) &= 0 \end{aligned}\right\} \tag{2}$$

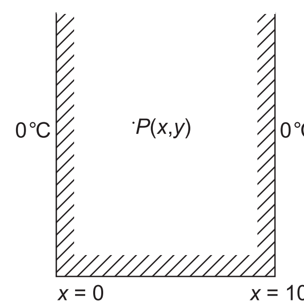


Fig. 15.7

Also, the initial condition is

$$u(x, 0) = 20\,x;\ 0 < x \le 5$$
$$= 20(10 - x);\ 5 < x < 10 \tag{3}$$

Suppose (1) has the solution of the form

$$u(x, y) = X(x) \cdot Y(y) \tag{4}$$

Substituting the values of $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ in (1), we get

$$\frac{X''}{X} = -\frac{Y''}{Y} = K \text{ (say)}$$

Then $$X'' - KX = 0 \text{ and } Y'' + KY = 0$$

It can be shown that for $K \ge 0$, only the trivial solution exists. So for non-trivial solutions, consider $K < 0$, i.e., $K = -\lambda^2$; then we get

$$X'' + \lambda^2 X = 0 \tag{5}$$

$$Y'' - \lambda^2 Y = 0 \tag{6}$$

The boundary conditions (2) reduce to

$$u(0, y) = 0 = X(0) \cdot Y(y), \text{ i.e., } X(0) = 0$$
$$u(10, y) = 0 = X(10) \cdot Y(y), \text{ i.e., } X(10) = 0$$
$$u(x, \infty) = 0 = X(x) \cdot Y(\infty), \text{ i.e. } Y(\infty) = 0$$

Now, the solution of (5), with the condition $X(0) = 0 = X(10)$ is

$$X(x) = A \cos \lambda x + B \sin \lambda x$$
$$X(0) = 0 \Rightarrow A = 0 \text{ and } X(10) = 0 \Rightarrow B \sin 10\lambda = 0$$

Since $B \neq 0$, $\sin 10\lambda = 0 = \sin n\pi$

or $$\lambda = \frac{n\pi}{10}, n = 1, 2, 3, \ldots$$

$\therefore$ $$X(x) = B \sin\left(\frac{n\pi x}{10}\right)$$

Now, the solution of (6), with $Y(\infty') = 0$ is

$$Y(y) = Ce^{\lambda y} + De^{-\lambda y}$$
$$Y(\infty) = \Rightarrow C = 0$$

$\therefore$ $$Y(y) = De^{-\lambda y}$$
$$= De^{-\frac{n\pi y}{10}}$$

Thus, $$u(x, y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right) \cdot e^{-\frac{n\pi y}{10}} \text{ (where } E_n = \text{BD)} \tag{7}$$

Using $$u(x, 0) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right)$$

Using (3), this requires the expansion of u in Fourier series, and we get

$$E_n = \frac{2}{10}\int_0^5 20\, x \sin \frac{n\pi x}{10}\, dx + \frac{2}{10}\int_5^{10} 20\,(10 - x) \sin \frac{n\pi x}{10}\, dx$$

$$= 4\int_0^5 x \sin \frac{n\pi x}{10}\, dx + 4\int_5^{10} (10 - x) \sin \frac{n\pi x}{10}\, dx$$

$$= 4\left[x \cdot \left(\frac{10}{n\pi}\right)\left(-\cos \frac{n\pi x}{10}\right) - \left(\frac{10}{n\pi}\right)^2 \cdot \left(-\sin \frac{n\pi x}{10}\right)\right]_0^5$$

$$+ 4\left[10 \cdot \frac{(10 - x)}{n\pi}\left(-\cos \frac{n\pi x}{10}\right) - (-1) \cdot \left(\frac{10}{n\pi}\right)^2 \cdot \left(-\sin \frac{n\pi x}{10}\right)\right]_5^{10}$$

$$= 4\left[-\frac{50}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2} \cdot \sin\left(\frac{n\pi}{2}\right)\right] + 4\left[\frac{100}{b^2\pi^2} \sin(n\pi)\right.$$

$$\left. + \frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= 4\left[\frac{200}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{100}{n^2\pi^2}\cdot\sin(n\pi)\right]$$

$$= \frac{800}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}\right) - 0$$

$$= \frac{800}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} \dfrac{800}{n^2\pi^2}(-1)^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

or
$$E_n = \frac{(-1)^{n+1}\cdot 800}{(2n-1)^2\cdot\pi^2}$$

Thus, the required solution is

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\times\sin\frac{(2n-1)\,\pi\,x}{100} : e^{\frac{(2n-1)\pi y}{10}}$$

EXERCISE 15.5

1. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ within the rectangle $0 \le x \le a$, $0 \le y \le b$ given that $u(0, y) = u(a, y) = u(x, b) = 0$ and $u(x, 0) = x(a - x)$.
2. Find the steady-state temperature distribution in a thin rectangular metal plate $0 < x < a$, $0 < y < b$ with its two faces insulated with the following boundary conditions prescribed on the four edges.

   $u(0, y) = 0 = u(x, 0) = u(x, b)$ and $u(a, y) = g(y)$, $0 < y < b$
3. Solve the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ with the boundary conditions $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$, $u(a, y) = f(y) = ky(b - y)$, $0 < y < b$.
4. A thin rectangular plate whose surface is impervious to heat flow, has at $t = 0$ an arbitrary distribution of temperature $f(x, y)$. If four edges $x = 0$, $x = a$, $y = 0$, and $y = b$ are kept at zero temperature, determine the temperature at a point of the plate as t increases.
5. The edges of a thin plate of π cm side are kept at 0°C and the faces are perfectly insulated. The initial temperature is $u(x, y, 0) = f(x, y) = xy(\pi - x)(\pi - y)$. Determine the temperature in the plate at time t.

6. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin\left(\dfrac{\pi x}{8}\right)$, $0 < x < 8$, while the two long edges $x = 0$ and $x = 8$, as well as the other short edge are kept at 0°C, find the steady state temperature $u(x, y)$.

Answers

(1) $$u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin \frac{(2n+1)\pi x}{a} \times \frac{\sin h \dfrac{(2n+1)\pi x\,(b-y)}{a}}{\sin h \cdot \dfrac{(2n+1)\,\pi b}{a}}$$

(2) $$u(x, y) = \sum_{n=1}^{\infty} A_n . \sin h\left(\frac{n\pi x}{b}\right) \cdot \sin\left(\frac{n\pi x}{b}\right)$$

where $$A_n = \frac{2}{b} \operatorname{cosec} h\left(\frac{n\pi a}{b}\right) \int_0^b g(y) \cdot \sin\left(\frac{n\pi y}{b}\right) dy$$

(3) $$u(x, y) = \frac{-8kb^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3 \sin h \dfrac{(2n-1)\,\pi a}{b}} \cdot \sin h \cdot \left(\frac{(2n-1)\pi x}{b}\right) \cdot \left(\frac{(2n-1)\pi y}{b}\right)$$

(4) $$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right) e^{-c^2 \lambda_{mn}^2 t}$$

where, $$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \cdot \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right) dxdy$$

(5) $$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{64}{m^3 n^3 \pi^2} \sin mx \cdot \sin ny$$

(6) $$u(x, y) = 100 \sin\left(\frac{\pi x}{8}\right) \cdot e^{-\pi y/8}$$

15.11 TRANSMISSION-LINE EQUATIONS

15.11.1 Introduction

The purpose of a transmission-line network is to transfer electric energy from generating units at various locations to the distribution system which ultimately supplies the load.

All transmission lines in a power system exhibit the electrical properties of the resistance R, inductance L capacitance to ground C, and conductance to ground G of the cable per unit length. Consider a long cable or telephone wire that is imperfectly insulated so that leaks occur along the entire length l km of the cable.

The sources are at $x = 0$ and the terminal T at $x = l$. Let P be any point x km from the source S. The instantaneous current and voltage at the point P are $i(x, t)$ and $v(x, t)$, where t is the time.

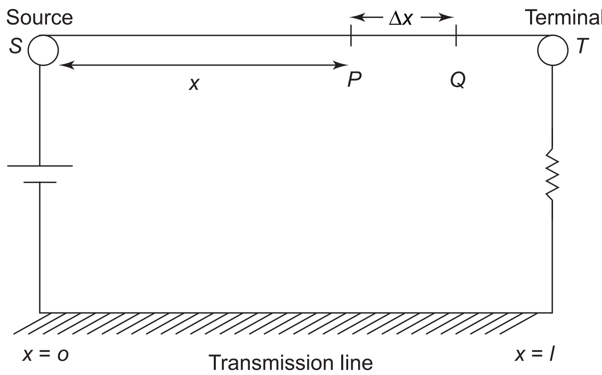


Fig. 15.8

15.11.2 Derivation of Transmission Line Equations

Let Q be a distance Δx from p. Applying Kirchhoff's voltage law to a small portion PQ of the cable,

The potential drop across the segment PQ = potential drop due to resistance + potential drop due to inductance

i.e., $$-\Delta v = Ri\,\Delta x + L\Delta x\frac{\partial i}{\partial t}$$

Dividing by Δx and taking limit as $\Delta x \to 0$, we have

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \tag{69}$$

which is known as the ***first transmission-line equation***.

Similarly, applying Kirchhoff's current law difference of the current in crossing the segment PQ = loss in current due to capacitance + leakage

i.e., $$-\Delta i = G\,v\Delta x + C\,\Delta x\frac{\partial v}{\partial t}$$

Dividing by Δx and taking limit $\Delta x \to 0$, we have

$$-\frac{\partial i}{\partial x} = Gv + C\frac{\partial v}{\partial t} \tag{70}$$

which is known as the ***second transmission-line equation***.

(i) Telephone Equations

Differentiating (69), and (70) partially w.r.t. x and t respectively, we have

$$\frac{\partial^2 v}{\partial x^2} = R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x\,\partial t} \tag{71}$$

and $$-\frac{\partial^2 i}{\partial x\,\partial t} = G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2} \tag{72}$$

Eliminating i from (70), (71), and (72), we get

$$-\frac{\partial^2 v}{\partial x^2} = R\left[-Gv - C\frac{\partial v}{\partial t}\right] + \left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]$$

or
$$+\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (RC + LG)\frac{\partial v}{\partial t} + RGv \tag{73}$$

Now, differentiating(69), and (70) w.r.t. t and x respectively, we have

$$-\frac{\partial^2 v}{\partial x\,\partial t} = R\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} \tag{74}$$

And
$$-\frac{\partial^2 i}{\partial x^2} = G\frac{\partial v}{\partial x} + C\frac{\partial^2 v}{\partial x\,\partial t} \tag{75}$$

Eliminating v from (69), (74), and (75), we get

$$-\frac{\partial^2 i}{\partial x^2} = G\left[-RL - L\frac{\partial i}{dt}\right] + V\left[-R\frac{\partial i}{dt} - L\frac{\partial^2 i}{dt^2}\right]$$

or
$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \tag{76}$$

The equations (73) and (76) are known as the ***telephone equations***.

(ii) Telegraph Equations

If $L = G = 0$, the equations (73) and (76) reduce to

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \tag{77}$$

$$\frac{\partial^2 i}{\partial t^2} = RC\frac{\partial i}{\partial t} \tag{78}$$

known as *submarine cable equations* or ***telegraph equations***. Equations (77) and (78) are similar to the one-dimensional heat equations.

(iii) Radio Equations

If $R = G = 0$, the equations (73) and (76) reduce to

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \tag{79}$$

and
$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \tag{80}$$

which are known as the ***radio equations***. Equations (79) and (80) are similar to the one-dimensional wave equations.

(v) Transmission Lines

If R and G are negligible, the transmission lines reduce to

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}$$

and
$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}$$

Example 21 A 1000 km long transmission line is initially under steady-state conditions with a potential of 1300 volts at the sending end, $x = 0$, and 1200 volts at the receiving end, $x = 1000$. The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential $E(x, t)$.

Solution Since L and G are negligible, we use the telegraph equation:

$$\frac{\partial^2 E}{\partial x^2} = RC\frac{\partial E}{\partial t} \tag{1}$$

Here, E_s = initial steady voltage satisfying $\dfrac{\partial^2 E}{\partial x^2} = 0$

$$= 1300 - \left(\frac{1300 - 1200}{1000}\right)x$$

$$= 1300 - 0.1x = E(x, 0) \tag{2}$$

E'_s = steady voltage when steady conditions are ultimately reached

$$= 1300 - \left(\frac{1300 - 0}{1000}\right)x$$

$$= 1300 - 1.3x$$

$\therefore$ $E(x, t) = E'_s - E_r(x, t)$, where $E_r(x, t)$ is the transient part

$$= (1300 - 1.3x) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cdot e^{-\frac{n^2\pi^2 t}{l^2RC}}, \text{ where } l = 1000 \text{ km} \tag{3}$$

Putting $t = 0$ in (3), we get

$$E(x, 0) = (1300 - 1.3x) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$(1300 - 0.1\,x) = (1300 - 1.3x) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

or
$$1.2x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where
$$b_n = \frac{2}{l}\int 1.2x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2.4}{l}\int_0^l x\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2.4}{l}\left[x\cdot\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\left(-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2.4}{l}\frac{l}{n\pi}\times[-l\cos n\pi] \qquad [\because l = 1000, \cos n\pi = (-1)^n]$$

$$= \frac{2400}{\pi}\frac{(-1)^{n+1}}{n}$$

Hence, $$E(x, t) = (1300 - 1.3x) + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{l}\right)\cdot e^{-\frac{n^2\pi^2 t}{l^2 RC}}$$

Example 22 Neglecting R and G, find the emf $v(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = E_1 \sin\frac{\pi x}{l} + E_5\cdot\sin\frac{5\pi x}{l}$.

Solution Since R and G are negligible, so we will use the radio equation

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \tag{1}$$

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, v(l, t) = 0 \tag{2}$$

Also, the initial conditions are

$$\left.\begin{aligned} &i(x, 0) = i_0 \text{ and} \\ &v(x, 0) = E_1 \sin\frac{\pi x}{l} + E_5 \sin\frac{5\pi x}{l}\end{aligned}\right\} \tag{3}$$

$\therefore$ $$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \text{ gives}$$

$$\left(\frac{\partial v}{\partial t}\right)_{t=0} = 0 \tag{4}$$

Let $v(x, t) = X(x)\cdot T(t)$ be the solution of (1).

Substituting the values of $\frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial t^2}$ in (1), we have

$$TX'' = 2CX\,T''$$

or $$\frac{X''}{X} = LC\frac{T''}{T} = -\lambda^2 \text{ (say)}$$

$\therefore$ $$X'' + \lambda^2 X = 0 \tag{5}$$

and $$T'' + \left(\frac{\lambda^2}{LC}\right)T = 0 \tag{6}$$

Solutions of (5) and (6) are given by

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$T(t) = C_3 \cos\left(\frac{\lambda}{\sqrt{LC}}\right)t + C_4 \sin\left(\frac{\lambda}{\sqrt{\lambda C}}\right)t$$

Thus, $$v(x, t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)\cdot\left(C_3 \cos \frac{\lambda t}{\sqrt{LC}} + C_4 \sin \frac{\lambda t}{\sqrt{LC}}\right) \tag{7}$$

Using (2), we get

$$C_1 = 0 \text{ and } \lambda = \frac{n\pi}{l}, n = 1, 2, 3 \ldots$$

$$\therefore \quad v(x, t) = \sin \frac{n\pi x}{l}\left[a_n \cdot \cos \frac{n\pi t}{l\sqrt{LC}} + b_n \sin \frac{n\pi t}{l\sqrt{LC}}\right] \tag{8}$$

Using the initial condition (4), we get

$$b_n = 0$$

$\therefore$ (8) becomes, $v(x, t) = a_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi t}{l\sqrt{LC}}$

Thus, the general solution of (1) is

$$v(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos \frac{n\pi t}{l\sqrt{LC}} \tag{9}$$

Using $v(x, 0) = E_1 \sin \dfrac{\pi x}{l} + E_5 \sin \dfrac{5\pi t}{l}$

$$\therefore \quad E_1 \sin \frac{\pi x}{l} + E_5 \sin \frac{5\pi t}{l} = \sum_{n=1}^{\infty} a_x \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$\therefore \quad a_1 = E_1$ and $a_5 = E_5$, while all other a's are zero

Hence, $$v(x_1, t) = E_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + E_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

EXERCISE 15.6

1. Assuming R and G are negligible, find the voltage $v(x, t)$ and current $i(x, t)$ in a transmission line of length l, t seconds after the ends are suddenly grounded. The initial conditions are $v(x, 0) = v_0 \sin\left(\dfrac{\pi x}{l}\right)$ and $i(x, 0) = i_0$.

2. Solve $\dfrac{\partial^2 v}{\partial x^2} = LC \dfrac{\partial^2 v}{\partial t^2}$, assuming that the initial voltage $v(x, 0) = v_0 \sin\left(\dfrac{\pi x}{l}\right)$. $v_t(x, 0) = 0$ and $v = 0$ at the ends, $x = 0$ and $x = l$ for all t.

3. Obtain the solution of the radio equation $\dfrac{\partial^2 v}{\partial x^2} = LC\,\dfrac{\partial^2 v}{\partial t^2}$. When a periodic e.m.f. $E_0 \cos \lambda t$ is applied at the end $x = 0$ of the line.
4. In a telephone wire of length l, a steady voltage distribution of 20 volts at the source end and 12 volts at the terminal end is maintained. At time $t = 0$, the terminal end is grounded. Determine the voltage and current assuming that $L = 0 = G$.

Answers

1. $v(x, t) = v_0 \sin\left(\dfrac{\pi x}{l}\right)\cdot\cos\left(\dfrac{\pi t}{l\sqrt{LC}}\right)$; $i(x, t) = i_0 - v_0\sqrt{\dfrac{C}{L}}\cdot\cos\left(\dfrac{\pi x}{l}\right)\cdot\sin\left(\dfrac{\pi t}{l\sqrt{LC}}\right)$
2. $v(x, t) = v_0 \cos\left(\dfrac{\pi t}{l\sqrt{LC}}\right).\sin\left(\dfrac{\pi x}{l}\right)$
3. $v(x, t) = E_0 \cos(t - x\sqrt{LC})$
4. $v(x, t) = \dfrac{20}{l}(l - x) + \sum\limits_{n=1}^{\infty} A_n \cdot e^{-\left[\frac{n^2\pi^2}{l^2 RC}\right]t}$; $i(x, t) = \dfrac{20}{Rl} + \dfrac{24}{Rl}\sum\limits_{n=1}^{\infty}(-1)^n\cdot\cos\left(\dfrac{n\pi x}{l}\right)\cdot e^{-\left[\frac{n^2\pi^2}{l^2 RC}\right]t}$

SUMMARY

1. One-Dimensional Wave Equation

Consider an elastic string, placed along the X-axis, stretched to a length l between two fixed points A at $x = 0$ and B at $x = l$.

Let $u(x, t)$ denote the deflection (displacement from equilibrium position). Then the vibrations of the string is governed by the one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where c is the phase velocity of the string with boundary conditions

$$u\,(0, t) = 0,\ u\,(l, t) = 0 \tag{2}$$

The form of motion of the string will depend on the initial displacement or deflection at time $t = 0$

$$u\,(x, 0) = f\,(x) \tag{3}$$

and the initial velocity

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = u_t(x, 0) = g(x) \tag{4}$$

The initial boundary-value problem can be solved by the method of separation of variables.

Under the above conditions, the equation (1) has the solution of the form

$$u\,(x, t) = \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\lambda d}{l} + B_n \sin\frac{n\lambda d}{l}\right]\sin\frac{n\pi x}{l} \tag{5}$$

The unknown constants A_n and B_n are determined using initial conditions.

Note I. When the initial displacement $u\ (x, 0) = f\ (x)$ is given.
Putting $t = 0$ in (5), we get

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

Thus, A_n are the Fourier coefficients in the half-range Fourier sine-series expansion of $f(x)$ in (0, l).

Hence, $$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx,\ n = 1, 2, 3, \ldots \tag{6}$$

Note II. When the initial velocity $u_t\ (x, 0) = g\ (x)$ is given then differentiating (5) w.r.t. 't'

$$u_t(n, 0) = \sum_{n=1}^{\infty}\left[-\frac{A_n l}{n\lambda c}\sin\frac{n\pi ct}{l} + B_n\frac{l}{n\pi c}\cos\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l}.$$

Put $t = 0$

$$u_t\ (x, 0) = g\ (x) = \sum_{n=1}^{\infty} B_n\frac{l}{n\pi c}\sin\frac{n\pi x}{l}$$

or $$B_n = \frac{2}{n\pi c}\int_0^l g(x)\cdot\sin\frac{n\pi x}{l}\,dx \tag{7}$$

Hence, the general solution of the one-dimensional wave equation (1) is given by

$$u\ (x, t) = \sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l},\ \text{where}$$

A_n and B_n are given by (6) and (7).

2. Solution of One-dimensional Heat Equation

The one-dimensional heat-flow equation is given by

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \tag{8}$$

Suppose $u(x, t)$ is the temperature. Since the ends $x = 0$ and $x = l$ are kept at zero temperature. The boundary conditions are given by $u(0, t)$, $u(l, t) = 0$ for all t, and the initial condition is given by $u(x, 0) = f\ (x)$.
Therefore, the solution of the equation (8) is of the form

$$u\ (x, t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n\sin\left(\frac{n\pi x}{l}\right)\cdot e\frac{-n^2\pi^2c^2t}{l^2} \tag{9}$$

Putting $t = 0$ in (9) and using $u\ (x, 0) = f\ (x)$, we get

$$f(x) = \sum_{n=1}^{\infty} E_n\cdot\sin\left(\frac{n\pi x}{l}\right)$$

which is a Fourier sine series. So the constant equations are given by

$$E_n = \frac{2}{l}\int_0^l f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx,\ n = 1, 2, 3, \ldots \tag{10}$$

Hence, (9) is the required solution whose equation is given by (10).

3. Solution of Two-dimensional Wave Equation

Consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{11}$$

subject to the boundary conditions

$$u(0, y, t) = 0,\ u(a, y, t) = 0,\ u(x, 0, t) = 0,\ u(x, b, t) = 0$$

and initial conditions $u(x, y, 0)\, f(x, y)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y)$

Consider that the more general solution of equation (12) is given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\cos\lambda_{mn} t + B_{mn}\cdot\sin\lambda_{mn} t\right]\cdot\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{12}$$

Differentiating (12) partially w.r.t. 't', we get

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-A_{mn}\cdot\lambda_{mn}\sin\lambda_{mn} t + B_{mn}\cdot\lambda_{mn}\cos\lambda_{mn} t\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{13}$$

Putting $t = 0$ in (12) and (13) using IC $u(x, y, 0) = f(x, y)$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y)$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{14}$$

$$g(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (B_{mn}\lambda_{mn})\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{15}$$

which are double Fourier sine series. We get

$$A_{mn} = \frac{4}{ab}\int_{n=0}^{a}\int_{y=0}^{b} f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \tag{16}$$

and

$$B_{mn} = \frac{4}{ab\lambda_{mn}}\int_{n=0}^{a}\int_{y=0}^{b} g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \tag{17}$$

Thus, the desired deflection $u\ (x, y, t)$ of the given membrane is given by (12) wherein A_{mn} and B_{mn} are given by (16) and (17) respectively.

4. Solution of Two-Dimensional Heat Equation by the Method of Separation of Variables

The diffusion of heat in a rectangular metal plate of uniform, isotropic material, with both faces insulated and with the four edges kept at zero temperature is given by the transient temperature $u\ (x, y, t)$ which satisfies the following initial boundary-value problem consisting of

$$\text{PDE}\quad \frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{18}$$

in the region of the plate $0 \le x \le a,\ 0 \le y \le b,\ t > 0$.

with boundary conditions $u(x, 0, t) = 0 = u(x, b, t)$,

$0 < x < a, t > 0$ (19)

$u(0, y, t) = 0 = u(a, y, t), 0 < y < b, t > 0$ (20)

and initial conditions $u(x, y, 0) = f(x, y), 0 < x < a, 0 < y < b$ (21)

We can apply the method of separation of variables on (18), because PDE (18) and the BC (19) and (20) are homogeneous

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\cdot\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)dx\,dy \tag{22}$$

with $m = 1, 2, 3, \ldots$and $n = 1, 2, 3, \ldots$

Now consider the general solution is given by

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)e^{-\lambda_{mn}^2 c^2 t} \tag{23}$$

Using the initial condition (21) in (23), we have

$$f(x, y) = y(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)$$

which is a double Fourier sine series. Here, A_{mn} are determined by (22). Thus the complete solution to the BVP (18), (19), (20), and (21), is given by (23) with coefficient A_{mn} determined by (22).

5. Solution of Two-Dimensional Laplace's Equation by the Method of Separation of Variables

The two-dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{24}$$

in a rectangle is the xy-plane, $0 < x < a$ and $0 < y < b$ satisfying the following boundary conditions:

$u(x, 0) = 0$ (25)

$u(x, b) = 0$ (26)

$u(x, y) = 0$ (27)

$u(a, y) = f(y)$ (28)

i.e., u is zero on three lines OA, OB, BC and is prescribed by the given function $f(y)$ on the fourth side AC of the rectangle $OACB$.

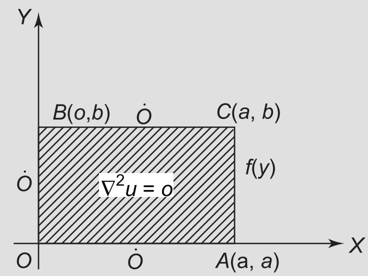


Fig. 15.9

To solve (24) by the separation of variables,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin h\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{29}$$

Using (28), $f(y) = u(a, y)$

$$\therefore \qquad f(y) = \sum\left[A_n\cdot\sin h\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\right]$$

Thus, A_n's are Fourier coefficients of the Fourier half-range sine series of $f(y)$ in the interval $(0, b)$ and given by

$$A_n \cdot \sin h\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\cdot \sin\left(\frac{x\pi y}{b}\right)dy$$

or $$A_n = \frac{2}{b\cdot \sin h\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\cdot \sin\left(\frac{n\pi y}{b}\right)dy \tag{30}$$

$n = 1, 2, 3, \ldots$

Hence, the harmonic function $u(x, \text{y})$ satisfying the Laplace's equation (24) and four boundary conditions (25), (26), (27) and (28) is given by (29) with A_n's determined by (30).

OBJECTIVE-TYPE QUESTIONS

1. The one-dimensional heat flow equation is

(a) $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ (b) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial u}{\partial x}$

(c) $\frac{\partial u}{\partial t} = 0$ (d) $\frac{\partial^2 u}{\partial x^2} = c^2$

2. Two-dimensional heat equation is

(a) $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\right)$

(b) $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

(c) $\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial u}{\partial t}$

(d) $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

3. Two-dimensional wave equation is

(a) $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

(b) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

(c) $\frac{\partial u'}{\partial t} + \frac{\partial u'}{\partial x} = 0$

(d) $\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} = 0$

4. The one-dimensional wave equation is

(a) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x}$ (b) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

(c) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ (d) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

5. In two dimensional heat flow the temperature along the normal to xy plane is

(a) 1 (b) c^2
(c) 0 (d) ∞

6. The partial differential equation of two-dimensional heat flow in steady state conditions is

(a) $\frac{\partial u}{\partial t} = 0$ (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(c) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ (d) none of above

7. The boundary conditions for one dimensional heat flow when the ends $x = 0$ and $x = l$ are insulated are

(a) $u_x(0, t) = 0, u_x(l, t) = 0 \ \forall\ t$
(b) $u_t(l, t) = 0, u_x(0, t) = 0 \ \forall\ t$
(c) $u(0, t) = 0 = u(x, 0) \ \forall\ t$
(d) none of the above

ANSWERS

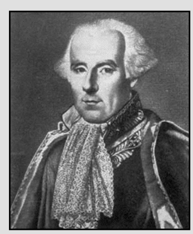
1. (a)	2. (b)	3.(a)	4. (b)	5. (c)	6. (b)	7. (a)

16 Laplace Transform

16.1 INTRODUCTION

Laplace transform is a powerful tool in engineering and science. This method is applicable for solving differential equations and corresponding initial-value and boundary-value problems. When this method is applied to a given initial-value or boundary-value problem consisting of a single or a system of linear ordinary differential equations, it converts it into a single or a system of linear algebraic equations called the *subsidiary equation*. The solution of the subsidiary equation is expressed for the Laplace transform of the dependent variable. Taking the inverse Laplace transform method, we obtain the solution of the original problem. Partial Differential Equations (PDE) can be solved by the Laplace transform method in which two or more than two independent variables are involved. The Laplace transform method is applied with respect to one of the variables, say t. The resulting Ordinary Differential Equation (ODE) in terms of the second variable solved by the usual method of solving the ODE, and taking the inverse Laplace transform method of this solution, we obtain the solution of the PDE. The other important applications are in mathematical modeling of the physical problems and solutions of integral equations.

Pierre-Simon Laplace (23 March 1749–5 March 1827) was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics. Laplace formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in mathematics, is also named after him. He restated and developed the nebular hypothesis of the origin of the solar system and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.



16.2 DEFINITION OF LAPLACE TRANSFORM

Let $F(t)$ be a real-valued function defined over the interval $(-\infty, \infty)$ such that $F(t) = 0$, $t < 0$. The Laplace transform of $F(t)$, denoted by $L\{F(t)\}$, is defined as

$$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt \tag{1}$$

We also write $$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} \cdot F(t)\, dt$$

Here, L is called the *Laplace transformation operator*. The parameter s is a real or complex number. In general, the parameters are taken to be real positive numbers.

The Laplace transform is said to exist if the integral (1) is convergent for some value of s. The operation of multiplying $F(t)$ by e^{-st} and integrating from 0 to ∞ is called *Laplace transformation*.

Notation

We follow two types of notations:

(i) Functions are denoted by capital letters, $F(t)$, $G(t)$, $H(t)$,… and their Laplace transforms are denoted by corresponding lowercase letters $f(s)$, $g(s)$, $h(s)$, … or by $f(p)$, $g(p)$, $h(p)$,…

(ii) Functions are denoted by lowercase letters $f(t)$, $g(t)$, $h(t)$,… and their Laplace transforms are denoted by $\overline{f}(s), \overline{g}(s), \overline{h}(s), \ldots$, respectively or $\overline{f}(p), \overline{g}(p), \overline{h}(p), \ldots$

Existence of Laplace Transform

Theorem 1

If $F(t)$ is a piecewise continuous function which is defined on every finite interval in the range $t \geq 0$ and satisfies

$|F(t)| \leq Me^{\alpha t}$; $\forall t \geq 0$, and for some constants α and M then the Laplace transform of $F(t)$ exists for all $s > \alpha$.

Proof We know that

$$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt$$

$$= \int_0^{t_0} e^{-st} F(t)\, dt + \int_{t_0}^\infty e^{-st} F(t)\, dt \qquad (2)$$

The first integral $\int_0^{t_0} e^{-st} F(t)\, dt$ exists, since $F(t)$ is piecewise continuous on every finite interval $0 \leq t \leq t_0$.

Now, $\left| \int_{t_0}^\infty e^{-st} F(t)\, dt \right| \leq \int_{t_0}^\infty \left| e^{-st} F(t)\, dt \right|$

$$\leq \int_{t_0}^\infty e^{-st} M e^{\alpha t}\, dt \qquad [\because \ |F(t)| \leq Me^{\alpha t}]$$

$$= \int_{t_0}^\infty e^{-(s-\alpha)t} \cdot M dt$$

$$= \frac{Me^{-(s-\alpha)t_0}}{(s-\alpha)}; \qquad s > \alpha$$

$$\left| \int_{t_0}^\infty e^{-st} F(t)\, dt \right| \leq \frac{Me^{-(s-\alpha)t_0}}{(s-\alpha)}; \qquad s > \alpha$$

Thus, the Laplace transform exists for $s > \alpha$.

Note

1. It should be noted that the above theorem gives only the sufficient conditions for the existence of the Laplace transform. That is, a function may have a Laplace transform even it violates the existence condition.

   Example: Let $F(t) = t^{-\frac{1}{2}}$. $F(t)$ is not continuous in [0, T], because it has limit ∞ as $t \to 0$.

   $$\therefore \qquad \int_0^T e^{-st}\, t^{-1/2}\, dt \text{ exists } \forall\, T > 0$$

   $$\text{Thus,} \quad L\left\{t^{-\frac{1}{2}}\right\} = \int_0^\infty e^{-st} t^{-\frac{1}{2}} dt$$

   $$= \frac{2}{\sqrt{s}} \int_0^\infty e^{-z^2}\, dz \qquad \left[\text{Put } \sqrt{st} = z\right]$$

   $$= \sqrt{\frac{\pi}{s}}; \quad s > 0$$

   Thus $L\{t^{-1/2}\}$ exists for $s > 0$ even if $t^{-1/2}$ is not continuous for $t \geq 0$.

(2) If the function $F(t)$ satisfies the conditions of the existence theorem and $L[F(t)] = f(s)$, then (i) $\lim_{s\to\infty} f(s) = 0$, and (ii) $\lim_{s\to\infty} (s \cdot f(s))$ is bounded.

   These two conditions indicate that not all functions of s are Laplace transforms of some function $F(t)$.

16.3 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

1. $L\{1\} = \dfrac{1}{s}$; $s > 0$

Proof Here, $F(t) = 1$

$$L\,\{F(t)\} = \int_0^\infty e^{-st}\, F(t)\, dt \tag{3}$$

[Putting the value of $F(t)$ in Eq. (3)]

$$\therefore \qquad L[1] = \int_0^\infty e^{-st} \cdot 1 dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \left[-\frac{e^{-s0}}{s} + \frac{e^{0s}}{s}\right] = \left[0 + \frac{1}{s}\right]$$

Hence, proof of $L\{1\} = \dfrac{1}{s}$, if $s > 0$

2. $L\{t^n\} = \dfrac{\overline{|(n+1)}}{s^{n+1}}$, if $s > 0$ and $n > -1$

Proof Here, $F(t) = t^n$

$$L\,\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt \tag{4}$$

$$\therefore \qquad L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s} \qquad \text{[Put } st = x\text{]}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^{n+1-1}\, dx$$

$$= \frac{1}{s^{n+1}} \cdot \overline{|n+1} \qquad \left[\because \int_0^\infty e^{-x} \cdot x^{n-1}\, dx = \overline{|n}\right]$$

$$L\{t^n\} = \frac{\overline{|n+1}}{s^{n+1}} \text{ if } s > 0 \text{ and } n + 1\ 0 \text{ or } n > -1$$

In a particular case, if n is a +ve integer then

$$\overline{|(n+1)} = n\,!, \text{ and, hence,}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. $\boldsymbol{L\{e^{at}\} = \dfrac{1}{s-a}; \quad s > a}$

Proof Here, $F(t) = e^{at}$

$$L\{F(t)\} = \int_0^\infty e^{-st}\, F(t)\, dt = \int_0^\infty e^{-st} e^{at} \cdot dt \tag{5}$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$L\{e^{at}\} = \frac{1}{s-a}; \text{ if } s > a$$

4. $\boldsymbol{L\{e^{-at}\} = \dfrac{1}{s+a}; s > -a}$

Proof Here, $F(t) = e^{-at}$

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^\infty e^{-st} \cdot e^{-at}\, dt \tag{6}$$

$$= \int_0^\infty e^{-(s+a)t}\, dt = \left[\frac{1}{-(s+a)}\right]_0^\infty$$

$$L\{e^{-at}\} = \frac{1}{s+a};\ \text{if } s > -a$$

5. $\boldsymbol{L(\cos at) = \dfrac{s}{s^2 + a^2},\ L(\sin at) = \dfrac{a}{s + a^2}}$

Proof

$$L\{e^{iat}\} = \int_0^\infty e^{-st} \cdot e^{iat}\, dt \tag{7}$$

$$= \int_0^\infty e^{-(s-ia)t}\, d\tilde{t}$$

$$= \left[\frac{e^{-(s-ia)}}{-(s-ia)}\right]_0^\infty = \frac{1}{s - ia}$$

$$= \frac{s+ia}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Comparing real and imaginary part of both sides, we get the required result.

$$L\{\cos at\} = \frac{s}{s^2+a^2} \text{ and } L\{\sin at\} = \frac{a}{s^2+a^2}$$

6. $\boldsymbol{L\{\sin h\, at\} = \dfrac{a}{s^2 - a^2}}$

Proof Here, $F(t) = \sin h\, at$.

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^\infty e^{-st} \cdot \sin h\, at\, dt \tag{8}$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st} \cdot e^{at}\, dt - \frac{1}{2}\int_0^\infty e^{-st} \cdot e^{-at}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\, dt - \frac{1}{2}\int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{s+a}\right] = \frac{1}{2}\,\frac{2a}{s^2 - a^2}$$

$$L\{\sin h\, at\} = \frac{a}{s^2 - a^2}$$

7. $\boldsymbol{L\{\cos h\,at\} = \dfrac{s}{s^2 - a^2}}$

Proof $L\{\cos h\ at\} = \int_0^\infty e^{-st} \cdot \cos h\,at\,dt$ (9)

$$= \int_0^\infty e^{-st} \cdot \frac{e^{at} + e^{-at}}{2} dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-st} \cdot e^{at}\,dt + \int_0^\infty e^{-st} \cdot e^{-at}\,dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt + \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2} \cdot \frac{2s}{s^2 - a^2}$$

$$L\{\cos h\ at\} = \frac{s}{s^2 - a^2}$$

16.4 LINEARITY PROPERTY OF LAPLACE TRANSFORMS

Suppose $f(s)$ and $g(s)$ are Laplace transforms of $F(t)$ and $G(t)$ respectively.

Then

$$L\{aF(t) + bG(t)\} = aL\{F(t)\} + bL\{G(t)\}$$

where a and b are any constants.

Proof Let $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$

And $L\{G(t)\} = g(s) = \int_0^\infty e^{-st} G(t)\,dt$

Also, let a and b be arbitrary constants.

$\therefore$ $L\{aF(t) + bG(t)\} = \int_0^\infty e^{-st} \{a\,F(t) + b\,G(t)\} dt$ (10)

$$= a\int_0^\infty e^{-st} F(t)\,dt + b\int_0^\infty e^{-st} G(t)\,dt$$

$$= aL\{F(t)\} + b\,L\{G(t)\}$$

Hence, proved.

Note Generalizing this result, we obtain

$$L\left\{\sum_{r=1}^{n} C_r\,F_r(t)\right\} = \sum_{r=1}^{n} C_r\,L\{F_r(t)\}$$

Theorem 2: First Shifting (or First Translation) Theorem

If $L\{F(t)\} = f(s)$, then $L\{e^{at}\, F(t)\} = f(s - a)$

Proof Let $L\{F(t)\} = f(s) = \int_0^\infty e^{-st}\, F(t)\, dt$ (11)

Then
$$L\{e^{at}\, F(t)\} = \int_0^\infty e^{-st} \cdot e^{at}\, F(t)\, dt$$
$$= \int_0^\infty e^{-(s-a)t}\, F(t)\, dt$$
$$= \int_0^\infty e^{-ut}\, F(t)\, dt \text{ where } u = (s-a) > 0$$
$$= f(u)$$
$$= f(s-a)$$
Proved.

Remarks This theorem can also be restated as follows:

If $f(s)$ is the Laplace transform of $F(t)$ and a is any real or complex number then $f(s + a)$ is the Laplace transform of $e^{-at}\, F(t)$. That is to say,

$$f(s) = L\{F(t)\} \Rightarrow f(s+a) = L\{e^{-at}\, F(t)\}$$

Theorem 3: Second Translation or Heaviside's Shifting Theorem

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$

Then $L\{G(t)\} = e^{-as} f(s)$

Proof
$$L\{G(t)\} = \int_0^\infty e^{-st}\, G(t)\, dt \tag{12}$$
$$= \int_0^a e^{-st}\, G(t)\, dt + \int_a^\infty e^{-st}\, G(t)\, dt$$
$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot F(t-a)\, dt$$
$$= \int_a^\infty e^{-st}\, F(t-a)\, dt$$
Put $t - a = u$
$dt = du$
$$= \int_0^\infty e^{-s(u+a)}\, F(u)\, du$$
$$= \int_0^\infty e^{-su} \cdot e^{-as}\, F(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} F(u)\, du = e^{-as} f(s)$$ **Hence, proved.**

Another Form

If $f(s)$ is the Laplace transform of $F(t)$ and $a > 0$ then $e^{-as} f(s)$ is the Laplace transform of $F(t - a)$ $H(t - a)$ when

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Proof Suppose $f(s) = L\{F(t)\}$ and $a > 0$.

Also, suppose $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$

To prove that $L\{F(t - a).\ H(t - a)\} = e^{-as} .f(s)$.

$$\therefore \quad L\{F(t-a)\, H(t-a)\} = \int_0^\infty e^{-st} F(t-a) H(t-a)\, dt \tag{13}$$

Put $t - a = x$ so that $dt = dx$

$$t - a = x,\ t = 0 \Rightarrow x = -a$$
$$t - a = x,\ t = \infty \Rightarrow x = \infty$$

Then $L\{F(t-a)\, H(t-a)\} = \int_{-a}^\infty F(x)\cdot H(x)\, e^{-s(a+x)}\, dx$ [by (1)]

$$= \int_{-a}^0 F(x)\cdot H(x)\, e^{-s(a+x)}\, dx + \int_0^\infty F(x)\cdot H(x)\, e^{-s(a+x)}\, dx$$

$$= \int_{-a}^0 F(x)\cdot 0\cdot e^{-s(a+x)}\, dx + \int_0^\infty F(x) 1\cdot e^{-s(a+x)}\, dx$$

$$= 0 + \int_0^\infty F(x)\, e^{-s(a+x)}\, dx$$

$$= e^{-as} \int_0^\infty F(t)\cdot e^{-st}\, dt$$

$$= e^{-as} L\{F(t)\} = e^{-as} \cdot f(s)$$

Theorem 4: Change-of-Scale Property

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \dfrac{1}{a} f(s/a)$

$$L\{F(at)\} = \int_0^\infty e^{-st} \cdot F(at)\, dt \tag{14}$$

$$= \int_0^\infty e^{-\frac{sx}{a}} \cdot F(x)\frac{dx}{a} \qquad \begin{bmatrix} \text{put } at = x \\ a\,dt = dx \end{bmatrix}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{sx}{a}} \cdot F(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{st}{a}} \cdot F(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-pt} \cdot F(t)\frac{dx}{a}; \qquad p = \frac{s}{a}$$

$$= \frac{1}{a} f(p)$$

$$= \frac{1}{a} f(s/a)$$

Example 1 Find the Laplace transform of $F(t) = 3e^{3t} + 5t^4 - 4\cos 3t + 3\sin 4t$.

Solution $L\{F(t)\} = L\{3e^{3t} + 5t^4 - 4\cos 3\,t + 3\sin 4t\}$

$$= 3L\{e^{3t}\} + 5L\{t^4\} - 4\,L\{\cos 3t\} + 3L\{\sin 4t\}$$

$$= 3\cdot\frac{1}{s-3} + 5\frac{4!}{s^5} - 4\frac{s}{s^2+9} + 3\frac{4}{s^2+16}$$

Example 2 Find the LT of $F(t) = e^{-3t} + 3\cosh st - 4\sinh 4t$.

Solution $L\{F(t)\} = L\{e^{-3t} + 3\cosh 5t - 4\sinh 4t\}$

$$= L\{e^{-3t}\} + 3L\{\cosh 5t\} - 4L\{\sinh 4t\}$$

$$= \frac{1}{s+3} + 3\cdot\frac{s}{s^2-25} - 4\cdot\frac{4}{s^2-16}$$

$$= \frac{1}{s+3} + \frac{3s}{s^2-25} - \frac{16}{s^2-16}$$

Example 3 Find $L\{t^3\, e^{2t}\}$.

Solution

$$L\{t^3\} = \frac{3!}{s^4} = \frac{12}{s^4} \equiv f(s) \quad \text{say}$$

Then using the first translation theorem,

$$L\{t^3 \cdot e^{2t}\} = f(s-2)$$

$$= \frac{12}{(s-2)^4}$$

Example 4 Find $L\{e^{-3t} \sin 2t\}$.

Solution

$$L\{\sin 2t\} = \frac{2}{s^2 + 4} \equiv f(s) \quad \text{say}$$

Then $L\{e^{-3t} \sin 2t\} = f(s + 3)$

$$= \frac{2}{(s+3)^2 + 4}$$

$$= \frac{2}{s^2 + 6s + 13}$$

Example 5 Find $L\{e^{-2t}(3 \cos 6t - 5 \sin 6t\}$.

Solution

$$L\{3 \cos 6\,t - 5 \sin 6t\} = 3L\{\cos 6\,t\} - 5\,L\{\sin 6t\}$$

$$= \frac{3s}{s^2 + 36} - \frac{30}{s^2 + 36}$$

$$= \frac{3s - 30}{s^2 + 36}$$

$$= \frac{3(s - 10)}{s^2 + 36} \equiv f(s)$$

$\therefore \quad L\{e^{-2t}(3 \cos 6t - 5 \sin 6t)\} = f(s + 2)$

$$= \frac{3\{(s+2) - 10\}}{(s+2)^2 + 36}$$

$$= \frac{3(s - 8)}{s^2 + 4s + 40}$$

Example 6 Find $L\{F(t)\}$ if $F(t) = \begin{bmatrix} \sin(t - \pi/3); & t > \pi/3 \\ 0; & t < \pi/3 \end{bmatrix}$.

Solution

$$L\{F(t)\} = \int_0^\infty e^{-st}\, F(t)\, dt$$

$$= \int_0^{\pi/3} e^{-st} . 0\, dt + \int_{\pi/3}^{\infty} e^{-st} \sin(t - \pi/3)\, dt$$

$$= \int_0^\infty e^{-s\left(u + \frac{\pi}{3}\right)} . \sin u\, du \qquad \begin{bmatrix} \text{Put } t - \dfrac{\pi}{3} = u \\ dt = du \end{bmatrix}$$

$$= e^{-\frac{\pi s}{3}} \int_0^\infty e^{-su} \sin u du = e^{-\frac{\pi s}{3}} \cdot L\{\sin u\}$$

$$= e^{-\frac{\pi s}{3}} \cdot \frac{1}{s^2+1}$$

Example 7 Find $L\{\cos 3t\}$.

Solution

$$L\{\cos t\} = \frac{s}{s^2+1} \equiv f(s) \quad \text{say}$$

$$\therefore \qquad L\{\cos 3t\} = f(s/3)$$

$$= \frac{1}{3} \frac{s/3}{(s/3)^2+1}$$

$$= \frac{1}{3} \frac{s \times 9}{3(s^2+9)}$$

$$= \frac{s}{s^2+9}$$

16.5 A FUNCTION OF CLASS A

A function which is sectionally continuous over every finite interval in the range $t \geq 0$ and is of exponential order as $t \to \infty$ is termed as 'a function of class A'.

16.6 SECTIONALLY CONTINUOUS

A function $F(t)$ is said to be sectionally continuous in a closed interval $a \leq t \leq b$ if that interval can be subdivided into a finite number of intervals such that in each function is continuous and has finite right- and left- hand limits.

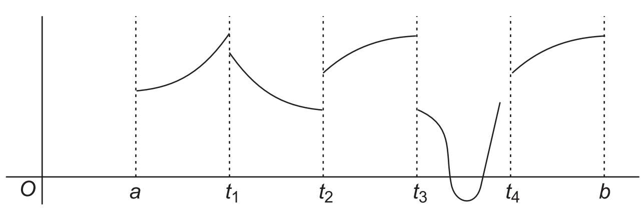


Fig. 16.1

In Fig. 16.1, $F(t)$ is discontinuous at t_1, t_2, t_3, t_4 but is sectionally continuous in the interval (a, b).

(i) Here, $F(t)$ is continuous in the open intervals
$a < t < t_1$, $t_1 < t < t_2$, $t_3 < t < t_4$, $t_4 < t < b$

(ii) The right-hand and left-hand limits at t_1 are

$\lim\limits_{\in \to 0} F(t_1 + \in)$ and $\lim\limits_{\in \to 0} F(t_1 - \in)$ which exist as is evident from the above diagram.

Functions of Exponential Order

A function $F(t)$ is said to be of exponential order b as $t \to \infty$ if there exists a positive constant M, a number b and a finite number t_0 such that

$$|F(t)| < Me^{bt};\ t \geq t_0 \tag{15}$$

Sometimes we write

$$F(t) = O(e^{bt}),\ t \to \infty$$

and read it as $F(t)$ is of exponential order b, as $t \to \infty$

Note If a function is of exponential order b, it is also of order a such that $a > b$.

Example 8 Show that $F(t) = t^2$ is of exponential order 3.

Solution Let us find

$$\lim_{t \to \infty} \{e^{-bt} \cdot f(t)\} = \lim_{t \to \infty} \{e^{-bt} \cdot t^2\}$$

$$= \lim_{t \to \infty} \left\{\frac{t^2}{e^{bt}}\right\} = 0$$

Therefore, it is of exponential order

Now, since $|t^2| = t^2 < e^{3t}$ for all $t > 0$, it is of order 3.

Example 9 Show that $F(t) = e^{t^2}$ is not of exponential order as $t \to \infty$.

Solution

$$\lim_{t \to \infty} \left[e^{-bt}\ F(t)\right] = \lim_{t \to \infty} \left[e^{-bt}\ e^{t^2}\right]$$

$$= \lim_{t \to \infty} \left[\frac{e^{t^2}}{e^{bt}}\right]$$

$$= \lim_{t \to \infty} e^{t^2 - bt}$$

Case I: If $b \leq 0$, this limit is infinite.

Case II: If $b > 0$, $\lim\limits_{t \to \infty} e^{t(t-b)} = \infty$

Thus, whatever be the value of b, this limit is not finite, Hence, we cannot find a number M such that

$$e^{t^2} < Me^{bt}$$

Hence, the given function is not of exponential order as $t \to \infty$.

16.7 LAPLACE TRANSFORMS OF DERIVATIVES

Theorem 5(a)

If $L\{F(t)\} = f(s)$ **then**

$L\{F'(t)\} = sf(s) - F(0)$: $s > 0$

Proof
$$L\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t)\,dt \tag{16}$$
$$= \left[e^{-st} F(t)\right]_0^{\infty} + s\int e^{-st} F(t)\,dt \quad \text{[Integrating by parts]}$$
$$= -F(0) + sf(s)$$
$$= sf(s) - F(0)$$

Result I By applying the above theorem to $F''(t)$, we have
$$L\{F''(t)\} = sL\{F'(t)\} - F'(0)$$
$$= s\{sf(s) - F(0)\} - F'(0)$$
$$= s^2 f(s) - sF(0) - F'(0)$$

Result II Similarly, for LT of derivatives of order n, we have
$$L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) + \ldots - F^{(n-1)}(0)$$

Example 10 Find $L\{-3 \sin 3t\}$.

Solution Let $F(t) = \cos 3t$

Then $F'(t) = -3 \sin 3t$ and $L\{\cos 3t\} = \dfrac{s}{s^2 + 9} \equiv f(s)$ say
$$\therefore \quad L\{-3 \sin 3t\} = L\{F'(t)\}$$
$$= sf(s) - F(0)$$
$$= s\cdot\frac{s}{s^2 + 9} - \cos 0$$
$$= \frac{s}{s^2 + 9} - 1$$
$$= \frac{-9}{s^2 + 9}$$

16.8 DIFFERENTIATION OF TRANSFORMS

Theorem 5(b)

Multiplication by t

If $L\{F(t)\} = f(s)$ **then**
$$L\{t \cdot F(t)\} = (-1)\frac{d}{ds} f(s) = (-1)\, f'(s)$$

In general, $L\{t^n \cdot F(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s)$ (17)

Proof We know that $f(s) = \int_0^\infty e^{-st} F(t)\,dt$

By Leibnitz's rule for differentiating under the integral sign

$$\frac{d}{ds} f(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t)\,dt$$

$$\frac{d}{ds} f(s) = \int_0^\infty F(t) \cdot \frac{\partial}{\partial s}(e^{st})\,dt$$

$$= \int_0^\infty F(t) \cdot (-te^{-st})\,dt$$

$$= -\int_0^\infty F(t) \cdot (te^{-st})\,dt = -L\{t \cdot f(t)\}$$

$$\therefore \quad L\{t \cdot F(t)\} = -\frac{d}{ds} f(s) = (-1)\, f'(s) \tag{18}$$

Similarly, $L\{t^2 F(t)\} = (-1)^2 f''(s)$

By mathematical induction, the result for the n^{th} derivative follows:

$$L\{t^n \cdot F(t)\} = (-1)^n f^{(n)}(s)$$

Example 11 Find $L\{t \cdot \sin at\}$.

Solution Here, $F(t) = \sin at$

$$L\{F(t)\} = L\{\sin at\} = \frac{a}{s^2 + a^2} \equiv F(s)$$

$$\therefore \quad L\{t \cdot \sin at\} = -\frac{d}{ds} f(s)$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right) = \frac{2as}{(s^2 + 9^2)^2}$$

16.9 LAPLACE TRANSFORM OF THE INTEGRAL OF A FUNCTION

Theorem 6

If $L\{F(t)\} = f(s)$ then

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}; \quad s > 0$$

Proof Let $G(t) = \int_0^t F(u)\,du$

Then $G'(t) = F(t)$ and $G(0) = 0$

We know that

$$f(s) = L\{F(t)\} = L\{G'(t)\}$$
$$= s\,L\{G(t)\} - G(0)$$
$$= s \cdot L\{G(t)\}$$

$$\therefore \qquad L\{G(t)\} = \frac{1}{s} f(s)$$

or $$L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s) \tag{19}$$

Result Similarly, if $L\{F(t)\} = f(s)$ then

$$L\left\{\int_0^t dt_1 \int_0^{t_1} F(u)\,du\right\} = \frac{1}{s^2} f(s)$$

The double integral can also be written as

$$L\left\{\int_0^t\int_0^t F(t)\,dt^2\right\}$$

Generalization for n^{th} integral is

$$L\left\{\int_0^t \int_0^t {}_{n\text{-times}} \int_0^t F(t)\,dt^n\right\} = \frac{1}{s^n} f(s) \tag{20}$$

16.10 INTEGRATION OF TRANSFORM! (DIVISION BY *t*)

Theorem 7

If $L\{F(t)\} = f(s)$, then

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

Proof $$f(s) = L\{F(t)\} = \int_0^\infty e^{-st}\,F(t)\,dt \tag{21}$$

Integrating on both side w.r.t. s from 0 to ∞, we have

$$\int_0^\infty f(u)\,du = \int_s^\infty \int_0^\infty e^{-st}\,F(t)\,dt \cdot ds$$

Since s and t are independent variables, interchanging the order of integration, we get

$$\begin{aligned}\int_s^\infty f(u)\,du &= \int_0^\infty F(t)\left\{\int_s^\infty e^{-st}\,ds\right\}dt\\ &= \int_0^\infty F(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt\\ &= \int_0^\infty F(t)\left(\frac{-e^{-st}}{-t}\right)dt\\ &= \int_0^\infty e^{-st}\cdot\left(\frac{F(t)}{t}\right)dt\end{aligned}$$

or $$\int_s^\infty F(u)\,du = L\left\{\frac{F(t)}{t}\right\}$$

Example 12 Find $L\left\{\dfrac{\cos at-\cos bt}{t}\right\}$.

Solution Here, $F(t) = \cos at - \cos bt$

$$\begin{aligned}L\{F(t)\} &= L\{\cos at - \cos bt\}\\ &= L\{\cos at\} - L\{\cos bt\}\\ &= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \equiv f(s)\end{aligned}$$

Using division by t,

$$L\left\{\frac{F(t)}{t}\right\} = \int_S^\infty f(u)\,du$$

$$\begin{aligned}L\left\{\frac{\cos at-\cos bt}{t}\right\} &= \int_s^\infty\left[\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right]du\\ &= \frac{1}{2}\log\left(\frac{u^2+a^2}{u^2+b^2}\right)\Bigg]_s^\infty\\ &= \frac{1}{2}\left[0-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]\\ &= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\end{aligned}$$

16.11 HEAVISIDE'S UNIT FUNCTION (UNIT-STEP FUNCTION)

If a function is of the form

$$F(t) = 0;\ t < a$$
$$= 1;\ t > a$$

then $H(t - a)$ is defined as the Heaviside's unit function. See Fig. 16.2.

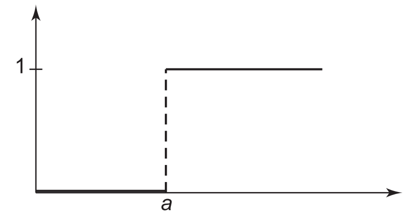


Fig. 16.2

Let us consider a function $H(t)$ given by

$$H(t) = \begin{cases} F(t) & , \text{ for } \ t \geq 0 \\ 0 & , \text{ for } \ t < a \end{cases}$$

This shows that $H(t) = 0$ when t is negative and is $F(t)$ when t is positive or zero. It follows that

$$H(t - a) = \begin{cases} F(t-a) & \text{for } \ t \geq a \\ 0 & \text{for } \ t < a \end{cases}$$

The graph of $y = F(t)$, $t \geq 0$ is shown in Fig. 16.3, and the graph of $y = F(t - a)$. $H(t - a)$ $t \geq a$ is shown in Fig. 16.4. (1)

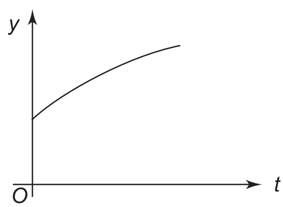


Fig. 16.3

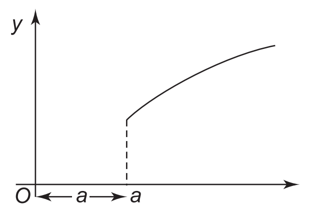
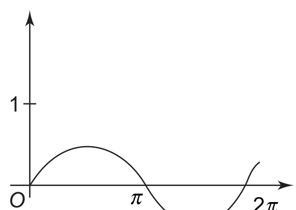


Fig. 16.4

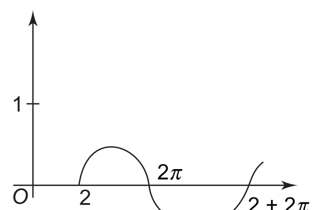
Now, if $F(x)$ is defined for $-a \leq x < 0$ then $F(t - a)$ is defined for $0 \leq t < a$ and y of equation (1) is 0 for $0 \leq t < a$ because of $H(t - a)$.

The following diagrams will make this idea more clear.



Graph of $y = \sin t$

(a)



Graph of $y = \sin(t - 2)\, H(t - 2)$

(b)

Fig. 16.5

16.12 DIRACS DELTA FUNCTION (OR UNIT-IMPULSE FUNCTION)

Forces (like earthquakes) that produce large effects on a system, when applied for a very short time interval, can be represented by an impulse function which is a discontinuous function and is highly irregular from the mathematical point of view.

Impulse of a force $F(t)$ in the interval $(a, a+\in) = \int_a^{a+\in} F(t)\,dt$

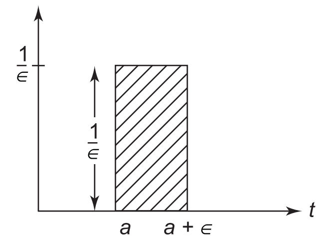


Fig. 16.6

Now, we define the function

$$F_\in(t-a)\begin{bmatrix} 0 & \text{for } t<a \\ \dfrac{1}{\in} & \text{for } a \le t \le a+\in \\ 0 & \text{for } t>a \end{bmatrix}$$

This can also be represented in terms of two unit-step functions as follows:

$$F_\in(t-a) = \frac{1}{\in}\left[H(t-a) - H(t-(a+\in))\right] \tag{22}$$

Note that

$$\int_0^\infty F_\in(t-a)\,dt = \int_0^a 0\,dt + \int_a^{a+\in} \frac{1}{\in}\,dt + \int_{a+\in}^\infty 0 = 1$$

Thus, the impulse $I_\in$ is 1.
Taking Laplace transform,

$$L\{F_\in(t-a)\} = \frac{1}{\in} L\left\{H(t-a) - H(t-(a+\in)\right\}$$

$$= \frac{1}{\in s}\left[e^{-as} - e^{-(a+\in)s}\right]$$

$$= e^{-as} \cdot \frac{(1-e^{-\in s})}{\in s}$$

Dirac delta function (or unit-impulse function) is denoted by $\delta(t-a)$ as the limit of $F_\in(t-a)\}$ as $\in \to 0$

i.e., $$\delta(t-a) = \lim_{\in\to 0} \cdot F_\in(t-a)$$

Then the Laplace transform of the Dirac delta function is obtained as

$$L\{\delta(t-a)\} = \lim_{\in\to 0} \cdot L\{F_\in(t-a)\}$$

$$= \lim_{\in\to 0} e^{-as} \cdot \frac{(1-e^{-\in s})}{\in s}$$

$$\boldsymbol{L\{\delta(t-a)\} = e^{-as}}$$

Thus, the Dirac delta function is a "generalized function" defined as

$$\delta(t-a) = \begin{cases} \infty, & \text{if } t=a \\ 0, & \text{otherwise} \end{cases}$$

subject to $$\int_0^\infty \delta(t-a)\,dt = 1 \tag{23}$$

Example 13 Find $L\{F(t-a)\}$, where $F(t-a)$ is Heaviside's unit-step function.

Solution

$$F(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

$$\therefore \quad L\{F(t-a)\} = \int_0^\infty e^{-st} F(t-a)dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-sa}}{-s}\right]_a^\infty$$

$$= \frac{e^{-sa}}{s}$$

Example 14 Find $L\{F_\in(t)\}$, where $F_\in(t)\}$ is a Dirac delta function.

Solution

$$F_\in(t) = \begin{cases} \dfrac{1}{\in} & \text{of} \quad 0 \le t \le \in \\ 0 & \text{if} \quad t > \in \end{cases}$$

$$\therefore \quad L\{F_\in(t)\} = \int_0^\infty e^{-st} F_\in(t)\, dt$$

$$= \int_0^\in e^{-st} . \frac{1}{\in} dt + \int_\in^\infty e^{-st} \cdot 0\, dt$$

$$= \frac{1}{\in}\left[\frac{e^{-st}}{-s}\right]_0^\in + 0$$

$$= \frac{1}{\in s}(1 - e^{-\in s})$$

16.13 LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

A function $F(t)$ is said to be a periodic function of period $T > 0$ if

$$F(t) = F(t+T) = F(t+2T) = \ldots = F(t+nT)$$

sin t, cos t are periodic functions of period 2π.

Theorem 8

The Laplace transform of a periodic function $F(t)$ with period T is

$$L\{F(t)\} = \frac{1}{(1-e^{-Ts})}\int_0^T e^{-sT}\, F(t)\, dt \qquad s > 0$$

Proof $L\{F(t)\} = \int_0^\infty e^{-sT}\, F(t)\, dt$

$$= \int_0^T e^{-sT}\, F(t)\, dt + \int_T^{2T} e^{-sT}\, F(t)\, dt + \int_{2T}^{3T} e^{-sT} F(t)\, dt + \cdots + \int_{(n-1)T}^{nT} e^{-sT} F(t)\, dt \tag{24}$$

Put $t = u + T$ in the second integral
$t = u + 2T$ in the third integral
$\vdots$
$t = u + (n-1)T$ in the n^{th} integral.

Then the new limits for each interval are 0 to T and by periodicity,

$$F(t + T) = F(t),\ T(t + 2T) = F(t) \text{ and so on}$$

$$\therefore \qquad L\{F(t)\} = \int_0^T e^{-su}\, F(u)\, du + \int_0^T e^{-s(u+T)}\, F(u)\, du + \int_0^T e^{-s(u+2T)} F(u)\, du + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots\right]\int_0^T e^{-su}\, F(u)\, du$$

$$L\{F(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-sT}\, F(t)\, dt, \quad s > 0$$

Since the bracketed quantity in RHS is GP,

i.e., $\qquad 1 + r + r^2 + r^3 \cdots = \dfrac{1}{1-r}\ |r| < 1$, with $r = e^{-sT}$

Example 15 Find the Laplace transform of the sawtooth wave (Fig. 16.7).

$$F(t) = \frac{K}{T}\cdot t; \qquad 0 < t < T$$

and $\qquad F(t + T) = F(t)$

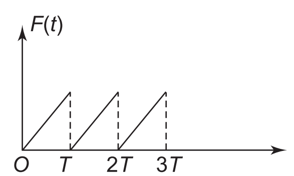


Fig. 16.7

Solution

$$L\{F(t)\} = \frac{\int_0^T e^{-st}\, F(t)\, dt}{(1-e^{-sT})} \tag{1}$$

Now, $\qquad \int_0^T e^{-st}\, F(t)\, dt = \int_0^T e^{-st}\cdot\frac{K}{T}\, t\, dt = \frac{K}{T}\int_0^T e^{-st}\cdot t\, dt$

$$= \frac{K}{T}\left[\left(-\frac{t}{s}e^{-st}\right)_0^T + \frac{1}{s}\int_0^T e^{-st}\,dt\right]$$

$$= \frac{K}{T}\cdot\left[-\frac{T}{s}e^{-sT} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^T\right]$$

$$= \frac{K}{T}\left[-\frac{T}{s}e^{-sT} - \frac{1}{s^2}(e^{-st} - 1)\right]$$

$\therefore$ Eq. (1), becomes

$$L\{F(t)\} = \frac{K}{T\times(1-e^{-sT})}\times\left[-\frac{T}{s}e^{-sT} - \frac{1}{s^2}(e^{-sT}-1)\right]$$

$$= \frac{K}{Ts^2} - \frac{K\,e^{-sT}}{s(1-e^{-sT})}, \quad s > 0$$

16.14 THE ERROR FUNCTION

The error function, abbreviated as 'erf' or 'E', is defined as

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx$$

and *erfc x* read as '**the complement of error function**' is defined as

$$\operatorname{erfc} x = 1 - \operatorname{erf} x$$

$$= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}\,dx$$

The numerical values of erf x have been tabulated in most books on the subject. It is convenient to note that

$$\lim_{x\to 0}\operatorname{erf} x = 0, \quad \lim_{x\to\infty}\operatorname{erf} x = 1 \tag{25}$$

Example 16 Find $L\{\operatorname{erf}\sqrt{t}\}$.

Solution By definition,

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right]dx$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \cdots\right]$$

$$\therefore \quad L\{\text{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}} L\left\{t^{1/2} - \frac{1}{3}t^{3/2} + \frac{1}{5.2!}t^{5/2} - \frac{1}{7.3!}t^{7/3} + \cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7.3!\,s^{9/2}} + \cdots\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\cdot\frac{1}{s^{5/2}} + \frac{1.3}{2.4}\cdot\frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^{9/2}} + \cdots$$

$$= \frac{1}{s^{3/2}}\left\{1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1.3}{2.4}\cdot\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^3} + \cdots\right\}$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2}$$

$$\boldsymbol{L\{\text{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}}$$

16.15 LAPLACE TRANSFORM OF BESSEL'S FUNCTIONS $J_0(t)$ AND $J_1(t)$

We know that

$$J_0(t) = \left[1 - \frac{t^2}{2^2} + \frac{t^2}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2 . 6^2} + \cdots\right]$$

$$\therefore \quad L\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2!}{s^3} + \frac{1}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}\cdot\frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\cdot\frac{1}{s^2} + \frac{1.3}{2\cdot 4}\cdot\frac{1}{s^4} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^6} + \cdots\right]$$

$$= \frac{1}{s}\left[\left(1 + \frac{1}{s^2}\right)^{-1/2}\right] = \frac{1}{\sqrt{1+s^2}} \tag{26}$$

We know that

$$L\{F(at)\} = \frac{1}{a}F(s/a)$$

$$\therefore \quad L\{J_0(at)\} = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{s^2}{a^2}+1\right)}} = \frac{1}{\sqrt{s^2+a^2}}$$

Now, we know that

$$J_0'(t) = -J_1(t)$$

$$\therefore \quad L\{J_1(t)\} = -L\{J'_0(t)\} \qquad \text{[Using LT of derivatives]}$$

$$= -[\{sL(J_0(t)\} - 1]$$

$$= 1 - \frac{s}{\sqrt{1+s^2}}$$

16.16 INITIAL AND FINAL-VALUE THEOREMS

Theorem 9: Initial Value Theorem

If $L\{F(t)\} = f(s)$ then

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} s\, f(s)$$

Proof We know that

$$L\{F'(t)\} = sf(s) - F(0)$$

or $$\int_0^\infty e^{-st}\, F'(t)\, dt = sF(s) - F(0) \tag{27}$$

Taking $s \to \infty$,

$$\lim_{s \to \infty} \int_0^\infty e^{-st}\, F'(t)\, dt = \lim_{s \to \infty} [s\, f(s) - F(0)]$$

or $$\lim_{s \to \infty} s\, f(s) = F(0) + \int_0^\infty \left(\lim_{s \to \infty} e^{-st} \right) \cdot F'(t)\, dt$$

$$= F(0) + \int_0^\infty 0 . F'(t)\, dt \qquad \left[\because \lim_{s \to \infty} \cdot e^{-st} = 0 \right]$$

$$= F(0) + 0$$

or $$\lim_{S \to \infty} sf(t) = F(0) = \lim_{t \to \infty} F(t)$$

Theorem 10: Final-value Theorem

If $L\{F(t)\} = f(s)$ then

$$\lim_{t \to \infty} F(t) = \lim_{S \to 0} s \cdot f(s)$$

Proof We know that

$$L\{F'(t)\} = s \cdot f(s) - F(0)$$

or $$\int_0^\infty e^{-st} \cdot F'(t)\, dt = s \cdot f(s) - F(0) \tag{28}$$

$$\lim_{s \to 0} [sf(s) - F(0)] = \lim_{s \to 0} \int_0^\infty e^{-st}\, F'(t)\, dt$$

$$= \int_0^\infty \left(\lim_{s \to 0} \cdot e^{-st} \right) \cdot F'(t)\, dt$$

$$= \int_0^\infty F'(t)\, dt$$

$$= [F(t)]_{t=0}^{\infty}$$

$$\lim_{s\to 0} sf(s) - F(0) = \lim_{t\to\infty} F(t) - F(0)$$

$$\lim_{s\to 0} s\cdot f(s) = \lim_{t\to\infty} F(t)$$ **Hence, proved.**

16.17 LAPLACE TRANSFORM OF THE LAPLACE TRANSFORM

$$L[L\{F(t)\}] = L\int_0^\infty e^{-st}\, F(t)\, dt$$

$$\int_0^\infty e^{-us} ds \int_0^\infty e^{-st}\, F(t)\, dt \quad (29)$$

The area of integration being the whole positive quadrant. Now changing the order of integration, we get

$$L[L\{F(t)\} = \int_0^\infty F(t)\,dt \int_0^\infty e^{-s(u+t)}\, ds$$

$$= \int_0^\infty F(t)\,dt \cdot \left[\frac{e^{-s(u+t)}}{-(u+t)}\right]_{s=0}^{\infty} = \int_0^\infty \frac{F(t)}{(u+t)}\, dt$$

Hence, $$L\{L\{F(t)\}] = \int_0^\infty \frac{F(t)}{(u+t)}\, dt \quad (30)$$

Example 17 Find the LT of the function $F(t) = \dfrac{e^{at} - 1}{a}$.

Solution $$L\{F(t)\} = L\left\{\frac{e^{at} - 1}{a}\right\}$$

$$= \frac{1}{a} L\,\{e^{at} - 1\}$$

$$= \frac{1}{a} L\,\{e^{at}\} - \frac{1}{a} L\{1\}$$

$$= \frac{1}{a}\,\frac{1}{s-a} - \frac{1}{a}.\frac{1}{s} = \frac{1}{s(s-a)}$$

Example 18 Find the LT of $F(t)$, if

$$F(t) = \begin{cases} (t-1)^2 & ; t > 1 \\ 0 & ; 0 < t < 1 \end{cases}$$

Solution

$$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt$$

$$= \int_0^1 e^{-st} \cdot 0\, dt + \int_1^\infty e^{-st}\, (t-1)^2\, dt$$

$$= 0 + \left[\frac{(t-1)^2 \cdot e^{-st}}{-s}\right]_1^\infty + \frac{2}{s}\int_1^\infty (t-1)\, e^{-st}\, dt$$

$$= 0 + 0 + \frac{2}{s}\left[-\frac{(t-1)\, e^{-st}}{2}\right]_1^\infty + \frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_{t=1}^\infty$$

$$= \frac{2}{s}(0) + \frac{2}{s^2}\left(\frac{e^{-s}}{s}\right) = \frac{2}{s^3}\, e^{-s}$$

Example 19 Find $L[(1 + te^{-t})^3]$.

Solution Here, $F(t) = (1 + te^{-t})^3$

$$F(t) = 1 + t^3\, e^{-3t} + 3te^{-t} + 3t^2\, e^{-2t}$$

$$\therefore \quad L\{F(t)\} = L\{1 + t^3\, e^{-3t} + 3t\, e^{-t} + 3t^2\, e^{-2t}\}$$

$$= \{1\} + L\{t^3\, e^{-3t}\} + 3L\{t\, e^{-t}\} + 3\, L\{t^2\, e^{-2t}\}$$

$$= \frac{1}{s} + \frac{3!}{(s+3)^4} + \frac{3.1!}{(s+1)^2} + \frac{3.2!}{(s+2)^3} \qquad \left[\text{Using } L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}\right]$$

$$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3}$$

Example 20 Find the Laplace transforms of (i) $\sin\sqrt{t}$ (ii) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

Solution (i) $F(t) = \sin\sqrt{t}$

$$= \sqrt{t} + \frac{t^{3.2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

$$\therefore \quad L\{F(t)\} = L\{\sin\sqrt{t}\}$$

$$= L\left\{\sqrt{t} - \frac{t^{3.2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots\right\}$$

$$= L\,\{t^{1/2}\} - \frac{1}{3!}L\,\{t^{3/2}\} + \frac{1}{5!}L\,\{t^{5/2}\} - \frac{1}{7!}L\,\{t^{7/2}\} + \cdots$$

$$= \frac{\sqrt{3/2}}{s^{3/2}} - \frac{1}{3!}\frac{\sqrt{5/2}}{s^{5/2}} + \frac{1}{5!}\frac{\sqrt{7/2}}{s^{7/2}} - \frac{\sqrt{9/2}}{s^{9/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{s^{3/2}}\left[\frac{1}{2} - \frac{1}{3!}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s} + \frac{1}{s^2.5!}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2} - \frac{1}{s^3.7!}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2} + \cdots\right]$$

$$\left[\text{Using } L\{t^n\} = \overline{|(n+1)}/s^{n+1}, \left|\frac{1}{2} = \sqrt{\pi}\right.\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left[1-\frac{1}{2^2 s}+\frac{1}{(2^2 s)^2\cdot 2!}-\frac{1}{(2^2 s)^2\cdot 3!}+\cdots\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left(e^{-\frac{1}{2^2 s}}\right)$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

(ii) Let $L\{F(t)\} = \{\sin\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s} \equiv f(s)$ say

$$F(0) = \sin 0 = 0$$

$$F'(t) = \frac{1}{2\sqrt{t}}\cos\sqrt{t} = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$L\{F'(t)\} = sf(s) - F(0)$$

$$= s\,f(s) - 0 = s\,f(s)$$

$$= s\cdot\frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-1/4s}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

or $$\frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

or $$L\left\{\cos\frac{\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Example 21 Find $L\left\{\int_0^t \frac{\sin x}{x}\,dx\right\}$.

Solution We know that $L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{s} f(s)$

Here, $F(t) = \dfrac{\sin t}{t}$

$$L\{\sin t\} = \frac{1}{1+s^2}$$

$\therefore$ $$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{1+s^2}\,ds = \{\tan^{-1} s\}_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \equiv f(s)$$

$$\therefore \qquad L\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \frac{1}{s} \cot^{-1} s$$

Example 22 Find the LT of $F(t)$, where

$$F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{if } t > \frac{2\pi}{3} \\ 0 & \text{if } t < \frac{2\pi}{3} \end{cases}$$

Solution

Method (*I*)

$$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt$$

$$= \int_0^{\frac{2\pi}{3}} e^{-st} \cdot F(t)\, dt + \int_{\frac{2\pi}{3}}^\infty e^{-st} \cdot F(t)\, dt$$

$$= \int_0^{\frac{2\pi}{3}} e^{-st} \cdot 0\, dt + \int_{\frac{2\pi}{3}}^\infty e^{-st} \,.\, \cos\left(t - \frac{2\pi}{3}\right) dt$$

$$= 0 + \int_0^\infty e^{-s\left(x + \frac{2\pi}{3}\right)} \cos x\, dx \quad \left[\text{put } t - \frac{2\pi}{3} = x\right]$$

$$= e^{-\frac{2\pi s}{3}} \int_0^\infty e^{-sx} \cos x\, dx$$

$$= e^{\frac{-2\pi s}{3}} \int_0^\infty e^{-st} \cos t\, dt$$

$$= e^{\frac{-2\pi s}{3}} L\{\cos t\} = e^{-\frac{2\pi s}{3}} \frac{s}{s^2 + 1}$$

Method II $\quad L\{F(t)\} = e^{-as} f(s)$

Here, $\quad a = \frac{2\pi}{3},\ F(t) = \cos t,\ f(s) = L\{\cos t\} = \frac{s}{s^2 + 1}$

$$\therefore \qquad L\{F(t)\} = e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}$$

Example 23 Find the LT of $F(t)$, where

$$F(t) = \begin{cases} \dfrac{t}{T} & \text{if} \quad 0 < t < T \\ 1 & \text{if} \quad t > T \end{cases}$$

Solution

$$\begin{aligned} L\{F(t)\} &= \int_0^\infty e^{-st} \cdot F(t)\, dt \\ &= \int_0^T e^{-st} F(t)\, dt + \int_T^\infty e^{-st}\; F(t)\, dt \\ &= \int_0^T e^{-st} \cdot \frac{t}{T}\, dt + \int_T^\infty e^{-st} \cdot 1\, dt \\ &= \frac{1}{T}\left[\left(\frac{t\,e^{-st}}{-s}\right) - \frac{1}{s^2}\, e^{-st}\right]_{t=0}^{T} - \frac{1}{s}\,(e^{-st})_{t=T}^{\infty} \\ &= \frac{1}{T}\left[\left(\frac{T\,e^{-st}}{-s} - 0\right) - \frac{1}{s^2}\,(e^{-sT} - 1)\right] + \frac{e^{-sT}}{s} \\ &= \frac{-1}{s^2 T}\,(1 - e^{-sT}) \end{aligned}$$

Example 24 Show that $\displaystyle\int_0^\infty t \cdot e^{-3t} \sin t\, dt = \frac{3}{50}$.

Solution Since $L\{\sin t\} = \dfrac{1}{s^2+1} \equiv f(s)$

$$\begin{aligned} \therefore \qquad L\{t \cdot \sin t\} &= (-1)\,\frac{d}{ds} \cdot f(s) \\ &= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} \end{aligned}$$

$$\Rightarrow \qquad \int_0^\infty e^{-st}\,(t \sin t)\, dt = \frac{2s}{(s^2+1)^2}$$

Putting $s = 3$,

$$\therefore \qquad \int_0^\infty t e^{-3t} \sin t = \frac{2.3}{(3^2+1)^2} = \frac{3}{50}$$

Example 25 Given $L\left(2\sqrt{\dfrac{t}{\pi}}\right) = \dfrac{1}{s^{\frac{3}{2}}}$, show that

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$$

Solution Let $F(t) = 2\sqrt{\left(\dfrac{t}{\pi}\right)}, F(0) = 0$

$$F'(t) = \frac{2}{2\sqrt{\pi t}} = \frac{1}{\sqrt{\pi t}}$$

$$\therefore \quad L\{F'(t)\} = s\,f(s) - f(0) = s\,f(s) - 0$$

$$= s \cdot \frac{1}{s^{3/2}} = \frac{1}{s^{1/2}}$$

$$\therefore \quad L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$$

Hence, proved.

Example 26 Evaluate $\displaystyle\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt.$

Solution If we write $F(t) = e^{-t} - e^{-3t}$ then

$$f(s) = L\{F(t)\} = L\{e^{-t} - e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\therefore \quad L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^\infty \left[\frac{1}{s+1} - \frac{1}{s+3}\right] ds$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \log\left(\frac{s+3}{s+1}\right)$$

Putting $s = 0$, we get

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt = \log 3$$

Example 27 Show that $\displaystyle\int_{t=0}^\infty \int_{u=0}^t \frac{e^{-t}\sin u}{u}\,du\,dt = \frac{\pi}{4}.$

Solution The given double integral on the LHS is the Laplace transform of $\displaystyle\int_0^t \frac{\sin u}{u}\,du$ with $s = 1$. We know that

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \int_{t=0}^\infty \int_{u=0}^t \frac{e^{-st}\cdot \sin u}{u}\,du\,dt$$

$$= \left[\frac{1}{s}\tan^{-1}\frac{1}{s}\right]_{s=1}^{\infty} \qquad \left[\because\ L\int_0^t \frac{\sin x}{x}\,dx = \frac{1}{s}\tan^{-1}\frac{1}{s}\right]$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

Hence, proved.

Example 28 Express in terms of Heaviside's unit-step function:

$$F(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$$

Solution Consider $F_1(t) = \sin t$, $F_2(t) = \sin 2t$, $F_3(t) = \sin 3t$, so that

$$F(t) = F_1(t) + (F_2 - F_1)\, H(t - \pi) + (F_3 - F_2)\, H(t - 2\pi)$$

$$= \sin t + (\sin 2t - \sin t)\, H(t - \pi) + (\sin 3t - \sin 2t)\, H(t - 2\pi)$$

Example 29 Find the LT of $F(t)$, where (see Fig. 16.8)

$$F(t) = 1;\ 0 \le t < 2$$

$$= -1;\ 2 \le t < 4$$

$$F(t+4) = F(t)$$

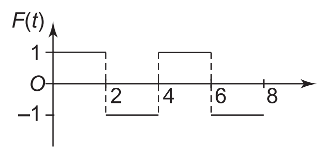


Fig. 16.8

Solution Here, period $T = 4$,

Then

$$L\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-sT}} = \frac{\int_0^4 e^{-st} F(t)\,dt}{1 - e^{-4T}}$$

$$= \frac{1}{1 - e^{-4s}}\left[\int_0^2 e^{-st}\cdot(1)\,dt + \int_2^4 e^{-st}\cdot(-1)\,dt\right]$$

$$= \frac{1}{1 - e^{-4s}}\left[\left(\frac{e^{-st}}{-s}\right)_0^2 + \left(\frac{-e^{-st}}{-s}\right)_2^4\right]$$

$$= \frac{1}{1 - e^{-4s}}\left[\left(\frac{1}{s}\right)\{-e^{-2s} + 1 + e^{-4s} - e^{-2s}\}\right]$$

$$= \frac{1 + e^{-4s}}{s(1 - e^{-4s})}$$

EXERCISE 16.1

Find the Laplace transform of the following:

1. $F(t) = t^3 + t^2 + t$
2. $F(t) = e^{-t} \cos^2 t$
3. $F(t) = \cos t \cdot \cos 2t$
4. $F(t) = \sin^3 2t$
5. $F(t) = \begin{cases} \sin\left(t - \dfrac{2\pi}{3}\right) & \text{if} \quad t > \dfrac{2\pi}{3} \\ 0 & \text{if} \quad t < \dfrac{2\pi}{3} \end{cases}$
6. $F(t) = \begin{cases} (t-1)^2 & \text{if} \quad t > 1 \\ 0 & \text{if} \quad 0 < t < 1 \end{cases}$
7. $F(t) = \begin{cases} \sin t & \text{if} \quad 0 < t < \pi \\ 0 & \text{if} \quad \pi < t < 2\pi \end{cases}$
8. $F(t) = t \cos h\, at$
9. $F(t) = t^2 \sin t$
10. $F(t) = t\, e^{-t} \cos ht$
11. $F(t) = (\sin t - \cos t)^2$
12. $F(t) = \sin at - at \cos at$
13. $F(t) = \begin{cases} 0 & ; \quad 0 < t < 1 \\ 1 & ; \quad 1 < t < 2 \\ 2 & ; \quad t > 2 \end{cases}$
14. $F(t) = a + bt + \dfrac{c}{\sqrt{t}}$
15. $F(t) = \dfrac{\sin 2t}{t}$
16. $F(t) = \dfrac{e^{at} - \cos b\, t}{t}$
17. Express the following function in terms of unit-step function:

    $F(t) = \begin{cases} (t-1) & 1 < t < 2 \\ (3-t) & 2 < t < 3 \end{cases}$

    Also, find its LT.
18. Prove that $\displaystyle\int_0^\infty e^{-t} \frac{\sin t}{t}\, dt = \frac{\pi}{4}$.
19. Use LT to prove that $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{4}$.
20. Prove that $L\{H(t)\} = \dfrac{2(1 - e^{-\pi s})}{s^2 + 4}$

    where $H(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$
21. If $erf(t) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^t e^{-x^2}\, dx$ then prove that $L\{erf(\sqrt{t})\} = \dfrac{1}{s\sqrt{s+1}}$.
22. Find $L\{erf\sqrt{t}\}$, and hence, deduce that $L\{t \cdot erf\,(2\sqrt{t})\} = \dfrac{3s+8}{s^2(s+4)^{3/2}}$.

23. Show that $L\{F(t)\} = \left(\frac{1}{s^2} + \frac{1}{s}\right) e^{-S} - \left(\frac{1}{s^2} + \frac{2}{s}\right) e^{-2s}$.

where $F(t) = \begin{bmatrix} 0 & \text{if} & 0 < t < 1 \\ t & \text{if} & 1 < t < 2 \\ 0 & \text{if} & t > 2 \end{bmatrix}$

24. Find $L\{9 \cos^2 3t\}$.

25. Show that $L\{e^{3t} \cdot erf\,\sqrt{t}\} = \frac{1}{(s-3)\sqrt{(s-2)}}$.

26 Show that $L\left\{t \cdot (3 \sin 2t - 2 \cos 2t\right\} = \frac{8 + 12s - 2s^2}{(s^2+4)^2}$.

27. Prove that $\int_0^\infty e^{-t} \cdot erf\,\sqrt{t}\, dt = \frac{1}{\sqrt{2}}$.

28. Find the LT of $F(t) = \begin{cases} (1+t) & 0 \le t < 1 \\ 3-t & 1 \le t \le 2 \end{cases}$ $F(t+2) = F(t)$.

29. Express $F(t)$ in terms of Heaviside's unit-step function;

(i) $F(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4t & t > 2 \end{cases}$ (ii) $F(t) = \begin{cases} \sin t & t > \pi \\ \cos t & t < \pi \end{cases}$

30. Find LT of $F(t)$, where $F(t) = \begin{cases} t & 0 < t < 3 \\ 3 & t > 3 \end{cases}$

Answers

1. $\frac{6}{s^4} + \frac{2}{s^3} + \frac{1}{s^2}$

2. $\frac{1}{2s+2} + \frac{s+1}{2s^2+4s+10}$

3. $\frac{s(s^2+5)}{(s^2+1)(s^2+9)}$

4. $\frac{48}{(s^2+4)(s^2+36)}$

5. $e^{-\frac{2\pi s}{3}}\left(\frac{1}{s^2+1}\right)$

6. $\frac{2e^{-s}}{s^8}$

7. $\frac{1}{(1+s^2)(1-e^{-\pi s})}$

8. $\frac{s^2+a^2}{(s^2-a^2)^2}$

9. $\frac{2(3s^2-1)}{(s^2+1)^3}$

10. $\frac{s^2+2s+2}{(s^2+2s)^2}$

11. $\frac{s^2-2s+4}{3(s^2+4)}$

12. $\frac{2a^3}{(s^2+a^2)^2}$

13. $\frac{1}{s}[e^{-2s} + e^{-s}]$

14. $\frac{a}{s} + \frac{b}{s^2} + c\sqrt{\left(\frac{\pi}{s}\right)}$

15. $\cot^{-1}\left(\frac{s}{2}\right)$

16. $\frac{1}{2}\log\left[\frac{s^2+b^2}{(s-a)^2}\right]$

17. $F(t) = (t-1)\,H(t-1)$

24. $\frac{9(s^2+18)}{s(s^2+36)}$

28. $\frac{1}{s} + \frac{1-e^{-s}}{1+e^{-s}} \cdot \frac{1}{s^2}$

29. (i) $t^2 + (4t - t^2)\,H(t-2)$ (ii) $\cos t + (\sin t - \cos t) \cdot H(t - \pi)$

30. $\frac{1}{s^2}(1 - e^{-3s})$

16.18 THE INVERSE LAPLACE TRANSFORM (ILT)

Definition

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e., $L\{F(t)\} = f(s)$ then $F(t)$ is called an inverse Laplace transform (ILT) of $f(s)$.

We also write $F(t) = L^{-1}\{f(s)\}$.

L^{-1} is called the *inverse Laplace transformation operator*.

Thus, $$L^{-1}\left\{\frac{1}{s}\right\} = 1,\ L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{31}$$

16.19 NULL FUNCTION

Let $N(t)$ be a function of t, such that

$$\int_0^{t_0} N(t)\,dt = 0\ \forall\ t_0 > 0 \tag{32}$$

Then $N(t)$ is called a null function.

16.20 UNIQUENESS OF INVERSE LAPLACE TRANSFORM

Since the Laplace transform of null function is zero, and hence, if

$$L\{F(t)\} = f(s), \text{ then}$$

$$L\{F(t) + N(t)\} = f(s)$$

Consequently, $$L^{-1}\{f(s)\} = F(t)$$

$$L^{-1}\{f(s)\} = F(t) + N(t)$$

This implies that we can have two different functions with the same Laplace transform.

As a result of which the inverse Laplace transform of a function is not unique if we allow null functions. Hence, the inverse Laplace transform of a function is unique if we do not allow null functions which do not appear in cases of physical interest. This result is explained by Lerch's theorem as given below.

Theorem 11: Lerch's Theorem

Let $L\{F(t)\} = f(s)$. Let $F(t)$ be piecewise continuous in every finite interval $0 \le t \le a$ and of exponential order for $t > a$, then the inverse Laplace transform of $F(t)$ is $f(s)$ unique.

From some of the functions for which Laplace transforms have been obtained, given below the inverse Laplace transforms:

Table 16.1

$F(s)$	$F(t) = L^{-1}\{f(s)\}$
$1/s$	1
$1/s^2$	t
$\dfrac{1}{s^{n+1}}$	$\dfrac{t^n}{\lfloor n+1}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{a}{s^2+a^2}$	$\sin at$
$\dfrac{s}{s^2+a^2}$	$\cos at$
$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sin h\, at$
$\dfrac{s}{s^2-a^2}$	$\cos h\, at$

Theorem 12: Linearity Property

If $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$ and if a, b are constants then

$$\begin{aligned} L^{-1}\{af(s) + bg(s)\} &= aL^{-1}\{f(s)\} + bL^{-1}\{g(s)\} \\ &= aF(t) + bG(t) \end{aligned}$$

The result can easily be proved.

Example 30 Obtain $L^{-1}\left\{\dfrac{1}{s-2} + \dfrac{2}{s+5} + \dfrac{6}{s^4}\right\}$.

Solution

$$L^{-1}\left\{\frac{1}{s-2} + \frac{2}{s+5} + \frac{6}{s^4}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 2\,L^{-1}\left\{\frac{1}{s+5}\right\} + L^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= e^{2t} + 2e^{-5t} + t^3$$

Example 31 Find $L^{-1}\left\{\dfrac{1}{s^{n+1}}\right\}$ for $n > -1$.

Solution We know that

$$L = \left\{\frac{t^n}{\left|\overline{n+1}\right.}\right\} = \left\{\frac{1}{\left|\overline{n+1}\right.}\cdot L\{t^n\}\right\}$$

$$= \frac{1}{\left|\overline{n+1}\right.}\cdot\frac{\left|\overline{n+1}\right.}{s^{n+1}} = \frac{1}{s^{n+1}}$$

$$\therefore \qquad L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\left|\overline{n+1}\right.}$$

Theorem 13: First Shifting Property

If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\{f(s-a)\} = e^{at}\cdot F(t)$$

Proof

$$L\{F(t)\} = f(s) = \int_0^{\infty} e^{-st}\, F(t)\, dt \tag{33}$$

$$\therefore \qquad f(s-a) = \int_0^{\infty} e^{-(s-a)t}\, F(t)\, dt$$

$$= \int_0^{\infty} e^{-st}\cdot(e^{at}\, F(t))\, dt$$

$$= L\{e^{at}\cdot F(t)\}$$

$$\therefore \qquad L^{-1}\{f(s-a)\} = e^{at}\cdot F(t)$$

Hence, proved.

Note The result of this theorem is also expressible as

$$L^{-1}\{f(s)\} = e^{-at}\, L^{-1}\{f(s-a)\}$$

Also

$$L^{-1}\{f(s)\} = e^{at}\, L^{-1}\,\{f(s+a)\} \tag{34}$$

This result is of vital importance for further study.

Example 32 Find $L^{-1}\left\{\dfrac{1}{(s-4)^5} + \dfrac{5}{(s-2)^2+5^2} + \dfrac{s+3}{(s+3)^2+6^2}\right\}$.

Solution We know that $L^{-1}\,\{t^n\} = \dfrac{n!}{s^{n+1}}$

$$\therefore \qquad L\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}} \quad \text{or} \quad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

Now, $L^{-1}\left\{\frac{1}{(s-4)^5}+\frac{5}{(s-2)^2+5^2}+\frac{s+3}{(s+3)^2+6^2}\right\}$

$$= L^{-1}\left\{\frac{1}{(s-4)^5}\right\}+L^{-1}\left\{\frac{5}{(s-2)^2+5^2}\right\}+L^{-1}\left\{\frac{s+3}{(s+3)^2+6^2}\right\}$$

$$= e^{4t}L^{-1}\left\{\frac{1}{s^5}\right\}+e^{2t}L^{-1}\left\{\frac{5}{(s^2+5^2)}\right\}+e^{-3t}L^{-1}\left\{\frac{s}{(s^2+6^2)}\right\}$$

$$= e^{4t}\cdot\frac{t^4}{4!}+e^{2t}\cdot\sin 5t+e^{-3t}\cos 6t$$

Example 33 Evaluate $L^{-1}\left\{\frac{1}{(s-4)^5}+\frac{5}{(s-2)^2+5^2}\right\}$.

Solution

$$L^{-1}\left\{\frac{1}{(s-4)^5}+\frac{5}{(s-2)^2+5^2}\right\} = L^{-1}\left\{\frac{1}{(s-4)^5}\right\}+L^{-1}\left\{\frac{5}{(s-2)^2+5^2}\right\}$$

$$= e^{4t}L^{-1}\left\{\frac{1}{s^5}\right\}+e^{2t}L^{-1}\left\{\frac{5}{s^2+5^2}\right\}$$

$$= e^{4t}\cdot\frac{t^4}{4!}+e^{2t}\sin 5t$$

$$= e^{4t}\cdot\frac{t^4}{24}+e^{2t}\sin 5t$$

Theorem 14: Second Shifting Property

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{e^{-as} f(s)\} = G(t)$, where

$$\boldsymbol{G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}}$$

Proof $$L\{G(t)\} = \int_0^\infty e^{-st}\,G(t)\,dt \tag{35}$$

$$= \int_0^a e^{-st}\,G(t)\,dt+\int_a^\infty e^{-st}\,G(t)\,dt$$

$$= \int_0^a e^{-st}\,0\,dt+\int_a^\infty e^{-st}\cdot F(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^\infty e^{-s(a+u)} F(u)\,du \qquad \text{[Put } t-a=u \Rightarrow dt = du\text{]}$$

$$= \int_0^\infty e^{-as} . e^{-su} F(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-su} F(u)\,du$$

$$= e^{-as} L\{F(t)\} = e^{-as} f\{(s)\}$$

$$\therefore \qquad L\{G(t)\} = e^{-as} f(s) \Rightarrow G(t) = L^{-1}\{e^{-as} \cdot f(s)\} \tag{36}$$

Remark This result is also expressible as

$$L^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

or
$$L^{-1}\{e^{-as} f(s)\} = F(t-a) \cdot H(t-a) \tag{37}$$

where $H(t-a)$ is Heaviside's unit-step function.

Example 34 Find the inverse Laplace transform of $\dfrac{e^{-\pi s}}{s^2+1}$.

Solution

$$L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t = F(t) \quad \text{say}$$

$$\therefore \qquad L^{-1}\left\{e^{-\pi s} \cdot \frac{1}{s^2+1}\right\} = \begin{cases} \sin(t-\pi) & \text{if } t > \pi \\ 0 & \text{if } t < \pi \end{cases} \qquad \text{[By second shifting theorem]}$$

$$= \sin(t-\pi) \cdot H(t-\pi)$$

$$= -\sin t \cdot H(t-\pi)$$

Example 35 Find the inverse Laplace transform of

$$\frac{s e^{-as}}{s^2-\omega^2};\ a > 0$$

Solution

$$L^{-1}\left\{\frac{s}{s^2-\omega^2}\right\} = \cosh \omega t = F(t) \quad \text{say}$$

$$\therefore \qquad L^{-1}\left\{e^{-as} \frac{s}{s^2-\omega^2}\right\} \begin{cases} \cosh \omega(t-a), & \text{if } t > a \\ 0, & \text{if } t < a \end{cases} \qquad \text{[By second shifting theorem]}$$

$$= \cosh \omega(t-a).\ H(t-a)$$

Example 36 Evaluate $L^{-1}\left[\dfrac{e^{-3s}}{(s-4)^2}\right]$.

Solution Let $L\{F(t)\} = f(s) = \dfrac{1}{(s-4)^2}$

Then
$$F(t) = L^{-1}[f(s)] = L^{-1}\left\{\frac{1}{(s-4)^2}\right\}$$
$$= e^{4t} \cdot L^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= te^{4t}$$

By the second shifting theorem,
$$L^{-1}\left[e^{-3s} \cdot \frac{1}{(s-4)^2}\right] = \begin{bmatrix} (t-3)\cdot e^{4(t-3)} & \text{if } t > 3 \\ 0 & \text{if } t < 3 \end{bmatrix}$$
$$= (t-3)e^{4(t-3)} \cdot H(t-3)$$

Example 37 Evaluate $L^{-1}\left[\dfrac{s \cdot e^{-\frac{2s\pi}{3}}}{s^2+9}\right]$.

Solution Let $L\{F(t)\} = f(s) = \dfrac{s}{s^2+9}$

Then
$$F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$$

$$\therefore \quad L^{-1}\left\{e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+9}\right\} = \begin{bmatrix} \cos 3\left(t - \dfrac{2\pi}{3}\right) & \text{if } t > \dfrac{2\pi}{3} \\ 0 & \text{if } t < \dfrac{2\pi}{3} \end{bmatrix}$$
$$= \cos 3t \cdot H\left(t - \frac{2\pi}{3}\right)$$

Theorem 15: Change-of-scale Property

If $L^{-1}\{f(s)\} = F(t)$,

then $$L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof Since $f(s) = \int_0^\infty e^{-st} \cdot F(t)\,dt$, we have

$$f(a\,s) = \int_0^\infty e^{-ast}\ F(t)\,dt \qquad \text{[Putting } y = at \Rightarrow dt = adt]$$

$$= \frac{1}{a}\int_0^\infty e^{-sy} F\left(\frac{y}{a}\right) dy$$

$$= \frac{1}{a} L\left\{F\left(\frac{t}{a}\right)\right\}$$

$$dy = adt$$

$$\therefore \qquad L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right) \qquad (38)$$

Example 38 Find $L^{-1}\left\{\dfrac{s}{2s^2+8}\right\}$.

Solution Since $L^{-1}\left\{\dfrac{s}{s^2+16}\right\} = \cos 4t$

$$\therefore \qquad L^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\frac{4t}{2}$$

$$= \frac{1}{2}\cos 2t$$

i.e.,

$$L^{-1}\left\{\frac{s}{2s^2+8}\right\} = \frac{1}{2}\cos 2t$$

Example 39 If $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{1}{2} t\cdot\sin t$, find $L^{-1}\left\{\dfrac{32s}{(16s^2+1)^2}\right\}$.

Solution Given $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{1}{2} t\cdot\sin t$

Writing *as* for *s*,

$$L^{-1}\left\{\frac{as}{(a^2 s^2+1)^2}\right\} = \frac{1}{2}\cdot\frac{t}{a}\cdot\sin\frac{t}{a}$$

Again writing $a = 4$.

$$L^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\} = \frac{t}{8}\sin\frac{t}{4}$$

$$\therefore \qquad \frac{1}{8} L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = \frac{t}{8}\sin\frac{t}{4}$$

$$\therefore \qquad L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = t\sin\frac{t}{4}$$

Example 40 Find $L^{-1}\left\{\dfrac{64}{81s^4-256}\right\}$.

Solution We know that

$$L^{-1}\left\{\frac{a^3}{s^4-a^4}\right\} = \frac{1}{2}L^{-1}\left[\frac{a}{s^2-a^2}-\frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{2}(\sin h\,at - \sin at)$$

Rewriting $L^{-1}\left\{\dfrac{64}{81s^4-256}\right\} = L^{-1}\left\{\dfrac{4^3}{(3s)^4-4^4}\right\}$

$= L^{-1}\{f(3s)\}$ with $a = 4$.

where $\qquad f(s) = \dfrac{a^3}{s^4-a^4}$ and $F(t) = \dfrac{1}{2}(\sinh at - \sin at)$

Thus, applying change-of-scale property,

$$L^{-1}\{f(3s)\} = \frac{1}{3}F\left(\frac{t}{3}\right)$$

$$= \frac{1}{3}\left[\frac{1}{2}\sinh 4\frac{t}{3} - \sin 4\frac{t}{3}\right]$$

Example 41 If $L^{-1}\left\{\dfrac{e^{-\frac{1}{s}}}{\sqrt{s}}\right\} = \dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ then find $L^{-1}\left\{\dfrac{e^{-\frac{a}{s}}}{\sqrt{s}}\right\}$.

Solution Since $L^{-1}\{f(as)\} = \dfrac{1}{a}F\left(\dfrac{t}{a}\right)$

Hence, $\qquad L^{-1}\left\{\dfrac{e^{-\frac{1}{s}}}{\sqrt{s}}\right\} = \dfrac{\cos(2\sqrt{t})}{\sqrt{\pi}\,t}$ gives $L^{-1}\left\{\dfrac{e^{-\frac{1}{sk}}}{\sqrt{(sk)}}\right\} = \dfrac{1}{k}\cdot\dfrac{\cos 2\sqrt{\left(\frac{t}{k}\right)}}{\sqrt{\left(\frac{\pi t}{k}\right)}}$

or $\qquad L^{-1}\left\{\dfrac{e^{-\frac{1}{sk}}}{\sqrt{(s)}}\right\} = \dfrac{\sqrt{k}}{k}\cdot\dfrac{\sqrt{k}\cos 2\left(\frac{t}{k}\right)^{1/2}}{(\pi t)\frac{1}{2}}$

Putting $k = \dfrac{1}{a}$, we get

$$L^{-1}\left\{\frac{e^{-\frac{a}{s}}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

Theorem 16: Inverse Transforms of Derivatives

If $L^{-1}\{f(s)\} = F(t)$, then

$$\boldsymbol{L^{-1}\{f^{(n)}(s)\} = L^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\}}$$

$$\boldsymbol{= (-1)^n \cdot t^n \cdot F(t)}$$

Proof $L\{t^n F(t)\} = (-1)^n \left\{\dfrac{d^n}{ds^n} f(s)\right\} = (-1)^n f^{(n)}(s)$

$$\therefore \quad L^{-1}\{f^{(n)}(s)\} = (-1)^n t^n \cdot F(t) \qquad (39)$$

Hence, proved.

Example 42 Evaluate $L^{-1}\left[\log\left(\dfrac{s+3}{s+2}\right)\right]$.

Solution Let $f(s) = \log\left(\dfrac{s+3}{s+2}\right) = \log(s+3) - \log(s+2)$

Then $$f'(s) = \frac{1}{s+3} - \frac{1}{s+2}$$

$$L^{-1}\{f'(s)\} = e^{-3t} - e^{-2t} \equiv F(t)$$

But $$L^{-1}\{f'(s)\} = (-1)' \, t \cdot F(t)$$

$$= -tL^{-1}\{f(s)\}$$

or $$L^{-1}\{f(s)\} = -\frac{1}{t} L^{-1}\{f'(s)\}$$

$$= -\frac{1}{t} \cdot \left[e^{-3t} - e^{-2t}\right]$$

$$= \frac{1}{t}\left[e^{-2t} - 3e^{-3t}\right]$$

Example 43 Find $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right)$.

Solution We have $\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{-2s}{(s^2+a^2)^2}$

$$\therefore \qquad \frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)$$

$$\text{But} \qquad L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$$\therefore \qquad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{t}{2a}\sin at$$

Theorem 17: Division by S

If $L^{-1}\{f(s)\} = F(t)$ then

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

Proof Let $G(t) = \int_0^t F(u)\,du$ then $G'(t) = F(t)$, $G(0) = 0$

$$\therefore \qquad L\{G'(t)\} = sL\{G(t)\} - G(0) = sL\{G(t)\} \tag{40}$$

$$\text{But} \qquad L\{G'(t)\} = L\{F(t)\} = f(s) \tag{41}$$

From (40) and (41), we get

$$sL\{G(t)\} = f(s)$$

$$\therefore \qquad L\{G(t)\} = \frac{f(s)}{s}$$

$$\text{or} \qquad L^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(u)\,du$$

Theorem 18: $L^{-1}\left\{\frac{f(s)}{s^2}\right\} \int_0^t\int_0^v F(u)\,du\,dv$

Proof Let $G(t) = \int_0^t\int_0^v F(u)\,du\,dv$. Then $G'(t) = \int_0^t F(u)\,du$

$$G''(t) = F(t), \text{ Also } G(0) = G'(0) = 0$$

$$\therefore \qquad L\{G''(t)\} = s^2L\{G(t)\} - sG(0) - G'(0) \tag{42}$$

$$= s^2L\{G(t)\} = f(s), \text{ as } L\{G''(t)\} = L\{F(t)\}$$

$$\therefore \qquad L\{G(t)\} = \frac{f(s)}{s^2}$$

$$\text{or} \qquad L^{-1}\left\{\frac{f(s)}{s^2}\right\} = G(t) = \int_0^t\int_0^v F(u)\,du\,dv$$

The result can also be written as

$$L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t\int_0^v F(t)\,dt^2$$

In general, $$L^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t\int_0^t \cdots \int_0^t F(t)\,dt^n \qquad (43)$$

Example 44 Find $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.

Solution

$$L\{e^{at}\} = \frac{1}{s-a} \quad \therefore \quad L\{e^{-at}\} = \frac{1}{s+a}$$

or $$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \text{ or } L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s+1}\right)\right\} = (-1)^1 \cdot t^1 \cdot e^{-t} = -te^{-t}$$

or $$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

or $$L^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \int_0^t u \cdot e^{-u}\,du$$

$$= \left[-ue^{-u} + \int e^{-u}\,du\right]_0^t$$

$$= \left[-e^{-u}(u+1)\right]_0^t = 1 - e^{-t}(t+1)$$

or $$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \int_0^t \left[1 - e^{-u}(u+1)\right]du = \left[u + e^{-u}(u+1) + e^{-u}\right]_0^t$$

$$= t + e^{-t} + e^{-t}(t+1) - (1+1)$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Theorem 19: Inverse Laplace Transforms of Integrals

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\left\{\displaystyle\int_s^\infty f(u)\,du\right\} = \dfrac{F(t)}{t}$

Example 45 Evaluate $L^{-1}\left\{\displaystyle\int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right)du\right\}$.

Solution Let $f(s) = \dfrac{s}{s^2 + a^2} - \dfrac{s}{s^2 + b^2}$

$$F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2 + a^2}\right\} - L^{-1}\left\{\frac{s}{s^2 + b^2}\right\}$$

$$= \cos at - \cos bt$$

By Inverse LT of integrals,

$$L^{-1}\left\{\int_s^\infty f(u)\,du\right\} = L^{-1}\left\{\int_s^\infty \left(\frac{u}{u^2 + a^2} - \frac{u}{u^2 + b^2}\right) du\right\}$$

$$= \frac{F(t)}{t}$$

$$= \frac{\cos at - \cos bt}{t}$$

Example 46 Evaluate $L^{-1}\left\{\displaystyle\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\}$.

Solution Consider $f(s) = \dfrac{1}{s} - \dfrac{1}{s+1}$

$$F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} = 1 - e^{-t}$$

By ILT of integrals,

$$L^{-1}\left\{\int_s^\infty f(u)\,du\right\} = L^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\} = \frac{F(t)}{t}$$

$$= \frac{1 - e^{-t}}{t}$$

Theorem 20: I.L.T. of Multiplication by *s*

If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$ then $L^{-1}\{sf(s)\} = F'(t)$

Proof We know that

$$L\{F'(t)\} = sf(s) - F(0)$$

$$= sf(s)$$

Then $$F'(t) = L^{-1}\{s.f(s)\} \tag{44}$$

Thus, multiplication by s amounts to differentiating $F(t)$ w.r.t. 't'.

Example 47 Find $L^{-1}\left\{\dfrac{s}{s^2-a^2}\right\}$.

Solution We know that $\dfrac{1}{a}L^{-1}\left\{\dfrac{a}{s^2-a^2}\right\}=\dfrac{\sinh at}{a}$

and $\qquad F(0)=\sinh 0=0$

$$\therefore \qquad L^{-1}\left\{s\cdot\frac{1}{s^2-a^2}\right\}=\frac{d}{dt}\frac{\sinh at}{a}=\frac{a\cosh at}{a}$$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh at$$

16.21 USE OF PARTIAL FRACTIONS TO FIND ILT

Application of LT to a differential equation results in a subsidiary equation which usually comes out as a rational function $Y(s)=\dfrac{F(s)}{G(s)}$, where $F(s)$ and $G(s)$ are polynomials in s. When the degree of $F(s)$ $\le$ degree $G(s)$ then the rational function $\dfrac{F(s)}{G(s)}$ can be written as the sum of simpler rational functions called partial fractions, depending on the nature of factors of the denominator $G(s)$ as follows:

Table 16.2

Factor in denominator	Corresponding partial fraction
1. Nonrepeated linear factor $ax+b$ (occurring once)	$\dfrac{A}{ax+b}$ with $A\neq 0$
2. Repeated linear factor $(ax+b)^r$ (occurring r times)	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b^2)}+\cdots\dfrac{A_r}{(ax+b)^r}$ with $A_r\neq 0$
3. Nonrepeated quadratic factor ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$ with at least one of A, B nonzero
4. Repeated quadratic factor $(ax^2+bx+c)^r$ (occurring r times)	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots\dfrac{A_r\,x+B_r}{(ax^2+bx+c)^r}$ with at least one of A_r, B_r nonzero

By finding the ILT of each of these partial fractions, $L^{-1}\{Y(s)\}=L^{-1}\left\{\dfrac{F(s)}{G(s)}\right\}$ can be determined.

Example 48 Find $L^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$.

Solution Using partial fractions, we have

$$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)}$$

$$= \frac{A}{s-3} + \frac{B}{s+1}$$

or $$3s + 7 = A\,(s + 1) + B\,(S - 3)$$

$$3s + 7 = (A + B)\,s + A - 3\,B$$

Equating the corresponding coefficients on both sides

$$A + B = 3,\ \ A - 3\,B = 7$$

Then $$A = 4,\ B = -1.$$

$$\therefore \quad \frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$$

$$\therefore \quad L^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = 4\,L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 4\,e^{3t} - e^{-t}.$$

Example 49 Find $L^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$.

Solution

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{(s^2+2s+5)} + \frac{Cs+D}{(s^2+2s+5)}$$

$$s^2 + 2s + 3 = (A\,s + B)\,(s^2 + 2s + 5) + (C\,s + D)\,(s^2 + 2\,s + 2)$$

$$= (A + C)\,s^3 + (2A + B + 2C + D)\,s^2 + (5A + 2B + 2C + 2D)\,s + 5B + 2D$$

Comparing coefficients of s on either side,

$$A + C = 0,\ \ 2\,A + B + 2C + D = 1$$

$$5\,A + 2\,B + 2\,C + 2\,D = 2$$

$$5\,B + 2\,D = 3$$

On solving these equations, we get

$$A = 0,\ \ B = \frac{1}{3},\ C = 0,\ D = \frac{2}{3}$$

$$\therefore \quad \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{1/3}{(s^2+2s+2)} + \frac{2/3}{(s^2+2s+s)}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + \frac{2}{3}L^{-1}\left(\frac{1}{s^2+2s+5}\right)$$

$$= \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t$$

$$= \frac{1}{3} e^{-t} (\sin t + \sin 2t)$$

Example 50 Find $L^{-1}\left\{\frac{a(s^2-2a^2)}{s^4+4a^4}\right\}$.

Solution Rewriting $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2as)^2$

$$= (s^4 + 2as + 2a^2)\,(s^2 - 2as + 2a^2)$$

$$\frac{a(s^2-2a^2)}{s^4+4a^4} = \frac{a(s^2-2a^2)}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$$= \frac{1}{2}\left(\frac{-(s+a)}{s^2+2as+2a^2}\right) + \frac{1}{2}\left(\frac{(s-a)}{s^2-2as+2a^2}\right)$$

$$= \frac{1}{2}\left[\frac{-(s+a)}{(s+a)^2+a^2} + \frac{(s-a)}{(s-a)^2+a^2}\right]$$

$$\therefore \quad L^{-1}\left\{\frac{a(s^2-2a^2)}{s^4+4a^4}\right\} = \frac{1}{2} L^{-1}\left\{\frac{-(s+a)}{(s+a)^2+a^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{(s-a)}{(s-a)^2+a^2}\right\}$$

$$= -\frac{1}{2} e^{-at} \cos at + \frac{1}{2} e^{at} \cos at$$

$$= \cos at \left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \cos at \sinh at$$

Example 51 Find $L^{-1}\left\{\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}\right\}$.

Solution $\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2} = \frac{As+B}{(s^2-2s+2)^2} + \frac{Cs+D}{(s^2-2s+2)}$

$$s^3 - 3s^2 + 6s - 4 = (As + B) + (Cs + D)(s^2 - 2s + 2)$$

$$= Cs^3 + (D - 2C)s^2 + (A + 2C - 2D)s + B + 2D$$

Equating the corresponding coefficients and solving,

$$A = 2,\ B = -2,\ C = 1,\ D = -1$$

Now, $s^2 - 2s + 2 = (s-1)^2 + 1$

$$\therefore \qquad L^{-1}\left\{\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}\right\} = L^{-1}\left\{\frac{2(s-1)}{\left[(s-1)^2+1\right]^2}\right\}+L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}$$

$$= 2e^t L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}+e^t.\cos t$$

$$= 2e^t\cdot\frac{t}{2}\cdot\sin t+e^{-t}.\cos t$$

$$= e^t[t\sin t+\cos t]$$

16.22 CONVOLUTION

Convolution is used to find ILT in solving differential equations and integral equations.

Suppose two Laplace transforms $f(s)$ and $g(s)$ are given. Let $F(t)$ and $G(t)$ be their inverse Laplace transforms respectively, i.e., $F(t) = L^{-1}\{f(s)\}$ and $G(t) = L^{-1}\{g(s)\}$. Then the inverse $H(t)$ of the product of transforms $h(s) = f(s)\cdot g(s)$ can be calculated from the known inverse $F(t)$ and $G(t)$.

Convolution $H(t)$ of $f(t)$ and $G(t)$ is denoted by $(F*G)(t)$ and is defined as

$$H(t) = (F*G)(t) = \int_0^t F(u)\cdot G(t-u)\,du$$

$$= \int_0^t G(u)\cdot F(t-u)\,du$$

$$= (G*F)(t) \tag{45}$$

$F*G$ is called the *convolution or falting of F and G* and can be regarded as a “generalized product” of these functions.

Theorem 21: Convolution Theorem

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$$\boldsymbol{L^{-1}\{f(s)\cdot g(s)\} = F*G = \int_0^t F(u)\cdot G(t-u)\,du}$$

Proof From the definition of Laplace transform,

$$L\{F*G\} = \int_0^\infty e^{-st}(F*G)\,dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t F(r)\cdot G(t-r)\,dr\right]dt$$

$$= \int_0^\infty\int_0^t e^{-st}F(r)G(t-r)\,dr\,dt$$

$$= \iint_R e^{-st}F(r)\cdot G(t-r)\,dr\,dt \tag{7}$$

where R is the 45° wedge bounded by the lines $r = 0$ and $t = r$ {see Fig. 16.9}. Change the variables r, t to the new variables u, v by the transformation

$$u = t - r,\ v = r$$

$$J(u, v) = \frac{\partial(u, v)}{\partial(t, r)} = \begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial r} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \qquad (46)$$

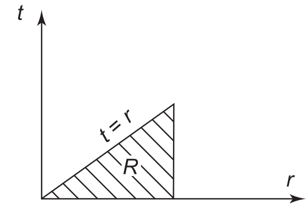


Fig. 16.9

The Jacobian $J = 1$. Thus, the double integral over R transforms to a double integral over R the first quadrant of the new uv plane [see Fig. 16.10].

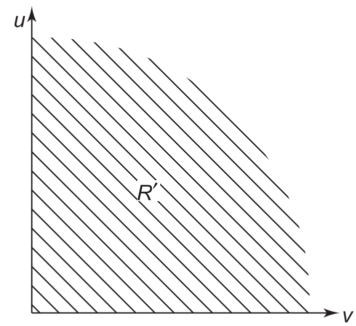


Fig. 16.10

$$\therefore \quad L\{F * g\} = \iint_{R'} e^{-s(u+v)} F(v)\cdot G(u)\, du\, dv$$

$$L\{F * G\} = \int_0^\infty \int_0^\infty e^{-s(u+v)} F(v)\cdot G(u)\, du\, dv$$

$$= \int_0^\infty e^{-sv} F(v)\, dv \cdot \int_0^\infty e^{-su} G(u)\, du$$

$$= L\{F(t)\} \cdot L\{G(t)\}$$

$$= f(s).\ g(s)$$

$$\therefore \quad F * G = L^{-1}\{f(s)\cdot g(s)\} \qquad (47)$$

Hence, proved.

Properties of Convolution

(P_1)	$F * G = G * F$	Commutative property	(48)
(P_2)	$(F * G) * H = F * (G * H)$	Associative property	(49)
(P_3)	$F * (G + H) = F * G + F * H$	Distributive property	(50)
(P_4)	$F * O = O * F = O$		

Example 52 Use convolution theorem to find

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$$

Solution We know that $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$ and $L^{-1}\left(\frac{1}{s+b}\right) = e^{-bt}$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = e^{-at} * e^{-bt} = \int_0^t e^{-au}\cdot e^{-b(t-u)}\, du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}\,du = e^{-bt}\left[\frac{e^{-(a-b)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-at}-e^{-bt}}{b-a}$$

Example 53 Find $L^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\}$.

Solution

$$L^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} * L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{1}{\sqrt{\pi t}} * e^t \qquad \text{[By convolution theorem]}$$

$$= \int_0^t (\pi u)^{-1/2}\cdot e^{t-u}\,du \qquad [\text{put } u = x^2 \Rightarrow du = 2x\,dx]$$

$$= e^t\cdot\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= e^t\cdot\operatorname{erf}(\sqrt{t})$$

Example 54 Find $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Solution

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2}\cdot\frac{1}{s^2+1} = f(s)\cdot g(s)$$

so that $F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t$

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$\therefore$ by convolution theorem,

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = F*G = \int_0^t (t-u)\sin u\,du$$

$$= -t\cos t + t + t\cos t - \sin t$$

$$= (t - \sin t)$$

Example 55 Find $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$.

Solution

$$\frac{1}{(s^2+a^2)^2} = \left(\frac{1}{(s^2+a^2)}\right)\cdot\left(\frac{1}{(s^2+a^2)}\right) = f(s)\cdot g(s)$$

$$\therefore \qquad F(t) = L^{-1}\{f(s)\} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{2}\sin at$$

$$\text{and} \qquad G(t) = L^{-1}\{g(s)\} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{2}\sin at$$

By convolution theorem,

$$\begin{aligned}
L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] &= F*G = \int_0^t \frac{\sin au}{a}\cdot\frac{\sin a(t-u)}{a}\,du \\
&= \frac{1}{2a^2}\int_0^t [\cos a(2u-t) - \cos at]\,du \\
&= \frac{1}{2a^2}\left[\frac{\sin a(2u-t)}{2a} - \cos at\cdot u\right]_{u=0}^{t} \\
&= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \frac{\sin(-at)}{2a}\right] \\
&= \frac{1}{2a^3}[\sin at - at\cos at]
\end{aligned}$$

16.23 THE HEAVISIDE EXPANSION FORMULA

Theorem 22

If $y(s) = \dfrac{F(s)}{G(s)}$, where $F(s)$ and $G(s)$ are polynomials in s, the degree of $F(s)$ < degree $G(s)$ and if $G(s)$ $= (s-\alpha_1)(s-\alpha_2)\,(s-\alpha_3)\,\ldots\,(s-\alpha_n)$
where $\alpha_1, \alpha_2, \ldots \alpha_r$ are distinct constants real or complex.

$$\text{Then } y(t) = L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{r=1}^{n}\frac{F(\alpha_r)}{G'(\alpha_r)}\cdot e^{\alpha_r t} \qquad (51)$$

Example 56 Use the Heaviside expansion theorem to find

$$L^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s}\right]$$

Solution Here, $G(s) = s^3 + s^2 - 2s$, $F(s) = 2s^2 + 5s - 4$

$$G'(s) = 3s^2 + 2s - 2$$

Put $G(s) = 0 \Rightarrow s(s^2 + s - 2) = 0$

$$s(s-1)(s+2) = 0$$

$$s = 0, 1, -2$$

$$\therefore \quad L^{-1}\left[\frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s}\right] = \sum_{r=1}^{3} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$$

$$= \frac{F(0)}{G'(0)} e^{0t} + \frac{F(1)}{G'(1)} e^{t} + \frac{F(-2)}{G'(-2)} e^{-2t}$$

$$= \frac{-4}{-2} + \frac{3}{2} e^{t} - \frac{6}{6} e^{-2t}$$

$$= 2 + \frac{3}{2} e^{t} - e^{-2t}$$

16.24 METHOD OF FINDING RESIDUES

(M_1) Residue at Simple Pole

(i) If $f(z)$ has a simple pole at $z = z_0$ then $\text{Res}\,(z = z_0) = \lim_{z \to z_0} \left[(z - z_0) \cdot f(z)\right]$

(ii) If $f(z)$ is of the form $f(z) = \dfrac{f_1(z)}{f_2(z)}$ where $f_1(z_0) = 0$ but $f_2(z_0) \neq 0$ then

$$\text{Res}\,(z = z_0) = \frac{f_1(z_0)}{f'_2(z_0)} \tag{52}$$

(M_2) Residue at a Pole of Order 'n'

$$\text{Res}\,(z = z_0) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}(z - z_0)^n \cdot f(z)\right]_{z = z_0} \tag{53}$$

(M_3) Residue at a Pole of Any Order

(M_3) residue at a pole $z = z_0$, of any order where $z = a + t$.

Then $\text{Res}\,(z = z_0) = \text{Res}\, f(z_0) =$ coefficient of $\left(\dfrac{1}{t}\right)$. (54)

16.25 INVERSION FORMULA FOR THE LAPLACE TRANSFORM

If $F(t)$ has a continuous derivative and is of exponential order, then

$$F(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s)\, ds; \; t > 0 \tag{55}$$

$=$ sum of the residues of $e^{st} f(s)$ at poles of $f(s)$

where $f(s) = L\{F(t)\}$

Example 57 Use the method of residues to show that

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \frac{1}{9}e^{-t} + \frac{t}{3}e^{2t} - \frac{1}{9}e^{2t}$$

Solution

$$L^{-1}\left[\frac{1}{(s+1)(s-2)^2}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\frac{1}{(s+1)(s-2)^2}\,ds$$

$$= \frac{1}{2\pi i}\int_c e^{st}\,\frac{1}{(s+1)(s-2)^2}\,ds$$

$$= \text{sum of residues of } e^{st}\,\frac{1}{(s+1)(s-2)^2}$$

At simple pole $s = -1$ and double pole $s = 2$.

$$= \lim_{s\to -1}\left[(s+1)\cdot\frac{e^{st}}{(s+1)(s-2)^2}\right] + \lim_{s\to 2}\frac{d}{ds}\left[\frac{(s-2)^2\cdot e^{st}}{(s+1)(s-2)^2}\right]$$

$$= \lim_{s\to -1}\left[\frac{e^{st}}{(s-2)^2}\right] + \lim_{s\to 2}\frac{d}{ds}\left[\frac{e^{st}}{(s+1)}\right]$$

$$= \frac{e^{-t}}{9} + \lim_{s\to 2}\frac{(s+1)\cdot t\,e^{st} - e^{st}}{(s+1)^2}$$

$$= \frac{e^{-t}}{9} + \frac{3t\,e^{2t} - e^{2t}}{9}$$

$$= \frac{1}{9}e^{-t} + \frac{t}{3}e^{2t} - \frac{1}{9}e^{2t}$$

Hence, proved.

Example 58 Use the method of residue, find the inverse transform of $\dfrac{(s+1)}{s^2+2s}$.

Solution

$$f(s) = \frac{(s+1)}{s^2+2s} = \frac{s+1}{s(s+2)} \quad \text{for pole, put } s(s+2) = 0$$

$$s = 0 - 2$$

$$\therefore \quad L^{-1}\left[\frac{s+1}{s(s+2)}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,.\,\frac{s+1}{s(s+1)}\,ds$$

$$= \frac{1}{2\pi i}\int_c e^{st}\cdot\frac{s+1}{s(s+1)}\,ds$$

= sum of the residues of $e^{st} f(s)$ at the poles $s = 0, -2$.

$$= \lim_{s\to 0}\left[(s-0)\frac{e^{st}(s+1)}{s(s+2)}\right] + \lim_{s\to -2}\left[\frac{(s+2)e^{st}\cdot(s+1)}{s(s+2)}\right]$$

$$= \lim_{s\to 0}\left[\frac{e^{st}(s+1)}{(s+2)}\right] + \lim_{s\to -2}\left[\frac{e^{st}(s+1)}{s}\right]$$

$$= \frac{1}{2} + \frac{1}{2}e^{-2t}$$

EXERCISE 16.2

Find the Inverse Laplace transform for each of the following functions:

1. $\dfrac{1}{s^2+4}$

2. $\dfrac{8s}{s^2+16}$

3. $\dfrac{3s-12}{s^2+8}$

4. $\dfrac{3s-8}{4s^2+25}$

5. If $L^{-1}\left\{\dfrac{e^{-\frac{1}{5}}}{\sqrt{s}}\right\} = \dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, show that $L^{-1}\left\{\dfrac{e^{-\frac{a}{5}}}{\sqrt{s}}\right\} = \dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$, when $a > 0$.

6. If $L^{-1}\left\{\dfrac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$, show that $L^{-1}\left\{\dfrac{9s^2-1}{(9s^2+1)^2}\right\} = \dfrac{t}{9}\cdot\cos\dfrac{t}{3}$

7. $\dfrac{s}{(s+1)^5}$

8. $\dfrac{3s+2}{4s^2+12s+9}$

9. $\tan^{-1}\dfrac{2}{s^2}$

10. $\log\left(\dfrac{s(s+1)}{s^2+4}\right)$

11. $\cot^{-1}\left(\dfrac{s+3}{2}\right)$

12. $\dfrac{s^2}{s^2+1}$

13. $\dfrac{1}{s(s^2+a^2)}$

14. $\dfrac{e^{-3s}}{(s-4)^2}$

15. $\dfrac{3}{s} - \dfrac{4e^{-s}}{s^2} + \dfrac{4e^{-3s}}{s^2}$

16. $\dfrac{2s+12}{s^2+6s+13}$

17. $\dfrac{2s^2+15s+7}{(s+1)^2(s^2+4)}$

18. $\dfrac{s^2+9s-9}{s^3-9s}$

19. Show that $L^{-1}\left\{\left(\frac{\sqrt{s-1}}{s}\right)^2\right\} = 1 + t - 4\sqrt{\frac{t}{\pi}}$

20. Show that $L^{-1}\left\{\frac{3s-2}{s^{5/2}} - \frac{7}{3s+2}\right\} = 6\sqrt{\frac{t}{\pi}} - \frac{8}{3}t\sqrt{\frac{t}{\pi}} - \frac{7}{3}e^{-\frac{2t}{3}}$

21. Show that $L^{-1}\left\{\frac{s}{(s+1)^{\frac{5}{2}}}\right\} = \frac{2}{3}e^{-t}\sqrt{\frac{t}{\pi}}(3-2t)$

22. Show that $L^{-1}\left\{\frac{s}{(s+3)^{\frac{7}{2}}}\right\} = e^{-3t}\left[\frac{4}{3\sqrt{\pi}}t^{3/2} - \frac{8}{5\sqrt{\pi}}t^{5/2}\right]$

23. Show that $L^{-1}\left\{\frac{1}{s\sqrt{s+4}}\right\} = \frac{1}{2}erf(2\sqrt{t})$

24. Show that $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$

25. Show that $1*1*1*1 = \frac{t^3}{3!}$

26. Show that $1*1*1*1*\cdots*1(n \text{ times}) = \frac{t^{n-1}}{(n-1)!}$

27. Apply the convolution theorem to show that $\int_0^t \sin u.\cos(t-u)\,du = \frac{t}{2}\sin t$

28. Show that $L^{-1}\left\{\frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right\} = \frac{3}{50}e^{3t} - \frac{1}{25}e^{-2t}$

$$-\frac{1}{50}e^{-t}\cos 2t + \frac{9}{25}e^{-t}\sin 2t$$

Apply Heaviside's expansion theorem to prove the following:

29. $L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$

30. $L^{-1}\left\{\frac{s+5}{(s+1)(s^2+1)}\right\} = 2e^{-t} + 3\sin t - 2\cos t$

31. Show that $\int_0^\infty x\cos x^3\,dx = \frac{\pi}{3\sqrt{3}\left\lceil\frac{1}{3}\right.}$

32. Show that $L^{-1}\left\{\dfrac{e^{-x\sqrt{s}}}{\sqrt{s}}\right\} = \dfrac{e^{-x^2/4t}}{\sqrt{\pi t}}$

33. Show that $\int_0^t J_0\,(u)\sin(t-u)\,du = t\cdot J_1(t)$

34. Show that $L^{-1}\left\{\dfrac{1}{s}\cdot\sin\dfrac{1}{s}\right\} = t - \dfrac{t^3}{(3!)^2} + \dfrac{t^5}{(5!)^2} - \dfrac{t^7}{(7!)^2} + \cdots$

35. Show that $L^{-1}\left\{\dfrac{1}{s}\cos\dfrac{1}{s}\right\} = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \cdots.$

36. Show that $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\} = t\,e^{-t} + 2\,e^{-t} + t - 2$

37. By use of the complex inversion Formula, show that

$$L^{-1}\left\{e^{-\frac{a\sqrt{s}}{s}}\right\} = \text{erf}\,c\left(\frac{a}{2\sqrt{t}}\right)$$

Answers

1. $\dfrac{1}{2}\sin 2t$

2. $8\cos 4\,t$

3. $3\cos 2\sqrt{2}\,t - 3\sqrt{2}\sin 2\sqrt{2}\,t$

4. $\dfrac{3}{4}\cos\dfrac{5t}{2} - \dfrac{4}{5}\sin\dfrac{5t}{2}$

7. $\dfrac{e^{-t}}{24}(4t^3 - t^4)$

8. $\dfrac{3}{4}e^{-\frac{3t}{2}} - \dfrac{5}{8}t\cdot e^{-\frac{3t}{2}}$

9. $\dfrac{2}{t}\cdot\sin h\,t\cdot\sin t$

10. $\dfrac{2\cos 2t - e^{-t} - 1}{t}$

11. $\dfrac{2}{t}e^{-3t}\sin 2t$

12. $\cos t$

13. $(1 - \cos at)/a^2$

14. $(t-3)\,e^{4(t-3)}.\,H\,(t-3)$

15. $3 - 4\,(t-1)\,H\,(t-1) + 4\,(t-3)\,H\,(t-3)$

16. $e^{-3t}\,(2\cos 2t + 3\sin 2t)$

17. $(2\,t - 3)e^{-t} + 5\,e^{2t}$

18. $(1 + 3\sin h\,3\,t)$

16.26 APPLICATIONS OF LAPLACE TRANSFORM

16.26.1 Introduction

The Laplace transform is a mathematical tool that greatly facilitates the solution of linear differential equations. Its application leads the differential equation to be transformed into a relatively simple algebraic equation. They can be transformed back into a complete solution. It eliminates one of the variables (say t) when used partial differential equation. This method is very useful specially when the boundary conditions are actually initial conditions, i.e., those at $t = 0$.

16.26.2 Solution of Ordinary Linear Differential Equations with Constant Coefficients

Let the differential equation

$$\frac{d^n y}{dt^n} + C_1 \frac{d^{n-1} y}{d t^{n-1}} + \cdots + C_{n-1} \frac{dy}{dt} + C_n y = f(t) \tag{56}$$

be a linear differential equation with constant coefficients when $t > 0$ and $f(t)$ is a given function of the variable t.

Suppose we find a solution of this equation satisfying the conditions.

$$y = y_0, \frac{dy}{dt} = y_1, \ldots, \frac{d^{n-1}y}{dt^{n-1}} = y_{n-1} \tag{57}$$

when $t = 0$

Applying Laplace transform to each term of (56), we get

$$L\left\{\frac{d^n y}{dt^n}\right\} + C_1 L\left\{\frac{d^{n-1} y}{d t^{n-1}}\right\}^{+} + \cdots + C_{n-1} L\left\{\frac{dy}{dt}\right\} + C_n L\{y\} = L\{f(t)\} \tag{58}$$

Let $L\{f(t)\} = \overline{f}(s)$ and $L\{y\} = \overline{y}$

Now,

$$L\left\{\frac{d^n y}{d t^n}\right\} = s^n \overline{y} - s^{n-1} (y)_{t=0} - s^{n-2} \left(\frac{dy}{dt}\right)_{t=0} - \cdots$$

$$- s\left(\frac{d^{n-2} y}{d t^{n-2}}\right)_{t=0} - \left(\frac{d^{n-1} y}{d t^{n-1}}\right)_{t=0}$$

$$= s^n \overline{y} - s^{n-1} y_0 - s^{n-2} y_1 - \cdots - s y_{n-2} - y_{n-1}$$

$$L\left\{\frac{d^{n-1} y}{d t^{n-1}}\right\} = s^{n-1} \overline{y} - s^{n-2} y_0 - \cdots - y_{n-2}$$

Substituting these expressions in (58), we get.

$$s^n \overline{y} - (s^{n-1} y_0 + s^{n-2} y_1 + \cdots + s y_{n-2} + y_{n-1})$$

$$+ C_1 \{s^{n-1} \overline{y} - (s^{n-2} y_0 + s^{n-3} y_1 + \cdots + y_{n-2})\} + \cdots + C_{n-1} (s\overline{y} - y_0) + c_n \overline{y} = \overline{f}(s)$$

or

$$(s^n + c_1 s^{n-1} + \cdots + c_{n-1} s + c_n)\overline{y} = \overline{f}(s) + c_{n-1} y_0 + C_{n-2} (s y_0 + y_1) + \cdots$$

$$+ (s^{n-1} y_0 + s^{n-2} y + \cdots + y_{n-1}) \tag{59}$$

Equation (59) is called the *subsidiary equation*.

Note that without going through all this process, Eq. (58) can be directly written from (56) by keeping the following points in view.

1. On the LHS, write $\dfrac{d^r y}{d t^r} = s^r \bar{y}$ for $r = 0, 1, 2, \ldots, n$.
2. On the RHS write the Laplace transform of $\bar{f}(t)$, say $\breve{f}(s)$. The terms involving the initial conditions (57) are to be written according to the following rule.

Write $\quad y_0$ for $\dfrac{dy}{dt}$

Write $\quad s\,y_0 + y_1$ for $\dfrac{d^2 y}{d t^2}$

Write $\quad s^2\, y_0 + s\, y_1 + y_2$ for $\dfrac{d^3 y}{d t^3}$

Write $\quad s^{r-1}\, y_0 + s^{r-2}\, y_1 + \cdots + y_{r-1}$ for $\dfrac{d^r y}{d t^r}$

The solution of (91) will give $\bar{y}(s)$ and the I.L.T. will give the value of y.

By Lerch's theorem, which you have already read in the previous section, this solution is taken as unique solution.

Example 59 Solve $(D^2 + 1)\, y = 0$, $t > 0$ given that $y = 1$ $Dy = 1$ when $t = 0$.

Solution Given $(D^2 + 1)\, y = 0$ (1)

$$y_0 = 1,\ y_1 = 1 \tag{2}$$

Taking L.T. both sides on (1), we have the subsidiary equation is

$$(s^2 + 1)\,\bar{y} = s + 1$$

$$\bar{y} = \frac{s+1}{s^2+1}$$

$$\bar{y} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

Taking ILT, we get

$$L^{-1}\{y\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y = \cos t + \sin t$$

Example 60 Solve $\dfrac{d^2 y}{d t^2} + 2\dfrac{dy}{dt} + y = t$; given that $y = -3$, when $t = 0$, $y = -1$, when $t = 1$.

Solution We can write the given equation into the form of

$$(D^2 + 2\,D + 1)\, y = t \tag{1}$$

Consider $y'(0) = A$

Taking L.T. both sides on (1), we get the subsidiary equation

$$(s^2 + 2s + 1)\,\bar{y} = \frac{1}{s^2} + 2\, y_0 + 5\, y_0 + y'(0)$$

$$= \frac{1}{s^2} + 2(-3) + \{s(-3) + A\}$$

$$= \frac{1}{s^2} - 6 - 3s + A$$

$$\therefore \quad \bar{y} = \frac{1}{s^2(s+1)^2} - \frac{6+3s-A}{(s+1)^2}$$

$$= -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3(s+1)+3-A}{(s+1)^2}$$

$$= -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{s+1} - \frac{3}{(s+1)^2} + \frac{A}{(s+1)^2}$$

$$= \frac{1}{s^2} - \frac{2}{s} - \frac{1}{s+1} + \frac{A-2}{(s+1)^2}$$

Taking I.L.T. on both sides, we get

$$y = t - 2 - e^t + (A-2)\, t \cdot e^{-t}$$

Now, we are to impose the other condition that $y = -1$ when $t = 1$.

$\therefore \quad -1 = 1 - 2 - e^{-t} + (A-2)e^{-1}$

i.e., $\quad A = 3$

$\therefore$ the complete solution is

$$y = t - 2 - e^t + t\, e^{-t}$$

Example 61 Solve $(D^2 + a^2)\, y = \cos at$, $t > 0$, given that $y = y_0$, $Dy = y$, when $t = 0$.

Solution Taking Laplace transform on both sides, we get

$$L\{(D^2 + a^2)y\} = L\{\cos at\}$$

$$(s^2 + a^2)\bar{y} = \frac{s}{s^2 + a^2} + s\, y_0 + y_1$$

or

$$\bar{y}(s) = \frac{s}{(s^2 + a^2)^2} + \frac{s\, y_0}{s^2 + a^2} + \frac{y_1}{s^2 + a^2}$$

Taking I.L.T., we get

$$y(t) = \frac{t}{2a}\sin at + y_0 \cos at + \frac{y_1}{a}\sin at$$

Example 62 Solve $(D^2 + 1)\, y = 6 \cos 2\, t$, give that $y = 3$, $y' = 1$ when $t = 0$.

Solution Taking LT on both sides on the given equation then, the subsidiary equation is

$$(s^2 + 1)\bar{y} = L\{6 \cos 2t\} + s\, y_0 + y_1$$

$$= \frac{6s}{s^2 + 4} + 3s + 1$$

or
$$\bar{y} = \frac{6s}{(s^2+1)(s^2+4)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1}$$
$$= \frac{2s}{s^2+1} - \frac{2s}{s^2+4} + \frac{3s}{s^2+1} + \frac{1}{s^2+1}$$

Taking I.L.T., we get
$$y(t) = 2\cos t - 2\cos 2t + 3\cos t + \sin t$$
$$= 5\cos t - 2\cos 2t + \sin t$$

Example 63 Solve $(D^2 + 1)\, y = x \,.\, \cos 2x$, $x > 0$, given that $y = 0 = y'$ when $x = 0$.

Solution Here, $L\{x \cdot \cos 2x\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$
$$= \frac{1}{s^2+4} - \frac{8}{(s^2+4)^2}$$
$$y_0 = 0,\ y_1 = 0$$

$\therefore$ The subsidiary equation is
$$(s^2+1)\,\bar{y}(s) = \frac{1}{s^2+4} - \frac{8}{(s^2+4)^2}$$

or
$$\bar{y} = \frac{1}{(s^2+1)(s^2+4)} - \frac{8}{(s^2+1)(s^2+4)^2}$$
$$= \frac{1}{3}\frac{1}{(s^2+1)} - \frac{1}{3(s^2+4)} - \frac{8}{9}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4} - \frac{3}{(s^2+4)^2}\right]$$
$$= -\frac{5}{9}\frac{1}{s^2+1} + \frac{5}{9}\frac{1}{(s^2+4)} + \frac{8}{3}\frac{1}{(s^2+4)^2}$$

Taking I.L.T., we get
$$y(x) = -\frac{5}{9}\sin x + \frac{5}{18}\sin 2x + \frac{1}{3}\left(\frac{1}{2}\sin 2x - x\cos 2x\right)$$

Example 64 Solve $(D^2 + 25)\, y = 10 \cos 5t$, given that $y = 2$, $y' = 0$ when $t = 0$.

Solution Taking LT of the given DE, we get

The subsidiary equation
$$\left[s^2\,\bar{y} - s\,y(0) - y'(0)\right] + 25\,\bar{y} = 10 \cdot \frac{s}{s^2+25}$$
$$(s^2+25)\,\bar{y} = \frac{10s}{s^2+25} + 2s$$

or
$$\bar{y} = \frac{10s}{(s^2+25)^2} + \frac{2s}{s^2+25}$$

Taking I.L.T., we get

$$y(t) = t \sin 5t + 2 \cos 5t$$

Example 65 Solve $(D^2 + 9)\, y = \cos 2t$, given that

$$y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1$$

Solution $(D^2 + 9)\, y = \cos 2t$ (1)

Taking LT of both sides of(1), we get

The subsidiary equation.

$$s^2 \bar{y} - s\,y(0) - y'(0) + 9\bar{y} = \frac{s}{s^2+4}$$

or $$(s^2+9)\bar{y} = \frac{s}{s^2+4} + s + A \quad [y'(0) = A \text{ say}]$$

or $$\bar{y} = \frac{s^3+5s}{(s^2+4)(s^2+9)} + \frac{A}{s^2+9}$$

$$\bar{y} = \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{A}{s^2+9}$$

Taking I.L.T., we get

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t \tag{2}$$

On putting $y\left(\frac{\pi}{2}\right) = -1$ in (2), we get

$$-1 = -\frac{1}{5} + 0 - \frac{A}{3}$$

or $$A = \frac{12}{5}$$

Putting the value of A in (2), we get

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$

$$y(t) = \frac{1}{5}[\cos 2t + 4\cos 3t + 4\sin 3t]$$

EXERCISE 16.3

Solve the following differential equations:

1. $(D^2 + 1)\, y = 0$, given that $y = 1$, $y' = 1$ at $t = 0$.
2. $(D^2 - 3D + 2)y = 4\, e^{2t}$, given $y = -3$, $y' = 5$ at $t = 0$

3. $(D^2 + 2D + 5)y = 0$, given $y = 2$, $y' = -4$, when $t = 0$.
4. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 3\cos 3t - 11\sin 3t$; given $y(0) = 0$ and $y'(0) = 6$
5. $\frac{d^3y}{dt^3} 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$; given $y(0) = y'(0) = 0$ and $y''(0) = 6$
6. $\frac{d^2y}{dt^2} + 9y = 18t$; given $y(0) = 0 = y\left(\frac{\pi}{2}\right)$
7. $\frac{d^2y}{dt^2} - y = a\cosh t$; given $y(0) = 0 = y'(0)$
8. $(D^2 + \omega^2)y = \cos \omega t$, $t > 0$; given $y(0) = 0 = y'(0)$
9. $\frac{dy}{dt} - y = e^t$; given $y(0) = 1$

Answers

1. $y = \sin t + \cos t$
2. $-7e^t + 4e^{2t} + 4te^{2t}$
3. $y(t) = e^{-t}(2\cos 2t - \sin 2t)$
4. $y(t) = e^t - e^{-2t} + \sin 3t$
5. $y(t) = e^t - 3e^{-t} + 2e^{-2t}$
6. $y(t) = \pi \sin 3t + 2t$
7. $y(t) = \left(\frac{at}{2}\right)\cdot \sin h\, t$
8. $y(t) = \left(\frac{1}{2\omega}\right)\sin \omega t$
9. $y(t) = e^t + te^t = (1 + t)e^t$

16.27 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

The Laplace transform technique is very useful in solving the equations having the terms of the form $t^m y^n(t)$ whose Laplace transform is

$$(-1)^m \frac{d^m}{ds^m} L\{y^n(t)\} \tag{60}$$

Example 66 Solve $\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0$; given $y(0) = 0$, $y'(0) = 1$.

Solution Taking LT on both sides of the given equations, we get

$$L\{y''\} + L\{ty'\} - L\{y\} = L\{0\}$$

$$s^2\bar{y} - s\,y(0) - y'(0) - \frac{d}{ds}L\{y'\} - L\{y\} = 0$$

or $$s^2\bar{y} - 1 - \frac{d}{ds}[s\bar{y} - y(0)] - \bar{y} = 0$$

or $$-s\frac{d\bar{y}}{ds}+(s^2-2)\bar{y}-1 = 0$$

or $$\frac{d\bar{y}}{ds}+\left(\frac{2}{s}-s\right)\bar{y}-\frac{1}{s} = 0$$

$$\frac{d\bar{y}}{ds}+\left(\frac{2}{s}-s\right)\bar{y} = \frac{1}{s}$$

which is linear in $\bar{y}$ and s

$\therefore$ $$\text{IF} = e^{\int P\,ds} = e^{\int\left(\frac{2}{s}-s\right)ds}$$

$$\text{IF} = e^{2\log s-\frac{s^2}{2}}$$

$$= s^2\cdot e^{-\frac{s^2}{2}}$$

The complete solution is

$$\bar{y}\cdot s^2\cdot e^{-\frac{s^2}{2}} = \int\left(-\frac{1}{s}\right)s^2\cdot e^{-\frac{s^2}{2}}\,ds+c$$

$$\bar{y}\cdot s^2\cdot e^{-\frac{s^2}{2}} = e^{-\frac{s^2}{2}}+c;\ c \text{ is any constant}$$

c must vanish if $\bar{y}(s)$ is a transform since, $\bar{y}(s)\to 0$ as $s\to\infty$

$\therefore$ $$\bar{y}(s) = \frac{1}{s^2} \text{ or } L^{-1}\left\{\frac{1}{s^2}\right\}.$$

or $y(t) = t$ which is the required solution.

EXERCISE 16.4

Solve the following differential equations:

1. $ty''+y'+4ty=0$, given $y(0)=3$, $y'(0)=0$.
2. $\dfrac{d^2y}{dt^2}-t\dfrac{dy}{dt}+y=1$, given $y(0)=1$, $y'(0)=2$.
3. $t\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+ty=0$, given $y(0)=0$, $y'(0)=0$

Answers

1. $y=3J_0(2t)$ 2. $y=1+2t$ 3. $y=2J_0(t)$

16.28 SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

In this case also the same transformations procedure on each of the equations is applied.

Example 67 Solve $\dfrac{dx}{dt} - y = e^t, \dfrac{dy}{dt} + x = \sin t$; given $x(0) = 1$, $y(0) = 0$.

Solution We can write the above equations into the form of

$$D x - y = e^t \text{ and } Dy + x = \sin t$$

Taking LT of the given equations, we get

$$[s\tilde{x} - x(0)] - \tilde{y} = \frac{1}{s-1}$$

or $$s\tilde{x} - 1 - \tilde{y} = \frac{1}{s-1}$$

or $$s\tilde{x} - \tilde{y} = \frac{s}{s-1} \qquad (1)$$

and $$s\tilde{y} - y(0) + \tilde{x} = \frac{1}{s^2+1}$$

or $$\tilde{x} + s\tilde{y} = \frac{1}{s^2+1} \qquad (2)$$

Solving (1) and (2) for $\tilde{x}$ and $\tilde{y}$, we have

$$\tilde{x} = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$\tilde{x} = \frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2}$$

and $$\tilde{y} = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

$$\tilde{y} = \frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

Taking I.L.T. of both sides, we get

$$x = \frac{1}{2}L^{-1}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}\left[e^t + \cos t + \sin t\right] + \frac{1}{2}(\sin t - t\cos t)$$

$$\left[\because\ L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^2}(\sin at - at\cos at)\right]$$

$$= \frac{1}{2}\left[e^t + \cos t + 2\sin t - t\cos t\right]$$

and $$y = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}t\sin t - \frac{1}{2}\left[e^t - \cos t - \sin t\right]$$

$$= \frac{1}{2}\left[t\sin t - e^t + \cos t - \sin t\right] \qquad \left[\because\ L^{-1}\left[\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^2}t\sin \text{at}\right]\right]$$

Hence $$x = \frac{1}{2}\left[e^t + \cos t + 2\sin t - t\cos t\right]$$

$$y = \frac{1}{2}\left[t\sin t - e^t + \cos t - \sin t\right]$$

Example 68 Solve

$(D^2 + 2)\,x - D\,y = 1,$

$D\,x + (D^2 + 2)\,y = 0,$ with $t > 0$, $x\,(0) = 0$, $x'\,(0) = 0$,

$y\,(0) = 0 = y'\,(0)$, x and y both being functions of t.

Solution Taking the LT of the given equations, we get

$$\left[s^2\,\tilde{x} - s\,x(0) - \tilde{x}(0)\right] - \left[s\,\tilde{y} - y(0)\right] + 2\tilde{x} = \frac{1}{s}$$

$$s^2\,\tilde{x} - 2\tilde{x} - s\,\tilde{y} = \frac{1}{s}$$

or $$(s^2+2)\,\tilde{x} - s\,\tilde{y} = \frac{1}{s} \qquad (1)$$

and $$\left[s\,\tilde{x} - x(0)\right] + \left[s^2\,\tilde{y} - s\,y(0) - y'(0)\right] + 2\,\tilde{y} = 0$$

$$s\,\tilde{x} + (s^2+2)\,\tilde{y} = 0 \qquad (2)$$

Solving (1) and (2) for $\tilde{x}, \tilde{y}$, we get

$$\tilde{x} = \frac{s^2+2}{s(s^2+1)(s^2+4)} = \frac{1}{2s} - \frac{s}{3(s^2+1)} - \frac{s}{6(s^2+4)}$$

$$\tilde{y} = -\frac{1}{(s^2+1)(s^2+4)} = -\frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

Taking ILT of both sides, we get

$$x = \frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t$$

and $$y = -\frac{1}{3}\sin t + \frac{1}{6}\sin 2t$$

EXERCISE 16.5

Solve the following simultaneous Differential equations:

1. $3\frac{dx}{dt}+3\frac{dy}{dt}+5x=25\cos t$

   $2\frac{dx}{dt}-3\frac{dy}{dt}=5\sin t$ given that $x(0) = 2$, $y(0) = 3$.

2. $(D^2 + 3).\, x - 2y = 0$, $(D^2 - 3)x + (D^2 + 5)\, y = 0$
   given that $x(0) = 0 = y(0)$, $x'(0) = 3$, $y'(0) = 2$.
3. $Dx + y = \sin t$, $x + Dy = \cos t$; given that $x(0) = 2$, $y(0) = 0$.
4. $(D - 2)\, x - (D + 1)\, y = 6e^{3t}$.
   $(2D - 3)\, x + (D - 3)\, y = 6e^{3t}$; given that $x(0) = 3$, $y(0) = 0$
5. $Dx + 4y = 0$, $Dy - 9x = 0$; given that $x(0) = 2$, $y(0) = 1$.

Answers

1. $x = 2\cos t + 3\sin t$, $y = 3\cos t + 2\sin 2t$
2. $x=-\frac{11}{4}\sin t+\frac{1}{12}\sin 3t$, $y=\frac{11}{4}\sin t-\frac{1}{4}\sin 3t$.
3. $x = e^{t} + e^{-t}$, $y = e^{-t} - e^{t} + \sin t$.
4. $x = e^{t} + 2te^{t} + 2e^{3t}$, $y = \sin h\, t + \cos ht - e^{-3t} - te^{t}$.
5. $x=-\frac{2}{3}\sin 6t+2\cos 6t$, $y = \cos 6t + 3\sin 6t$.

16.29 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS (PDE)

Let y be a function of x and t.

Let the PDE be

$$\frac{\partial^2 y}{\partial x^2}+c_1\frac{\partial^2 y}{\partial x\,\partial t}+c_2\frac{\partial^2 y}{\partial t^2}+c_3\frac{\partial y}{\partial x}+c_4\frac{\partial y}{\partial t}+c_5\, y=f(x,t) \tag{61}$$

Let t be the principal variable and x be the secondary variable independent of t.

Consider $L\{y(x, t)\} = \tilde{y}(x, s)$

Then $$L\left\{\frac{\partial y}{\partial t}\right\} = s\tilde{y}(x,s)-y(x,0)$$

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = s^2\tilde{y}(x,s)-sy(x,0)-\left(\frac{dy}{dt}\right)_{t=0}$$

Also, we have

$$L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d}{dx}\tilde{y}(x,s)$$

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2}{dx^2}\tilde{y}(x, s)$$

$$L\left\{\frac{\partial^2 y}{\partial x \partial t}\right\} = s\frac{d}{dx}\tilde{y}(x, s) - y'(x, 0)$$

Applying LT to every term of (61) and using the above value, we get

$$\frac{d^2}{dx^2}\tilde{y}(x, s) + (c_1 s + c_2)\frac{d}{dx}\tilde{y}(x, s) + [c_2 s^2 + c_4 s + c_5]\tilde{y}(x, s)$$

$$= \overline{f}(x, s) + (c_1 + c_2 s + c_4)y(x, 0) + c_2\left(\frac{dy}{dt}\right)_{t=0} \quad (62)$$

Here, $\tilde{y}(x, s)$ is the unknown function and the functions of x, $y(x, 0)$ and $\left(\dfrac{dy}{dt}\right)_{t=0}$ constitute the initial conditions. Since Eq. (62) is of second order, it is necessary to know two additional limiting conditions. If these conditions are of the form

$y(x_0, t) = a$ then Eq. (62) is in LT Having determined $\tilde{y}(x, s)$, we can determine $y(x, t)$ by Inverse Laplace Transforms (ILT).

Example 69 Solve $\dfrac{\partial y}{\partial t} = 2\dfrac{\partial^2 y}{\partial x^2}$; given that $y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = -10 \sin 4\pi x$.

Solution Taking Laplace transform of the given equation, we get

$$L\left\{\frac{\partial y}{\partial t}\right\} = L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

$$s\,\tilde{y}(x, s) - y(x, 0) = 2\frac{d^2\tilde{y}(x, s)}{dx^2}$$

or $$s\,\tilde{y} - 10\sin 4\pi x = 2\frac{d^2\tilde{y}}{dx^2}$$

or $$2\frac{d^2\tilde{y}}{dx^2} - s\,\tilde{y} = -10 \sin 4\,\pi x$$

or $$\frac{d^2\tilde{y}}{dx^2} - \frac{1}{2}s\,\tilde{y} = -5 \sin 4\,\pi x$$

$$\left(D^2 - \frac{s}{2}\right)\tilde{y} = -5 \sin 4\pi x;\ D \equiv \frac{d}{dx}$$

AE is $$m^2 - \frac{s}{2} = 0 \Rightarrow m = \pm\sqrt{\frac{s}{2}}$$

$$\text{CF} = c_1 e^{\left(\sqrt{\frac{s}{2}}\right)x} + c_2 e^{-\left(\sqrt{\frac{s}{2}}\right)x}$$

$$\text{PI} = \frac{1}{\left(D^2 - \frac{s}{2}\right)}(-5\sin 4\pi x)$$

$$= \frac{-5}{-16\pi^2 - \frac{s}{2}}\sin 4\pi x = \frac{10}{32\pi^2 + s}\sin 4\pi x$$

The general solution is

$$\tilde{y}(x, s) = C.F. + P.I. = c_1 e^{x\sqrt{\frac{s}{2}}} + c_2 e^{-x\sqrt{\frac{s}{2}}} + \frac{10}{32\pi^2 + s}\sin 4\pi x \qquad (1)$$

Now using boundary conditions:

$$y(0, t) = 0 \Rightarrow \tilde{y}(0, t) = 0$$

$\therefore$ $$0 = c_1 e^{ox} + c_2 e^{ox} + 0$$

or $$c_1 + c_2 = 0 \qquad (2)$$

$$y(5, t) = 0 \Rightarrow \tilde{y}(5, t) = 0$$

$$0 = c_1 e^{+5\sqrt{s/2}} + c_2 e^{-5\sqrt{s/2}} + \frac{10}{32\pi^2 + s}\sin 20\pi$$

Since $\sin 20\pi = 0$,

$\therefore$ this is possible only when $c_1 = 0 = c_2$

Then (1), becomes

$$\tilde{y}(x, s) = \frac{10}{32\pi^2 + s}\sin 4\pi x$$

Taking ILT, we get

$$y(x, t) = L^{-1}\left[\frac{10}{32\pi^2 + s}\sin 4\pi x\right]$$

$$y(x, t) = 10e^{-32\pi^2 t}\cdot\sin 4\pi x$$

Example 70 Solve $\frac{\partial y}{\partial x} - \frac{\partial y}{\partial t} = 1 - e^{-t}; 0 < x < 1, t > 0$

Given that $y(x, 0) = x$

Solution Taking LT of the given equation, we get

$$L\left(\frac{\partial y}{\partial x}\right) - L\left(\frac{\partial y}{\partial t}\right) = L(1 - e^{-t})$$

$$\frac{d\tilde{y}}{dx} - [s\tilde{y} - y(x, 0)] = \frac{1}{s} - \frac{1}{s+1}$$

or $$\frac{d\tilde{y}}{dx} - s\tilde{y} + x = \frac{1}{s} - \frac{1}{s+1}$$

or $$\frac{d\tilde{y}}{dx} - s\tilde{y} = \left(\frac{1}{s} - \frac{1}{s+1}\right) - x$$ which is LDE in $\tilde{y}$ and x

$$\text{IF} = e^{\int -s dx} = e^{-sx}$$

Complete solution is

$$\tilde{y}\cdot e^{-sx} = \int\left(\frac{1}{s}e^{-sx} - \frac{1}{s+1}e^{-sx} - xe^{-sx}\right)dx$$

or $$\tilde{y} = \frac{1}{s}x + \frac{1}{s(s+1)}$$

$$\tilde{y} = \frac{1}{s}x + \frac{1}{s} - \frac{1}{s+1}$$

Taking ILT, we get

$$y(x, t) = x + 1 - e^{-t}$$

EXERCISE 16.6

Solve the following PDE:

1. $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$; given that $y\left(\dfrac{\pi}{2}, t\right) = 0$
   $\left(\dfrac{\partial y}{\partial x}\right)_{x=0} = 0$ and $y\,(x, 0) = 30 \cos 5x$.
2. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$; given $u\,(x, 0) = \sin \pi x$.
3. $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$; given $u\,(0, t) = 0$, $u(5, t) = 0$
   $u(x, 0) = 10 \sin 4\,\pi x - 5 \sin 6\,\pi x$.
4. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - 4u$, given $u\,(0, t) = 0, = u(\pi, t)$
   $u(x, 0) = 6 \sin x - 4 \sin 2\,x$.
5. $\dfrac{\partial^2 y}{\partial t^2} = 9\dfrac{\partial^2 y}{\partial x^2}$; given $y\,(0, t) = 0 = y\,(2, t)$
   $y\,(x, 0) = \sin 5\,\pi x$

Answers

1. $y\,(x, t) = 30\, e^{-75t} \cos 5x$.
2. $u\,(x, t) = \sin \pi x \cdot e^{-s^2 t}$
3. $u(x, t) = -5e^{-72\pi^2 t} \sin 6\pi x + 10e^{-32\pi^2 t} \sin 4\pi x$
4. $u(x, t) = -6e^{-4t} \sin x - 4e^{-8t} \sin 2x$
5. $y\,(x, t) = \sin 5\pi x \cdot \cos 5\pi t$

SUMMARY

1. Definition of Laplace Transform

The Laplace transform of a function F(t), denoted by $L\{F(t)\}$ is defined as

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} \cdot F(t)dt$$

Here, L is called the *Laplace transformation operator*. The parameter s is a real or complex number and t is a variable independent of s. In general, the parameters are taken to be a real positive number.

2. Laplace Transforms of Elementary Functions

(i) $L\{1\} = \frac{1}{s}; s > 0$

(ii) $L\{t^n\} = \frac{\overline{|(n+1)}}{s^{n+1}}$, if $s > 0$ and $n > -1$

In a particular case, if n is a positive integer then

$\overline{|n+1} = n!$ and hence $L\{t^n\} = \frac{n!}{s^{n+1}}$

(iii) $L\{e^{at}\} = \frac{1}{s-a}, s > a$

(iv) $L\{e^{-at}\} = \frac{1}{s+a}; s > -a$

(v) $L\{\cos at\} = \frac{s}{s^2+a^2}, L\{\sin at\} = \frac{a}{s^2+a^2}$

(vi) $L\{\sin h\, at\} = \frac{a}{s^2-a^2}$

(vii) $L\{\cos h\, at\} = \frac{s}{s^2-a^2}$

3. Linearity Property of Laplace Transform

Suppose $f(s)$ and $g(s)$ are Laplace transforms of $F(t)$ and $G(t)$ respectively. Then

$L\{aF(t) + bG(t)\} = aL\{F(t)\} + bL\{G(t)\}$ where a and b are any constants.

Note: Generalizing this result, we obtain

$$L\left\{\sum_{r=1}^{n} C_r F_r(t)\right\} = \sum_{r=1}^{n} C_r L\{F_r(t)\}$$

4. Theorem: First Shifting (or First Translation) Theorem

If $L\{F(t)\} = f(s)$ then $L\left\{e^{at} F(t)\right\} = f(s-a)$

5. Theorem: Second Translation or Heaviside's Shifting Theorem

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$

then $\quad L\{G(t)\} = e^{-as} f(s)$

Another form

If $f(s)$ is the Laplace transform of $F(t)$ and $a > 0$ then $e^{-as}f(s)$ is the Laplace transform of $F(t-a)\,H(t-a)$, where

$$H(t) = \begin{cases} 1 & \text{if } \; t > 0 \\ 0 & \text{if } \; t < 0 \end{cases}$$

6. Theorem: Change-of-Scale Property

If $L\{F(t)\} = f(s)$ then $L\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$

7. Laplace Transform of Derivatives

If $L\{F(t)\} = f(s)$ then $L\{F'(t)\} = sf(s) - F(0)$: $s > 0$

By applying the above theorem to $F''(t)$, we have

Result I $\quad L\{F''(t)\} = sL\{F'(t)\} - F(0)$

Result II $\quad = s\{sf(s) - F(0)\} - F'(0)$

$= s^2 f(s) - sF(0) - F'(0)$

Similarly, for LT of derivatives of order n,

$$L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \ldots - F^{(n-1)}(0)$$

8. Laplace Transform of (Multiplication by *t*)

If $L\{F(t)\} = f(s)$ then

$$L\{tF(t)\} = (-1)\frac{d}{ds} f(s) = (-1)\, f'(s)$$

In general, $\quad L\{t^n F(t)\} = (-1^n)\dfrac{d^n}{ds^n} f(s) = (-1)^n\, f^{(n)}(s)$

9. Laplace Transform of the Integral of a Function

If $L\{F(t)\} = f(s)$ then

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}; \quad s > 0$$

Result: Similarly if $L\{F(t)\} = f(s)$ then

$$L\left\{\int_0^t dt_1 \int_0^{t_1} F(u)\,du\right\} = \frac{1}{s^2} f(s)$$

Generalization for n^{th} integral

$$L\left\{\int_0^t \int_0^t \text{n-times} \int_0^t F(t)\,dt^n\right\} = \frac{1}{s^n} f(s)$$

10. Integration of Laplace Transform of Division by *t*

If $L\{F(t)\} = f(s)$ then

$$L\left\{\frac{F(t)}{t}\right\} = \int_S^\infty f(u)\, du$$

11. Heaviside's Unit Function (Unit-Step Function)

If a function is of the form

$F(t) = 0;\ t < a$

$\quad = 1;\ t > a$

then $H(t - a)$ is defined as the Heaviside's unit function. See the following figure.

Let us consider a function $H(t)$ given by

$$H(t) = \begin{cases} F(t) & , \text{ for } t \geq 0 \\ 0 & , \text{ for } t < a \end{cases}$$

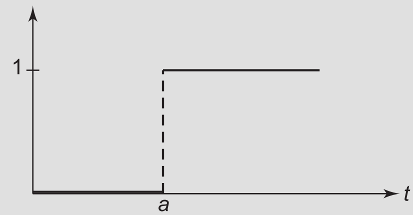


Fig. 16.11

12. Dirac' Delta Function (or Unit-Impulse Function)

Dirac delta function (or unit-impulse function) denoted by $\delta(t - a)$ as $\in \to 0$, i.e.,

$$\delta(t-a) = \lim_{\in \to 0} \cdot\ F_\in(t-a)$$

Then the Laplace transform of the Dirac delta function is obtained as

$$L\{\delta(t-a)\} = \lim_{\in \to 0} \cdot L\{F_\in(t-a)\}$$

$$= \lim_{\in \to 0} e^{-as} \cdot \frac{(1 - e^{-\in s})}{\in s}$$

$$L\{\delta(t-a)\} = e^{-as}$$

Thus, the Dirac delta function is a "generalized function" defined as

$$\delta(t-a) = \begin{cases} \infty, & \text{if } t = a \\ 0, & \text{otherwise} \end{cases}$$

subject to $\int_0^\infty \delta(t-a)\, dt = 1$

13. Laplace Transform of a Periodic Function

A function $F(t)$ is said to be a periodic function of period $T > 0$ if

$$F(t) = F(t + T) = F(t + 2T) = \ldots = F(t + nT).$$

The Laplace Transform of a periodic function $F(t)$ with period T is

$$L\{F(t)\} = \frac{1}{(1 - e^{-sT})} \int_0^T e^{-st}\, F(t)\, dt, \qquad s > 0$$

14. The Error Function

The error function, abbreviated as 'erf'' or 'E', is defined as

$$erf\ x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx$$

and *erfc x*, read as 'the complement of error function', is defined as

$$\text{erfc } x = 1 - \text{erf } x$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2}\, dx$$

15. Initial- and Final-Value Theorems

(i) Initial-Value Theorem

If $L\{F(t)\} = f(s)$ then

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} s\, f(s)$$

(ii) Final-Value Thorem

If $L\{F(t)\} = f(s)$ then

$$\lim_{t \to \infty} F(t) = \lim_{s \to \infty} s\, f(s)$$

16. The Inverse Laplace Transform

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e., $L\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace transform (ILT) of $f(s)$.

We also write $F(t) = L^{-1}\{f(s)\}$.

L^{-1} is called the *inverse Laplace transformation operator*.

The inverse Laplace transforms of the functions $F(s)$ are given in the following table:

Table 16.3

$F(s)$	$F(t) = L^{-1}\{f(s)\}$
$1/s$	1
$1/s^2$	t
$\frac{1}{s^{n+1}}$	$\frac{t^n}{\lceil n+1}$
$\frac{1}{s-a}$	e^{at}
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a} \sin h\, at$
$\frac{s}{s^2-a^2}$	$\cos h\, at$

17. Linearity Property

If $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$ and if a, b are constants then

$$L^{-1}\{a\, f(s) + b\, g(s)\} = a\, L^{-1}\{f(s)\} + b\, L^{-1}\{G(s)\} = a\, F(t) + b\, G(t)$$

18. First Shifting Property

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f(s-a)\} = e^{at} \cdot F(t)$.

19. Second Shifting Property

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{e^{-as} f(s)\} = G(t)$, where

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t > a \end{cases}$$

Remarks: This result is also expressible as

$$L^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

or $$L^{-1}\{e^{-as} f(s)\} = F(t-a) \cdot H(t-a)$$

20. Change-of-scale Property

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f(as)\} = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$

21. Inverse Laplace Transforms of Derivatives

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f^{(n)}(s)\} = L^{-1}\left\{\dfrac{d^n}{ds^n} f(s)\right\} = (-1)^n \cdot t^n \cdot F(t)$

22. Inverse Laplace Transform of Division by s

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\dfrac{f(s)}{s}\right\} = \displaystyle\int_0^t F(u)\, du$

Also, $L^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = \displaystyle\int_0^t \int_0^t F(u)\, du\, du$

In general, $L^{-1}\left\{\dfrac{f(s)}{s^n}\right\} = \displaystyle\int_0^t \int_0^t \ldots \int_0^t F(u)\, du^n$

23. Inverse Laplace Transforms of Integrals

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\displaystyle\int_s^\infty f(u)\, du\right\} = \dfrac{F(t)}{t}$

24. Inverse Laplace Transforms of Multiplication by *s*

If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$ then $L^{-1}\{s\, f(s)\} = F'(t)$

25. Convolution

Suppose two Laplace transforms $f(s)$ and $g(s)$ are given. Let $F(t)$ and $G(t)$ be their inverse Laplace transforms respectively, i.e., $F(t) = L^{-1}\{f(s)\}$ and $G(t) = L^{-1}\{g(s)\}$. Then the inverse $H(t)$ of the product of transforms $h(s) = f(s) \cdot g(s)$ can be calculated from the known inverse $F(t)$ and $G(t)$.
Convolution $H(t)$ of $F(t)$ and $G(t)$, denoted by $(F * G)(t)$ is defined as

$$H(t) = (F*G)(t) = \int_0^t F(u) \cdot G(t-u)\, du = \int_0^t G(u) \cdot F(t-u)\, du = (G * F)(t)$$

$F*G$ is called the convolution of F and G, and can be regarded as a generalized product of these functions.

26.

$$L^{-1}\{f(s)\} = F(t) \text{ and } L^{-1}\{g(s)\} = G(t) \text{ then}$$

$$L^{-1}\{f(s) \cdot g(s)\} = F*G = \int_0^t F(u) \cdot G(t-u)\, du$$

27. The Heaviside Expansion Formula

If $y(s) = \dfrac{F(s)}{G(s)}$, where $F(s)$ and $G(s)$ are polynomials in s, the degree of $F(s) \leq$ degree $G(s)$ and if

$$G(s) = (s - \alpha_1)(s - \alpha_2)(s - \alpha_3) \ldots (s - \alpha_n)$$

where $\alpha_1, \alpha_2, \ldots \alpha_n$ are distinct constants real or complex

then $$y(t) = L^{-1}\left\{\frac{f(s)}{g(s)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} \cdot e^{\alpha_r t}$$

28. Method of Finding Residues

Method I: Residue at a simple pole

(i) If $f(z)$ has a simple pole at $z = Z$ then Res $(z = a_0) = \lim\limits_{z \to z_0} \left[(z - z_0) \cdot f(z)\right]$

(ii) If $f(z)$ is of the form $f(z) = \dfrac{f_1(z)}{f_2(z)}$ where $f_1(z_0) = 0$ but $f_2(z_0) \neq 0$ then

$$\text{Res } (z = z_0) = \frac{f_1(z_0)}{f'_2(z_0)}$$

Method II: Residue at a pole of order 'n'

$$\text{Res } (z = z_0) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}(z - z_0)^n \cdot f(z)\right]_{z=z_o}$$

Residue at a pole $z = z_0$, of any order where $z = a + t$.

Then Res $(z = z_0) =$ Res $f(z_0) =$ coefficient of $\left(\dfrac{1}{t}\right)$.

29. Inversion Formula for the Laplace Transform

If $F(t)$ has a continuous derivative and is of exponential order then

$$F(t) = \frac{1}{2\pi i}\int_{r+i\infty}^{r+i\infty} e^{st} f(s)\, ds\ ;\ t > 0$$

= sum of the residues of $e^{st} f(s)$ at poles of $f(s)$

where $f(s) = L\{F(t)\}$.

OBJECTIVE-TYPE QUESTIONS

1. The Laplace transform of a unit step ramp function starting at $t = a$, is

(a) $\frac{1}{(s+a)^2}$ (b) $\frac{e^{-as}}{(s+a)^2}$

(c) $\frac{e^{-as}}{s^2}$ (d) $\frac{a}{s^2}$

[GATE (ECE) 1994]

2. The inverse Laplace transform of the function $\frac{s+5}{(s+1)(s+3)}$ is

(a) $2e^{-t} - e^{-3t}$ (b) $2e^{-t} + e^{-3t}$

(c) $e^{-t} - 2e^{-3t}$ (d) $e^{t} + e^{-3t}$

[GATE (ECE) 1996]

3. The Laplace transform of $e^{\alpha t}$ cos at is equal to

(a) $\frac{s-\alpha}{(s-\alpha)^2 + a^2}$ (b) $\frac{s+\alpha}{(s-\alpha)^2 + a^2}$

(c) $\frac{1}{(s-\alpha)^2}$ (d) none of these

[GATE (ECE) 1997]

4. $f(s) = (s+1)^{-2}$ is the Laplace transform of

(a) t^2 (b) t^3

(c) te^{-t} (d) e^{-2t}

[GATE (ME) 1998]

5. The Laplace transform of $(t^2 - 2t)\, u(t-1)$ is

(a) $\frac{2e^{-s}}{s^3} - \frac{2}{s^2}e^{-s}$ (b) $\frac{2}{s^3}e^{-2s} - \frac{2}{s^2}e^{-s}$

(c) $\frac{2}{s^3}e^{-s} - \frac{1}{s}e^{-s}$ (d) none of these

[GATE (EE) 1998]

6. The Laplace transform of a unit step function $u_x(t)$ defined as

$$u_a(t) = \begin{cases} 0, & \text{for } 1 < a \\ 1, & \text{for } t > a \end{cases}$$

(a) $\frac{e^{-as}}{s}$ (b) se^{-as}

(c) $s - u(0)$ (d) $se^{-as} - 1$

[GATE (Civil Engineering) 1998]

7. Laplace transform of$(a + bt)^2$, where 'a' and 'b' are constants is given by

(a) $(a + bs)^2$ (b) $\frac{1}{(a+bs)^2}$

(c) $\frac{a^2}{s} + \frac{2ab}{s^2} + \frac{2b^2}{s^3}$ d) $\frac{a^2}{s} + \frac{2ab}{s^2} + \frac{b^2}{s^3}$

[GATE (ME) 1998]

8. If $L[f(t)] = \frac{s}{s^2 + \omega^2}$, then the value of $\lim_{t\to\infty} f(t)$ is

(a) cannot be determined

(b) is zero

(c) is unity

(d) is infinite

[GATE (ECE) 1998]

9. If $L[f(t)] = F(s)$, then $L[f(t - T)]$ is equal to

(a) $e^{+sT}F(s)$ (b) $e^{-sT}F(s)$

(c) $\frac{F(s)}{1 + e^{sT}}$ (d) $\frac{F(s)}{1 - e^{-sT}}$

[GATE (ECE) 1999]

10. The Laplace transform of the function

$$f(t)=\begin{cases}k, & 0<t<c\\ 0, & c<t<\infty\end{cases}\text{ is}$$

(a) $\frac{k}{s}e^{-ks}$ (b) $\frac{k}{s}e^{ks}$

(c) ke^{-ks} (d) $\left(\frac{k}{s}\right)(1-e^{-cs})$

[GATE (CE) 1999]

11. The Laplace transform of the function $\sin^2 2t$ is

(a) $\frac{(1/s)-s}{2(s^2+16)}$ (b) $\frac{s}{s^2+16}$

(c) $\frac{(1/s)-s}{s^2+4}$ (d) $\frac{s}{s^2+4}$

[GATE (ME) 2000]

12. Given $L[f(t)]=\frac{s+1}{s^2+1}$, $L[g(t)]$ $=\frac{s^2+1}{(s+3)(s+2)}$, $h(t)=\int_0^t f(t)g(t-\tau)d\tau$, $L[h(t)]$ is

(a) $\frac{s^2+1}{s+3}$

(b) $\frac{1}{s+3}$

(c) $\frac{s^2+1}{(s+3)(s+2)}+\frac{s+2}{s^2+1}$

(d) none of these

[GATE (ECE) 2000]

13. Let $F(s)=L[f(t)]$ denotes the Laplace transform of the function $f(t)$ which of the following statement is correct

(a) $L\left[\frac{df}{dt}\right]=\frac{1}{s}F(s), L\left[\int_0^t f(\tau)d\tau\right]$ $=-sF(s)-f(0)$

(b) $L\left[\frac{df}{dt}\right]=sF(s)-f(0)$, $L\left[\int_0^t f(\tau)d\tau\right]=-\frac{dF}{ds}$

(c) $L\left[\frac{df}{dt}\right]=sF(s)-F(0), L\left[\int_0^t f(\tau)d\tau\right]$ $=F(s-a)$

(d) $L\left[\frac{df}{dt}\right]=sF(s)-f(0), L\left[\int_0^t f(\tau)d\tau\right]$ $=\frac{F(s)}{s}$

[GATE (Civil Engg.) 2000]

14. The inverse Laplace transform of $\frac{1}{s^2+2s}$ is

(a) $(1-e^{2t})$ (b) $1-e^{-2t}$

(c) $\frac{1-e^{2t}}{2}$ (d) $\frac{1-e^{-2t}}{2}$

[GATE (Civil Engg.) 2001]

15. The Laplace transform of the following function

$$f(t)=\begin{cases}\sin t, & 0\le t<\pi\\ 0, & t>\pi\end{cases}\text{ is}$$

(a) $\frac{1}{1+s^2}, s>0$ (b) $\frac{1}{1+s^2}, s<\pi$

(c) $\frac{1+e^{-\pi s}}{1+s^2}, s>0$ (d) $\frac{e^{-\pi s}}{1+s^2}, s>0$

[GATE (Civil Engg.) 2002]

16. Let $Y(s)$ be the Laplace transform of the function $y(t)$, then the final value of the function is

(a) $\lim_{s\to 0} Y(s)$ (b) $\lim_{s\to\infty} Y(s)$

(c) $\lim_{s\to 0} sY(s)$ (d) $\lim_{s\to\infty} sY(s)$

[GATE (EE) 2002]

17. If L defines the Laplace transform of a function, $L[\sin at]$ will be equal to

(a) $\frac{a}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$

(c) $\frac{s}{s^2+a^2}$ (d) $\frac{s}{s^2-a^2}$

[GATE (Civil Engg.) 2003]

18. The Laplace transform of a function $f(t)$ is

$$F(s) = \frac{5s^2 + 23s + 6}{s(s^2 + 2s + 2)}$$

As $t \to \infty$, $f(t)$ approaches to

(a) 3 (b) 5

(c) $\frac{17}{2}$ (d) ∞

[GATE (EE) 2005]

19. Consider the function $f(t)$ having Laplace transform $F(s) = \frac{\omega_0}{s^2 + \omega_0^2}, R_e[s] > 0$. The final value of $f(t)$ would be

(a) 0 (b) 1

(c) $-1 \le f(\infty) \le 1$ (d) ∞

[GATE (ECE) 2006]

20. The Dirac-Delta function $\delta(t)$ is defined as

(a) $\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$

(b) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$

(c) $\delta(t) = \begin{cases} 1, & t > 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

(d) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

[GATE (ECE) 2006]

21. Given that $F(s)$ is the one-sided Laplace transform of $f(t)$, the Laplace transform of $\int_0^t (f(\tau)\, d\tau$ is

(a) $sF(s) - f(0)$ (b) $\frac{F(s)}{s}$

(c) $\int_0^s F(\tau)\, d\tau$ (d) $\frac{1}{s}[F(s) - f(0)]$

[GATE (ECE) 2009]

22. The inverse Laplace transform of $1/(s^2 + s)$ is

(a) $1 + e^t$ (b) $1 - e^t$

(c) $1 - e^{-t}$ (d) $1 + e^{-t}$

[GATE (ME) 2009]

23. The Laplace transform of $f(x) = \cosh ax$ is

(a) $\frac{a}{s^2 - a^2}$ (b) $\frac{s}{s^2 - a^2}$

(c) $\frac{a}{s^2 + a^2}$ (d) $\frac{s}{s^2 + a^2}$

[GATE (Civil Engg.) 2009]

24. The Laplace transform of the function $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ -1, & 2 \le t < 4 \end{cases}$, $f(t + 4) = f(t)$ is given as

(a) $\frac{1 - e^{-2t}}{s(1 + e^{-2s})}$ (b) $\frac{1 + e^{-2s}}{s(1 + e^{-2s})}$

(c) 0 (d) $\frac{s + 1}{s - 1}$

[UPTU 2009]

25. The inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$ is given by

(a) $\frac{2\cosh t}{t}$ (b) $\frac{2 \sinh t}{t}$

(c) $2t \cos t$ (d) $2t \sin t$

[UPTU 2009]

26. A delayed unit step function is defined as

$$u(t - a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \ge a \end{cases}$$

Its Laplace transform is

(a) ae^{-as} (b) $e^{-as/s}$

(c) e^{as}/s (d) $e^{-as/a}$

[GATE (ME) 2004]

27. An input $x(t) = \exp(-2t)u(t) + \delta(t - 6)$ is applied to an LTI system with impulse response $h(t) = u(t)$. The output is

(a) $[1 - \exp(-2t)]\, u(t) + u(t + 6)$
(b) $[1 - \exp(-2t)]\, u(t) + u(t - 6)$
(c) $0.5[1 - \exp(-2t)]\, u(t) + u(t + 6)$
(d) $0.5[1 - \exp(-2t)]\, u(t) + u(t - 6)$

[GATE (EC) 2011]

28. If $F(s) = L\{f(t)\} = \frac{2(s+1)}{s^2 + 4s + 7}$ then the initial and final values of $f(t)$ are respectively

(a) 0, 2 (b) 2, 0

(c) $0, \frac{2}{7}$ (d) $\frac{2}{7}, 0$

[GATE (EC) 2011]

29. The inverse Laplace transform of the funtion $F(s) = \frac{1}{s(s+1)}$ is given by

(a) $f(t) = \sin t$ (b) $f(t) = e^{-t} \sin t$

(c) $f(t) = e^{-t}$ (d) $f(t) = 1 - e^{-t}$

[GATE (ME) 2012]

30. The unilateral Laplace transform of $f(t)$ is $\frac{1}{s^2+s+1}$. The unilateral Laplace transform of $t\,f(t)$ is

(a) $-\frac{s}{(s^2+s+1)^2}$ (b) $-\frac{2s+1}{(s^2+s+1)^2}$

(c) $\frac{s}{(s^2+s+1)^2}$ (d) $\frac{2s+1}{(s^2+s+1)^2}$

[GATE (EE) 2012]

31. The solution to the differential equation $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$ where k is a constant, subject to the boundary, conditions $u(0) = 0$ and $u(L) = U$ is

(a) $u = U\frac{x}{L}$ (b) $u = U\left(\frac{1-e^{kx}}{1-e^{kL}}\right)$

(c) $u = U\left(\frac{1-e^{-kx}}{1-e^{-kL}}\right)$ (d) $u = U\left(\frac{1+e^{-kx}}{1+e^{-kL}}\right)$

[GATE (ME) 2013]

32. The function $f(t)$ satisfies the differential equation $\frac{d^2f}{dt^2} + f = 0$ and the auxiliary conditions, $f(0) = 0$, $\frac{\partial f}{\partial t}(0) = 4$. The Laplace transform of $f(t)$ is given by

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$

(c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^4+1}$

[GATE (ME) 2013]

33. The Laplace transform of $f(t) = 2t + 6$ is

(a) $\frac{1}{s} + \frac{2}{s^2}$ (b) $\frac{3}{4} - \frac{6}{s^2}$

(c) $\frac{6}{s} + \frac{2}{s^2}$ (d) $-\frac{6}{s} + \frac{2}{s^2}$

[GATE (BT) 2013]

34. The Laplace Transform representation of the triangular pulse shown below is

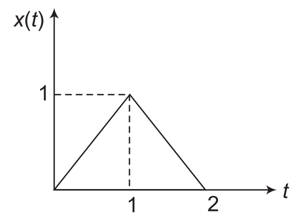


Fig. 16.12

(a) $\frac{1}{s^2}[1 + e^{-2s}]$

(b) $\frac{1}{s^2}[1 - e^{-s} + e^{-2s}]$

(c) $\frac{1}{s^2}[1 - e^{-s} + 2e^{-2s}]$

(d) $\frac{1}{s^2}[1 - 2e^{-s} + e^{-2s}]$

[GATE (IN) 2013]

35. A system is described by the differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = x(t)$. Let $x(t)$ be a rectangular pulse given by

$$x(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

Assuming that $y(0) = 0$ and $\frac{dy}{dt} = 0$ at $t = 0$, the Laplace transform of $y(t)$ is

(a) $\frac{e^{-2s}}{s(s+2)(s+3)}$ (b) $\frac{1-e^{-2s}}{s(s+2)(s+3)}$

(c) $\frac{e^{-2s}}{(s+2)(s+3)}$ (d) $\frac{1-e^{-2s}}{(s+2)(s+3)}$

[GATE (EC) 2013]

36. Laplace transform of cos (ωt) is $\dfrac{s}{s^2+\omega^2}$.

The Laplace transform of $e^{-2t}\cos(4t)$ is

(a) $\dfrac{s-2}{(s-2)^2+16}$ (b) $\dfrac{s+2}{(s-2)^2+16}$

(c) $\dfrac{s-2}{(s+2)^2+16}$ (d) $\dfrac{s+2}{(s+2)^2+16}$

[GATE (ME) 2014]

37. For the time domain function, $f(t) = t^2$, which ONE of the following is the Laplace transform of $\int_0^t f(t)\,dt$?

(a) $\dfrac{3}{s^4}$ (b) $\dfrac{1}{4s^2}$

(c) $\dfrac{2}{s^3}$ (d) $\dfrac{2}{s^4}$

[GATE (CH) 2014]

38. The unilateral Laplace transform of $f(t)$ is $\dfrac{1}{s^2+s+1}$. Which one of the following is the unilateral Laplace transform of $g(t) = t\,f(t)$?

(a) $\dfrac{-s}{(s^2+s+1)}$ (b) $\dfrac{-(2s+1)}{(s^2+s+1)^2}$

(c) $\dfrac{s}{(s^2+s+1)^2}$ (d) $\dfrac{2s+1}{(s^2+s+1)^2}$

[GATE (EC) 2014]

39. A system is described by the following differential equation, where $u(t)$ is the input to the system and $y(t)$ is the output of the system.

$$y(t) + 5y(t) = u(t)$$

When $y(0) = 1$ and $u(t)$ is a unit step function, $y(t)$ is

(a) $0.2 + 0.8e^{-5t}$ (b) $0.2 - 0.2e^{-5t}$

(c) $0.8 + 0.2e^{-5t}$ (d) $0.8 - 0.2e^{-5t}$

[GATE (EC) 2014]

ANSWERS

1. (c)	2. (a)	3. (a)	4. (c)	5. (c)	6. (a)	7. (c)	8. (a)	9. (b)	10. (d)
11. (a)	12. (d)	13. (d)	14. (d)	15. (c)	16. (c)	17. (b)	18. (a)	19. (c)	20. (a)
21. (b)	22. (c)	23. (b)	24. (a)	25. (b)	26. (a)	27. (d)	28. (b)	29. (d)	30. (d)
31. (b)	32. (c)	33. (c)	34. (d)	35. (b)	36. (d)	37. (d)	38. (d)	39. (a)	

Appendix: Basic Formulae and Concepts

1. Greek Letters

α	alpha	β	beta
γ	gamma	δ	delta
ι	iota	λ	lambda
ζ	zeta	η	eta
ξ	xi	ψ	psi
ϕ	phi	$\in$	epsilon
θ	theta	κ	kappa
π	pi	μ	mu
ν	nu	ρ	rho
σ	sigma	τ	tau
χ	chi	ω	omega
Γ	capital gamma	Δ	capital delta
Σ	capital sigma	Ω	capital omega

2. Notation

$\cup$	union	$\cap$	intersection
$\in$	belongs to	$\notin$	does not belong to
$\Rightarrow$	implies	$\Leftrightarrow$	implies and implied by
$\ni$	such that		

3. Properties of Logarithms

(i) $\log_a a = 1$, $\log_a 1 = 0$, $\log_a 0 = -\infty$ if $a > 1$,
$\log_{10} 10 = 1$, $\log_e 2 = 0.6931$, $\log_e 10 = 2.3060$, $\log_{10} e = 0.4343$

(ii) $\log_a m^n = n \log_a m$

(iii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iv) $\log_a(mn) = \log_a m + \log_a n$

(v) $\log_a m = \log_b b \cdot \log_b n = \frac{\log_b n}{\log_b a}$

4. Differential Calculus

(i) $\frac{d}{dx}(e^x) = e^x$

(ii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(iii) $\frac{d}{dx}(\sin x) = \cos x$

(iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(viii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(ix) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(x) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(xi) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(xiii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xiv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xv) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xvi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(xvii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(xviii) $\frac{d}{dx}(\sinh x) = \cosh x$

(xix) $\frac{d}{dx}(\cosh x) = \sinh x$

(xx) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(xxi) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(xxii) $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

(xxiii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(xxiv) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (chain rule)

(xxv) $dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds$ (total derivative rule)

(xxvi) $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

5. n^{th} Derivative

(i) $\frac{d^n}{dx^n} e^{ax+b} = a^n e^{ax+b}$

(ii) $\frac{d^n}{dx^n}(ax+b)^m = m(m-1)(m-2)\cdots(m-n+1)\, a^n (ax+b)^{m-n}$

(iii) $\dfrac{d^n}{dx^n}[\log(ax+b)] = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

(iv) $\dfrac{d^n}{dx^n}\{\sin(ax+b)\} = a^n \sin\left(ax+b+\dfrac{n\pi}{2}\right)$

(v) $\dfrac{d^n}{dx^n}\{\cos(ax+b)\} = a^n \cos\left(ax+b+\dfrac{n\pi}{2}\right)$

(vi) $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

(vii) $\dfrac{d^n}{dx^n}\{e^{ax}\sin(bx+c)\} = e^{ax}\cdot(a^2+b^2)^{n/2}\sin(bx+c+n\tan^{-1} b/a)$

(viii) $\dfrac{d^n}{dx^n}\{e^{ax}\cos(bx+c)\} = e^{ax}(a^2+b^2)^{\frac{n}{2}}\cos(bx+c+n\tan^{-1} b/a)$

(ix) **Leibnitz's Theorem:**

$$\frac{d^n}{dx^n}(u\cdot v) = n_{c_0}u_n v + n_{c_1}u_{n-1}v_1 + n_{c_2}u_{n-2}v_2 + \cdots + n_{c_r}u_{n-r}v_r + \cdots + n_{c_n}uv_n$$

6. Integral Calculus

(i) $\int x^n dx = \dfrac{x^{n+1}}{n+1}\ (n \ne 1)$

(ii) $\int e^x dx = e^x$

(iii) $\int \dfrac{1}{x} dx = \log_e x$

(iv) $\int a^x dx = a^x / \log_e a$

(v) $\int \cos x\, dx = \sin x$

(vi) $\int \sin x\, dx = -\cos x$

(vii) $\int \tan x\, dx = -\log\cos x = \log\sec x$

(viii) $\int \cot x\, dx = \log\sin x$

(ix) $\int \sec x\, dx = \log(\sec x + \tan x)$

(x) $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x)$

(xi) $\int \sec^2 x\, dx = \tan x$

(xii) $\int \operatorname{cosec}^2 x\, dx = -\cot x$

(xiii) $\int \sqrt{(a^2-x^2)}\, dx = \dfrac{x\sqrt{(a^2-x^2)}}{2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$

(xiv) $\int \sqrt{(a^2+x^2)}\, dx = \dfrac{x\sqrt{(a^2+x^2)}}{2} + \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right)$

(xv) $\int \sqrt{(x^2-a^2)}\, dx = \dfrac{x\sqrt{(x^2-a^2)}}{2} - \dfrac{a^2}{2}\cosh^{-1}\left(\dfrac{x}{a}\right)$

(xvi) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(xvii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

(xviii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

(xix) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

(xx) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

(xxi) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

(xxii) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$

(xxiii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$

(xxiv) $\int \sinh x\,dx = \cosh x$

(xxv) $\int \cosh x\,dx = \sinh x$

(xxvi) $\int \tanh x\,dx = \log\cosh x$

(xxvii) $\int \coth x\,dx = \log\sinh x$

(xxviii) $\int \operatorname{sech}^2 x\,dx = \tanh x$

(xxix) $\int \operatorname{cosech}^2 x\,dx = -\coth x$

7. Definite Integral

(i) $\int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$

where $\phi(x)$ is the primitive or antiderivative of a function $f(x)$ defined on $[a, b]$.

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

(v) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

(vi) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(vii) If $f(x)$ is a periodic function with period T then

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$

(viii) If $f(x)$ is a periodic function with period T then $\int_a^{a+T} f(x)\,dx$ is independent of a.

8. Integral Function

Let $f(x)$ be a continuous function defined on $[a, b]$ then a function $F(x)$ defined by

$$F(x) = \int_a^b f(t)\,dt,\ x \in [a, b]$$

is called the integral function of the function f.

9. Summation of Series using Definite Integral as the Limit of a Sum

If $f(x)$ is an integrable function defined on $[a, b]$ then we define

$$\int_a^b f(x)\,dx = \lim_{n\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)n)$$

where $h = \dfrac{b-a}{n}$

As $h \to 0$, we have $n \to \infty$

$$\therefore \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} h\,[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h] \tag{1}$$

Putting $a = 0$, $b = 1$, $h = \dfrac{b-a}{n} = \dfrac{1-0}{n} = \dfrac{1}{n}$ in (1), we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right]$$

$$r\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) \tag{2}$$

Equation (2) is very useful in finding the summation of infinite series, which is expressible in the form $\dfrac{1}{n}\sum f\left(\dfrac{r}{n}\right)$.

To find the value of the given infinite series, first of all express the series in the form $\lim\limits_{n\to\infty}\left[\dfrac{1}{n}\sum f\left(\dfrac{r}{n}\right)\right]$, and replace $\sum$ by $\int$, $\dfrac{r}{n}$ by x and $\dfrac{1}{n}$ by dx. Obtain lower and upper limits by computing $\lim\limits_{n\to\infty}\left(\dfrac{r}{n}\right)$ for the least and greatest values of r respectively. Then evaluate the integral.

10. Vectors

(i) If vector $\vec{r} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$ then

$|\vec{r}| = \sqrt{(x^2 + y^2 + z^2)}$ and direction-cosines of $\vec{r}$ will be

$$\frac{x}{\sqrt{(x^2+y^2+z^2)}}, \frac{y}{\sqrt{(x^2+y^2+z^2)}}, \frac{z}{\sqrt{(x^2+y^2+z^2)}}$$

(ii) Unit vector along $\vec{r}$ is

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

(iii) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1, \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ and

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0; \hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$ (clockwise product)

$\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}, \hat{i}\times\hat{k} = -\hat{j}$ (anticlockwise product)

(iv) Properties of Scalar Product

(a) The scalar product of two vector's is commutative, i.e.,

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(b) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ and $(\vec{b}+\vec{c})\vec{a} = \vec{b}\cdot\vec{a} + \vec{c}\cdot$

(c) Let $\vec{a}, \vec{b}$ be two vectors inclined at an angle θ. Then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

(d) Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}\perp\vec{b}$

[read as vector $\vec{a}$ perpendicular to $\vec{b}$]

(v) Properties of Vector Product

(a) Vector product is not commutative, i.e., if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$, however, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

(b) If $\vec{a}$ and $\vec{b}$ are two vectors and λ is a scalar then $\lambda\vec{a}\times\vec{b} = \lambda(\vec{a}\times\vec{b}) = \vec{a}\times\lambda\vec{b}$

(c) If $\vec{a}$ and $\vec{b}$ are two vectors and α, β are scalars then

$\alpha\vec{a}\times\beta\vec{b} = \alpha\beta(\vec{a}\times\vec{b}) = \alpha(\vec{a}\times\beta\vec{b}) = \beta(\alpha\vec{a}\times\vec{b})\cdot$

(d) Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors; then

$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ and $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$

(e) If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$ then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

(f) The vector product of two nonzero vectors is a zero vector iff they are parallel (collinear), i.e., $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b}, \vec{a}, \vec{b}$ are nonzero vectors.

(g) If $\vec{a}$ and $\vec{b}$ are adjacent sides of a triangle then the area of the triangle = $\frac{1}{2}|\vec{a}\times\vec{b}|$

(h) If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram then the area of the parallelogram = $|\vec{a}\times\vec{b}|$.

(i) The area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.

(j) The area of a plane quadrilateral $ABCD$ is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where AC and BD are its diagonals.

(vi) Scalar Triple Product

Let $\vec{a},\vec{b},\vec{c}$ be three vectors. Then the scalar $(\vec{a}\times\vec{b})\cdot\vec{c}$ is called the scalar triple products of $\vec{a},\vec{b}$, and $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ or $[\vec{b}\ \vec{c}\ \vec{a}]$ or $[\vec{c}\ \vec{a}\ \vec{b}]$.

Thus, $[\vec{a}\ \vec{b}\ \vec{c}]=(\vec{a}\times\vec{b})\cdot\vec{c}$

(vii) Coplanar Vectors

The necessary and sufficient condition for three nonzero, noncollinear vectors $\vec{a},\vec{b},\vec{c}$ to be coplanar is that $[\vec{a}\ \vec{b}\ \vec{c}]=0$ i.e., $\vec{a},\vec{b},\vec{c}$ coplanar $\Leftrightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

(viii) The volume of the parallelpiped with $\vec{a},\vec{b},\vec{c}$ as coterminous edges.

$= [\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})$

(ix) The volume of a tetrahedron, whose three coterminous edges in the right-handed system are $\vec{a},\vec{b},\vec{c}$ is $\frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\frac{1}{6}\left[\vec{a}\cdot(\vec{b}\times\vec{c})\right]$

(x) Vector Triple Product

Let $\vec{a},\vec{b},\vec{c}$ be any three vectors then the vectors $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are called vector triple product of $\vec{a},\vec{b},\vec{c}$, i.e., $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

11. Algebraic Formula

(i) Quadratic Equation

$$ax^2+bx+c=0 \tag{1}$$

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

Let $\alpha=\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $\beta=\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$ are the roots of the quadratic equation (1).

Then

(a) $\alpha+\beta=\dfrac{-b}{a}$

(b) $\alpha\beta=\dfrac{c}{a}$

(c) If $b^2-4ac>0$ then roots are real and distinct.

(d) If $b^2-4ac<0$ then roots are complex.

(e) If $b^2-4ac=0$ then roots are equal.

(f) If b^2-4ac is a perfect square then roots are rational.

(ii) Progressions

(a) Arithmetic progression (AP) $a, a + d, a + 2d, a + 3d, \dots$

its n^{th} term $T_n = a + (n - 1)d$ and sum of n terms = $S_n = \frac{n}{2}[2a + (n-1)d]$

(b) Geometric Progression (GP)

$a, ar, ar^2, ar^3, ar^4, \dots$

Its n^{th} term $T_n = ar^{n-1}$ and sum of n terms $S_n = \dfrac{a(1-r^n)}{1-r}$, $S_\infty = \dfrac{a}{1-r}; r < 1$

(c) Harmonic Progression (HP)

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \cdots$$

Its n^{th} term $T_n = \dfrac{1}{a+(n-1)d}$, $S_n = \dfrac{1}{\frac{n}{2}[(2a+(n-1)d)]}$

(d) Let a and b be any two numbers then their

Arithmetic Mean (AM) = $\frac{1}{2}(a+b)$

Harmonic Man (HM) = $\dfrac{2ab}{(a+b)}$

Geometric Mean (GM) = $\sqrt{(ab)}$

(e) Let $1, 2, 3, 4, 5, \dots, n$ be the set of natural numbers then

$$\sum n = \frac{n(n+1)}{2}$$

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

(iii) Binomial Expansion

(a) When n is a positive integer, $(1+x)^n = 1 + n_{c_1} + n_{c_2}x^2 + n_{c_3}x^3 + \cdots + n_{c_n}x^n$

(b) When n is a negative integer or a fraction

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty$$

(c) $T_{r+1} = n_{c_r} x^{n-r}$

(iv) Some Important Expansions

(a) Exponential Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

$$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \cdots \infty, (a > 0)$$

(b) Logarithmic Series

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$$

$$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \infty$$

(v) Sine and Cosine Series

(a) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \infty$$

(b) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \infty$$

12. Trigonometric Products and Sums

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(iii) $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

(iv) $$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$$

(v) $$\sin C - \sin D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$$

(vi) $$\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$$

(vii) $$\cos C - \cos D = 2 \sin \frac{C + D}{2} \sin \frac{D - C}{2}$$

(viii) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(ix) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

(x) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(xi) $2 \cos A \cos B = \cos (A + B) - \cos (A - B)$

(xii) $$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

(xiii) $\cos 2A = \cos^2 A - \sin^2 A$

$$= 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(xiv) $\tan 2A = \dfrac{2\tan A}{1+\tan^2 A}$

(xv) $\sin 3A = 3\sin A - 4\sin^3 A$

(xvi) $\cos 3A = 4\cos^3 A - 3\cos A$

(xvii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

13. Complex Numbers

(i) If $z = x + iy$
$= r\cos\theta + ir\sin\theta$; $\left(\text{Put} \quad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right)$

$z = r(\cos\theta + i\sin\theta)$

(ii) Euler's theorem: $e^{i\theta} = \cos\theta + i\sin\theta$

(iii) De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

14. Hyperbolic Functions

(i) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(ii) $\cosh x = \dfrac{ex^2 + e^{-x}}{2}$

(iii) $\tanh x = \dfrac{\sinh x}{\cosh x}, \coth x = \dfrac{\cosh x}{\sinh x}$

(iv) $\sin(ix) = i\sinh x$, $\cos(ix) = \cosh x$, $\tan(ix) = i\tanh x$

(v) $\sinh^{-1} x = \log[x + \sqrt{(x^2+1)}]$

(vi) $\cosh^{-1} x = \log[x + \sqrt{(x^2-1)}]$

(vii) $\tanh^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$

Index

A

Absolute convergence 5.62, 8.27
Alternating series 8.7
Analytic function 13.1
Applications of double integrals 5.81
Applications of integral calculus 5.43
Applications of Laplace transform 16.56
Applications of matrices 1.54
Applications of ordinary differential equations 10.41
Area of closed curves 5.17
Areas of curves 5.9
Asymptotes to a curve 4.19

B

Basic definitions 10.1
Bernoulli's Equation 10.22
Bessel's equation 6.1
Beta function 6.29
Binomial series 8.35
Bounds of a sequence 8.1

C

Calculation of *n*th order differential coefficients 2.3
Canonical form 1.51
Canonical form (or normal form) of a matrix 1.18
Cauchy's root (or radical) test 8.21
Cayley–Hamilton theorem 1.40
Centre of gravity 5.43
Change of interval 9.20
Change of order of integration 5.75
Characteristic roots and vectors (or eigenvalues and eigenvectors) 1.33
Charpit's method 14.50
Chemical reactions and solutions 10.46
Clairaut's equation 14.21
Classification of partial differential equations of order one 14.2
Classification of partial differential equations of second order 14.48
Classification of singularities 13.1
Comparison test 8.11
Complete solutions of $y'' + Py' + Qy = R$ in terms of one known solution belonging to the CF 12.1
Complex quadratic form 1.50
Composite function 3.21
Concavity 4.18
Condition for consistency theorem 1.28
Condition for inconsistent solution 1.28
Conditional convergence 8.27
Conservative field and scalar potential 7.15
Continuity 7.2
Convergence of a sequence 8.2
Convergence of improper integrals 5.55
Convexity 4.18

Convolution 16.48
Curve tracing 4.21
  cartesian coordinates 4.22
  parametric form 4.31
  in polar coordinates 4.27

D

D'Alembert's ratio test 8.16
Deduction from Euler's theorem 3.14
Definite integral 5.2
Definition of Laplace transform 16.1
Derivation of transmission line equations 15.50
Diagonalization matrix 1.43
Differentiability of a Vector Function 7.2
Differential operator D 11.1
Differentiation 2.1
Differentiation of transforms 16.13
Diracs delta function (or unit-impulse function) 16.17
Dirichlet's conditions for a fourier series 9.4
Dirichlet's integral 5.93
Divergence and curl 7.3
Double integrals 5.63
Duplication formula 6.35

E

Echelon form of a Matrix 1.17
Elementary operations 1.13
Elementary transformation 1.13
Equality of two matrices 1.5
Equation reducible to linear equations with constant coefficients 14.44
Error function 16.21
Euler's formulae 9.3
Euler's theorem on homogeneous functions 3.12
Exact differential equations 10.28
Expansion of functions of several variables 3.41
Expansion of functions of two variables 3.41
Exponential series 8.36

F

Final-value theorems 16.23
First-order and first-degree differential equations 10.3
Formation of an ordinary differential equation 10.2
Formation of partial differential equations 14.2
Fourier half-range series 9.21
Fourier series 9.2
Fourier series for discontinuous functions 9.4
Fourier series for even and odd functions 9.12
Frobenius method 13.12
Function of Class A 16.11
Functional dependence 3.30

G

Gamma function 6.31
Gauss's divergence theorem (relation between volume and surface integers) 7.28
General methods of finding particular integrals (pi) 11.6
Generating function for $J_n(x)$ 6.6
Generating function of Legendre's polynomials 6.16
Geometric interpretation of partial derivatives 3.2
Geometric series 8.6
Geometrical interpretation of definite integral 5.2
Geometrical meaning of derivative at a point 2.1
Gradient 7.3
Green's Theorem in the plane: Transformation between line and double integral 7.24

H

Harmonic analysis and its applications 9.39
Heaviside expansion formula 16.51
Heaviside's unit function 16.17
Hermitian 1.9
Homogeneous equations 10.6

Homogeneous function 3.11, 10.7
Homogeneous systems of linear equations 1.27

I

Important properties of Jacobians 3.28
Important vector identities 7.10
Improper integrals 5.54
Indefinite integral 5.1
Infinite series 8.2
Initial-value theorems 16.23
Integral form of Bessel's function 6.8
Integrating factors by inspection (grouping of terms) 10.38
Integration of transform! (Division by t) 16.15
Intrinsic equations 5.27
Inverse Laplace Transform (I.L.T.) 16.33
Inverse of a Matrix 1.13
Inversion formula for the Laplace transform 16.52

J

Jacobian 3.27
Jacobians of implicit functions 3.30

K

Kinds of improper integrals 5.55

L

Lagrange's method of multipliers 4.12
Lagrange's method of solving the linear partial differential equations of first order 14.10
Laplace definite integral for $P_n(x)$ 6.18
Laplace transform of Bessel's functions $J_0(t)$ and $J_1(t)$ 16.22
Laplace transform of the integral of a function 16.14
Laplace transform of the Laplace transform 16.24
Laplace transforms of derivatives 16.13
Laplace transforms of elementary functions 16.3
Laplace transforms of periodic functions 16.19
Legendre polynomials 6.13
Leibnitz test 8.7
Leibnitz's rule of differentiation under the sign of integration 5.2
Leibnitz's theorem 2.9
Liebnitz's Linear Equation 10.16
Limit 7.2
Line integrals 7.12
Linear differential equations 10.16
Linear partial differential equation 14.1
Linear partial differential equation with constant coefficients 14.23
Linear systems of equations 1.26
Linearity property of Laplace transforms 16.6
Liouville's extension of Dirichlet's theorem 5.94
Logarithmic series 8.36

M

Maxima and minima of functions of two independent variables 4.1
Maximum and minimum values for a function $f(x, z)$ 4.10
Method of finding CF when equations are of the Type (1) 14.38
Method of finding CF when equations are of the Type(2) 14.40
Method of finding particular integrals (PI) 14.40
Method of finding residues 16.52
Method of finding the complementary function (CF) of the 14.24
Method of finding the particular integral (PI or ZP) of the linear homogeneous PDE with constant coefficients 14.29
Method of separation of variables 15.1
Method of variation of parameters 12.23
Moment of inertia 5.51
Monge's method 14.53
Monge's method of integrating 14.57
Monotonic sequence 8.2

More on fourier series 9.32
Multiple integrals 5.63

N

Nature of the characteristic roots 1.34
Necessary conditions for the existence of maxima or minima of $f(x, y)$ at the point (a, b) 4.1
Newton's law of cooling 10.43
Non-homogeneous differential equations reducible to homogeneous form 10.10
Non-homogeneous linear partial differential equations with constant coefficients 14.38
Nonlinear equation reducible to linear form 10.22
Nonlinear partial differential equations of first order 14.15
Nonlinear partial differential equations of second order 14.53
Notation and terminology 1.1
Null function 16.33

O

One-dimensional heat equation 15.17
Ordinary and singular points 13.2
Orthogonal properties of Legendre's polynomial 6.20
Orthogonal trajectories and geometrical applications 10.48

P

Parametric representation of vector functions 7.1
Partial derivatives of first order 3.1
Partial derivatives of higher orders 3.2
Periodic function 9.1
Physical applications 10.41
Physical interpretation of curl 7.6
Point of inflection 4.18
Positive definite quadratic and Hermitian forms 1.51
Positive-term series 8.10
Power series 8.31, 13.3
Power-series solution about the ordinary point 13.4
Properties of matrices 1.5
Properties of matrix multiplication 1.7
p-series test 8.11

Q

Quadratic forms 1.48

R

Raabe's test 8.24
Range 8.1
Rank of a matrix 1.17
Rate of growth or decay 10.43
Rectification 5.19
Recurrence Formulae for $P_n(x)$ 6.22
Recurrence formulae/Relations of Bessel's equation 6.3
Reduction formula for the integrals 5.3
Relation between beta and gamma functions 6.32
Relations between second-order derivatives of homogeneous functions 3.13
Removal of the first derivative: reduction to normal form 12.10
Rodrigues' formula 6.17
Rules for finding an integral (solution) belonging 12.3

S

Sectionally continuous 16.11
Sequence 8.1
Short methods of finding the particular integral when 'R' is of a certain special forms 11.7
Similarity of matrices 1.43
Simple electric circuits 10.47
Simultaneous ordinary differential equations 16.64
Skew-Hermitian matrices 1.9
Solution of Bessel's differential equation 6.1

Solution of higher order homogeneous linear differential equations with constant coefficient 11.2
Solution of higher order non-homogeneous linear differential equation with constant coefficients 11.5
Solution of Legendre's equations 6.13
Solution of one-dimensional heat equation 15.18
Solution of one-dimensional wave equation 15.5
Solution of ordinary differential equations with variable coefficients 16.62
Solution of ordinary linear differential equations with constant coefficients 16.57
Solution of partial differential equations (PDE) 16.66
Solution of third-order LDE by Method of variation of parameters 12.25
Solution of two-dimensional heat equation by the method of separation of variables 15.34
Solution of two-dimensional Laplace's equation by the method of separation of variables 15.37
Solution of two-dimensional wave equation 15.27
Solutions of simultaneous linear differential equations 11.21
Some important remark's 1.51
Some important theorems on characteristic roots and characteristic vector 1.34
Some standard results on integration 5.1
Special types of matrices 1.2
Special waveforms 9.37
Stokes' theorem (Relation between line and surface integrals) 7.36
Successive differentiation 2.2
Sufficient conditions for maxima and minima (Lagrange's condition for two independent variables) 4.2
Surface integrals: surface area and flux 7.17
Surfaces of solids of revolution 5.39
Symmetric and skew-symmetric matrices 1.8
Systems of linear non-homogeneous equations 1.27

T

Taylor's and Maclaurin's theorems for three variables 3.43
Test for absolute convergence 8.27
Theorems on Jacobian (without proof) 3.30
To convert non-exact differential equations in to exact differential equations using integrating factors 10.32
Transformation of the equation by changing 12.16
Transmission-line equations 15.49
Transpose of a matrix 1.8
Transposed conjugate 1.9
Triple integral 5.87
Two-dimensional heat flow 15.33

U

Uniform Convergence 8.32
Uniqueness of inverse Laplace transform 16.33
Unit-step function 16.17
Use of Partial fractions to find ILT 16.45

V

Variable-separable 10.3
Vector integration 7.12
Velocity of escape from the Earth 10.52
Vibrating membrane—two-dimensional wave equation 15.26
Volume integrals 7.21
Volumes and surfaces of solids of revolution 5.31

W

Working rule for solving equations by changing 12.17
Working rule for Solving problems by using normal form 12.12
Working Rule for solving second order LDE by the method of variation of parameters 12.24