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SECOND EDITION

MEASUREMENT AND DATA ANALYSIS FOR ENGINEERING AND SCIENCE

PATRICK F DUNN

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Cover

Featured in the background on the cover are two photographs of instruments that earmark measurement chronology. Leeuwenhoek's microscope stands at its beginning, ushering in a golden age of experimentation. A modern temperature sensor symbolizes the present.

During the 17th century, Galileo Galilei's (1564–1642) telescope and Antony van Leeuwenhoek's (1632–1723) microscope, both made of lenses, allowed man to begin probing the intricacies of his macroscopic and microscopic worlds. The microscope pictured on the front cover is an exact replica of van Leeuwenhoek's original microscope made before 1673. It was constructed in 2006 by Mr. Leon Hluchota, tool and die maker, of the Department of Aerospace and Mechanical Engineering at the University of Notre Dame. One of Leeuwenhoek's original microscopes is at the University Museum, Utrecht, The Netherlands. That microscope's magnification was calibrated by Dr. J. van Zuylen in 1981 and found to be 266×, with a focal length of 0.94 mm and a resolution of 1.35 μm. This magnification was at least one order of magnitude better than any other contemporary device and was not exceeded until over a century later.

The temperature sensor shown on the front cover was developed by Eric Matlis, Ph.D., in 2008 at the Institute for Flow Physics and Control at the University of Notre Dame. This state-of-the-art sensor is part of a suite of highbandwidth sensors based on the use of miniature, AC-driven, weakly ionized plasmas. The sensors can be designed to measure surface pressure, shear stress, gas temperature, and gas species, either singly or in combination.

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Preface

This text covers the fundamental tools of experimentation that are currently used by both engineers and scientists. These include the *basics of experimentation* (types of experiments, units, and technical reporting), the *hardware of experiments* (electronics, measurement system components, system calibration, and system response), and the *methods of data analysis* (probability, statistics, uncertainty analysis, regression and correlation, signal characterization, and signal analysis). Historical perspectives also are provided in the text.

This second edition of **Measurement and Data Analysis for Engineering and Science** follows the original edition published by McGraw-Hill in 2005. Since its first publication, the text has been used annually by over 30 universities and colleges within the U.S., both at the undergraduate and graduate levels. The second edition has been condensed and reorganized following the suggestions of students and instructors who have used the first edition. The second edition differs from the first edition as follows:

- The number of text pages and the cost of the text have been reduced.
- All text material has been updated and corrected.
- The order of the chapters has been changed to reflect the sequence of topics usually covered in an undergraduate class. Former Chapters 2 and 3 are now Chapters 11 and 12, respectively. Their topics (units and technical communication) remain vital to the subject. However, they often can be studied by students without covering the material in lecture. Former Chapter 6 on measurement systems has been moved up to Chapter 3. This immediately follows electronics, now Chapter 2.
- Some sections within chapters have been reorganized to make the text easier to follow as an introductory undergraduate text. Some sections now are denoted by asterisks, indicating that they typically are not covered during lecture in an introductory undergraduate course. The complete text, including the sections denoted by an asterisk, can be used as an upper-level undergraduate or introductory graduate text.
- Over 150 new problems have been added, bringing the total to over 420 problems. A Problem Topic Summary now is included immediately before the review and homework problems at the end of each chapter to guide the instructor and student to specific problems by topic.

- The text is now complemented by an extensive text web site for students and instructors (www.nd.edu/∼pdunn/www.text/measurements.html). Most appendices and some chapter features of the first edition have been moved to this site. These include unit conversions (formerly Appendix C), learning objectives (formerly Appendix D), review crossword puzzles and solutions (formerly at the end of each chapter and Appendix F), differential equation derivations (formerly Appendix I), laboratory exercise descriptions (formerly Appendix H), MATLAB® sidebars with M-files (formerly in each chapter), and homework data files. Instructors who adopt the text for their course can receive a CD containing the review problem/homework problem solutions manual, the laboratory exercise solution manual, and a complete set of slide presentations for lecture from Taylor & Francis / CRC Press.

Many people contributed to the first edition. They are acknowledged in the first edition preface (see the text web site). Since then, further contributions have been made by some of my Notre Dame engineering students, my senior teaching assistants Dr. Michael Davis and Benjamin Mertz, and my colleagues Professor Flint Thomas, Dr. Edmundo Corona, Professor Emeritus Raymond Brach, Dr. Abdelmaged Ibrahim, and Professor David Go. Dr. Eric Matlis and Mr. Leon Hluchota provided the instruments shown on the cover. Jonathan Plant also has supported me as the editor of both editions.

Most importantly, each and every member of my family has always been there with me along the way. This extends from my wife, Carol, who is happy to see the second edition completed, to my grandson, Eliot, whose curiosity will make him a great experimentalist.

Patrick F. Dunn
University of Notre Dame

Written while at the University of Notre Dame, Notre Dame, Indiana; the University of Notre Dame London Centre, London, England; and Delft University of Technology, Delft, The Netherlands.

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Patrick F. Dunn, Ph.D., P.E., is a professor of aerospace and mechanical engineering at the University of Notre Dame, where he has been a faculty member since 1985. Prior to 1985, he was a mechanical engineer at Argonne National Laboratory from 1976 to 1985 and a postdoctoral fellow at Duke University from 1974 to 1976. He earned his B.S., M.S., and Ph.D. degrees in engineering from Purdue University (1970, 1971, and 1974). He is the author of over 160 scientific journal and refereed symposia publications and a licensed professional engineer in Indiana and Illinois. He is a Fellow of the American Society of Mechanical Engineers and an Associate Fellow of the American Institute of Aeronautics and Astronautics. He is the recipient of departmental, college, and university teaching awards.

Professor Dunn's scientific expertise is in fluid mechanics and microparticle behavior in flows. He is an experimentalist with over 40 years of experience involving measurement uncertainty. He is the author of the textbook **Measurement and Data Analysis for Engineering and Science** (first edition by McGraw-Hill, 2005; second edition by Taylor & Francis / CRC Press, 2010), and **Uncertainty Analysis for Forensic Science** with R.M. Brach (first and second editions by Lawyers & Judges Publishing Company, 2004 and 2009).

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1

Experiments

CONTENTS

...there is a diminishing return from increased theoretical complexity and ... in many practical situations the problem is not sufficiently well defined to merit an elaborate approach. If basic scientific understanding is to be improved, detailed experiments will be required ...

Graham B. Wallis. 1980. *International Journal of Multiphase Flow* 6:97.

The lesson is that no matter how plausible a theory seems to be, experiment gets the final word.

Robert L. Park. 2000. *Voodoo Science*. New York: Oxford University Press.

Experiments essentially pose questions and seek answers. A good experiment provides an unambiguous answer to a well-posed question.

Henry N. Pollack. 2003. *Uncertain Science ... Uncertain World*. Cambridge: Cambridge University Press.

1.1 Chapter Overview

Experimentation has been part of the human experience ever since its beginning. We are born with highly sophisticated data acquisition and computing systems ready to experiment with the world around us. We come loaded with the latest tactile, gustatory, auditory, olfactory, and optical sensor packages. We also have a central processing unit capable of processing data and performing highly complex operations at incredible rates with a memory far surpassing any that we can purchase.

One of our first rudimentary experiments, although not a conscious one, is to cry and then to observe whether or not a parent will come to the aid of our discomfort. We *change* the environment and *record* the result. We are *active* participants in the process. Our view of reality is formed by what we sense. But what really are experiments? What roles do they play in the process of understanding the world in which we live? How are they classified? Such questions are addressed in this chapter.

1.2 Role of Experiments

Perhaps the first question to ask is, "Why do we do experiments?" Some of my former students have offered the following answers:

"Experiments are the basis of all theoretical predictions. Without experiments, there would be no results, and without any tangible data, there is no basis for any scientist or engineer to formulate a theory. ... The advancement of culture and civilization depends on experiments which bring about new technology... ." (P. Cuadra)

"Making predictions can serve as a guide to what we expect, ... but to really learn and know what happens in reality, experiments must be done." (M. Clark)

"If theory predicted everything exactly, there would be no need for experiments. NASA planners could spend an afternoon drawing up a mission with their perfect computer models and then launch a flawlessly executed mission that evening (of course, what would be the point of the mission, since the perfect models could already predict behavior in space anyway?)." (A. Manella)

In the most general sense, man seeks to reach a better understanding of the world. In this quest, man relies upon the collective knowledge of his predecessors and peers. If one understood *everything* about nature, there would be no need for experiments. One could predict every outcome (at least for deterministic systems). But that is not the case. Man's understanding is imperfect. Man needs to experiment in the world.

So how do experiments play a role in our process of understanding? The Greeks were the earliest civilization that attempted to gain a better understanding of their world through observation and reasoning. Previous civilizations functioned within their environment by observing its behavior and then adapting to it. It was the Greeks who first went beyond the stage of simple observation and attempted to arrive at the underlying physical causes of what they observed [1]. Two opposing schools emerged, both of which still exist but in somewhat different forms. Plato (428-347 B.C.) advanced that the highest degree of reality was that which men *think* by reasoning. He believed that better understanding followed from rational thought alone. This is called **rationalism**. On the contrary, Aristotle (384-322 B.C.) believed that the highest degree of reality is that which man *perceives* with his senses. He argued that better understanding came through careful observation. This is known as **empiricism**. Empiricism maintains that knowledge originates from and is limited to concepts developed from sensory experience. Today it is recognized that both approaches play important roles in advancing scientific understanding.

There are several different roles that experiments play in the process of scientific understanding. Harré [2], who discusses some of the landmark experiments in science, describes three of the most important roles: **inductivism**, **fallibilism**, and **conventionalism**. Inductivism is the process whereby the laws and theories of nature are arrived at based upon the facts gained from the experiments. In other words, a greater theoretical understanding of nature is reached through induction. Taking the fallibilistic approach, experiments are performed to test the validity of a conjecture. The conjecture is rejected if the experiments show it to be false. The role of experiments in the conventionalistic approach is illustrative. These experiments do not induce laws or disprove hypotheses but rather show us a more useful or illuminating description of nature. Testings fall into the category of conventionalistic experiments.

All three of these approaches are elements of the **scientific method**. Credit for its formalization often is given to Francis Bacon (1561-1626). The seeds of experimental science were sown earlier by Roger Bacon (c. 1220-1292), who was not related to Francis. Roger attempted to incorporate experimental science into the university curriculum but was prohibited by Pope Clement IV. He wrote of his findings in secrecy. Roger is considered "the most celebrated scientist of the Middle Ages." [3] Francis argued that our understanding of nature could be increased through a disciplined and orderly approach in answering scientific questions. This approach involved experiments, done in a systematic and rigorous manner, with the goal of arriving at a broader theoretical understanding. Using the approach of Francis Bacon's time, first the results of positive experiments and observations are gathered and considered. A preliminary hypothesis is formed. All rival hypotheses are tested for possible validity. Hopefully, only one correct hypothesis remains. Today the scientific method is used mainly to validate a particular hypothesis or to determine the range of validity of a hypothesis. In the end, it is the constant interplay between experiment and theory that leads to advancing our understanding, as illustrated schematically in Figure 1.1. The concept of the real world is developed from the data acquired through experiment and the theories constructed to explain the observations. Often new experimental results improve theory and new theories guide and suggest new experiments. Through this process, a more refined and realistic concept of the world is developed. Anthony Lewis summarizes it well, "The whole ethos of science is that any explanation for the myriad mysteries in our universe is a theory, subject to challenge and experiment. That is the scientific method."

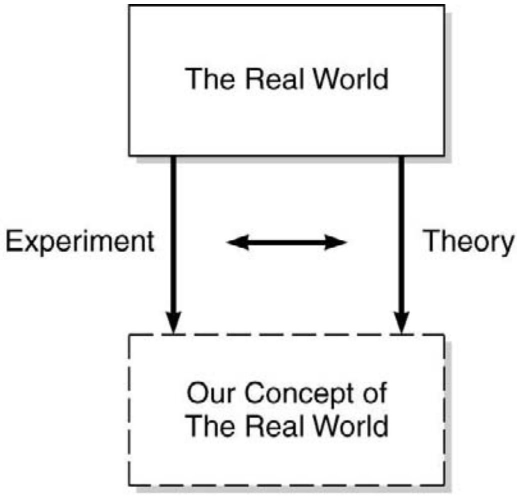

FIGURE 1.1
The interplay between experiment and theory.

Gale [4], in his treatise on science, advances that there are two goals of science: explanation and understanding, and prediction and control. Science is not absolute; it evolves. Its modern basis is the experimental method of proof. Explanation and understanding encompass statements that make causal connections. One example statement is that an increase in the temperature of a perfect gas under constant volume causes an increase in its pressure. These usually lead to an algorithm or law that relates the variables involved in the process under investigation. Prediction and control establish correlations between variables. For the previous example, these would result in the correlation between pressure and temperature. Science is a process in which false hypotheses are disproved and, eventually, the true one remains.

1.3 The Experiment

What exactly is an experiment? An **experiment** is an act in which one physically intervenes with the process under investigation and records the results. This is shown schematically in Figure 1.2. Examine this definition more closely. In an experiment one *physically changes in an active manner* the process being studied and then *records* the results of the change. Thus, computational simulations are *not* experiments. Likewise, sole observation of a process is *not* an experiment. An astronomer charting the heavens does not alter the paths of planetary bodies; he does not perform an experiment, rather he observes. An anatomist who dissects something does not physically change a process (although he physically may move anatomical parts); again, he observes. Yet, it is through the interactive process of observation-experimentation-hypothesizing that understanding advances. All elements of this process are essential. Traditionally, theory explains existing results and predicts new results; experiments validate existing theory and gather results for refining theory.

FIGURE 1.2
The experiment.

When conducting an experiment, it is imperative to identify all the variables involved. **Variables** are those physical quantities involved in the process under investigation that can undergo change during the experiment and thereby affect the process. They are classified as **independent**, **dependent**, and **extraneous**. An experimentalist manipulates the independent variable(s) and records the effect on the dependent variable(s). An extraneous variable cannot be controlled, but it affects the value of what is measured to some extent. A **controlled experiment** is one in which *all* of the variables involved in the process are identified and can be controlled. In reality, almost all experiments have extraneous variables and, therefore, strictly are not controlled. This inability to precisely control *every* variable is the primary source of experimental uncertainty, which is considered in Chapter 7. The measured variables are called **measurands**.

A variable that is either actively or passively fixed throughout the experiment is called a **parameter**. Sometimes, a parameter can be a specific function of variables. For example, the Reynolds number, which is a nondimensional number used frequently in fluid mechanics, can be a parameter in an experiment involving fluid flow. The Reynolds number is defined as $Re = \rho U d/\mu$, where U is the fluid velocity, d is a characteristic length, ρ is the fluid's density, and μ is the fluid's absolute viscosity. Measurements can be made by conducting a number of experiments for various U, d, ρ, and μ. Then the data can be organized for certain fixed values of Re, each corresponding to a different experiment.

Consider for example a fluid flow experiment designed to ascertain whether or not there is laminar (Poiseuille) flow through a smooth pipe. If laminar flow is present, then theory (conservation of momentum) predicts that $\Delta p = 8QL\mu/(\pi R^4)$, where Δp is the pressure difference between two locations along the length, L, of a pipe of radius, R, for a liquid with absolute viscosity, μ, flowing at a volumetric flow rate, Q. In this experiment, the volumetric flow rate is varied. Thus, μ, R, and L are parameters, Q is the independent variable, and Δp is the dependent variable. R, L, Q, and Δp are measurands. The viscosity is a dependent variable that is determined from the fluid's temperature (another measurand and parameter). If the fluid's temperature is not controlled in this experiment, it could affect the values of density and viscosity, and hence affect the values of the dependent variables.

Example Problem 1.1

Statement: An experiment is performed to determine the coefficient of restitution, e, of a ball over a range of impact velocities. For impact normal to the surface, $e = v_f/v_i$, where v_i is the normal velocity component immediately before impact with the surface and v_f is that immediately after impact. The velocity, v_i, is controlled by dropping the ball from a known initial height, h_a, and then measuring its return height, h_b. What are the variables in this experiment? List which ones are independent, dependent, parameter, and measurand.

Solution: A ball dropped from height h_a will have $v_i = \sqrt{2gh_a}$, where g is the local gravitational acceleration. Because $v_f = \sqrt{2gh_b}$, $e = \sqrt{h_b/h_a}$. So the variables are h_a, h_b, v_i, v_f, e, and g. h_a is an independent variable; h_b, v_i, v_f, and e are dependent variables; h_a and g are parameters; h_a and h_b are measurands.

Often, however, it is difficult to identify and control all of the variables that can influence an experimental result. Experiments involving biological systems often fall into this category. In these situations, repeated measurements are performed to arrive at statistical estimates of the measured variables, such as their means and standard deviations. **Repetition** implies that a set of measurements are repeated under the same, fixed operating conditions. This yields direct quantification of the variations that occur in the measured variables for the same experiment under fixed operating conditions. Often, however, the same experiment may be run under the same operating conditions at different times or places using the same or comparable equipment and facilities. Because uncontrollable changes may occur in the interim between running the experiments, additional variations in the measured variables may be introduced. These variations can be quantified by the **replication** (duplication) of the experiment. A **control** experiment is an experiment that is as nearly identical to the subject experiment as possible. Control experiments typically are performed to reconfirm a subject experiment's results or to verify a new experimental set-up's performance. Finally, experiments can be categorized broadly into **timewise** and **sample-to-sample** experiments [10]. Values of a measurand are recorded in a continuous manner over a period of time in timewise experiments. Values are obtained for multiple samples of a measurand in sample-to-sample experiments. Both types of experiments can be considered the same when values of a measurand are acquired at discrete times. Here, what distinguishes between the two categories is the time interval between samples.

In the end, performing a good experiment involves identifying and controlling as many variables as possible and making accurate and precise measurements. The experiment always should be performed with an eye out for discovery. To quote Sir Peter Medawar [6], "The merit of an experiment lies principally in its design and in the critical spirit in which it is carried out."

1.4 Experimental Approach

Park [7] remarks that "science is the systematic enterprise of gathering knowledge about the world and organizing and condensing that knowledge into testable laws and theories." Experiments play a pivotal role in this process. The general purpose of any experiment is to gain a better understanding about the process under investigation and, ultimately, to advance science. Many issues need to be addressed in the phases preceding, during, and following an experiment. These can be categorized as planning, design, construction, debugging, execution, data analysis, and reporting of results [10].

Prior to performing the experiment, a clear approach must be developed. The objective of the experiment must be defined along with its relation to the theory of the process. What are the assumptions made in the experiment? What are those made in the theory? Special attention should be given to assuring that the experiment correctly reflects the theory. The process should be observed with minimal intervention, keeping in mind that the experiment itself may affect the process. All of the variables involved in the process should be identified. Which can be varied? Which can be controlled? Which will be recorded and how? Next, what results are expected? Does the experimental set-up perform as anticipated? Then, after all of this has been considered, the experiment is performed.

Following the experiment, the results should be reviewed. Is there agreement between the experimental results and the theory? If the answer is yes, the results should be reconfirmed. If the answer is no, *both* the experiment and the theory should be examined carefully. Any measured differences should be explained in light of the uncertainties that are present in the experiment and in the theory.

Finally, the new results should be summarized. They should be presented within the context of uncertainty and the limitations of the theory and experiment. All this information should be presented such that another investigator can follow what was described and repeat what was done.

1.5 Classification of Experiments

There are many ways to classify experiments. One way is according to the intent or purpose of the experiment. Following this approach, most experiments can be classified as **variational**, **validational**, **pedagogical**, or **explorational**.

The goal of variational experiments is to establish (quantify) the mathematical relationships of the experiment's variables. This is accomplished by varying one or more of the variables and recording the results. Ideal variational experiments are those in which *all* the variables are identified and controlled. Imperfect variational experiments are those in which *some* of the variables are either identified or controlled. Experiments involving the determination of material properties, component behavior, or system behavior are variational. Standard testing also is variational.

Validational experiments are conducted to validate a specific hypothesis. They serve to evaluate or improve existing theoretical models. A critical validational experiment, which also is known as a Galilean experiment, is designed to refute a null hypothesis. An example would be an experiment designed to show that pressure does not remain constant when an ideal gas under constant volume is subjected to an increase in temperature.

Pedagogical experiments are designed to teach the novice or to demonstrate something that is already known. These are also known as Aristotelian experiments. Many experiments performed in primary and secondary schools are this type, such as the classic physics lab exercise designed to determine the local gravitational constant by measuring the time it takes a ball to fall a certain distance.

Explorational experiments are conducted to explore an idea or possible theory. These usually are based upon some initial observations or a simple theory. All of the variables may not be identified or controlled. The experimenter usually is looking for trends in the data in hope of developing a relationship between the variables. Richard Feynman [8] aptly summarizes the role of experiments in developing a new theory, "In general we look for a new law by the following process. First we guess it. Then we compute the consequences of the guess to see what would be implied if this law that we guessed is right. Then we compare the result of the computation to nature, with experiment or experience, compare it directly with observation, to see if it works. If it disagrees with experiment it is wrong. In that simple statement is the key to science. It does not make any difference how beautiful your guess is. It does not make any difference how smart you are, who made the guess, or what his name is – if it disagrees with experiment it is wrong. That is all there is to it."

An additional fifth category involves experiments that are far less common and lead to discovery. Discovery can be either anticipated by theory (an analytic discovery), such as the discovery of the quark, or serendipitous (a synthetic discovery), such as the discovery of bacterial repression by penicillin. There also are thought (*gedunken* or Kantian) experiments that are posed to examine what would follow from a conjecture. Thought experiments, according to our formal definition, are *not* experiments because they do not involve any *physical* change in the process.

1.6 Problem Topic Summary

Topic	Review Problems	Homework Problems
Experiments	2, 3, 5, 6	1, 2, 3, 4, 5, 6, 7, 8
Variables	1, 4, 7	2, 4, 6, 9

TABLE 1.1
Chapter 1 Problem Summary

1.7 Review Problems

1. Variables manipulated by an experimenter are (a) independent, (b) dependent, (c) extraneous, (d) parameters, or (e) presumed.

2. Immediately following the announcement by the University of Utah, on March 23, 1989, that Stanley Pons and Martin Fleischmann had "discovered" cold fusion, scientists throughout the world rushed to perform an experiment that typically would be classified as (a) variational, (b) validational, (c) pedagogical, (d) explorational, or (e) serendipitous.

3. If you were trying to perform a validational experiment to determine the base unit of mass, the gram, which of the following fluid conditions would be most desirable? (a) a beaker of ice water, (b) a pot of boiling water, (c) a graduated cylinder of water at room temperature, (d) a thermometer filled with mercury.

4. Match the following with the most appropriate type of variable (independent, dependent, extraneous, parameter, or measurand): (a) measured during the experiment, (b) fixed throughout the experiment, (c) not controlled during the experiment, (d) affected by a change made by the experimenter, (e) changed by the experimenter.

5. What is the main purpose of the scientific method?

6. Classify the following experiments: (a) estimation of the heating value of gasoline, (b) measuring the stress-strain relation of a new bio-material, (c) the creation of Dolly (the first sheep to be cloned successfully).

7. An experiment is performed to determine the velocity profile along a wind tunnel's test section using a pitot-static tube. The tunnel flow rate is fixed during the experiment. Identify the independent, dependent, extraneous, and parameter variables from the following list: (a) tunnel fan revolutions per minute, (b) station position, (c) environment pressure and temperature, (d) air density, (e) change in pressure measured by the pitot-static tube, (f) calculated velocity.

1.8 Homework Problems

1. Give one historical example of an inductivistic, a fallibilistic, and a conventionalistic experiment. State each of their significant findings.

2. Write a brief description of an experiment that you have performed or one with which you are familiar, noting the specific objective of the experiment. List and define all of the independent and dependent variables, parameters, and measurands. Also provide any equation(s) that involve the variables and define each term.

3. Give one historical example of an experiment falling into each of the four categories of experimental purpose. Describe each experiment briefly.

4. Write a brief description of the very first experiment that you ever performed. What was its purpose? What were its variables?

5. What do you consider to be the greatest experiment ever performed? Explain your choice. You may want to read about the 10 'most beautiful experiments of all time' voted by physicists as reported by George Johnson in the *New York Times* on September 24, 2002, in an article titled "Here They Are, Science's 10 Most Beautiful Experiments." Also see R.P. Crease, 2003. *The Prism and the Pendulum: The Ten Most Beautiful Experiments in Science*. New York: Random House.

6. Select one of the 10 most beautiful physics experiments. (See *http://physics-animations.com/Physics/English/top_ref.htm*). Briefly explain the experiment and classify its type. Then list the variables involved in the experiment. Finally, classify each of these variables.

7. Measure the volume of your room and find the number of molecules in it. Is this an experiment? If so, classify it.

8. Classify these types of experiments: (a) measuring the effect of humidity on the Young's modulus of a new 'green' building material, (b) demonstrating the effect of the acidity of carbonated soda by dropping a dirty

penny into it, (c) determining whether a carbon nanotube is stronger than a spider web thread.

9. Consider an experiment where a researcher is attempting to measure the thermal conductivity of a copper bar. The researcher applies a heat input q'', which passes through the copper bar, and four thermocouples to measure the local bar temperature $T(x)$. The thermal conductivity, k, can be calculated from the equation

$$q'' = -k\frac{dT}{dx}.$$

Variables associated with the experiment are the (a) thermal conductivity of the bar, (b) heater input, (c) temperature of points 1, 2, 3, and 4 from the thermocouples, (d) pressure and temperature of the surrounding air, (e) smoothness of copper bar at the interfaces with the heaters, and (f) position of the thermocouples. Determine whether each variable is dependent, independent, or extraneous. Then determine whether each variable is a parameter or a measurand.

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Electronics

CONTENTS

Nothing is too wonderful to be true, if it be consistent with the laws of nature and in such things as these, experiment is the best test of such consistency.
Michael Faraday (1791-1867) on March 19, 1849. From a display at the Royal Institution, London.

... the language of experiment is more authoritative than any reasoning: facts can destroy our ratiocination – not vice versa.
Alessandro Volta (1745-1827), quoted in *The Ambiguous Frog: The Galvani-Volta Controversy on Animal Electricity*, M. Pera, 1992.

2.1 Chapter Overview

We live in a world full of electronic devices. Stop for a minute and think of all the ones you encounter each day. The clock radio usually is the first. This electronic marvel contains a digital display, a microprocessor, an AM/FM radio, and even a piezoelectric buzzer whose annoying sound beckons us to get out of bed. Before we even leave for work we have used electric lights, shavers, toothbrushes, blow dryers, coffee pots, toasters, microwave ovens, refrigerators, and televisions, to name a few. At the heart of all these devices are electrical circuits. For us to become competent experimentalists, we need to understand the basics of the electrical circuits present in most instruments. In this chapter we will review some of the basics of electrical circuits. Then we will examine several more detailed circuits that comprise some common measurement systems.

2.2 Concepts and Definitions

Before proceeding to examine the basic electronics behind a measurement system's components, a brief review of some fundamentals is in order. This review includes the definitions of the more common quantities involved in electrical circuits, such as electric charge, electric current, electric field, electric potential, resistance, capacitance, and inductance. The SI dimensions and units for electric and magnetic systems are summarized in tables on the text web site. The origins of these and many other quantities involved in electromagnetism date back to a period rich in the ascent of science, the 17th through mid-19th centuries.

2.2.1 Charge

Electric **charge**, q, previously called *electrical vertue* [1], has the SI unit of coulomb (C) named after the French scientist Charles Coulomb (1736-1806). The effect of charge was observed in early years when two similar materials were rubbed together and then found to repel each other. Conversely, when two dissimilar materials were rubbed together, they became attracted to each other. Amber, for example, when rubbed, would attract small pieces of feathers or straw. In fact, *electron* is the Greek word for amber.

It was Benjamin Franklin (1706-1790) who argued that there was only one form of electricity and coined the relative terms *positive* and *negative* charge. He stated that charge is neither created nor destroyed, rather it is conserved, and that it only is transferred between objects. Prior to Franklin's declarations, two forms of electricity were thought to exist: *vitreous*, from glass or crystal, and *resinous*, from rubbing material like amber [1]. It now is known that positive charge indicates a deficiency of electrons and negative charge indicates an excess of electrons. Charge is not produced by objects, rather it is transferred between objects.

2.2.2 Current

The amount of charge that moves per unit time through or between materials is electric **current**, I. This has the SI unit of an ampere (A), named after the French scientist Andre Ampere (1775-1836). An ampere is a coulomb per second. This can be written as

$$I = dq/dt. \tag{2.1}$$

Current is a measure of the flow of electrons, where the charge of one electron is 1.602 177 33 $\times$ 10^{-19} C. Materials that have many free electrons are called conductors (previously known as *non-electrics* because they easily would lose their charge[1]). Those with no free electrons are known as insulators or dielectrics (previously known as *electrics* because they could remain charged) [1]. In between these two extremes lie the semi-conductors, which have only a few free electrons. By convention, current is considered to flow from the **anode** (the positively charged terminal that loses electrons) to the **cathode** (the negatively charged terminal that gains electrons) even though the *actual* electron flow is in the *opposite* direction. Current flow from anode to cathode often is referred to as **conventional current**. This convention originated in the early 1800's when it was assumed that positive charge flowed in a wire. **Direct current** (DC) is constant in time and **alternating current** (AC) varies cyclically in time, as depicted in Figure 2.1. When current is alternating, the electrons do not flow in one direction through a circuit, but rather back and forth in both directions. The symbol for a current source in an electrical circuit is given in Figure 2.2.

2.2.3 Force

When electrically charged bodies attract or repel each other, they do so because there is an electric **force** acting between the charges on the bodies. Coulomb's law relates the charges of the two bodies, q_1 and q_2, and the distance between them, R, to the electric force, F_e, by the relation

$$F_e = Kq_1q_2/R^2, \tag{2.2}$$

where $K = 1/(4\pi\epsilon_o)$, with the permittivity of free space $\epsilon_o = 8.854\ 187\ 817 \times 10^{-12}$ F/m. The SI unit of force is the newton (N).

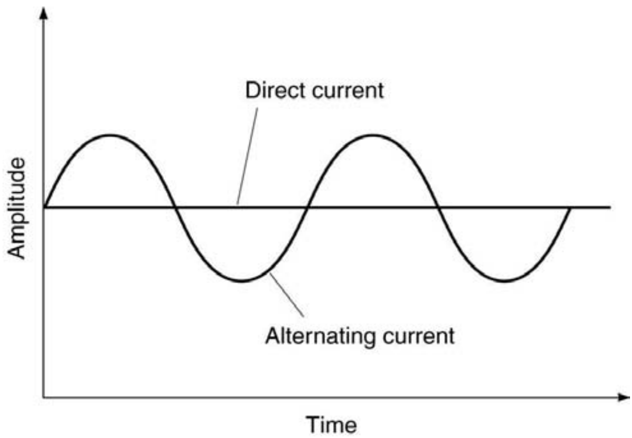

FIGURE 2.1
Direct and alternating currents.

2.2.4 Field

The **electric field**, E, is defined as the electric force acting on a positive charge divided by the magnitude of the charge. Hence, the electric field has the SI unit of newtons per coulomb. This leads to an equivalent expression $F_e = qE$. So, the work required to move a charge of 1 C a distance of 1 m through a unit electric field of 1 N/C is 1 N·m or 1 J. The SI unit of work is the joule (J).

2.2.5 Potential

The **electric potential**, Φ, is the electric field potential energy per unit charge, which is the energy required to bring a charge from infinity to an arbitrary reference point in space. Often it is better to refer to the **potential difference**, $\Delta\Phi$, between two electric potentials. It follows that the SI unit for electric potential is joules per coulomb. This is known as the volt (V), named after Alessandro Volta (1745-1827). Volta invented the voltaic pile, originally made of pairs of copper and zinc plates separated by wet paper, which was the world's first battery. In electrical circuits, a battery is indicated by a longer, solid line (the anode) separated over a small distance by a shorter, solid line (the cathode), as shown in Figure 2.3. The symbol for a voltage source is presented in Figure 2.2.

2.2.6 Resistance and Resistivity

When a voltage is applied across the ends of a conductor, the amount of current passing through it is linearly proportional to the applied voltage.

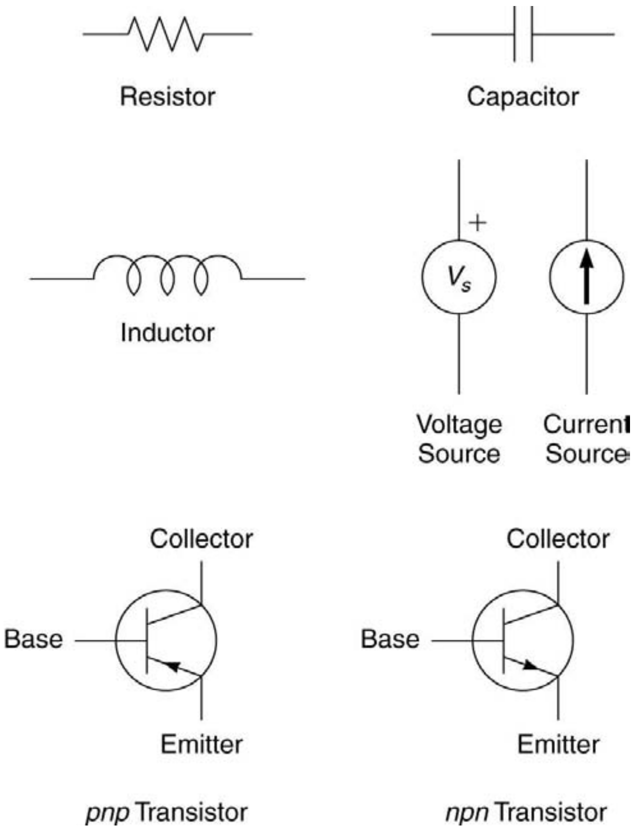

FIGURE 2.2
Basic circuit element symbols.

The constant of proportionality is the electric **resistance**, R. The SI unit of resistance is the ohm (Ω), named after Georg Ohm (1787-1854).

Electric resistance can be related to electric **resistivity**, ρ, for a wire of cross-sectional area A and length L as

$$R = \rho L/A. \tag{2.3}$$

The SI unit of resistivity is $\Omega{\cdot}$m. Conductors have low resistivity values (for example, Ag: 1.5×10^{-8} $\Omega{\cdot}$m), insulators have high resistivity values (for example, quartz: 5×10^{7} $\Omega{\cdot}$m), and semi-conductors have intermediate resistivity values (for example, Si: 2 $\Omega{\cdot}$m).

Resistivity is a property of a material and is related to the temperature of the material by the relation

$$\rho = \rho_0[1 + \alpha(T - T_0)], \tag{2.4}$$

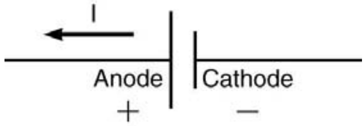

FIGURE 2.3
The battery.

where ρ_0 denotes the reference resistivity at reference temperature T_0 and α the coefficient of thermal expansion of the material. For conductors, α ranges from approximately 0.002/°C to 0.007/°C. Thus, for a wire

$$R = R_0[1 + \alpha(T - T_0)]. \tag{2.5}$$

2.2.7 Power

Electric power is electric energy transferred per unit time, $P(t) = I(t)V(t)$. Using Ohm's law, it can be written also as $P(t) = I^2(t)R$. This implies that the SI unit for electric power is J/s or watt (W).

2.2.8 Capacitance

When a voltage is applied across two conducting plates separated by an insulated gap, a charge will accumulate on each plate. One plate becomes charged positively ($+q$) and the other charged equally and negatively ($-q$). The amount of charge acquired is linearly proportional to the applied voltage. The constant of proportionality is the **capacitance**, C. Thus, $q = CV$. The SI unit of capacitance is coulombs per volt (C/V). The symbol for capacitance, C, should not be confused with the unit of coulomb, C. The SI unit of capacitance is the farad (F), named after the British scientist Michael Faraday (1791-1867).

2.2.9 Inductance

When a wire is wound as coil and current is passed through it by applying a voltage, a magnetic field is generated that surrounds the coil. As the current changes in time, a changing magnetic flux is produced inside the coil, which in turn induces a back **electromotive force** (emf). This back emf opposes the original current, leading to either an increase or a decrease in the current, depending upon the direction of the original current. The resulting magnetic flux, ϕ, is linearly proportional to the current. The constant of proportionality is called the electric **inductance**, denoted by L. The SI unit of inductance is the henry (H), named after the American Joseph Henry (1797-1878). One henry equals one weber per ampere.

Element	Unit Symbol	$I(t)$	$V(t)$	$V_{I=\text{const}}$
Resistor	R	$V(t)/R$	$RI(t)$	RI
Capacitor	C	$CdV(t)/dt$	$(1/C)\int_0^t I(\tau)d\tau$	It/C
Inductor	L	$(1/L)\int_0^t V(\tau)d\tau$	LdI/dt	0

TABLE 2.1
Resistor, capacitor, and inductor current and voltage relations.

Example Problem 2.1

Statement: 0.3 A of current passes through an electrical wire when the voltage difference between its ends is 0.6 V. Determine [a] the wire resistance, R, [b] the total amount of charge that moves through the wire in 2 minutes, q_{total}, and [c] the electric power, P.

Solution: [a] Application of Ohm's law gives $R = 0.6\ \text{V}/0.3\ \text{A} = 2\ \Omega$. [b] Integration of Equation 2.1 gives $q(t) = \int_{t_1}^{t_2} I(t)dt$. Because $I(t)$ is constant, q_{total} = (0.3 A)(120 s) = 36 C. [c] The power is the product of current and voltage. So, P = (0.3 A)(0.6 V) = 0.18 W = 0.2 W, with the correct number of significant figures.

2.3 Circuit Elements

At the heart of all electrical circuits are some basic circuit elements. These include the resistor, capacitor, inductor, transistor, ideal voltage source, and ideal current source. The symbols for these elements that are used in circuit diagrams are presented in Figure 2.2. These elements form the basis for more complicated devices such as operational amplifiers, sample-and-hold circuits, and analog-to-digital conversion boards, to name only a few (see [2]).

The resistor, capacitor, and inductor are **linear devices** because the complex amplitude of their output waveform is *linearly* proportional to the amplitude of their input waveform. A device is linear if [1] the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$ and [2] the response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant [4]. Thus, if the input waveform of a circuit comprised only of linear devices, known as a linear circuit, is a sine wave of a given frequency, its output will be a sine wave of the *same* frequency. Usually, however, its output amplitude will be different from its input amplitude and its output waveform will lag the input waveform in time. If the lag is between one-half to one cycle, the output waveform appears to lead the input waveform, although it always lags the input waveform. The response behavior of linear systems to various input waveforms is presented in Chapter 4. The current-voltage relations for the resistor, capacitor, and inductor are summarized in Table 2.1.

2.3.1 Resistor

The basic circuit element used more than any others is the resistor. Its current-voltage relation is defined through Ohm's law,

$$R = V/I. \tag{2.6}$$

Thus, the current in a resistor is related *linearly* to the voltage difference across it, or *vice versa*. The resistor is made out of a conducting material, such as carbon, carbon-film, or metal-film. Typical resistances range from a few ohms to more than 10^7 Ω.

2.3.2 Capacitor

The current flowing through a capacitor is related to the product of its capacitance and the time rate of change of the voltage difference, where

$$I = \frac{dq}{dt} = C\frac{dV}{dt}. \tag{2.7}$$

For example, 1 μA of current flowing through a 1 μF capacitor signifies that the voltage difference across the capacitor is changing at a rate of 1 V/s. If the voltage is not changing in time, there is no current flowing through the capacitor. The capacitor is used in circuits where the voltage varies in time. In a DC circuit, a capacitor acts as an open circuit. Typical capacitances are in the μF to pF range.

2.3.3 Inductor

Faraday's law of induction states that the change in an inductor's magnetic flux, ϕ, with respect to time equals the applied voltage, $d\phi/dt = V(t)$. Because $\phi = LI$,

$$V(t) = L\frac{dI}{dt}. \tag{2.8}$$

Thus, the voltage across an inductor is related *linearly* to the product of its inductance and the time rate of change of the current. The inductor is used in circuits in which the current varies in time. The simplest inductor is a wire wound in the form of a coil around a nonconducting core. Most inductors have negligible resistance when measured directly. When used in an AC circuit, the inductor's back emf controls the current. Larger inductances impede the current flow more. This implies that an inductor in an AC circuit acts like a resistor. In a DC circuit, an inductor acts as a short circuit. Typical inductances are in the mH to μH range.

2.3.4 Transistor

The transistor was developed in 1948 by William Shockley, John Bardeen, and Walter Brattain at Bell Telephone Laboratories. The common transistor consists of two types of semiconductor materials, *n*-type and *p*-type. The *n*-type semiconductor material has an excess of free electrons and the *p*-type material a deficiency. By using only two materials to form a *pn* junction, one can construct a device that allows current to flow in only one direction. This can be used as a **rectifier** to change alternating current to direct current. Simple junction transistors are basically three sections of semiconductor material sandwiched together, forming either *pnp* or *npn* transistors. Each section has its own wire lead. The center section is called the **base**, one end section the **emitter**, and the other the **collector**. In a *pnp* transistor, current flow is into the emitter. In an *npn* transistor, current flow is out of the emitter. In both cases, the emitter-base junction is said to be forward-biased or conducting (current flows forward from *p* to *n*). The opposite is true for the collector-base junction. It is always reverse-biased or non-conducting. Thus, for a *pnp* transistor, the emitter would be connected to the positive terminal of a voltage source and the collector to the negative terminal through a resistor. The base would also be connected to the negative terminal through another resistor. In such a configuration, current would flow into the emitter and out of both the base *and* the collector. The voltage difference between the emitter and the collector causing this current flow is termed the base bias voltage. The ratio of the collector-to-base current is the (current) gain of the transistor. Typical gains are up to approximately 200. The characteristic curves of a transistor display collector current versus the base bias voltage for various base currents. Using these curves, the gain of the transistor can be determined for various operating conditions. Thus, transistors can serve many different functions in an electrical circuit, such as current amplification, voltage amplification, detection, and switching.

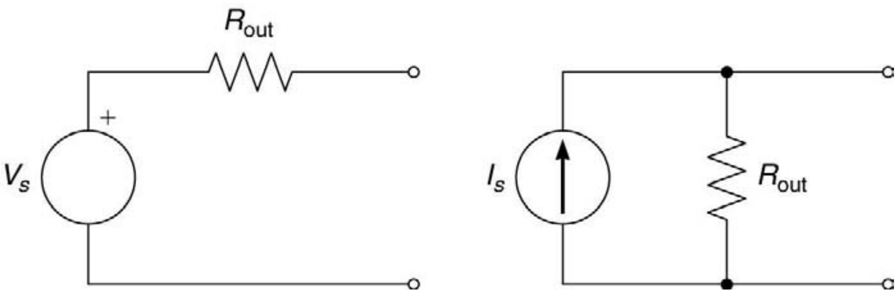

FIGURE 2.4
Voltage and current sources.

2.3.5 Voltage Source

An *ideal* voltage source, shown in Figure 2.4, with $R_{out} = 0$, maintains a fixed voltage difference between its terminals, independent of the resistance of the load connected to it. It has a zero output impedance and can supply infinite current. An *actual* voltage source has some internal resistance. So the voltage supplied by it is limited and equal to the product of the source's current and its internal resistance, as dictated by Ohm's law. A good voltage source has a very low output impedance, typically less than 1 Ω. If the voltage source is a battery, it has a finite lifetime of current supply, as specified by its capacity. Capacity is expressed in units of current times lifetime (which equals its total charge). For example, a 1200 mA hour battery pack is capable of supplying 1200 mA of current for 1 hour or 200 mA for 6 hours. This corresponds to a total charge of 4320 C (0.2 A × 21 600 s).

2.3.6 Current Source

An *ideal* current source, depicted in Figure 2.4, with $R_{out} = \infty$ maintains a fixed current between its terminals, independent of the resistance of the load connected to it. It has an infinite output impedance and can supply infinite voltage. An *actual* current source has an internal resistance less than infinite. So the current supplied by it is limited and equal to the ratio of the source's voltage difference to its internal resistance. A good current source has a very high output impedance, typically greater than 1 MΩ. Actual voltage and current sources differ from their ideal counterparts only in that the actual impedances are neither zero nor infinite, but finite.

2.4 *RLC* Combinations

Linear circuits typically involve resistors, capacitors, and inductors connected in various **series** and **parallel** combinations. Using the current-voltage relations of the circuit elements and examining the potential difference between two points on a circuit, some simple rules for various combinations of resistors, capacitors, and inductors can be developed.

First, examine Figure 2.5 in which the *series* combinations of two resistors, two capacitors, and two inductors are shown. The potential difference across an i-th resistor is IR_i, across an i-th capacitor is q/C_i, and across an i-th inductor is $L_i dI/dt$. Likewise, the total potential difference, V_T, for the series resistors' combination is $V_T = IR_T$, for the series capacitors' combination is $V_T = q/C_T$, and for the series inductors' combination is $V_T = L_T dI/dt$. Because the potential differences across resistors, capacitors, and inductors in series add, $V_T = V_1 + V_2$. Hence, for the resistors'

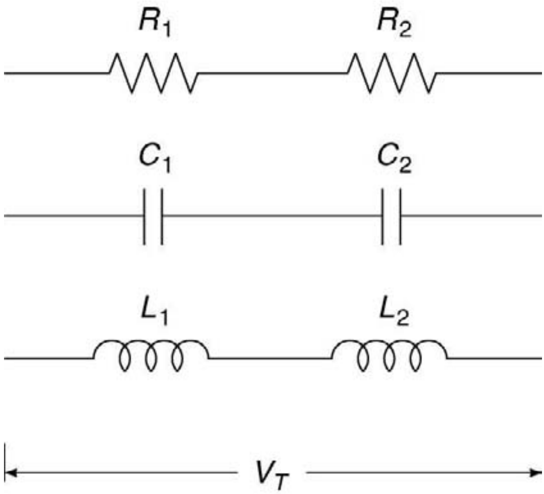

FIGURE 2.5
Series R, C, and L circuit configurations.

series combination, $V_T = IR_1 + IR_2 = IR_T$, which yields

$$R_T = R_1 + R_2. \tag{2.9}$$

For the capacitors' series combination, $V_T = q/C_1 + q/C_2 = q/C_T$, which implies

$$1/C_T = 1/C_1 + 1/C_2. \tag{2.10}$$

For the inductors' series combination, $V_T = L_1 dI/dt + L_2 dI/dt = L_T dI/dt$, which gives

$$L_T = L_1 + L_2. \tag{2.11}$$

Thus, when in series, resistances and inductances add, and the reciprocals of capacitances add.

Next, view Figure 2.6 in which the *parallel* combinations of two resistors, two capacitors, and two inductors are displayed. The same expressions for the i-th and total potential differences hold as before. Hence, for the resistors' parallel combination, $I_T = I_1 + I_2$, which leads to

$$1/R_T = 1/R_1 + 1/R_2. \tag{2.12}$$

For the capacitors' parallel combination, $q_T = q_1 + q_2$, which leads to

$$C_T = C_1 + C_2. \tag{2.13}$$

For the inductors' parallel combination, $I_T = I_1 + I_2$, which gives

$$1/L_T = 1/L_1 + 1/L_2. \tag{2.14}$$

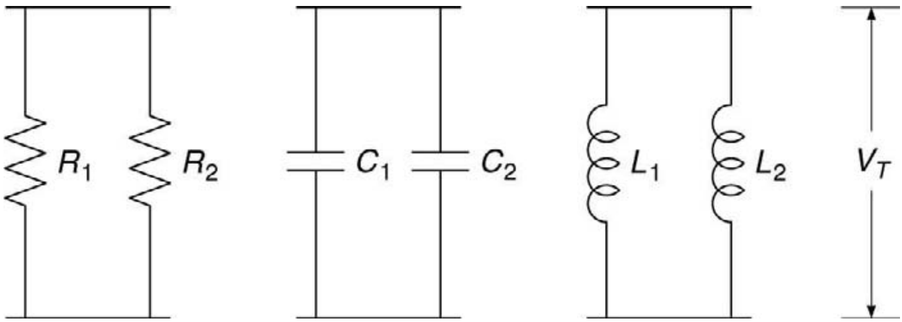

FIGURE 2.6
Parallel R, C, and L circuit configurations.

Thus, when in parallel, capacitances add, and the reciprocals of resistances and inductances add.

Example Problem 2.2

Statement: Determine the total equivalent resistance, R_T, and total equivalent capacitance, C_T, for the respective resistance and capacitance circuits shown in Figure 2.7.

Solution: For the two resistors in parallel, the equivalent resistance, R_a, is

$$\frac{1}{R_a} = \frac{1}{4} + \frac{1}{4} = 2\ \Omega.$$

The two other resistors are in series with R_a, so $R_T = R_a + 2 + 6 = 10\ \Omega$.

For the two capacitors in parallel, the equivalent capacitance, C_b, is $C_b = 3 + 3 = 6\ \mu F$. This is in series with the two other capacitors, which implies that

$$\frac{1}{C_T} = \frac{1}{2} + \frac{1}{3} + \frac{1}{C_b} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

So, C_T=1 μF.

To aid further with circuit analysis, two laws developed by G. R. Kirchhoff (1824 - 1887) can be used. **Kirchhoff's current (or first) law**, which is conservation of charge, states that at any junction (node) in a circuit, the current flowing into the junction must equal the current flowing out of it, which implies that

$$\sum_{node} I_{in} = \sum_{node} I_{out}. \tag{2.15}$$

A **node** in a circuit is a point where two or more circuit elements meet. **Kirchhoff's voltage (or second) law**, which is conservation of energy, says that around any loop in a circuit, the sum of the potential differences equals zero, which gives

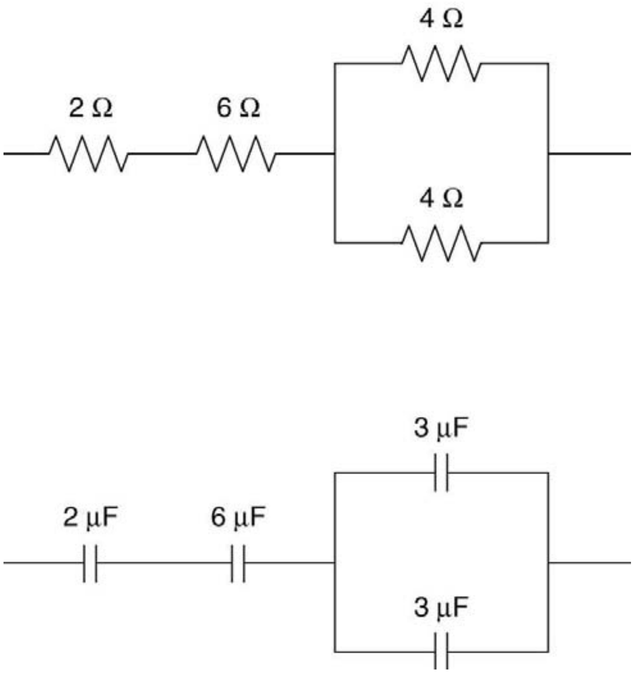

FIGURE 2.7
Resistor and capacitor circuits.

$$\sum_{i,cl.loop} V_i = 0. \tag{2.16}$$

A **loop** is a closed path that goes from one node in a circuit back to itself without passing through any intermediate node more than once. Any *consistent* sign convention will work when applying Kirchhoff's laws to a circuit. Armed with this information, some important DC circuits can be examined now.

2.5 Elementary DC Circuit Analysis

In DC circuits, current is steady in time. Thus, there is no inductance, even if an inductor is present. An actual inductor, however, has resistance. This typically is on the order of 10 Ω. Often, an inductor's resistance in a DC circuit is neglected.

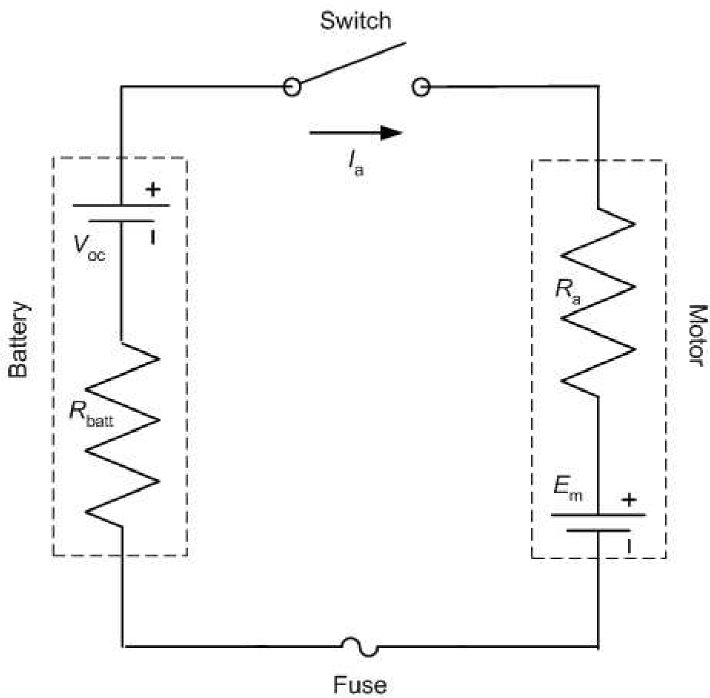

FIGURE 2.8
A battery and electric motor circuit.

The first elementary DC circuit to analyze is that of a DC electric motor in series with a battery, as shown in Figure 2.8. Examine the battery first. It has an internal resistance, R_{batt}, and an open circuit voltage (potential difference), V_{oc}. R_{batt} is the resistance and V_{oc} the potential difference that would be measured across the terminals of the battery if it were isolated from the circuit by not being connected to it. However, when the battery is placed in the circuit and the circuit switch is closed such that current, I_a, flows around the circuit, the situation for the battery changes. The measured potential difference across the battery now is less because current flows through the battery, effectively leading to a potential difference across R_{batt}. This yields

$$V_{batt} = V_{oc} - I_a R_{batt}, \tag{2.17}$$

in which V_{batt} represents the closed-circuit potential difference across the battery. Similarly, the DC motor has an internal resistance, R_a, which is mainly across its armature. It also has an opposing potential difference, E_m, when operating with the battery connected to it. To summarize, V_{oc} is measured across the battery terminals when the switch is open, and V_{batt} is measured when the switch is closed.

Now what is E_m in terms of the known quantities? To answer this, apply Kirchhoff's second law around the circuit loop when the switch is closed. Starting from the battery's anode and moving in the direction of the current around the loop, this gives

$$V_{oc} - I_a R_{batt} - E_m - I_a R_a = 0, \tag{2.18}$$

which immediately leads to

$$E_m = V_{oc} - I_a(R_{batt} + R_a). \tag{2.19}$$

This equation reveals a simple fact: the relatively high battery and motor internal resistances lead to a decrease in the motor's maximum potential difference. This consequently results in a decrease in the motor's power output to a device, such as the propeller of a remotely piloted aircraft.

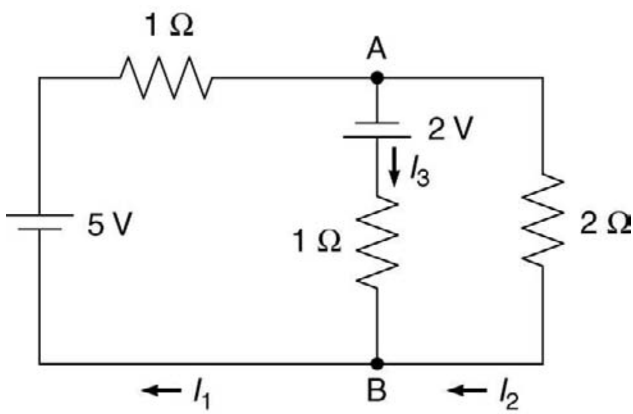

FIGURE 2.9
An electrical circuit.

Example Problem 2.3

Statement: For the electrical circuit shown in Figure 2.9, determine [a] the magnitude of the current in the branch between nodes A and B and [b] the direction of that current.

Solution: Application of Kirchhoff's second law to the left loop gives 5 V − (1 Ω)(I_1 A) + 2 V − (1 Ω)(I_3 A) = 0. Similar application to the right loop yields 2 V − (2 Ω)(I_2 A) + (1 Ω)(I_3 A) = 0. At node A, application of Kirchhoff's first law implies $I_1 - I_2 - I_3 = 0$. These three expressions can be solved to yield $I_1 = 3.8$ A, $I_2 = 0.6$ A, and $I_3 = 3.2$ A. Because I_3 is positive, the direction shown in Figure 2.9, from node A to node B, is correct.

The second elementary direct-current circuit is a **Wheatstone bridge**. The Wheatstone bridge is used in a variety of common instruments such

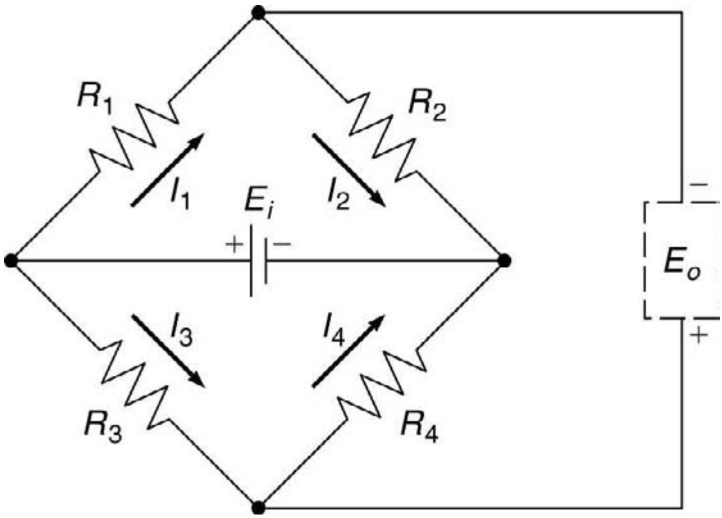

FIGURE 2.10
The Wheatstone bridge configuration.

as pressure transducers and hot-wire anemometers. Its circuit, shown in Figure 2.10, consists of four resistors (R_1 through R_4), each two comprising a pair (R_1 and R_2; R_3 and R_4) in series that is connected to the other pair in parallel, and a voltage source, E_i, connected between the R_1-R_3 and the R_2-R_4 nodes. The voltage output of the bridge, E_o, is measured between the R_1-R_2 and the R_3-R_4 nodes. E_o is measured by an ideal voltmeter with an infinite input impedance such that no current flows through the voltmeter.

An expression needs to be developed that relates the bridge's output voltage to its input voltage and the four resistances. There are four unknowns, I_1 through I_4. This implies that four equations are needed to reach the desired solution. Examination of the circuit reveals that there are four closed loops for which four equations can be written by applying Kirchhoff's second law. The resulting four equations are

$$E_i = I_1 R_1 + I_2 R_2, \tag{2.20}$$

$$E_i = I_3 R_3 + I_4 R_4, \tag{2.21}$$

$$E_o = I_4 R_4 - I_2 R_2, \tag{2.22}$$

and

$$E_o = -I_3 R_3 + I_1 R_1. \tag{2.23}$$

Kirchhoff's first law leads to $I_1 = I_2$ and $I_3 = I_4$, assuming no current flows through the voltmeter. These two current relations can be used in Equations 2.20 and 2.21 to give

$$I_1 = \frac{E_i}{R_1 + R_2} \tag{2.24}$$

and

$$I_3 = \frac{E_i}{R_3 + R_4}. \tag{2.25}$$

These two expressions can be substituted into Equation 2.23, yielding the desired result

$$E_o = E_i \left[\frac{R_1}{R_1 + R_2} - \frac{R_3}{R_3 + R_4} \right]. \tag{2.26}$$

Equation 2.26 leads to some interesting features of the Wheatstone bridge. When there is no voltage output from the bridge, the bridge is considered to be balanced even if there is an input voltage present. This immediately yields the balanced bridge equation

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}. \tag{2.27}$$

This condition can be exploited to use the bridge to determine an unknown resistance, say R_1, by having two other resistances fixed, say R_2 and R_3, and varying R_4 until the balanced bridge condition is achieved. This is called the **null method.** This method is used to determine the resistance of a sensor which usually is located remotely from the remainder of the bridge. An example is the hot-wire sensor of an anemometry system used in the constant-current mode to measure local fluid temperature.

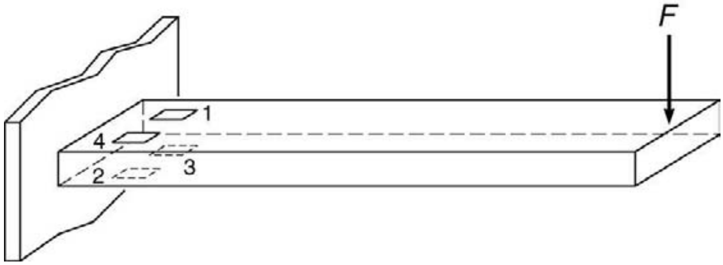

FIGURE 2.11
Cantilever beam with four strain gages.

The bridge can be used also in the **deflection method** to provide an output voltage that is proportional to a *change* in resistance. Assume that resistance R_1 is the resistance of a sensor, such as a fine wire or a strain gage. The sensor is located remotely from the remainder of the bridge circuit in an environment in which the temperature increases from some initial state. Its resistance will change by an amount δR from R_1 to R_1'. Application of Equation 2.26 yields

$$E_o = E_i \left[\frac{R_1'}{R_1' + R_2} - \frac{R_3}{R_3 + R_4} \right]. \tag{2.28}$$

Further, if all the resistances are initially the same, where $R_1 = R_2 = R_3 = R_4 = R$, then Equation 2.28 becomes

$$E_o = E_i \left[\frac{\delta R/R}{4 + 2\delta R/R} \right] = E_i f(\delta R/R). \quad (2.29)$$

Thus, when using the null and deflection methods, the Wheatstone bridge can be utilized to determine a resistance or a change in resistance.

One practical use of the Wheatstone bridge is in a force measurement system. This system is comprised of a cantilever beam, rigidly supported on one end, that is instrumented with four strain gages, two on top and two on bottom, as shown in Figure 2.11. The strain gage is discussed further in Chapter 3. A typical strain gage is shown in Figure 3.3 of that chapter. The electric configuration is called a four-arm bridge. The system operates by applying a known force, F, near the end of the beam along its centerline and then measuring the output of the Wheatstone bridge formed by the four strain gages. When a load is applied in the direction shown in Figure 2.11, the beam will deflect downward, giving rise to a tensile strain, ϵ_L, on the top of the beam and a compressive strain, $-\epsilon_L$, on the bottom of the beam. Because a strain gage's resistance increases with strain, $\delta R \sim \epsilon_L$, the resistances of the two tensile strain gages will increase and those of the two compressive strain gages will decrease. In general, following the notation in Figure 2.11, for the applied load condition,

$$R_1^{'} = R_1 + \delta R_1, \quad (2.30)$$

$$R_4^{'} = R_4 + \delta R_4, \quad (2.31)$$

$$R_2^{'} = R_2 - \delta R_2, \quad (2.32)$$

and

$$R_3^{'} = R_3 - \delta R_3. \quad (2.33)$$

If all four gages are identical, where they are of the same pattern with $R_1 = R_2 = R_3 = R_4 = R$, the two tensile resistances will *increase* by δR and the two compressive ones will *decrease* by δR. For this case, Equation 2.28 simplifies to

$$E_o = E_i(\delta R/R). \quad (2.34)$$

For a cantilever beam shown in Figure 2.11, the strain along the length of the beam on its top side is proportional to the force applied at its end, F. Thus, $\epsilon_L \sim F$. If strain gages are aligned with this axis of strain, then $\delta R \sim \epsilon_L$, as discussed in Section 3.3.2. Thus, the voltage output of this system, E_o, is *linearly* proportional to the applied force, F. Further, with this strain gage configuration, variational temperature and torsional effects are compensated for automatically. This is an inexpensive, simple yet elegant measurement system that can be calibrated and used to determine unknown forces. This configuration is the basis of most force balances used for aerodynamic and mechanical force measurements.

Example Problem 2.4

Statement: Referring to Figure 2.10, if $R_1 = 1\ \Omega$, $R_2 = 3\ \Omega$, and $R_3 = 2\ \Omega$, determine [a] the value of R_4 such that the Wheatstone bridge is balanced, and [b] the bridge's output voltage under this condition.

Solution: Equation 2.27 specifies the relationship between resistances when the bridge is balanced. Thus, $R_4 = R_2R_3/R_1 = (3)(2)/1 = 6\ \Omega$. Because the bridge is balanced, its output voltage is zero. This can be verified by substituting the four resistance values into Equation 2.26.

2.6 Elementary AC Circuit Analysis

As shown in Table 2.1, the expressions for the voltages and currents of AC circuits containing resistors, capacitors, and inductors involve differentials and integrals with respect to time. Expressions for $V(t)$ and $I(t)$ of AC circuits can be obtained directly by solving the first-order and second-order ordinary differential equations that govern their behavior. The differential equation solutions for RC and RLC circuits subjected to step and sinusoidal inputs are presented later in Chapter 4. At this point, a working knowledge of AC circuits can be gained through some elementary considerations.

When capacitors and inductors are exposed to time-varying voltages in AC circuits, they each create a **reactance** to the voltage. Reactance plus resistance equals **impedance**. Symbolically, $X + R = Z$. Often an RLC component is described by its impedance because it encompasses both resistance and reactance. Impedance typically is considered *generalized resistance* [2]. For DC circuit analysis, impedance is resistance because there is no reactance. For this case, $Z = R$.

Voltages and currents in AC circuits usually do not vary simultaneously in the same manner. An increase in voltage with time, for example, can be followed by a corresponding increase in current at some later time. Such changes of voltage and current in time are characterized best by using complex number notation. This notation is described in more detail in Chapter 9.

Assume that the voltage, $V(t)$, and the current, $I(t)$, are represented by the complex numbers $V_oe^{i\phi}$ and $I_oe^{i\phi}$, respectively. Here $e^{i\phi}$ is given by Euler's formula,

$$e^{i\phi} = \cos\phi + i\sin\phi, \tag{2.35}$$

where $i = \sqrt{-1}$. In electrical engineering texts, j is used to denote $\sqrt{-1}$

because i is used for the current. Throughout this text, i symbolizes the imaginary number. The real voltage and real current are obtained by multiplying each by the complex number representation $e^{i\omega t}$ and then taking the real part, Re, of the resulting number. That is,

$$V(t) = Re(Ve^{i\omega t}) = Re(V)\cos\omega t - Im(V)\sin\omega t \tag{2.36}$$

and

$$I(t) = Re(Ie^{i\omega t}) = Re(I)\cos\omega t - Im(I)\sin\omega t, \tag{2.37}$$

in which ω is the frequency in rad/s. The frequency in cycles/s is f, where $2\pi f = \omega$.

Expressions for capacitive and inductive reactances can be derived using the voltage and current expressions given in Equations 2.36 and 2.37 [2]. For a capacitor, $I(t) = CdV(t)/dt$. Differentiating Equation 2.36 with respect to time yields the current across the capacitor,

$$I(t) = -V_o C\omega \sin\omega t = Re[V_o \frac{e^{i/\omega C}}{1/i\omega C}]. \tag{2.38}$$

The denominator of the real component is the capacitive reactance,

$$X_C = 1/i\omega C. \tag{2.39}$$

In other words, $V(t) = I(t)X_C$.

For the inductor, $V(t) = LdI(t)/dt$. Differentiating Equation 2.37 with respect to time yields the voltage across the inductor,

$$V(t) = -I_o L\omega \sin\omega t = Re[i\omega L I_o e^{i/\omega C}]. \tag{2.40}$$

The numerator of the real component is the inductive reactance,

$$X_C = i/\omega L. \tag{2.41}$$

Simply put, $V(t) = I(t)X_L$.

Because the resistances of capacitors and inductors are effectively zero, their impedances equal the reactances. Further, the resistor has no reactance, so its impedance is its resistance. Thus, $Z_R = R$, $Z_C = 1/i\omega C$, and $Z_L = i\omega L$.

Ohm's law is still valid for AC circuits. It now can be written as $V = ZI$. Also the rules for adding resistances apply to adding impedances, where for impedances in series

$$Z_T = \sum Z_i, \tag{2.42}$$

and for impedances in parallel

$$Z_T = 1/\sum(\frac{1}{Z_i}). \tag{2.43}$$

Example Problem 2.5

Statement: An electrical circuit loop is comprised of a resistor, R, a voltage source, E_o, and a 1-μF capacitor, C, that can be added into the loop by a switch. When the switch is closed, all three electrical components are in series. When the switch is open, the loop has only the resistor and the voltage source in series. $E_o = 3$ V and $R = 2\ \Omega$. Determine the expression for the current as a function of time, $I(t)$, immediately after the switch is closed.

Solution: Applying Kirchhoff's second law to the loop gives

$$RI(t) + \frac{1}{C}\int I(t)dt = E_o.$$

This equation can be differentiated with respect to time to yield

$$R\frac{dI(t)}{dt} + \frac{I(t)}{C} = \frac{dE_o}{dt} = 0,$$

because E_o is a constant 3 V. This equation can be integrated to obtain the result $I(t) = C_1 e^{-t/RC}$, where C_1 is a constant. Now at $t = 0$ s, current flows through the loop and equals E_o/R. This implies that $C_1 = 3/2 = 1.5$ A. Thus, for the given values of R and C, $I(t) = 1.5e^{-t/(2\times10^{-6})}$. This means that the current in the loop becomes almost zero in approximately 10 μs after the switch is closed.

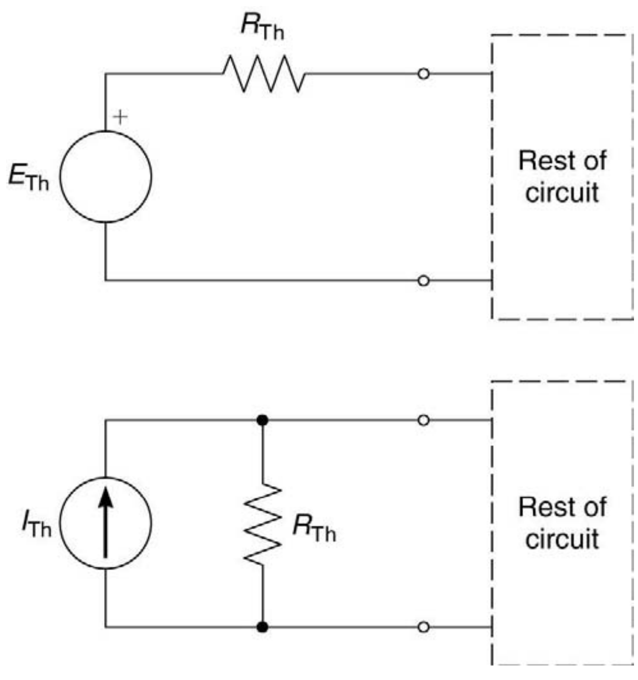

FIGURE 2.12
Thévenin and Norton equivalent circuits.

2.7 *Equivalent Circuits

Thévenin's equivalent circuit theorem states that any two-terminal network of linear impedances, such as resistors and inductors, and voltage sources can be replaced by an equivalent circuit consisting of an ideal voltage source, E_{Th}, in series with an impedance, Z_{Th}. This circuit is shown in the top of Figure 2.12.

Norton's equivalent circuit theorem states that any two-terminal network of linear impedances and current sources can be replaced by an equivalent circuit consisting of an ideal current source, I_{Th}, in parallel with an impedance, Z_{Th}. This circuit is illustrated in the bottom of Figure 2.12.

The voltage of the Thévenin equivalent circuit is the current of the Norton equivalent circuit times the equivalent impedance. Obtaining the equivalent impedance sometimes can be tedious, but it is very useful in understanding circuits, especially the more complex ones.

The **Thévenin equivalent voltage** and equivalent impedance can be determined by examining the open-circuit voltage and the short-circuit current. The Thévenin equivalent voltage, E_{Th}, is the open-circuit voltage, which is the potential difference that exists between the circuit's two terminals when nothing is connected to the circuit. This would be the voltage measured using an ideal voltmeter. Simply put, $E_{Th} = E_{oc}$, where the subscript oc denotes open circuit. The **Thévenin equivalent impedance**, Z_{Th}, is E_{th} divided by the short-circuit current, I_{sc}, where the subscript sc denotes short circuit. The short-circuit current is the current that would pass through an ideal ammeter connected across the circuit's two terminals.

An example diagram showing an actual circuit and its Thévenin equivalent is presented in Figure 2.13. In this figure, Z_{Th} is represented by R_{Th} because the only impedances in the circuit are resistances. R_m denotes the meter's resistance, which would be infinite for an ideal voltmeter and zero for an ideal ammeter.

The Thévenin equivalents can be found for the actual circuit. Kirchhoff's voltage law implies

$$E_i = I(R_1 + R_2). \tag{2.44}$$

Also, the voltage measured by the ideal voltmeter, E_m, using Ohm's law and noting that $R_2 << R_m$, is

$$E_m = E_{th} = IR_2. \tag{2.45}$$

Combining Equations 2.44 and 2.45 yields the open-circuit or Thévenin equivalent voltage,

$$E_{Th} = E_i \frac{R_2}{R_1 + R_2}. \tag{2.46}$$

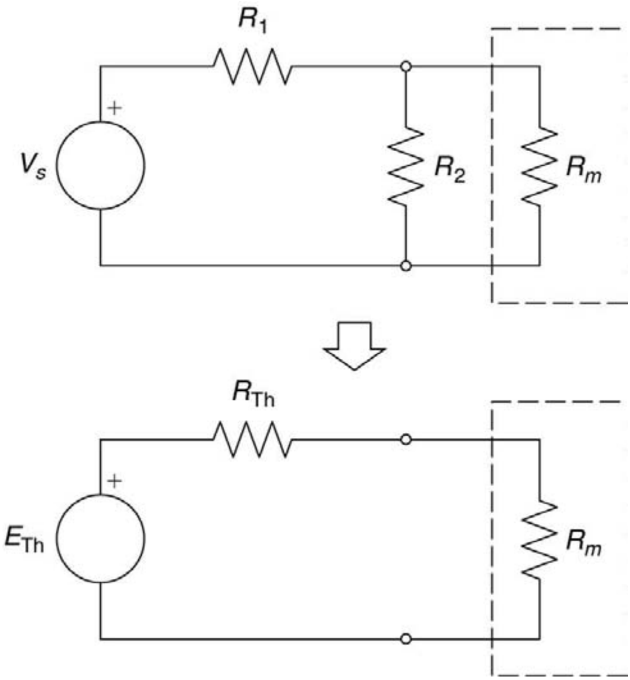

FIGURE 2.13
A circuit and its Thévenin equivalent.

Further, the short-circuit current would be

$$I_{sc} = E_i/R_1. \tag{2.47}$$

So, the Thévenin equivalent resistance is

$$R_{Th} \equiv \frac{E_{Th}}{I_{sc}} = \frac{R_1 R_2}{R_1 + R_2}. \tag{2.48}$$

The resulting Thévenin equivalent circuit is shown in the bottom of Figure 2.13.

An alternative approach to determining the Thévenin impedance is to replace all voltage sources in the circuit by their internal impedances and then find the circuit's output impedance. Usually the voltage sources' internal impedances are negligible and can be assumed to be equal to zero, effectively replacing all voltage sources by short circuits. For the circuit shown in Figure 2.13, this approach would lead to having the resistances R_1 and R_2 in parallel to ground, leading directly to Equation 2.48.

This alternative approach can be applied also when determining the Thévenin equivalent resistance, which is the output impedance, of the

Wheatstone bridge circuit shown in Figure 2.10. Assuming a negligible internal impedance for the voltage source E_i, R_{Th} is equivalent to the parallel combination of R_1 and R_2 in series with the parallel combination of R_3 and R_4. That is,

$$R_{Th} = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}. \tag{2.49}$$

Example Problem 2.6

Statement: For the circuit shown in the top of Figure 2.13, determine the Thévenin equivalent resistance and the Thévenin equivalent voltage, assuming that $V_s = 20$ V, $R_1 = 6\ \Omega$, $R_2 = 3\ \Omega$, and $R_m = 3$ MΩ.

Solution: Because $R_m >> R_2$, the Thévenin equivalent voltage is given by Equation 2.46 and the Thévenin equivalent resistance by Equation 2.48. Substitution of the given values for $V_s = E_i$, R_1, and R_2 into these equations yields $E_{th} = (20)\left[\frac{3}{6+3}\right] = 6.67$ V and $R_{th} = \frac{(6)(3)}{6+3} = 2\ \Omega$.

2.8 *Meters

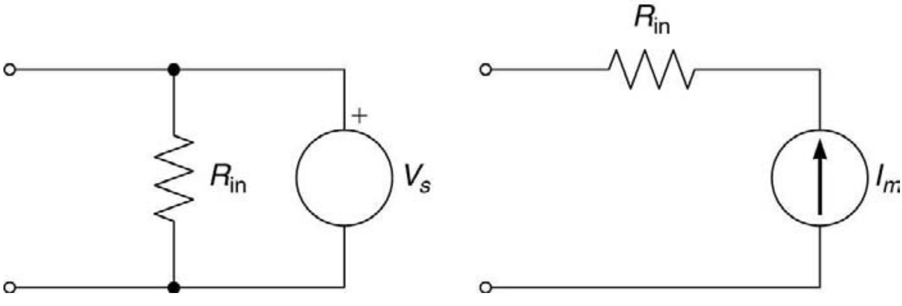

FIGURE 2.14
Voltage and current meters.

All voltage and current meters can be represented by Thévenin and Norton equivalent circuits, as shown in Figure 2.14. These meters are characterized by their input impedances. An *ideal* **voltmeter** has an infinite input impedance such that no current flows through it. An *ideal* **ammeter** has zero input impedance such that all the connected circuit's current flows through it. The actual devices differ from their ideal counterparts only in that the actual impedances are neither zero nor infinite, but finite.

A voltmeter is attached in parallel to the point of interest in the circuit. An ammeter is attached in series with the point of interest in the circuit. A good voltmeter has a very high input impedance, typically greater than 1 MΩ. Because of this, a good voltmeter connected to a circuit draws negligible current from the circuit and, therefore, has no additional voltage difference present between the voltmeter's terminals. Likewise, because a good ammeter has a very low input impedance, typically less than 1 Ω, almost all of the attached circuit's current flows through the ammeter.

Resistance measurements typically are made using an **ohmmeter**. The resistance actually is determined by passing a known current through the test leads of a meter and the unknown resistance and then measuring the total voltage difference across them. This is called the **two-wire method**. This approach is valid provided that the unknown resistance is much larger than the resistances of the test leads. In practice, this problem is circumvented by using a multimeter and the **four-wire method**. This method requires the use of two additional test leads. Two of the leads carry a known current through the unknown resistance and then back to the meter, while the other two leads measure the resulting voltage drop across the unknown resistance. The meter determines the resistance by Ohm's law and then displays it.

2.9 *Impedance Matching and Loading Error

When the output of one electronic component is connected to the input of another, the output signal may be altered, depending upon the component impedances. Each measurement circumstance requires a certain relation between the output component's output impedance and the input component's input impedance to avoid signal alteration. If this impedance relation is not maintained, then the output component's signal will be altered upon connection to the input component. A common example of impedance mismatch is when an audio amplifier is connected to a speaker with a high input impedance. This leads to a significant reduction in the power transmitted to the speaker, which results in a low volume from the speaker.

A **loading error** can be introduced whenever one circuit is attached to another. Loading error, e_{load}, is defined in terms of the difference between the true output impedance, R_{true}, the impedance that would be measured across the circuit's output terminals by an ideal voltmeter, and the impedance measured by an actual voltmeter, R_{meas}. Expressed on a percentage basis, the loading error is

$$e_{load} = 100\left[\frac{R_{true} - R_{meas}}{R_{true}}\right]. \tag{2.50}$$

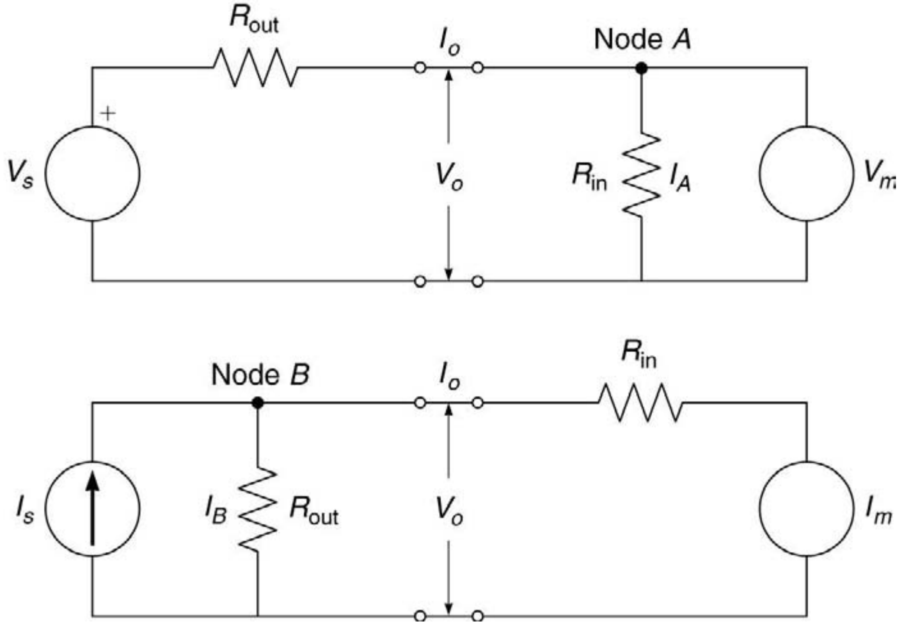

FIGURE 2.15
Voltage circuit (top) and current circuit (bottom) illustrating loading error.

Loading errors that occur when measuring voltages, resistances, or current can be avoided by following two simple rules. These rules, summarized at the end of this section, can be derived by considering two circuits, one in which an actual voltage source circuit is connected to an actual voltmeter, and the other in which an actual current source circuit is connected to an actual ammeter. These circuits are shown in Figure 2.15.

For the voltage circuit, Kirchhoff's voltage law applied around the outer circuit loop gives

$$V_m = V_s - I_o R_{out}. \tag{2.51}$$

Kirchhoff's current law applied at node A yields

$$I_o = I_A = V_m / R_{in}, \tag{2.52}$$

where all of the current flows through the voltmeter's R_{in}. Substituting Equation 2.52 into Equation 2.51 results in

$$V_m = V_s \left[\frac{1}{1 + \left(\frac{R_{out}}{R_{in}} \right)} \right] = V_s \left[\frac{R_{in}}{R_{in} + R_{out}} \right]. \tag{2.53}$$

When $R_{in} >> R_{out}$, $V_m = V_s$. Noting for this voltage measurement case that $R_{true} = R_{out}$ and $R_{meas} = (R_{in}R_{out})/(R_{in} + R_{out})$, the loading error becomes

$$e_{load,V} = \left[\frac{R_{out}}{R_{in} + R_{out}}\right]. \tag{2.54}$$

For the current circuit, Kirchhoff's current law applied at node B yields

$$I_s = I_B + I_o. \tag{2.55}$$

Kirchhoff's voltage law applied around the circuit loop containing R_{in} and R_{out} gives

$$I_m R_{in} = I_B R_{out}. \tag{2.56}$$

Substituting Equation 2.56 into Equation 2.55 results in

$$I_m = I_s \left[\frac{1}{1 + \left(\frac{R_{in}}{R_{out}}\right)}\right] = I_s \left[\frac{R_{out}}{R_{in} + R_{out}}\right]. \tag{2.57}$$

When $R_{in} << R_{out}$, $I_m = I_s$. Noting for the current measurement case that $R_{true} = R_{in}$ and $R_{meas} = (R_{in}R_{out})/(R_{in} + R_{out})$, the loading error becomes

$$e_{load,I} = \left[\frac{R_{in}}{R_{in} + R_{out}}\right]. \tag{2.58}$$

Loading errors can be avoided between two circuits by connecting them via a buffer that has near-infinite input and near-zero output impedances. This is one of the many uses of operational amplifiers. These are presented in Chapter 3.

Example Problem 2.7

Statement: Determine the minimum input impedance, R_{min}, of a voltage measurement circuit that would have less than 0.5 % loading error when connected to a circuit having an output impedance of 50 Ω.

Solution: Direct application of Equation 2.54 implies

$$\frac{0.5}{100} = \frac{50\ \Omega}{50\ \Omega + R_{min}}.$$

Solving for the minimum input impedance gives $R_{min} = 9950\ \Omega$, or approximately 10 kΩ. This condition can be met by using a unity-gain operational amplifier in the non-inverting configuration at the input of the voltage-measurement circuit (see Chapter 3).

The impedance relation for optimum power transmission between an output source and an input circuit can be determined [8]. For the voltage circuit in Figure 2.15, noting that the power received, P_{in}, equals V_{in}^2/R_{in}, Equation 2.53 becomes

$$P_{in} = V_s^2 \left[\frac{R_{in}}{(R_{in} + R_{out})^2} \right]. \tag{2.59}$$

Differentiating Equation 2.59 with respect to R_{in}, setting the result equal to zero and solving for R_{in} gives

$$R_{in} = R_{out}. \tag{2.60}$$

Substitution of Equation 2.60 into the derivative equation shows that this condition ensures a maximum transmission of power. Equation 2.60 represents true **impedance matching**, where the two impedances have the *same* value.

Example Problem 2.8

Statement: Determine the power that is transmitted, P_t, between two connected circuits if the output circuit impedance is 6.0 Ω, the input circuit impedance is 4.0 Ω, and the source voltage is 12 V.

Solution: Substitution of the given values into Equation 2.53 gives $V_m = 12\left[\frac{4}{6+4}\right]$ = 4.8 V. Now, the power transmitted is given by $P_t = V_{in}^2/R_{in} = 4.8^2/4 = 5.8$ W, with the correct number of significant figures.

Impedance matching also is critical when an output circuit that generates waveforms is connected by a cable to a receiving circuit. In this situation, the high-frequency components of the output circuit can reflect back from the receiving circuit. This essentially produces an input wave to the receiving circuit that is different from that intended. When a cable with characteristic impedance, R_{cable}, is connected to a receiving circuit of load impedance, R_{in}, and these impedances are matched, then the the input wave will not be reflected. The reflected wave amplitude, A_r, is related to the incident wave amplitude, A_i, by

$$A_r = A_i \left[\frac{R_{cable} - R_{in}}{R_{cable} + R_{in}} \right]. \tag{2.61}$$

When $R_{cable} < R_{in}$, the reflected wave is inverted. When $R_{cable} > R_{in}$, the reflected wave is not inverted [2].

The rules for impedance matching and for loading error minimization, as specified by Equations 2.53, 2.57, 2.60, and 2.61, are as follows:

- **Rule 1 – loading error minimization**: When measuring a voltage, the input impedance of the measuring device must be much greater than the equivalent circuit's output impedance.

- **Rule 2 – loading error minimization**: When measuring a current, the input impedance of the measuring device must be much less than the equivalent circuit's output impedance.
- **Rule 3 – impedance matching**: When transmitting power to a load, the output impedance of the transmission circuit must equal the input impedance of the load for maximum power transmission.
- **Rule 4 – impedance matching**: When transmitting signals having high frequency through a cable, the cable impedance must equal the load impedance of the receiving circuit.

2.10 *Electrical Noise

Electrical noise is defined as anything that obscures a signal [2]. Noise is characterized by its amplitude distribution, frequency spectrum, and the physical mechanism responsible for its generation. Noise can be subdivided into **intrinsic noise** and **interference noise**. Intrinsic noise is random and primarily the result of thermally induced molecular motion in any resistive element (Johnson noise), current fluctuations in a material (shot noise), and local property variations in a material ($1/f$ or pink noise). The first two are intrinsic and cannot be eliminated. The latter can be reduced through quality control of the material that is used.

Noise caused by another signal is called interference noise. Interference noise depends on the amplitude and frequency of the noise source. Common noise sources include AC-line power (50 Hz to 60 Hz), overhead fluorescent lighting (100 Hz to 120 Hz), and sources of radio-frequency (RF) and electromagnetically induced (EMI) interference, such as televisions, radios, and high-voltage transformers.

The causes of electrical interference include local electric fields, magnetic fields, and ground loops. These noticeably affect analog voltage signals with amplitudes less than one volt. A surface at another electric potential that is near a signal-carrying wire will establish an undesirable capacitance between the surface and the wire. A local magnetic field near a signal-carrying wire will induce an additional current in the wire. A current flowing through one ground point in a circuit will generate a signal in another part of the circuit that is connected to a different ground point.

Most interference noise can be attenuated to acceptable levels by proper shielding, filtering, and amplification. For example, signal wires can be shielded by a layer of conductor that is separated from the signal wire by an insulator. The electric potential of the shield can be driven at the same potential as the signal through the use of operational amplifiers and feedback, thereby obviating any undesirable capacitance [5]. Pairs of insulated wires carrying similar signals can be twisted together to produce signals with the same mode of noise. These signals subsequently can be conditioned using common-mode rejection techniques. Use of a single electrical ground point for a circuit almost always will minimize ground-loop effects. Signal amplification and filtering also can be used. In the end though, it is better to eliminate the sources of noise than to try to cover them up.

The magnitude of the noise is characterized through the signal-to-noise ratio (SNR). This is defined as

$$\text{SNR} \equiv 10 \log_{10} \left[\frac{V_s^2}{V_n^2} \right], \tag{2.62}$$

where V_s and V_n denote the source and noise voltages, respectively. The voltage values usually are rms values (see Chapter 9). Also, a center frequency and range of frequencies are specified when the SNR is given.

2.11 Problem Topic Summary

Topic	Review Problems	Homework Problems
Basics	1, 2, 4, 6, 7, 13, 14, 15 20, 21, 22, 23, 24, 25	8, 9, 10
Circuits	3, 5, 8, 9, 10, 11, 12 16, 17, 18, 19	4, 5, 7, 8, 11, 12
Systems	8, 9, 10, 11	1, 2, 3, 6
Op Amps	23	13, 14

TABLE 2.2
Chapter 2 Problem Summary

2.12 Review Problems

1. Three 11.9 μF capacitors are placed in series in an electrical circuit. Compute the total capacitance in μF to one decimal place.
2. Which of the following combination of units is equivalent to 1 J? (a) 1 C·A·W, (b) 1 W·s/C, (c) 1 N/C, (d) 1 C·V.

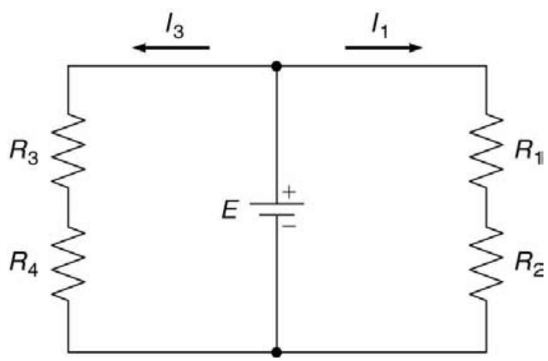

FIGURE 2.16
Electrical circuit.

3. For the electrical circuit depicted in Figure 2, given $R_1 = 160$ Ω, $R_3 = 68$ Ω, $I_1 = 0.9$ A, $I_3 = 0.2$ A, and $R_2 = R_4$, find the voltage potential, E, to the nearest whole volt.

Quantity	Famous Person
current	James Joule
charge	Charles Coulomb
electric field work	Georg Ohm
electric potential	James Watt
resistance	Andre Ampere
power	Michael Faraday
inductance	Joseph Henry
capacitance	Alessandro Volta

TABLE 2.3
Famous people and electric quantities.

4. The ends of a wire 1.17 m in length are suspended securely between two insulating plates. The diameter of the wire is 0.000 05 m. Given that the electric resistivity of the wire is 1.673×10^{-6} Ω·m at 20.00 °C and that its coefficient of thermal expansion is 56.56×10^{-5}/°C, compute the internal resistance in the wire at 24.8 °C to the nearest whole ohm.

5. A wire with the same material properties given in the previous problem is used as the R_1 arm of a Wheatstone bridge. The bridge is designed to be used in deflection method mode and to act as a transducer in a system used to determine the ambient temperature in the laboratory. The length of the copper wire is fixed at 1.00 m and the diameter is 0.0500 mm. $R_2 = R_3 = R_4 = 154$ Ω and $E_i = 10.0$ V. For a temperature of 25.8 °C, compute the output voltage, E_o, in volts to the nearest hundredth.

6. Which of the following effects would most likely *not* result from routing an AC signal across an inductor? (a) A change in the frequency of the output alternating current, (b) a back electromagnetic force on the input current, (c) a phase lag in the output AC signal, (d) a reduction in the amplitude of the AC signal.

7. Match each of the following quantities given in Table 2.3 with the famous person for whom the quantity's unit is named.

8. Given the electrical circuit in Figure 7, where $R_1 = 37$ Ω, $R_2 = 65$ Ω, $R_3 = 147$ Ω, $R_4 = 126$ Ω, and $R_5 = 25$ Ω, find the total current drawn by all of the resistors to the nearest tenth A.

   Questions 9 through 13 pertain to the electrical circuit diagram given in Figure 2.18.

9. A Wheatstone bridge is used as a transducer for a resistance temperature device (RTD), which forms the R_1 leg of the bridge. The coefficient of thermal expansion for the RTD is 0.005/°C. The reference resistance of

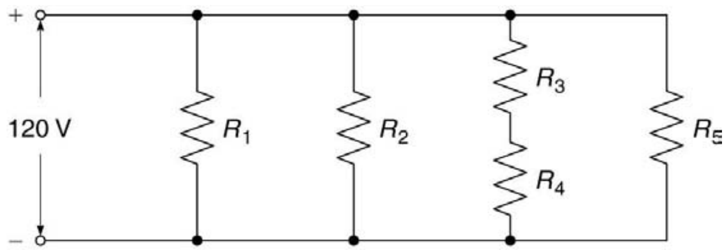

FIGURE 2.17
Resistor circuit.

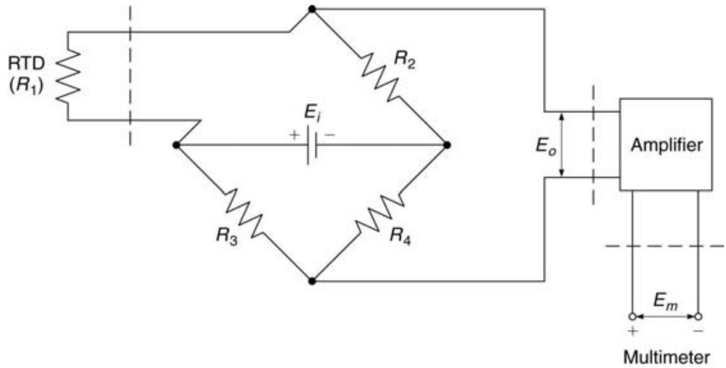

FIGURE 2.18
Temperature measurement system.

the device is 25 Ω at a reference temperature of 20 °C. Compute the resistance of the RTD at 67 °C to the nearest tenth of an ohm. Use this procedure to arrive at the answer in the next problem.

10. For the Wheatstone bridge shown in Figure 2.18, $R_2 = R_3 = R_4 = 25\ \Omega$ and $E_i = 5$ V. The maximum temperature to be sensed by the RTD is 78 °C. Find the maximum output voltage from the Wheatstone bridge to the nearest thousandth volt. The answer to this question will be used in the following problem. (Hint: The answer should be between 0.034 V and 0.049 V.)

11. A constant gain amplifier, with gain factor G, conditions the output voltage from the Wheatstone bridge shown in Figure 2.18. The multimeter used to process the output voltage from the amplifier, E_m, has a full-scale output of 10 V. Determine the maximum gain factor possible to the nearest hundred. The answer to this question will be used in the following problem.

12. The RTD shown in Figure 2.18 senses a temperature of 60 °C. Compute the voltage output to the multimeter, E_m, to the nearest hundredth volt.

13. What bridge method is used for the RTD measurement system shown in Figure 2.18? (a) Deflection method, (b) null method, (c) strain gage method, (d) resistance-temperature method.

14. Which of a following is a *consequence* of the conservation of energy?

    (a) Ohm's law, (b) Kirchhoff's first law, (c) potential differences around a closed loop sum to zero, (d) reciprocals of parallel resistances add.

15. Consider the cantilever-beam Wheatstone bridge system that has four strain gages (two in compression and two in tension). Which of the following statements is *not* true: (a) The change in resistance in each gage is proportional to the applied force, (b) temperature and torsional effects are automatically compensated for by the bridge, (c) the longitudinal (axial) strain in the beam is proportional to the output voltage of the bridge, (d) a downward force on the beam causes an increase in the resistance of a strain gage placed on its lower (under) side.

16. An initially balanced Wheatstone bridge has $R_1 = R_2 = R_3 = R_4 = 120$ Ω. If R_1 increases by 20 Ω, what is the ratio of the bridge's output voltage to its excitation voltage?

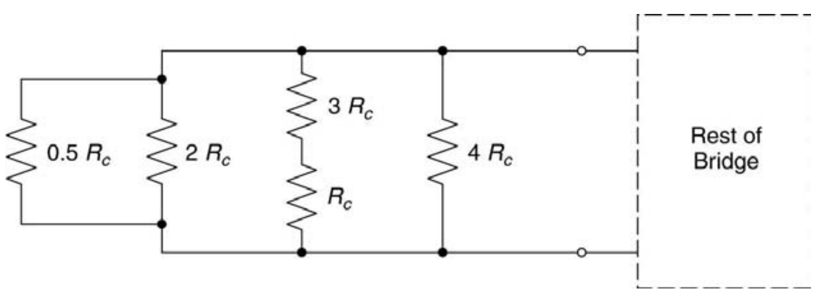

FIGURE 2.19
Wheatstone bridge circuit.

17. A Wheatstone bridge may be used to determine unknown resistances using the null method. The electrical circuit shown in Figure 2.19 (with no applied potential) forms the R_1 arm of the Wheatstone bridge. If $R_2 = R_3 = 31$ Ω and $R_c = 259$ Ω, find the necessary resistance of arm R_4 to balance the bridge. Resistances R_1, R_2, R_3, and R_4 refer to the resistances in the standard Wheatstone bridge configuration. Use the standard Wheatstone bridge. Round off the answer to the nearest ohm.

18. A Wheatstone bridge has resistances R_2=10 Ω, R_3=14 Ω, and R_4=3 Ω. Determine the value of R_1 in Ω when the bridge is used in the null method. Round off the answer to the nearest ohm.

19. Calculate the power absorbed by each resistor in Figure 2.20.

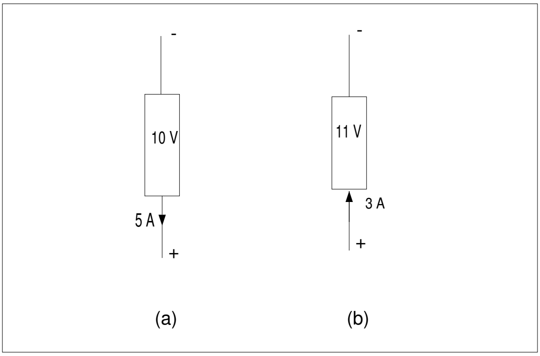

FIGURE 2.20
Two resistors.

20. A 2 mH inductor has a voltage $v(t) = 2\cos(1000t)$ V, with $i(t = 0) = 1.5$ A. Find the energy stored in the inductor at $t = \pi/6$ ms.

21. Determine the coefficient of thermal expansion (in Ω/°R) of a 1 mm-diameter wire whose resistance increases by 10 % when its temperature increases by 1.8 K.

22. Determine the current, in A, through a capacitor that is discharging at a rate of 10 C in 2.5 s.

23. The typical output impedance of an operational amplifier, in ohms, is (a) 0, (b) < 100, (c) $\sim$1000, or (d) $> 10^7$.

24. What is the unit of resistance (Ω) in the base units (kg, m, s, and C)?

2.13 Homework Problems

1. Consider the pressure measurement system shown in Figure 2.21. The Wheatstone bridge of the pressure transducer is initially balanced at $p = p_{atm}$. Determine (a) the value of R_x (in Ω) required to achieve this balanced condition and (b) E_o (in V) at this balanced condition. Finally, determine (c) the value of E_i (in V) required to achieve $E_o = 50.5$ mV

when the pressure is changed to p = 111.3 kPa. Note that $R_s(\Omega) = 100[1 + 0.2(p - p_{atm})]$, with p in kPa.

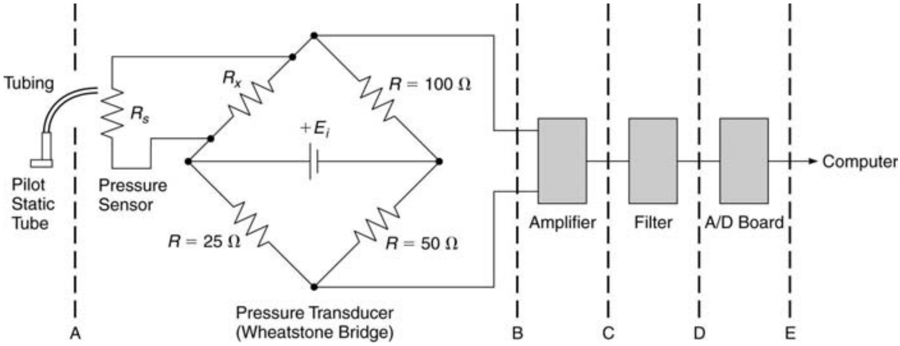

FIGURE 2.21
An example pressure measurement system configuration.

2. Consider the temperature measurement system shown in Figure 2.22. At station B determine (a) E_o (in V) when $T = T_o$, (b) E_o (in V) when T = 72 °F, and (c) the bridge's output impedance (in Ω) at T = 72 °F. Note that the sensor resistance is given by $R_s = R_o[1+\alpha(T-T_o)]$, with $\alpha = 0.004/°F$, and R_o = 25 Ω at T_o = 32 °F. Also E_i = 5 V.

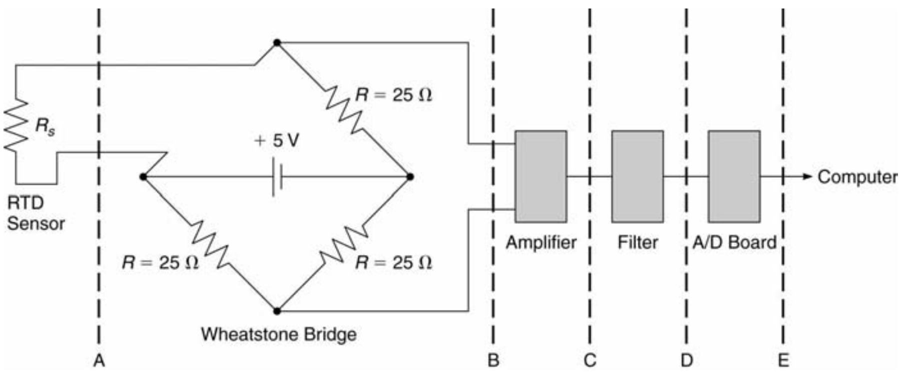

FIGURE 2.22
An example temperature measurement system configuration.

3. Consider the Wheatstone bridge that is shown in Figure 2.10. Assume that the resistor R_1 is actually a thermistor whose resistance, R, varies with the temperature, T, according to the equation

$$R = R_o \exp\left[\beta(\frac{1}{T} - \frac{1}{T_o})\right],$$

where $R_o = 1000\ \Omega$ at $T_o = 26.85\ °C = 300$ K (absolute) and $\beta = 3500$. Both T and T_o must be expressed in absolute temperatures in Equation 3. (Recall that the absolute temperature scales are either K or °R.) Assume that $R_2 = R_3 = R_4 = R_o$. (a) Determine the normalized bridge output, E_o/E_i, when $T = 400$ °C. (b) Write a program to compute and plot the normalized bridge output from $T = T_o$ to $T = 500$ °C.

(c) Is there a range of temperatures over which the normalized output is linear? (d) Over what temperature range is the normalized output very insensitive to temperature change?

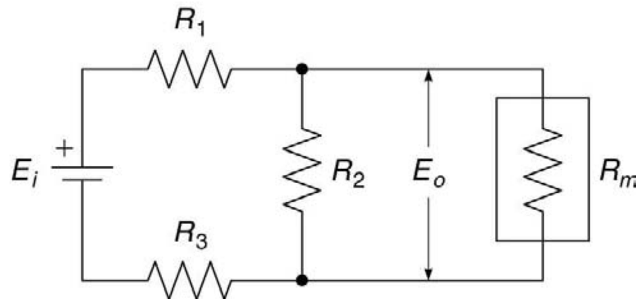

FIGURE 2.23
Test circuit.

4. For the test circuit shown in Figure 2.23, derive an expression for the output voltage, E_o, as a function of the input voltage, E_i, and the resistances shown for (a) the ideal case of the perfect voltmeter having $R_m = \infty$ and (b) the non-ideal voltmeter case when R_m is finite. Show mathematically that the solution for case (b) becomes that for case (a) when $R_m \rightarrow \infty$.

5. An inexpensive voltmeter is used to measure the voltage to within 1 % across the power terminals of a stereo system. Such a system typically has an output impedance of 500 Ω and a voltage of 120 V at its power terminals. Assuming that the voltmeter is 100 % accurate such that the instrument and zero-order uncertainties are negligible, determine the minimum input impedance (in Ω) that this voltmeter must have to meet the 1 % criterion.

6. A voltage divider circuit is shown in Figure 2.24. The common circuit is used to supply an output voltage E_o that is less than a source voltage E_i. (a) Derive the expression for the output voltage, E_o, measured by the meter, as a function of E_i, R_x, R_y, and R_M, assuming that R_m is **not** negligible with respect to R_x and R_y. Then, (b) show that the expression derived in part (a) reduces to $E_o = E_i(R_x/R_T)$ when R_M becomes infinite.

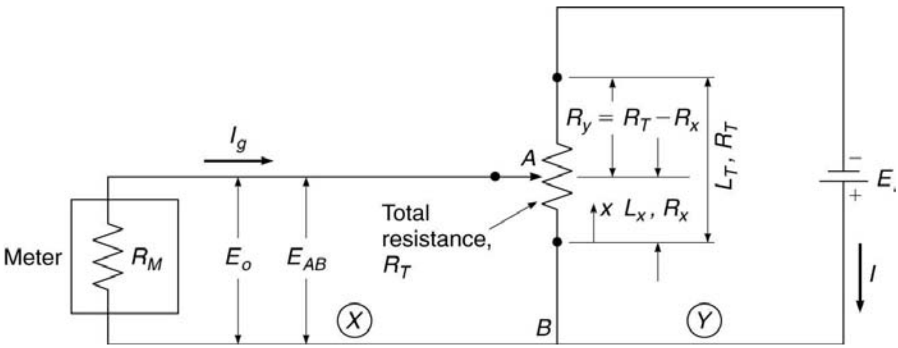

FIGURE 2.24
The voltage divider circuit.

7. Figure 2.25 presents the circuit used for a flash in a camera. The capacitor charges toward the source voltage through the resistor. The flash turns on when the capacitor voltage reaches 8 V. If $C = 10\ \mu$F, find R such that the flash will turn on once per second.

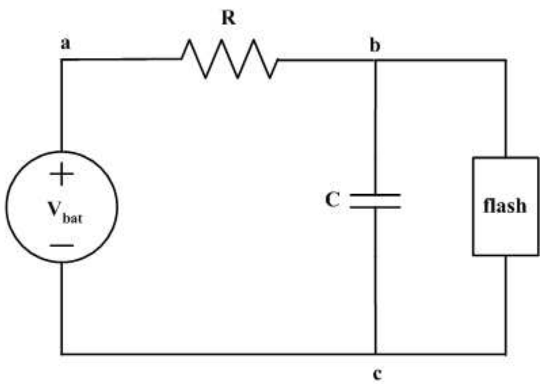

FIGURE 2.25
Camera flash circuit.

8. Find the differential equation for the current in the circuit shown in Figure 2.26.

9. Between what pair of points (A, B, C, D) shown in Figure 2.27 should one link up the power supply to charge all six capacitors to an equal capacitance?

10. A capacitor consists of two round plates, each of radius $r = 5$ cm. The gap between the plates is $d = 5$ mm. The capacity is given by $C =$

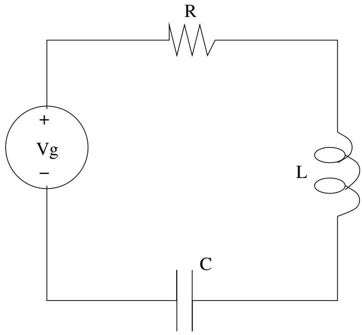

FIGURE 2.26
RLC circuit.

$\epsilon\epsilon_o S/d$ where S is the surface area, d is the gap between plates, ϵ_o is the permittivity of free space, and $\epsilon = 1$ for air. (a) Determine the maximum charge q_{max} of the capacitor in coulombs if the breakdown potential of the air is $V_{max} = 10$ kV. (b) Find the capacitor energy in both the International (SI) and the English Engineering (EE) systems.

11. Consider the flash circuit shown in Figure 2.25 for a camera supplied with a 9.0 V battery. The capacitor is used to modulate the flash of the camera by charging toward the battery through the resistor. When the capacitor voltage reaches 8.0 V, the flash discharges at a designed rate of once per 5 seconds. The resistor in this circuit is 25 kΩ. What is the capacitance of the capacitor for this design?

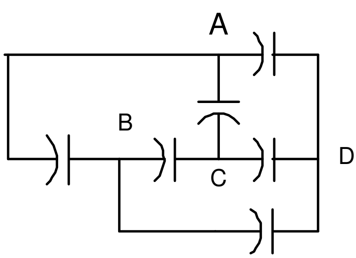

FIGURE 2.27
Six-capacitor circuit.

12. A researcher is attempting to decipher the lab notebook of a prior employee. The prior employee diagrammed the circuit shown in Figure 2.28 but gave no specification about the input voltage. Through advanced forensics you were able to find places where he recorded the measured current through the inductor I_L, at time t, the capacitor voltage V_C,

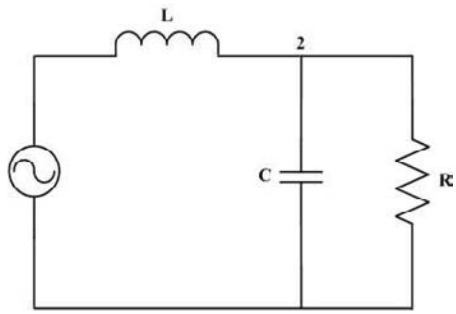

FIGURE 2.28
Notebook circuit.

and the capacitor capacitance C. Your boss has asked you to make sure you are using the right resistors, but the lab notebook does not specify the resistance. Formulate an expression to determine the resistance of the resistor. (Note: Assume that the time under consideration is small and that current through the inductor is constant when solving the differential equation. Also assume that the capacitor voltage was known at the beginning of the experiment when no current was flowing.)

13. Design an op-amp circuit with two input voltages, $E_{i,1}$ and $E_{i,2}$, such that the output voltage, E_o, is the sum of the two input voltages.

14. Consider the operational amplifier circuit shown in Figure 2.29 and the information in Table 2.1, in which R is resistance, C is capacitance, I is current, and t is time. The transfer function of the circuit can be written in the form where the output voltage, E_o, equals a function of the input voltage, E_i, and other variables. (a) List all of the other variables that would be in the transfer function expression. (b) Using Kirchhoff's laws, derive the actual transfer function expression. Identify any loops or nodes considered when applying Kirchhoff's laws.

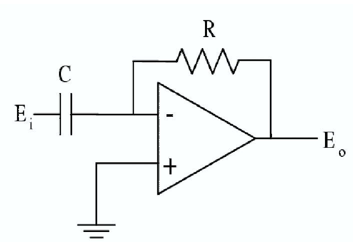

FIGURE 2.29
Op-amp circuit.

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3

Measurement Systems

CONTENTS

Measure what can be measured and make measurable what cannot be measured.
Galileo Galilei, c.1600.

If you can measure that of which you speak, and can express it by a number, you know something of your subject; but if you cannot measure it, your knowledge is meager and unsatisfactory.
Lord Kelvin, c.1850.

I profess to be a scientific man, and was exceedingly anxious to obtain accurate measurements of her shape; but... I did not know a word of Hottentot... Of a sudden my eye fell upon my sextant... I took a series of observations upon her figure in every direction, up and down, crossways, diagonally, and so forth... and thus having obtained both base and angles, I worked out the results by trigonometry and logarithms.
Sir Francis Galton, *Narrative of an Explorer in Tropical South Africa*, 1853.

3.1 Chapter Overview

The workhorse of an experiment is its measurement system. This is the equipment used from sensing an experiment's environment to recording the results. This chapter begins by identifying the main elements of a measurement system. The basic electronics behind most of these elements is covered in Chapter 2. The sensor and the transducer, the first two elements of a measurement system, will be examined first. Several sensor and transducer examples will be presented and discussed. Then the essentials of amplifiers will be covered, which include operational amplifiers that are the basic, active elements of all circuit boards today. Finally, filters and contemporary analog-to-digital processing methods will be considered. The chapter is concluded by examining a typical measurement system.

3.2 Measurement System Elements

A measurement system is comprised of the equipment used to sense an experiment's environment, to modify what is sensed into a recordable form, and to record its values. Formally, the elements of a measurement system include the sensor, the transducer, the signal conditioner, and the signal processor. These elements, acting in concert, sense the physical variable, provide a response in the form of a signal, condition the signal, process the signal, and store its value.

A measurement system's main purpose is to produce an accurate numerical value of the measurand. Ideally, the recorded value should be the exact value of the physical variable sensed by the measurement system. In practice, the perfect measurement system does not exist, nor is it needed. A result only needs to have a certain accuracy that is achieved using the most simple equipment and measurement strategy. This can be accomplished provided there is a good understanding of the system's response characteristics.

To accomplish the task of measurement, the system must perform several functions in series. These are illustrated schematically in Figure 3.1. First, the physical variable must be sensed by the system. The variable's stimulus determines a specific state of the sensor's properties. Any detectable physical property of the sensor can serve as the sensor's **signal**. When this signal changes rapidly in time, it is referred to as an **impulse**. So, by definition, the **sensor** is a device that senses a physical stimulus and converts it into a signal. This signal usually is electrical, mechanical, or optical.

For example, as depicted by the words in italics in Figure 3.1, the temperature of a gas (the physical stimulus) results in an electrical resistance (the signal) of a resistance temperature device (RTD, a temperature sensor)

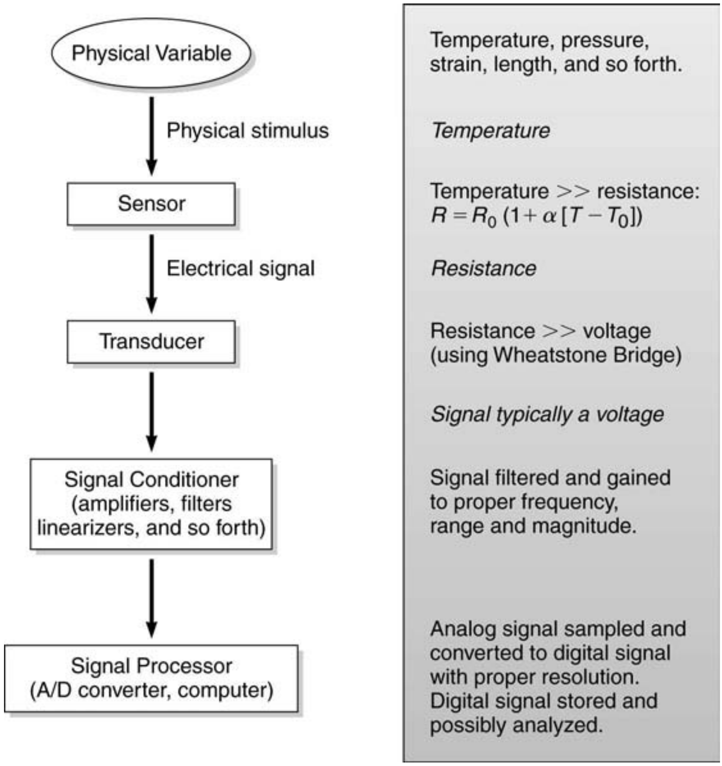

FIGURE 3.1
The general measurement system configuration.

that is located in the gas. This is because the resistance of the RTD sensor (typically a fine platinum wire) is proportional to the change in temperature from a reference temperature. Thus, by measuring the RTD's resistance, the local temperature can be determined. In some situations, however, the signal may not be amenable to direct measurement. This requires that the signal be changed into a more appropriate form, which, in almost all circumstances, is electrical. Most of the sensors in our bodies have electrical outputs.

The device that changes (transduces) the signal into the desired quantity (be it electrical, mechanical, optical, or another form) is the **transducer**. In the most general sense, a transducer transforms energy from one form to another. Usually, the transducer's output is an electrical signal, such as a voltage or current. For the RTD example, this would be accomplished by having the RTD's sensor serve as one resistor in an electrical circuit (a Wheatstone bridge) that yields an output voltage proportional to the sensor's resistance. Often, either the word *sensor* or the word *transducer* is used to describe the combination of the actual sensor and transducer. A transducer also can change an input into an output providing motion. In this case, the transducer is called an **actuator**. Sometimes, the term *transducer* is considered to encompass both sensors and actuators [2]. So, it is important to clarify what someone specifically means when referring to a transducer.

Often after the signal has been transduced, its magnitude still may be too small or may contain unwanted electrical noise. In this case, the signal must be conditioned before it can be processed and recorded. In the signal conditioning stage, an amplifier may be used to increase the signal's amplitude, or a filter may be used to remove the electrical noise or some unwanted frequency content in the signal. The **signal conditioner**, in essence, puts the signal in its final form to be processed and recorded.

In most situations, the conditioner's output signal is analog (continuous in time), and the **signal processor** output is digital (discrete in time). So, in the signal processing stage, the signal must be converted from analog to digital. This is accomplished by adding an analog-to-digital (A/D) converter, which usually is contained within the computer that is used to record and store data. That computer also can be used to analyze the resulting data or to pass this information to another computer.

A standard glass-bulb thermometer contains all the elements of a measurement system. The sensor is actually the liquid within the bulb. As the temperature changes, the liquid volume changes, either expanding with an increase in temperature or contracting with a decrease in temperature. The transducer is the bulb of the thermometer. A change in the volume of the liquid inside the bulb leads to a mechanical displacement of the liquid because of the bulb's fixed volume. The stem of the thermometer is a signal conditioner that physically amplifies the liquid's displacement, and the scale on the stem is a signal processor that provides a recordable output.

Thus, a measurement system performs many different tasks. It senses the physical variable, transforms it into a signal, transduces and conditions the signal, and then records and stores a corresponding numerical value. How each of these elements functions is considered in the following sections.

3.3 Sensors and Transducers

A sensor senses the process variable through its contact with the physical environment, and the transducer transduces the sensed information into a different form, yielding a detectable output. Contact does not need to be physical. The sensor, for example, could be an optical pyrometer located outside of the environment under investigation. This is a **non-invasive** sensor. An **invasive**, or *in situ*, sensor is located within the environment. Ideally, invasive sensors should not disturb the environment.

Usually the signals between the sensor and transducer and the detectable output are electrical, mechanical, or optical. Electrically based sensors and transducers can be **active** or **passive**. Passive elements require no external power supply. Active elements require an external power supply to produce a voltage or current output. Mechanically based sensors and transducers usually use a secondary sensing element that provides an electrical output. Often the sensor and transducer are combined physically into one device.

Sensors and transducers can be found everywhere. The sensor/transducer system in a house thermostat basically consists of a metallic coil (the sensor) with a small glass capsule (the transducer) fixed to its top end. Inside the capsule is a small amount of mercury and two electrical contacts (one at the bottom and one at the top). When the thermostat's set temperature equals the desired room temperature, the mercury is at the bottom of the capsule such that no connection is made via the electrically conducting mercury and the two contacts. The furnace and its blower are off. As the room temperature decreases, the metallic coil contracts, thereby tilting the capsule and causing the mercury to close the connection between the two contacts. The capsule transduces the length change in the coil into a digital (on/off) signal.

Another type of sensor/transducer system is in a land-line telephone mouthpiece. This consists of a diaphragm with coils housed inside a small magnet. There is one system for the mouth piece and one for the ear piece. The diaphragm is the sensor. Its coils within the magnet's field are the transducer. Talking into the mouth piece generates pressure waves causing the diaphragm with its coils to move within the magnetic field. This induces a current in the coil, which is transmitted (after modification) to another telephone. When the current arrives at the ear piece, it flows through the coils of the ear piece's diaphragm inside the magnetic field and causes the diaphragm to move. This sets up pressure waves that strike a person's eardrum as sound. Newer phones use piezo-sensors/transducers that generate an electric current from applied pressure waves, and, alternatively, pressure waves from an applied electric current. Today, most signals are digitally encoded for transmission either in optical pulses through fibers or in electromagnetic waves to and from satellites. Even with this new technology, the sensor still is a surface that moves, and the transducer still converts this movement into an electrical current.

3.3.1 Sensor Principles

Sensors are available today that sense almost anything imaginable. New ones are being developed constantly. Sensors can be categorized into **domains**, according to the type of physical variables that they sense [4], [2]. These domains and the sensed variables include

- chemical: chemical concentration, composition, and reaction rate,
- electrical: current, voltage, resistance, capacitance, inductance, and charge,
- magnetic: magnetic field intensity, flux density, and magnetization,
- mechanical: displacement or strain, level, position, velocity, acceleration, force, torque, pressure, and flow rate,
- radiant: electromagnetic wave intensity, wavelength, polarization, and phase, and
- thermal: temperature, heat, and heat flux.

The first step in understanding sensor functioning is to gain a sound knowledge about the basic principles behind sensor design and operation. This is especially true today because sensor designs change almost daily. Once these basic principles are understood, then any standard measurement textbook, for example [2], [3], [4], [5], and [18], can be consulted to obtain descriptions of innumerable devices based upon these principles. Most sensor and transducer manufacturers now provide information via the Internet that describes their product's performance characteristics.

Sensors always are based upon some physical principle or law [8]. The choice of either designing or selecting a particular sensor starts with identifying the physical variable to be sensed and the physical principle or law associated with that variable. Then, the sensor's input/output characteristics must be identified. These include, but are not limited to, the sensor's

- operational bandwidth,
- magnitude and frequency response over that bandwidth,
- sensitivity,
- accuracy,
- voltage or current supply requirements,
- physical dimensions, weight, and materials,
- environmental operating conditions (pressure, temperature, relative humidity, air purity, and radiation),
- type of output (electrical or mechanical),
- further signal conditioning requirements,
- operational complexity, and
- cost.

The final choice of sensor can involve some or all of these considerations. The following example illustrates how the design of a sensor can be a process that often involves reconsideration of the design constraints before arriving at the final design.

Example Problem 3.1

Statement: A design engineer intends to scale down a pressure sensor to fit inside an ultra-miniature robotic device. The pressure sensor consists of a circular diaphragm that is instrumented with a strain gage. The diaphragm is deflected by a pressure difference that is sensed by the gage and transduced by a Wheatstone bridge. The diaphragm of the full-scale device has a 1 cm radius, is 1 mm thick, and is made of stainless steel. The designer plans to make the miniature diaphragm out of silicon. The miniature diaphragm is to have a 600 μm radius, operate over the same pressure difference range, and have the same deflection. The diaphragm deflection, δ, at its center is

$$\delta = \frac{3(1-\nu^2)r^4\Delta p}{16Eh},$$

in which ν is Poisson's ratio, E is Young's modulus, r is the diaphragm radius, h is the diaphragm thickness, and Δp is the pressure difference. Determine the required diaphragm thickness to meet these criteria and comment on the feasibility of the new design.

Solution: Assuming that Δp remains the same, the new thickness is

$$h_n = h_o \left[\frac{(1-\nu_n^2)r_n^4 E_o}{(1-\nu_o^2)r_o^4 E_n}\right].$$

The properties for stainless steel are $\nu_o = 0.29$ and $E_o = 203$ GPa. Those for silicon are $\nu_n = 0.25$ and $E_n = 190$ GPa. Substitution of these and the aforementioned values into the expression yields $h_n = 1.41 \times 10^{-8}$ m = 14 nm. This thickness is too small to be practical. An increase in h_n by a factor of 10 will increase the Δp range likewise. Recall that this design required a similar deflection. A new design would be feasible if the required deflection for the same transducer output could be reduced by a factor of 1000, such as by the use of a piezoresistor on the surface of the diaphragm. This would increase h_n to 14 μm, which is reasonable using current micro-fabrication techniques. Almost all designs are based upon many factors, which usually require compromises to be made.

3.3.2 Sensor Examples

Sensor/transducers can be developed for different measurands and be based upon the same physical principle or law. Likewise, sensor/transducers can be developed for the same measurand and be based upon different physical principles or laws. A thin wire sensor's resistance inherently changes with strain. This wire can be mounted on various structures and used with a Wheatstone bridge to measure strain, force, pressure, or acceleration. A thin wire's resistance also inherently changes with temperature. This, as well as other sensors, such as a thermocouple, a thermistor, and a constant-current anemometer, can be used to measure temperature. Table 3.2 lists a

Measurand	Sensor	Transducer	Domain
strain	fine wire or strain gage	none	mechanical
force	strain gage on structure	Wheatstone bridge	mechanical
pressure	strain gage on structure	Wheatstone bridge	mechanical
acceleration	strain gage on structure	Wheatstone bridge	mechanical
acceleration	capacitance sensor on structure	Wheatstone bridge	mechanical
velocity	fine wire	constant-T anemometer	mechanical
velocity	microparticles in laser beams	photodetector	mechanical
temperature	dissimilar-wire junction	reference junction	thermal
temperature	fine wire	Wheatstone bridge	thermal
relative humidity	capacitance sensor	Wheatstone bridge	electrical

FIGURE 3.2
Example sensors. (L, length; R, resistance; δ, deflection; C, capacitance; U, velocity; T, temp
V, voltage; RH, relative humidity; ϵ, dielectric constant.)

number of sensor/transducers, their measurands, and other characteristics. Each type of sensor listed is considered next.

Fine Wire or Strain Gage Sensor

A sensor based upon the principle that a change in resistance can be produced by a change in a physical variable is, perhaps, the most common type of sensor. A resistance sensor can be used to measure displacement, strain, force, pressure, acceleration, flow velocity, temperature, and heat or light flux.

One simple sensor of this type is a pure metal wire or strip whose resistance changes with temperature. The resistance of a **resistance temperature device** (RTD) is related to temperature by

$$R = R_o[1 + \alpha(T - T_o) + \beta(T - T_o)^2 + \gamma(T - T_o)^3 + ...], \tag{3.1}$$

where $\alpha, \beta,$ and γ are coefficients of thermal expansion, and R_o is the resistance at the reference temperature T_o. A wire with a diameter on the order of 25 μm can be used to measure local velocity in a fluid flow. The fine wire is connected to a Wheatstone bridge-feedback amplifier circuit that is used to maintain the wire at a constant resistance, hence at a constant temperature above the fluid's temperature. As the wire is exposed to different velocities, the power required to maintain the wire at the constant temperature changes because of the changing heat transfer to the environment. The power is proportional to the square root of the fluid velocity. This system is called a **hot-wire anemometer**. Example problems involving the hot-wire anemometer are presented in Chapters 4 and 8.

If a semi-conductor is used instead of a conductor, a greater change in resistance with temperature can be achieved. This is a **thermistor**. Its resistance changes exponentially with temperature as

$$R = R_o \exp[\eta(\frac{1}{T} - \frac{1}{T_o})], \tag{3.2}$$

where η is a material constant. Thus, a thermistor usually gives better resolution over a small temperature range, whereas the RTD covers a wider temperature range. For both sensors, a transducer such as a Wheatstone bridge circuit typically is used to convert resistance to voltage.

The **strain gage** is the most frequently used resistive sensor. A typical strain gage is shown in Figure 3.3. The gage consists of a very fine wire of length L. When the wire is stretched, its length increases by ΔL, yielding a longitudinal strain of $\epsilon_L \equiv \Delta L/L$. This produces a change in resistance. Its width decreases by $\Delta d/d$, where d is the wire diameter. This defines the transverse strain $\epsilon_T \equiv \Delta d/d$. Poisson's ratio, ν, is defined as the negative of the ratio of transverse to longitudinal *local* strains, $-\epsilon_T/\epsilon_L$. The negative sign compensates for the *decrease* in transverse strain that accompanies an *increase* in longitudinal strain, thereby yielding positive values for ν. Poisson's ratio is a material property that couples these strains.

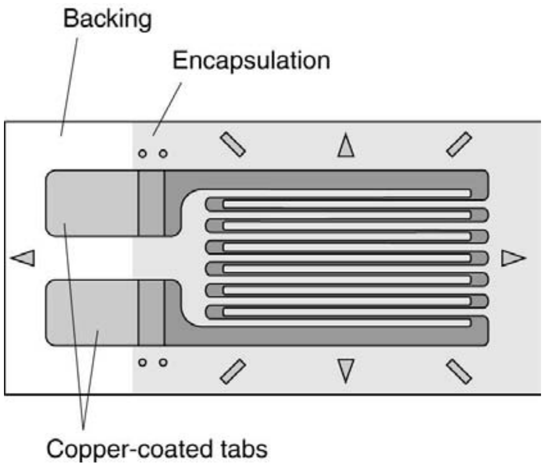

FIGURE 3.3
A strain gage with a typical sensing area of 5 mm × 10 mm.

For a wire, the resistance R can be written as

$$R = \rho \frac{L}{A}, \tag{3.3}$$

where ρ is the resistivity, L the length, and A the cross-sectional area. Taking the total derivative of Equation 3.3 yields

$$dR = \frac{\rho}{A} dL + \frac{L}{A} d\rho - \frac{\rho L}{A^2} dA. \tag{3.4}$$

Equation 3.4 can be divided by Equation 3.3 to give the relative change in resistance,

$$\frac{dR}{R} = (1 + 2\nu)\epsilon_L + \frac{d\rho}{\rho}. \tag{3.5}$$

Equation 3.5 shows that the relative resistance change in a wire depends on the strain of the wire and the resistivity change.

A **local gage factor**, G_l, can be defined as the ratio of the relative resistance change to the relative length change,

$$G_l = \frac{dR/R}{dL/L}. \tag{3.6}$$

This expression relates *differential* changes in resistance and length and describes a factor that is valid only over a very small range of strain.

An **engineering gage factor**, can be defined as

$$G_e = \frac{\Delta R/R}{\Delta L/L}. \tag{3.7}$$

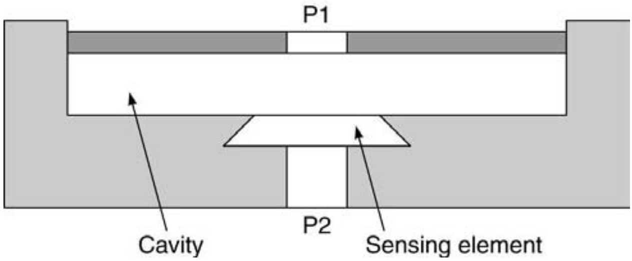

FIGURE 3.4
Schematic of an integrated-silicon pressure sensor.

This expression is based on small, finite changes in resistance and length. This gage factor is the slope based on the total resistance change throughout the region of strain investigated. The local gage factor is the instantaneous slope of a plot of $\Delta R/R$ versus $\Delta L/L$. Because it is very difficult to measure local changes in length and resistance, the engineering gage factor typically is used more frequently. Equation 3.5 can be rewritten in terms of the engineering gage factor as

$$G_e = 1 + 2\nu + [\frac{\Delta\rho}{\rho} \cdot \frac{1}{\epsilon_L}]. \tag{3.8}$$

For most metals, $\nu \approx 0.3$. The last term in brackets represents the strain-induced changes in the resistivity, which is a piezoresistive effect. This term is constant for typical strain gages and equals approximately 0.4. Thus, the value of the engineering gage factor is approximately 2 for most metallic strain gages.

An alternative expression for the relative change in resistance can be derived using statistical mechanics where

$$\frac{dR}{R} = 2\epsilon_L + \frac{dv_0}{v_0} - \frac{d\lambda}{\lambda} - \frac{dN_0}{N_0}. \tag{3.9}$$

Here v_0 is the average number of electrons in the material in motion between ions, λ is the average distance travelled by an electron between collisions, and N_0 is the total number of conduction electrons. Equation 3.9 implies that the differential resistance change and, thus, the gage factor, is independent of the material properties of the conductor. This also implies that the change in resistance only will be proportional to the strain when the sum of the changes on the right hand side of Equation 3.9 is either zero or directly proportional to the strain. Fortunately, most strain gage materials have this behavior. So, when a strain gage is used in a circuit such as a Wheatstone bridge, strain can be converted into a voltage. This system can be used as a **displacement sensor**.

Strain gages also can be mounted on a number of different flexures to yield various types of sensor systems. One example is four strain gages mounted on a beam to determine its deflection, as described in Chapter 2. As force is applied to the beam, it deflects, producing a strain. This strain is converted into a change in the resistance of a strain gage mounted on the beam. This is called a **force transducer**. Another example involves one or more strain gages mounted on the surface of a diaphragm that separates two chambers exposed to different pressures. As the diaphragm is deflected because of a pressure difference between the two chambers, a strain is produced. The resultant resistance usually is converted into a voltage using a Wheatstone bridge. This system is called a **pressure transducer**, although it actually contains both a sensor (the strain gage) and a transducer (the Wheatstone bridge). A schematic of a miniature, integrated-silicon pressure sensor is shown in Figure 3.4. The calibration and use of this type of pressure sensor in a model rocket's on-board measurement system is presented in Section 3.7.

An **accelerometer** uses a strain gage flexure arrangement. An accelerometer in the 1970's typically contained a small mass that was moved against a spring as the device containing them was accelerated. The displacement of the mass was calibrated against a known force. This information then was used to determine the acceleration from the displacement using Newton's second law. Accelerometers then used strain gages or piezoelectric transducers instead of a spring, although the size did not change much. Now micro-accelerometers are available [10]. These contain a very small mass attached to a silicon cantilever beam that is instrumented with a piezoresistor. As the device is accelerated, the beam deflects, the piezoresistor is deformed, and its resistance changes. The piezoresistor is incorporated into an on-board Wheatstone bridge circuit which provides a voltage output that is linearly proportional to acceleration. The entire micro-accelerometer and associated circuitry is several millimeters in dimension. The calibration and use of this type of accelerometer in a model rocket's on-board measurement system are presented in Section 3.7.

Capacitive Sensor

A **capacitive sensor** consists of two small conducting plates, each of area, A, separated by a distance, d, with a dielectric material in between. The capacitance between the two plates is

$$C = \epsilon_o \epsilon A/d, \tag{3.10}$$

where ϵ_o is the permittivity of free space and ϵ the relative permittivity. When used, for example, to measure pressure, the dielectric is air and one plate is held fixed. As the other plate moves because of the forces acting on it, the capacitance of the sensor changes. The change in capacitance is proportional to the difference in pressure from the reference pressure measured at zero plate deflection. When used in a capacitive Wheatstone bridge circuit, the pressure difference is converted into a voltage. This system forms a capacitive **pressure transducer**. A central plate fixed to a small mass can be used instead of air between the two capacitor plates. As the mass and its attached central plate are accelerated, the change in capacitance with respect to time is sensed. This is converted into a voltage that is proportional to the acceleration. This system constitutes an **accelerometer**. Another use of this type of sensor is to expose the dielectric material to moist air while keeping the distance between the two plates fixed. The permittivity of the dielectric material changes with relative humidity, which leads to a change in the sensor's capacitance. This type of sensor can be used as a **relative humidity sensor**.

Optical-based Sensor

An optically based measurement system can be designed to measure non-invasively the velocity and velocity fluctuations of a transparent fluid over the velocity range from $\sim$1 cm/s to $\sim$500 m/s with $\sim$1 % accuracy. This system is known as the **laser Doppler velocimeter** (LDV) and operates on the principle of the Doppler effect [9]. A coherent beam of laser light of a given frequency is directed into the moving fluid containing microparticles ($\sim$1 μm diameter), which ideally follow the flow. Because these microparticles are moving with respect to the beam, the frequency of light as received by the microparticles is Doppler shifted. A photodetector in the same reference frame as the laser receives the light that is scattered from the microparticles. This scattered light is frequency shifted once again at the receiver. The frequency of the scattered light, however, is too high to be detected using conventional detectors.

This limitation can be overcome by using two beams of equal frequency, intensity, and diameter, and crossing the beams inside the flow. This produces an ellipsoidal measurement volume with sub-millimeter dimensions. This method is called the dual-beam or Doppler frequency difference method. The crossed beams produce an ellipsoidal measurement volume, on the order of 0.5 to 1 mm in length and 0.1 to 0.3 mm in diameter. The velocity component, U, of the flow perpendicular to the bisector of the incident beams separated by an angle, θ, is related to the Doppler frequency difference, f_D, by

$$U = \frac{\lambda f_D}{2\sin(\theta/2)}, \tag{3.11}$$

where λ is the wavelength of the incident laser light. The Doppler frequency difference is the difference between the frequencies of the two scattered light beams, as received in the laboratory reference frame. Further modifications can be made by adding other beams of different frequencies in different Cartesian coordinate directions to yield all three components of the velocity. Also, frequency shifting usually is employed to compensate for insensitivity of Equation 3.11 to flow direction. If two additional, equal-spaced detectors are added, the phase lag between the signals of the three detectors is related to the diameter of the microparticle passing through the measurement volume. This system is called a **phase Doppler anemometer**.

3.3.3 *Sensor Scaling

Sensors have evolved considerably since the beginning of scientific instruments. Marked changes have occurred in the last 300 years. The temperature sensor serves as a good example. Daniel Gabriel Fahrenheit (1686-1736) produced the first mercury-in-glass thermometer in 1714 with a calibrated scale based upon the freezing point of a certain ice/salt mixture, the freezing point of water, and body temperature. This device was accurate to within several degrees and was approximately the length scale of 10 cm. In 1821, Thomas Johann Seebeck (1770-1831) found that by joining two dissimilar metals at both ends to form a circuit, with each of the two junctions held at a different temperature, a magnetic field was present around the circuit. This eventually led to the development of the thermocouple. Until very recently, the typical thermocouple circuit consisted of two dissimilar metals joined at each end, with one junction held at a fixed temperature (usually the freezing point of distilled water contained within a thermally insulated flask) and the other at the unknown temperature. A potentiometer was used to measure the mV-level emf. Presently, because of the advance in microcircuit design, the entire reference temperature junction is replaced by an electronic one and contained with an amplifier and linearizer on one small chip. Such chips even are being integrated with other micro-electronics and thermocouples such that they can be located in a remote environment and have the temperature signal transmitted digitally with very low noise to a receiving station. The simple temperature sensor has come a long way since 1700.

Sensor development has advanced rapidly since 1990 because of MEMS (microelectromechanical system) sensor technology [2]. The basic nature of sensors has not changed, although their size and applications have changed. Sensors, however, simply cannot be scaled down in size and still operate effectively. Scaling laws for micro-devices, such as those proposed by W.S.N. Trimmer in 1987, must be followed in their design [10]. As sensor sizes are reduced to millimeter and micrometer dimensions, their sensitivities to physical parameters can change. This is because some effects scale with the sensor's physical dimension. For example, the surface-to-volume ratio of a transducer with a characteristic dimension, L, scales as L^{-1}. So, surface area-active micro-sensors become more advantageous to use as their size is decreased. On the other hand, the power loss-to-onboard power scales as L^{-2}. So, as an actuator that carries its own power supply becomes smaller, power losses dominate, and the actuator becomes ineffective. Further, as sensors are made with smaller and smaller amounts of material, the properties of the material may not be isotropic. A sensor having an output that is related to its property values may be less accurate as its size is reduced. For example, the temperature determined from the change in resistance of a miniature resistive element is related to the coefficients of thermal expansion of the material. If property values change with size reduction, further error will be introduced if macro-scale coefficient values are used.

The scaling of most sensor design variables with length is summarized in Table 3.5. This can be used to examine the scaling of some conventional sensors. Consider the laminar flow element, which is used to determine a liquid flow rate. The element basically consists of many parallel tubes through which the bulk flow is subdivided to achieve laminar flow through each tube. The flow rate, Q, is related to the pressure difference, Δp, measured between two stations separated by a distance, L, as

$$Q = C_o \frac{\pi D^4 \Delta p}{128 \mu L}, \tag{3.12}$$

where D is the internal diameter of the pipe containing the flow tubes, μ the absolute viscosity of the fluid, and C_o the flow coefficient of the element. What happens if this device is reduced in size by a factor of 10 in both length and diameter? According to Equation 3.12, assuming C_o is constant, for the same Q, a Δp 1000 times greater is required! Likewise, to maintain the same Δp, Q must be reduced by a factor of 1000. The latter is most likely the case. Thus, a MEMs-scale laminar flow element is limited to operating with flow rates that are much smaller than a conventional laminar flow element.

Example Problem 3.2

Statement: Equation 3.12 is valid for a single tube when $C_o = 1$, where it reduces to the Hagen-Poiseuille law. How does the pressure gradient scale with a reduction in the tube's diameter if the same velocity is maintained?

Solution: The velocity, U, is the flow rate divided by the tube's cross-sectional area, $U = 4Q/(\pi D^2)$, where D is the tube diameter. Thus, Equation 3.12 can be written $\Delta p/L = 32\mu U D^{-2}$. This implies that the pressure gradient increases by a factor of 100 as the tube diameter is reduced by a factor of 10. Clearly, this presents a problem in sensors using micro-capillaries under these conditions. This situation necessitates the development of other means to move liquids in micro-scale sensors, such as piezoelectric and electrophoretic methods.

Decisions on the choice of a micro-sensor or micro-actuator are not based exclusively on length-scaling arguments. Other factors may be more appropriate. This is illustrated by the following example.

Example Problem 3.3

Statement: Most conventional actuators use electromagnetic forces. Are either electromagnetic or electrostatic actuators better for micro-actuators based upon force-scaling arguments?

Solution: Using Table 3.5, the electrostatic force scales as L^2 and the electromagnetic force as L^4. So, a reduction in L by a factor of 100 leads to a reduction in the electrostatic force by a factor of 1×10^4 and in the electromagnetic force by a factor of 1×10^8! If these forces are comparable at the conventional scale, then the electrostatic force is 10 000 times larger than the electromagnetic force at this reduced scale.

The final choice of which type of micro-actuator to use, however, may be based upon other considerations. For example, Madou [11] argues that energy density also could be the factor upon which to scale. Energy densities several orders of magnitude higher can be achieved using electromagnetics as compared to electrostatics, primarily because of limitations in electrostatic energy density. This could yield higher forces using electromagnetics as compared to electrostatics for comparable micro-volumes.

3.4 Amplifiers

An amplifier is an electronic component that scales the magnitude of an input analog signal, $E_i(t)$, producing an output analog signal, $E_o(t)$. In general, $E_o(t) = f\{E_i(t)\}$. For a linear amplifier $f\{E_i(t)\} = GE_i(t)$; for a logarithmic amplifier $f\{E_i(t)\} = G\log_x [E_i(t)]$, where G is the gain of the amplifier. Amplifiers are often used to increase the output signal of a transducer to a level that utilizes the full-scale range of an A/D converter that is between the transducer and the board. This minimizes errors that arise when converting a signal from analog to digital format.

The **common-mode rejection ratio** (CMRR) is another characteristic of amplifiers. It is defined as

$$\text{CMRR} = 20\log_{10}\frac{G_d}{G_c}, \tag{3.13}$$

in which G_d is the gain when different voltages are applied across the amplifier's positive and negative input terminals, and G_c is the gain when the same voltages are applied. Ideally, when two signals of the same voltage containing similar levels of noise are applied to the inputs of an amplifier, its output should be zero. Realistically, however, the amplifier's output for this case is not zero, but rather it is some finite value. This implies that the amplifier effectively has gained the signal difference by a factor of G_c, when, ideally, it should have been zero. Thus, the lower G_c is, and, consequently, the higher the CMRR is, the better it is. Typically, CMRR values greater than 100 are considered high and desirable for most applications.

Today, almost all amplifiers used in common measurement systems are operational amplifiers. An *op amp* is comprised of many transistors, resistors, and capacitors in the form of an integrated circuit. For example, the LM124 series op amp, whose schematic diagram is shown in Figure 3.6, consists of 13 transistors, 2 resistors, 1 capacitor, and 4 current sources.

Variable	Equivalent
displacement	distance
strain	length change/length
strain rate or shear rate	strain change/time
velocity	distance/time
surface	width × length
volume	width × length × height
force	mass × acceleration
line force	force/length
surface force	force/area
body force	force/volume
work, energy	force × distance
power	energy/time
power density	power/volume
electric current	charge/time
electric resistance	resistivity × length/cross-sectional area
electric field potential	voltage
electric field strength	voltage/length
electric field energy	permittivity × electric field strength^2
resistive power loss	voltage^2/resistance
electric capacitance	permittivity × plate area/plate spacing
electric inductance	voltage/change of current in time
electric potential energy	capacitance × voltage^2
electrostatic potential energy	capacitance × voltage^2 with $V \sim L$
electrostatic force	electrostatic potential energy change/distance
electromagnetic force	electromagnetic potential energy change/distance
flow rate	velocity × cross-sectional area
pressure gradient	surface force/area/length

FIGURE 3.5
Variable scaling with length, L.

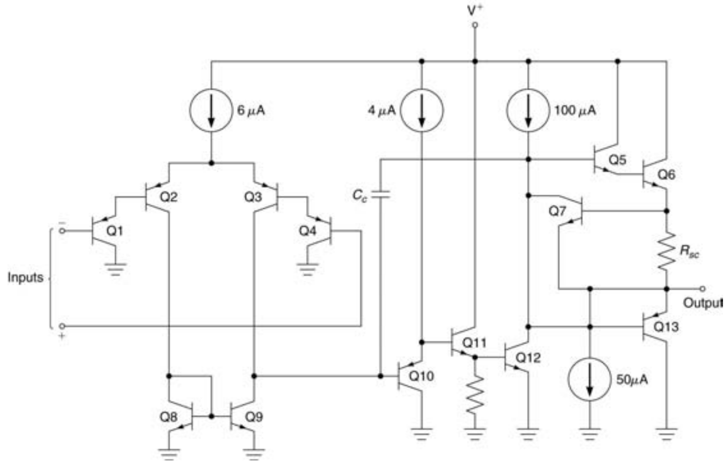

FIGURE 3.6
Internal layout of a low cost FET operational amplifier (National Semiconductor Corporation LM124 series).

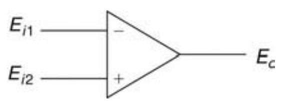

FIGURE 3.7
An operational amplifier in an open-loop configuration.

When used in an open-loop configuration, as shown in Figure 3.7, the output is not connected *externally* to the input. It is, of course, connected through the internal components of the op amp. For the open-loop configuration, $E_o(t) = A\left[E_{i2}(t) - E_{i1}(t) - V_o\right]$, where V_o is the op amp's offset voltage, which typically is zero. E_{i1} is called the *inverting* input and E_{i2} the *non-inverting* input. Because A is so large, this configuration is used primarily in situations to measure very small differences between the two inputs, when $E_{i2}(t) \cong E_{i1}(t)$.

The op amp's major attributes are as follows:

- Very high input impedance ($> 10^7\ \Omega$)
- Very low output impedance ($< 100\ \Omega$)
- High internal open-loop gain ($\sim 10^5$ to 10^6)

These attributes make the op amp an ideal amplifier. Because the input impedance is very high, very little current is drawn from the input circuits. Also, negligible current flows between the inputs. The high internal open-loop gain assures that the voltage difference between the inputs is zero. The

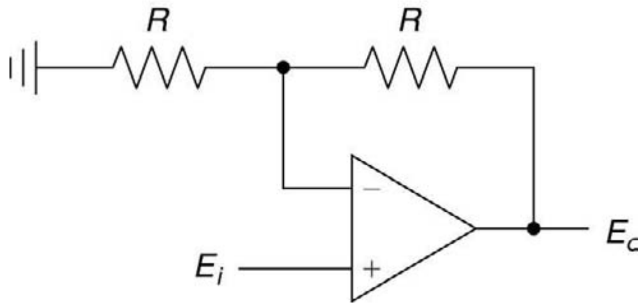

FIGURE 3.8
An operational amplifier in a closed-loop configuration.

very low output impedance implies that the output voltage is independent of the output current.

When used in the closed-loop configuration, as depicted in Figure 3.8, the output is connected externally to the input. That is, a feedback loop is established between the output and the input. The exact relation between $E_o(t)$ and $E_{i1}(t)$ and $E_{i2}(t)$ depends upon the specific feedback configuration.

Op amps typically can be treated as black boxes when incorporating them into a measurement system. Many circuit design handbooks provide equations relating an op amp's output to its input for a specified task. This can be a simple task such as inverting and gaining the input signal (the inverting configuration), not inverting but gaining the input signal (the non-inverting configuration), or simply passing the signal through it with unity gain (the voltage-follower configuration). An op amp used in the voltage-follower configuration serves as an impedance converter. When connected to the output of a device, the op amp effectively provides a very low output impedance to the device-op amp system. This approach minimizes the loading errors introduced by impedance mismatching that are described in Chapter 2. Op amps also can be used to add or subtract two inputs or to integrate or differentiate an input with respect to time, as well as many more complex tasks. The six most common op amp configurations and their input-output relations are presented in Figure 3.9.

Example Problem 3.4

Statement: Derive the expression given for the input-output relation of the differential amplifier shown in Figure 3.9.

Solution: Let node A denote that which connects R_1 and R_2 at the op amp's positive input and node B that which connects R_1 and R_2 at the op amp's negative input. Essentially no current passes through the op amp because of its very high input impedance. Application of Kirchhoff's first law at node A gives

$$\frac{E_{i2} - E_A}{R_1} = \frac{E_A - 0}{R_2}.$$

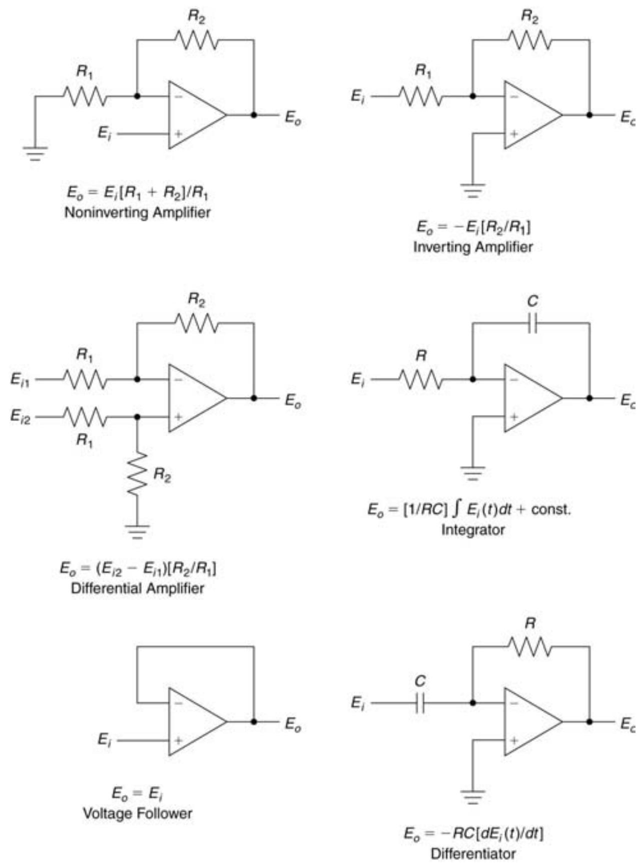

FIGURE 3.9
Other operational amplifier configurations.

This implies

$$E_A = \left[\frac{R_2}{R_1 + R_2}\right] E_{i2}.$$

Application of Kirchhoff's first law at node B yields

$$\frac{E_{i1} - E_B}{R_1} = \frac{E_B - E_o}{R_2}.$$

This gives

$$E_B = \left[\frac{R_1 R_2}{R_1 + R_2}\right]\left[\frac{E_{i1}}{R_1} + \frac{E_o}{R_2}\right].$$

Now $E_A = E_B$ because of the op amp's high internal open-loop gain. Equating the expressions for E_A and E_B gives the desired result, $E_o = (E_{i2} - E_{i1})(R_2/R_1)$.

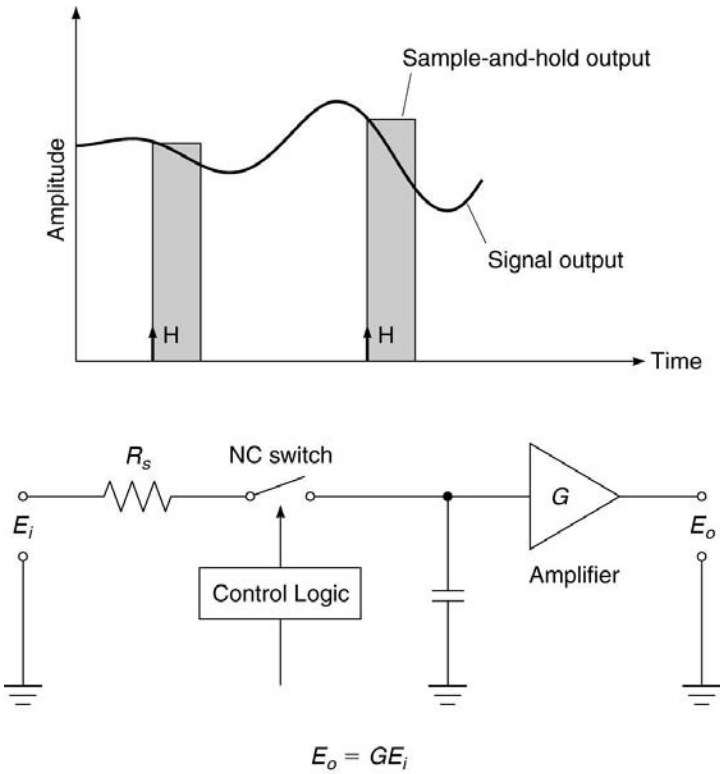

FIGURE 3.10
The sample-and-hold circuit.

In fact, op amps are the foundations of many signal-conditioning circuits. One example is the use of an op amp in a simple sample-and-hold circuit, as shown in Figure 3.10. In this circuit, the output of the op amp is held at a constant value ($= GE_i$) for a period of time (usually several microseconds) after the normally-closed (NC) switch is held open using a computer's logic control. Sample-and-hold circuits are common features of A/D converters, which are covered later in this chapter. They provide the capability to simultaneously acquire the values of several signals. These values are then held by the circuit for a sufficient period of time until all of them are stored in the computer's memory.

Quite often in measurement systems, a differential signal, such as that across the output terminals of a Wheatstone bridge, has a small (on the order of tens of millivolts), DC-biased (on the order of volts) voltage. When this is the case, it is best to use an **instrumentation amplifier**. An instrumentation amplifier is a high-gain, DC-coupled differential amplifier with a single output, high input impedance, and high CMRR [13]. This configuration assures that the millivolt-level differential signal is amplified sufficiently and that the DC-bias and interference-noise voltages are rejected.

3.5 Filters

Another measurement system component is the filter. Its primary purpose is to remove signal content at unwanted frequencies. Filters can be passive or active. Passive filters are comprised of resistors, capacitors, and inductors that require no external power supply. Active filters use resistors and capacitors with operational amplifiers, which require power. Digital filtering also is possible, where the signal is filtered *after* it is digitized.

The most common types of *ideal* filters are presented in Figure 3.11. The term *ideal* implies that the magnitude of the signal passing through the filter is not attenuated over the desired passband of frequencies. The term *band* refers to a range of frequencies and the term *pass* denotes the unaltered passing. The range of frequencies over which the signal is attenuated is called the stopband. The **low-pass** filter passes lower signal frequency content up to the cut-off frequency, f_c, and the **high-pass** filter passes content above f_c. A low-pass filter and high-pass filter can be combined to form either a **band-pass** filter or a **notch** filter, each having two cut-off frequencies, f_{cL} and f_{cH}. Actual filters do not have perfect step changes in amplitude at their cut-off frequencies. Rather, they experience a more gradual change, which is characterized by the roll-off at f_c, specified in terms of the ratio of amplitude change to frequency change.

The simplest filter can be made using one resistor and one capacitor. This is known as a **simple RC** filter, as shown in Figure 3.12. Referring to the top of that figure, if E_o is measured across the capacitor to ground, it serves as a low-pass filter. Lower frequency signal content is passed through the filter, whereas high frequency content is not. Conversely, if E_o is measured across the resistor to ground, it serves as a high-pass filter, as shown in the bottom of the figure. Here, higher frequency content is passed through the filter, whereas lower frequency content is not. For both filters, because they are not ideal, some fraction of intermediate frequency content is passed through the filter. The time constant of the simple RC filter, τ, equals RC. A unit balance shows that the units of RC are (V/A)·(C/V) or s. An actual filter differs from an ideal filter in that an actual filter alters both the *magnitude* and the *phase* of the signal, but it does *not* change its frequency.

Actual filter behavior can be understood by first examining the case of a simple sinusoidal input signal to a filter. This is displayed in Figure 3.13. The filter's input signal (denoted by A in the figure) has a peak-to-peak amplitude of E_i, with a one-cycle period of T seconds. That is, the signal's input frequency, f, is $1/T$ cycles/s or Hz. Sometimes the input

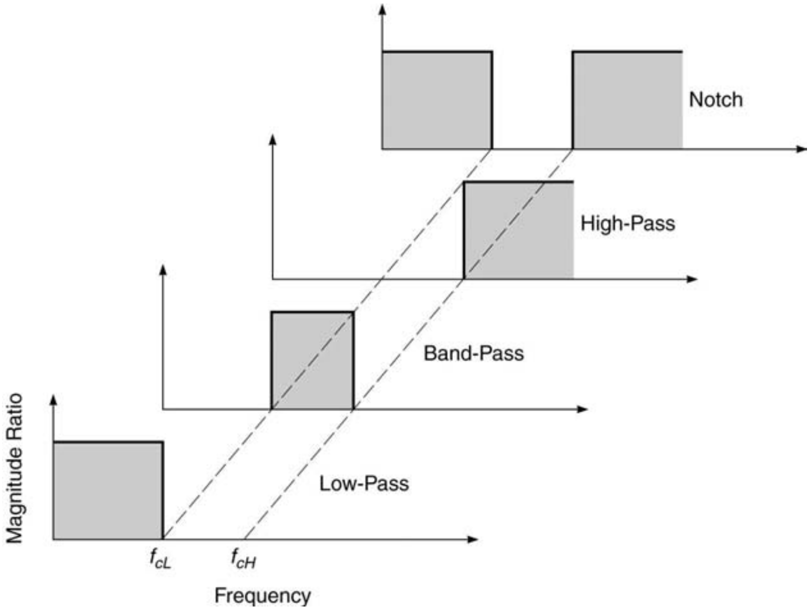

FIGURE 3.11
Ideal filter characteristics.

frequency is represented by the circular frequency, ω, which has units of rad/s. So, $\omega = 2\pi f$. If the filter only attenuated the input signal's amplitude, it would appear as signal B at the filter's output, having a peak-to-peak amplitude equal to E_0. In fact, however, an actual filter also delays the signal in time by Δt between the filter's input and output, as depicted by signal C in the figure. The output signal is said to lag the input signal by Δt. This **time lag** can be converted into a **phase lag** or phase shift by noting that $\Delta t/T = \phi/360°$, which implies that $\phi = 360°(\Delta t/T)$. By convention, the phase lag equals $-\phi$. The magnitude ratio, $M(f)$, of the filter equals $E_o(f)/E_i(f)$. For different input signal frequencies, both M and ϕ will have different values.

Analytical relationships for $M(f)$ and $\phi(f)$ can be developed for simple filters. Typically, M and ϕ are plotted each versus $\omega\tau$ or f/f_c, both of which are dimensionless, as shown in Figures 4.3 and 4.4 of Chapter 4. The **cutoff frequency**, ω_c, is defined as the frequency at which the power is one-half of its maximum. This occurs at $M = 0.707$, which corresponds to $\omega\tau = 1$ for first-order systems, such as simple filters [14]. Thus, for simple filters, $\omega_c = 1/(RC)$ or $f_c = 1/(2\pi RC)$. In fact, for a simple low-pass RC filter,

$$M(\omega) = 1/\sqrt{1 + (\omega\tau)^2} \tag{3.14}$$

and

$$\phi = -\tan^{-1}(\omega\tau). \tag{3.15}$$

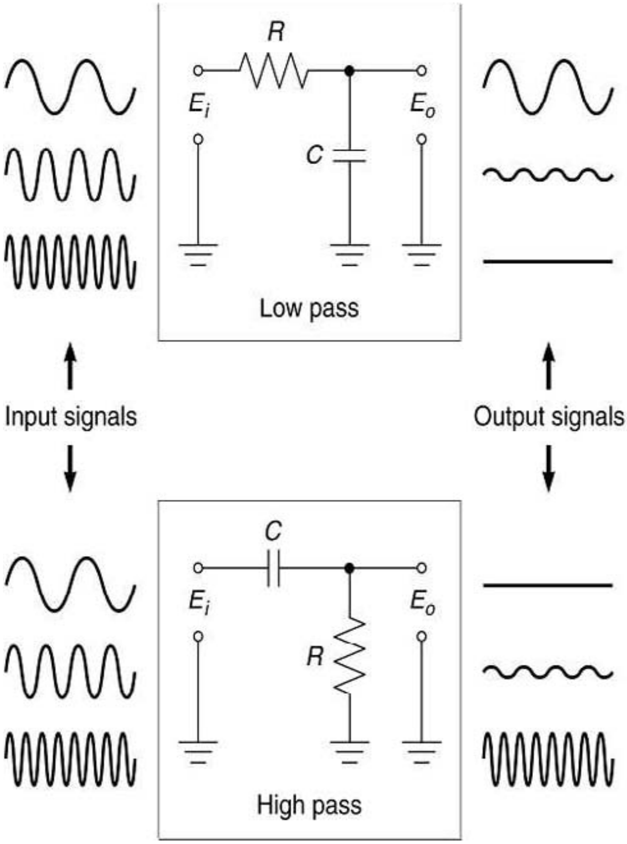

FIGURE 3.12
Simple *RC* low-pass and high-pass filters.

Using these equations, $M(\omega = 1/\tau) = 0.707$ and $\phi = -45°$. That is, at an input frequency equal to the cut-off frequency of an actual *RC* low-pass filter, the output signal's amplitude is 70.7 % of the signal's input amplitude and it lags the input signal by 45°. For an *RC* high-pass filter, the phase lag equation is given by Equation 3.15 and the magnitude ratio is

$$M(\omega) = \omega\tau/\sqrt{1 + (\omega\tau)^2}. \tag{3.16}$$

These equations are derived in Chapter 4.

An active low-pass Butterworth filter configuration is shown in Figure 3.14. Its time constant equals R_2C_2, and its magnitude ratio and phase lag are given by Equations 3.14 and 3.15, respectively. An active high-pass Butterworth filter configuration is displayed in Figure 3.15. Its time constant equals R_1C_1, and its phase lag is given by Equation 3.15. Its magnitude ratio

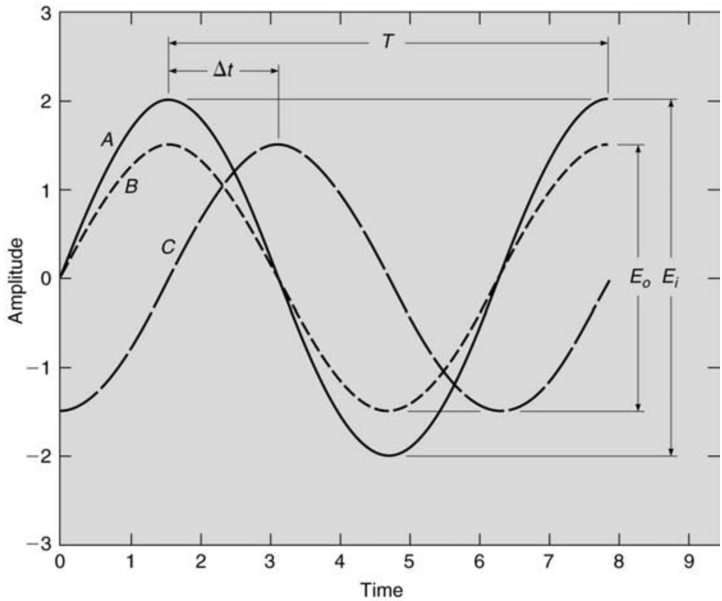

FIGURE 3.13
Generic filter input/output response characteristics.

is

$$M(\omega) = [R_2/R_1] \cdot [\omega\tau/\sqrt{1+(\omega\tau)^2}]. \tag{3.17}$$

Other classes of filters have different response characteristics. Refer to [13] for detailed descriptions or [4] for an overview.

Example Problem 3.5

Statement: For the circuit depicted in Figure 3.14, determine the equation relating the output voltage E_o to the input voltage E_i.

Solution: The op amp's major attributes assure that no current flows into the op amp and that the voltage difference between the two input terminals is zero. Assigning currents and nodes as shown in Figure 3.14 and applying Kirchhoff's current law and Ohm's law to node 1 gives

$$\begin{aligned} I_1 &= I_2 \\ \frac{E_1}{R_1} &= I_2. \end{aligned}$$

Applying Kirchhoff's current law and Ohm's law at node 2 results in

$$\begin{aligned} I_2 &= I_3 + I_4 \\ \frac{E_1}{R_1} &= -\frac{E_0}{R_2} - C_2\dot{E}_0. \end{aligned}$$

Dividing the above equation through by C_2 and rearranging terms yields

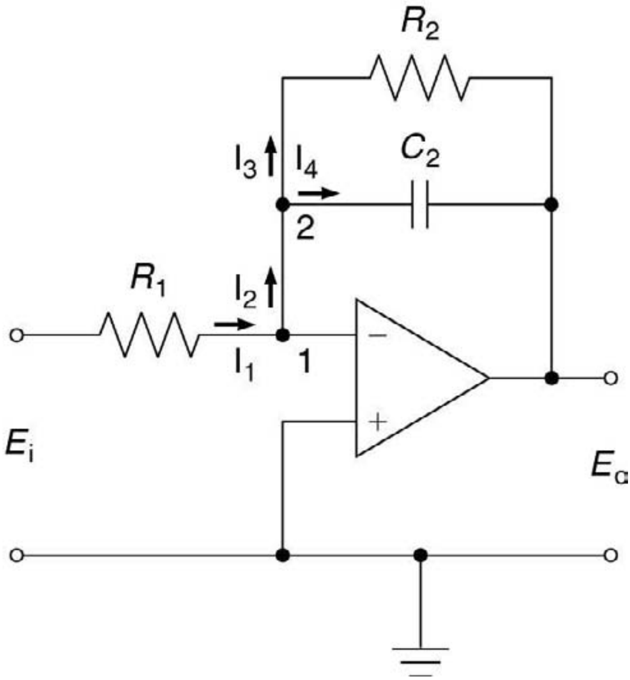

FIGURE 3.14
The active low-pass Butterworth filter.

$$\frac{E_1}{C_2R_1} = -\frac{E_0}{C_2R_2} - \dot{E}_0,$$
$$\dot{E}_0 + \frac{1}{C_2R_2}E_0 = -\frac{1}{C_2R_1}E_1.$$

This is a first-order, ordinary differential equation whose method of solution is presented in Chapter 4.

Digital filters operate on a digitally converted signal. The filter's cutoff frequency adjusts automatically with sampling frequency and can be as low as a fraction of a Hz [13]. An advantage that digital filters have over their analog counterparts is that digital filtering can be done *after* data has been acquired. This approach allows the original, unfiltered signal content to be maintained. Digital filters operate by successively weighting each input signal value that is discretized at equal-spaced times, x_i, with k number of weights, h_k. The resulting filtered values, y_i, are given by

$$y_i = \sum_{k=-\infty}^{\infty} h_k x_{i-k}. \tag{3.18}$$

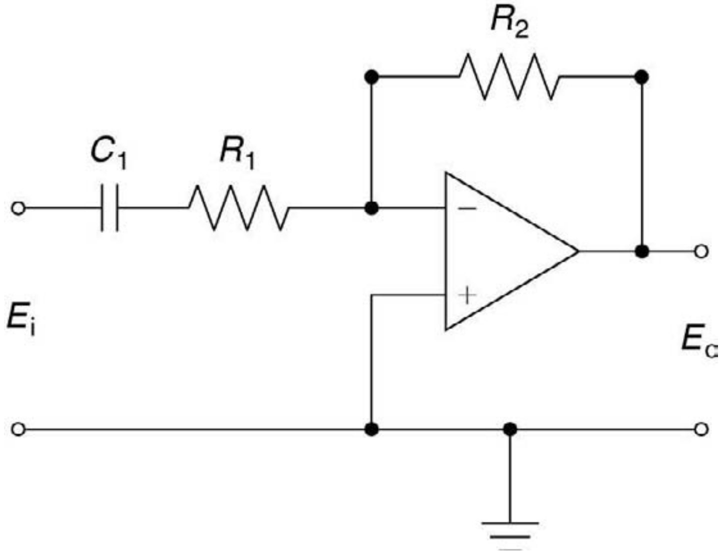

FIGURE 3.15
The active high-pass Butterworth filter.

The values of k are finite for real digital filters. When the values of h_k are zero except for $k \geq 0$, the digital filter corresponds to a real analog filter. Symmetrical digital filters have $h_{-k} = h_k$, which yield phase shifts of 0° or 180°.

Digital filters can use their output value for the i-th value to serve as an additional input for the (i+1)-th output value. This is known as a **recursive** digital filter. When there is no feedback of previous output values, the filter is a **nonrecursive** filter. A low-pass, digital recursive filter [13] can have the response

$$y_i = ay_{i-1} + (1 - a)x_i, \tag{3.19}$$

where $a = \exp(-t_s/\tau)$. Here, t_s denotes the time between samples and τ the filter time constant, which equals RC. For this filter to operate effectively, $\tau >> t_s$. Or, in other words, the filter's cut-off frequency must be much less than the Nyquist frequency. The latter is covered extensively in Chapter 10.

An example of this digital filtering algorithm is shown in Figure 3.16. The input signal of $\sin(0.01t)$ is sampled 10 times per second. Three output cases are plotted, corresponding to the cases of τ = 10, 100, and 1000. Because of the relatively high sample rate used, both the input and output signals appear as analog signals, although both actually are discrete. When $\omega\tau$ is less than one, there is little attenuation in the signal's amplitude. In fact, the filtered amplitude is 99 % of the original signal's amplitude. Also, the filtered signal lags the original signal by only 5°. At $\omega\tau = 1$, the amplitude attenuation factor is 0.707 and the phase lag is 45°. When $\omega\tau$ = 10, the

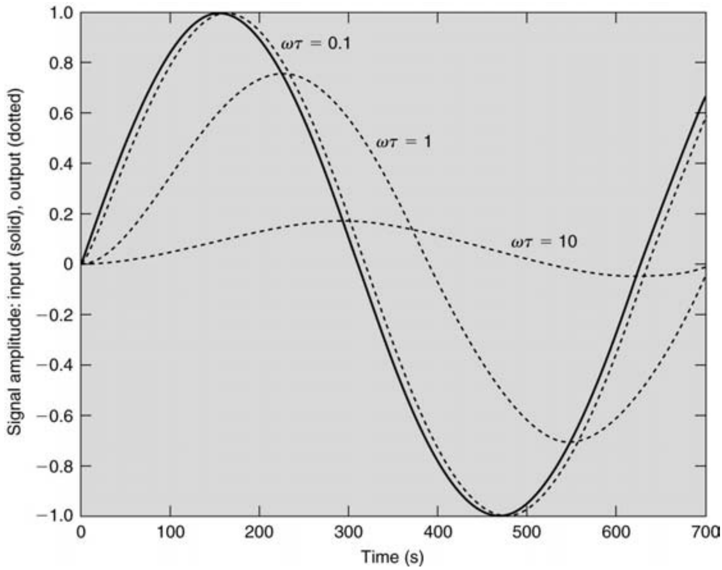

FIGURE 3.16
Digital low-pass filtering applied to a discretized sine wave.

attenuation factor is 0.90 and the phase lag is 85°. This response mirrors that of an analog filter, as depicted in Figure 3.13 and described further in Chapter 4.

Almost all signals are comprised of multiple frequencies. At first, this appears to complicate filter performance analysis. However, almost any input signal can be decomposed into the sum of many sinusoidal signals of different amplitudes and frequencies. This is the essence of Fourier analysis, which is examined in Chapter 9. For a linear, time-invariant system such as a simple filter, the output signal can be reconstructed from its Fourier component responses.

3.6 Analog-to-Digital Converters

The last measurement system element typically encountered is the A/D converter. This component serves to translate analog signal information into the digital format that is used by a computer. In the computer's binary world, numbers are represented by 0's and 1's in units called bits. A bit value of either 0 or 1 is stored physically in a computer's memory cell using a transistor in series with a capacitor. An uncharged or charged capacitor

On Decimal Value	4	2	1	Conversion	Decimal
Off Decimal Value	0	0	0	Process	Equivalent
Binary Representation	0	0	0	$0 \cdot 4 + 0 \cdot 2 + 0 \cdot 1$	0
	0	0	1	$0 \cdot 4 + 0 \cdot 2 + 1 \cdot 1$	1
	0	1	0	$0 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$	2
	0	1	1	$0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1$	3
	1	0	0	$1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1$	4
	1	0	1	$1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1$	5
	1	1	0	$1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$	6
	1	1	1	$1 \cdot 4 + 1 \cdot 2 + 1 \cdot 1$	7

TABLE 3.1
Binary to decimal conversion.

represents the value of 0 or 1, respectively. Similarly, logic gates comprised of *on-off* transistors perform the computer's calculations.

Decimal numbers are translated into binary numbers using a decimal-to-binary conversion scheme. This is presented in Table 3.1 for a 3-bit scheme. A series of locations, which are particular addresses, are assigned to a series of bits that represent decimal values corresponding from right to left to increasing powers of 2. The least significant (right-most) bit (LSB) represents a value of 2^0, whereas the most significant (left-most) bit (MSB) of an M-bit scheme represents a value of 2^{M-1}. For example, for the 3-bit scheme shown in Table 3.1, when the LSB and MSB are *on* and the intermediate bit is *off*, the binary equivalent 101 of the decimal number 5 is stored.

Example Problem 3.6

Statement: Convert the following decimal numbers into binary numbers: [a] 5, [b] 8, and [c] 13.

Solution: An easy way to do this type of conversion is to note that the power of 2 in a decimal number is equal to the number of zeros in the binary number. [a] 5 = 4 + 1 = 2^2 + 1. Therefore, the binary equivalent of 5 is 100 + 1 = 101. [b] 8 = 2^3. Therefore, the binary equivalent of 8 is 1000. [c] 13 = 8 + 4 + 1 = 2^3 + 2^2 + 1. Therefore, the binary equivalent of 13 is 1000 + 100 + 1 = 1101.

There are many methods used to perform analog-to-digital conversion electronically. The two most common ones are the successive-approximation and ramp-conversion methods. The **successive-approximation method** utilizes a D/A converter and a differential op amp that subtracts the analog input signal from the D/A converter's output signal. The conversion process begins when the D/A converter's signal is incremented in voltage steps from 0 volts using digital logic. When the D/A converter's signal rises to within ϵ volts of the analog input signal, the differential op amp's output,

Term	Formula	M = 8	M = 12
MSB Value	2^{M-1}	128	2048
LSB Value	2^0	1	1
Maximum Possible Value	$2^M - 1$	255	4095
Minimum Possible Value	0	0	0
Number of Possible Values	2^M	256	4096
MSB Weight	2^{-1}	1/2	1/2
LSB Weight	2^{-M}	1/256	1/4096
Resolution, Q (mV/bit) for $E_{FSR} = 10$ V	$E_{FSR}/2^M$	39.06	2.44
Dynamic Range (dB)	$20\log_{10}(Q/Q_o)$	-28	-52
Absolute Quantization Error (mV)	$\pm Q/2$	± 19.53	± 1.22

TABLE 3.2
M-bit terminology.

now equal to ϵ volts, causes the logic control to stop incrementing the D/A converter and tells the computer to store the converter's digital value. The **ramp-conversion method** follows a similar approach by increasing a voltage and comparing it to the analog input signal's voltage. The increasing signal is produced using an integrating op amp configuration, in which the op amp configuration is turned on through a switch controlled by the computer. In parallel, the computer starts a binary counter when the op amp configuration is turned on. When the analog input and op amp configuration signals are equal, the computer stops the binary counter and stores its values.

The terminology used for an M-bit A/D converter is summarized in Table 3.2. The values listed in the table for the LSB and MSB are when the bit is *on*. The bit equals 0 when it is *off*. The minimum decimal value that can be represented by the converter equals 0. The maximum value equals $2^M - 1$. Thus, 2^M possible values can be represented. The weight of a bit is defined as the value of the bit divided by the number of possible values. The resolution and absolute quantization error are based on an M-bit, unipolar A/D converter with a full-scale range (FSR) equal to 10.00 V, where $Q_o = 1000$ mV/bit. Most A/D converters used today are 12-bit or 16-bit converters, providing signal resolutions of 2.44 mV/bit and 0.153 mV/bit, respectively.

An analog signal is continuous in time and therefore comprised of an infinite number of values. An M-bit A/D converter, however, can only represent the signal's amplitude by a finite set of 2^M values. This presents a signal resolution problem. Consider the analog signal represented by the solid curve shown in Figure 3.17. If the signal is sampled discretely at δt time increments, it will be represented by the values indicated by the open

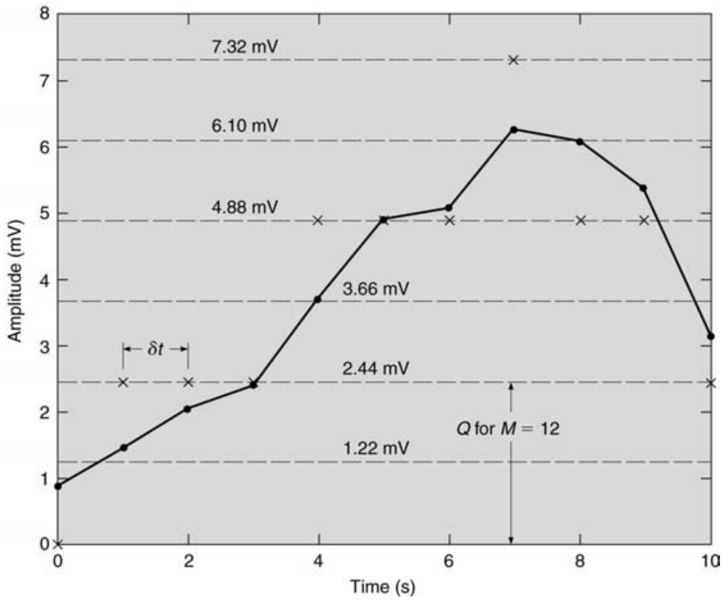

FIGURE 3.17
Schematic of analog-to-digital conversion.

circles. With **discrete sampling**, only the signal values between the sample times are lost, but the signal's exact amplitude values are maintained at each sample time. Yet, if this information is stored using the **digital sampling** scheme of the A/D converter, the signal's *exact* amplitude values also are lost. In fact, for the 12-bit A/D converter used to sample the signal shown in Figure 3.17, the particular signal is represented by only four possible values (0 mV, 2.44 mV, 4.88 mV, and 7.32 mV), as indicated by the $\times$'s in the figure. Thus, a signal whose amplitude lies within the range of $\pm Q/2$ of a particular bit's value will be assigned the bit's value. This error is termed the **absolute quantization error** of an A/D converter.

Quite often, if a signal's amplitude range is on the order of the A/D converter's resolution, an amplifier will be used before the A/D converter to gain the signal's amplitude and, therefore, reduce the absolute quantization error to an acceptable level. An alternative approach is to use an A/D board with better resolution, but, almost always, this is more expensive.

System no.	Variable (result)	Sensor/ Transducer	Signal Conditioner	Signal Processor
1	pressure (→velocity)	strain gage, Wheatstone bridge	amplifier, filter	A/D converter, computer
2	force (→thrust)	strain gage, Wheatstone bridge	amplifiers, filter	digital oscilloscope
3a	pressure (→velocity)	piezoresistive element	amplifier	microcontroller system
3b	acceleration (acceleration)	differential-capacitive structure	amplifier	microcontroller system

TABLE 3.3
The elements of three measurement systems.

3.7 Example Measurement Systems

The designs of three actual measurement systems are presented in this section to illustrate the different choices that can be made. The final design of each system involves many trade-offs between the accuracy and the cost of their components. The elements of each of these systems are summarized in Table 3.3. Several major differences can be noted. The most significant are the choices of the sensor/transducer and of the signal processing system. These are dictated primarily because of the environments in which each is designed to operate. System 1 is developed to be located near the test section of a subsonic wind tunnel, to be placed on a small table, and to use an existing personal computer. System 2 is designed to be located on one cart that can be moved to a remote location when the rocket motor is tested.

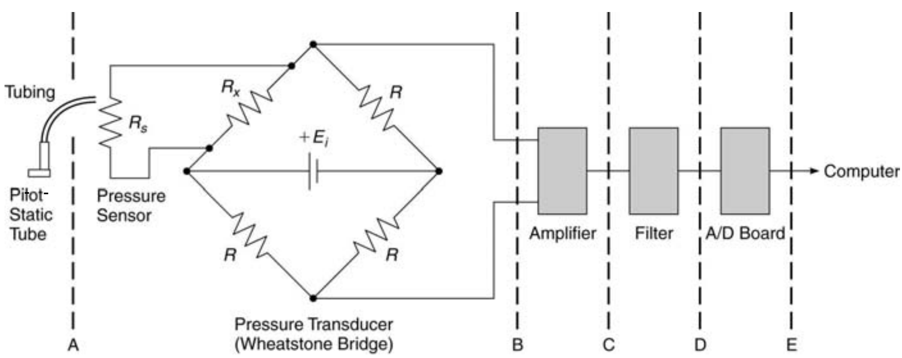

FIGURE 3.18
An example pressure measurement system.

Component	Conditions	Unknown
Environment	$T = 294$ K, $p = 1$ atm	ρ, Δp, F, ϵ_L, δR
Diaphragm	$\epsilon_L = 0.001 \times F$, $A = 1$ cm^2, $C_o = 0.001$	
Strain Gage	$R = 120\ \Omega$ at 294 K	
Wheatstone Bridge	All $R = 120\ \Omega$ at 294 K, $E_i = 5$ V	E_o
Amplifier	Non-inverting op amp, $R_1 = 1$ MΩ	R_2
Filter	Low-pass with $R = 1$ MΩ, $C = 1\ \mu$F	
A/D Converter	$E_{FSR} = 10$ V, $Q < 1$ mV/bit	M

TABLE 3.4
Velocity measurement system conditions.

A digital oscilloscope is chosen for its convenience of remote operation and its triggering and signal storage capabilities. System 3 is developed to be placed inside of a 2 in. internal diameter model rocket fuselage and then launched over 100 m into the air with accelerations and velocities as high as 60 m/s^2 ($\sim$6 g) and 50 m/s, respectively. These conditions constrain the size and weight of the measurement system package. A small, battery-powered, microcontoller-based data acquisition system with an A/D converter, amplifier, and memory is designed specifically for this purpose [15].

Measurement system 1 is designed to measure the velocity of air flowing in a wind tunnel, as shown in Figure 3.18. Its pitot-static tube is located within a wind tunnel. The pitot-static tube and tubing are passive and simply transmit the total and static pressures to a sensor located outside the wind tunnel. The actual sensor is a strain gage mounted on a flexible diaphragm inside a pressure transducer housing. The static pressure port is connected to one side of the pressure transducer's diaphragm chamber; the total pressure port to the other side. This arrangement produces a flexure of the diaphragm proportional to the dynamic pressure (the physical stimulus) that strains the gage and changes its resistance (the electrical impulse). This resistance change imbalances a Wheatstone bridge operated in the deflection mode, producing a voltage at station B. Beyond station B, the signal is amplified, filtered, converted into its digital format, and finally stored by the computer. Another example measurement system, used to measure temperature, is considered in this chapter's homework problems.

The velocity measurement system is to be designed such that the input voltage to the A/D converter, E_D, is 10 V when the wind tunnel velocity is 100 m/s. The design also is subject to the additional conditions specified in Table 3.4. Given these constraints, the desired input and output characteristics of each measurement system element can be determined for stations A through E, as denoted in Figure 3.18. Determination of the performance characteristics for each stage is as follows:

- Station A: The velocity, V, of 100 m/s yields a dynamic pressure, Δp, of 5700 N/m^2 using Bernoulli's equation, $\Delta p = 0.5\rho V^2$. The density, ρ, equals 1.14 kg/m^3, as determined using Equation 11.1.

- Station B: The dynamic pressure produces a force, F, on the diaphragm, which has an area, A, equal to 1 cm^2. The resulting force is 0.57 N, noting that the force equals the pressure difference across the diaphragm times its area. A longitudinal strain on the diaphragm, ϵ_L, is produced by F, where $\epsilon_L = C_o F$. The resulting strain is 5.7×10^{-4}. According to Equation 3.7, this gives $\delta R/R = 1.14 \times 10^{-3}$. The Wheatstone bridge is operated in the deflection method mode with all resistances equal to 120 Ω at 294 K and $V = 0$ m/s. The output voltage, $E_o = E_B$, is determined using Equation 2.29 and equals 1.42 mV.

- Station C: The relatively low output voltage from the Wheatstone bridge needs to be amplified to achieve the A/D input voltage, E_D, of 10 V. Assuming that the filter's magnitude ratio is unity, the gain of the amplifier equals E_D/E_B, which is 10/0.142 or 70.4. An op amp in the non-inverting configuration is used. Its input-output voltage relation is given in Figure 3.9. $E_o/E_i = 70.4$ and $R_1 = 1$ MΩ implies that R_2 equals 69.4 MΩ.

- Station D: The measurement system operates at steady state. The voltages are DC, having zero frequency. Thus, the filter's magnitude ratio is unity. Therefore, $E_D = E_C$.

- Station E: If the A/D converter has a full scale input voltage, E_{FSR}, of 10 V, then the converter is at its maximum input voltage when $V = 100$ m/s. The relationship between the A/D converter's E_{FSR}, Q, and the number of converter bits, M, is presented in Table 3.2. Choosing $M = 12$ does not meet the constraint. The next choice is $M = 16$. This yields $Q = 0.153$ mV/bit, which satisfies the constraint.

Many choices can be made in designing this system. For example, the supply voltage to the Wheatstone bridge could be increased from 5 V to 10 V or 12 V, which are common supply voltages. This would increase the output voltage of the bridge and, therefore, require less amplification to meet the 10 V constraint. Other resistances can be used in the bridge. A different strain gage can be used on the diaphragm. If the system will be used for non-steady velocity measurements, then the time responses of the tubing, the diaphragm, and the filter need to be considered. Each can affect the magnitude and the phase of the signal. The final choice of specific components truly is an engineering decision.

Next, examine measurement system 2 that is designed to acquire thrust as a function of time of a model rocket motor. The first element of the measurement system consists of an aluminum, cantilevered beam with four 120 Ω strain gages, similar to that shown schematically in Figure 2.11. These

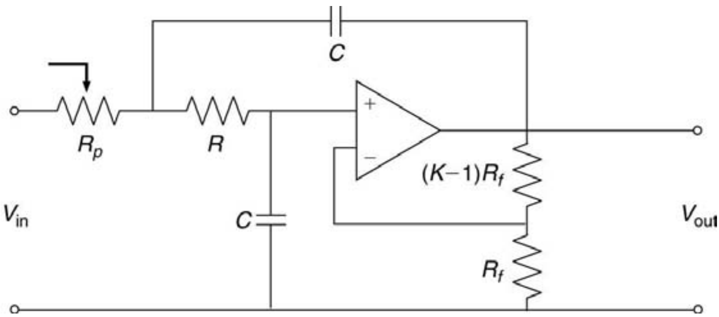

FIGURE 3.19
Two-pole, low-pass Sallen-and-Key filter.

strain gages comprise the legs of a Wheatstone bridge. The maximum output of the bridge is approximately 50 mV. So, the output of the bridge is connected to an instrumentation amplifier with a gain of 100 and then to a variable-gain, operational amplifier in the inverting configuration. This allows the signal's amplitude to be adjusted for optimum display and storage by the digital oscilloscope. A two-pole, low-pass Sallen-and-Key filter [13] receives the second amplifier's output, filters it, and then passes it to the digital oscilloscope. The filter's schematic is shown in Figure 3.19. Typical filter parameter values are $R = 200$ kΩ, $R_f = 200$ kΩ, $C = 0.1$ μF, and $K = 1.586$. A low-pass filter is used to eliminate the $\sim$30 Hz component that is the natural frequency of the cantilevered beam. The original and filtered rocket motor thrust data as a function of time are shown in Figure 3.20. The effect of the low-pass filter is clearly visible. Additional details about the experiment can be found on the text web site.

Finally, consider the design of measurement system 3, to be used remotely in a model rocket to acquire the rocket's acceleration and velocity data during ascent. The measurement system hardware consists of two sensor/transducers, one for pressure and the other for acceleration, and a board containing a microcontroller-based data acquisition system. The pressure transducer includes an integrated silicon pressure sensor that is signal conditioned, temperature compensated, and calibrated on-chip. A single piezoresistive element is located on a flexible diaphragm. Total and static pressure ports on the rocket's nose cone are connected with short tubing to each side of the flexible diaphragm inside the transducer's housing. The difference in pressure causes the diaphragm to deflect, which produces an output voltage that is directly proportional to the differential pressure, which for this case is the dynamic pressure. The single-axis ± 5 g accelerometer contains a polysilicon surface sensor. A differential capacitor structure attached to the surface deflects under acceleration, causing an imbalance in its the capacitor circuit. This produces an output voltage that is linearly proportional to the acceleration. The accelerometer and pressure transducer calibration curves are

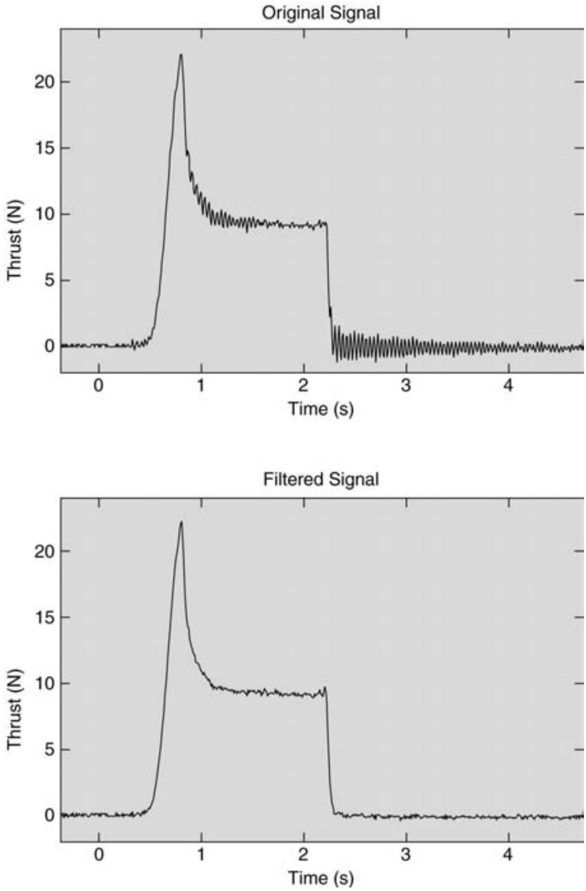

FIGURE 3.20
Original and filtered rocket motor thrust signal.

shown in Figures 3.21 and 3.22, respectively. Both sensor/tranducer outputs are each routed into an amplifier with a gain of 16 and then to the inputs of a 12-bit A/D converter. The output digital signals are stored directly into memory (256 kB). The measurement system board has a mass of 33 g and dimensions of 4.1 cm by 10.2 cm. The on-board, 3.3 V, 720 mAh Li-battery that powers the entire system has a mass of 39 g. All of the on-board data is retrieved after capture and downloaded into a laptop computer. A sample of the reconstructed data is displayed in Figure 3.23 The rocket's velocity in time can be determined from the pressure transducer's output. This is compared to the time integral of the rocket's acceleration in Figure 3.24. Finally, this information can be used with the information on the rocket's drag to determine the maximum altitude of the rocket.

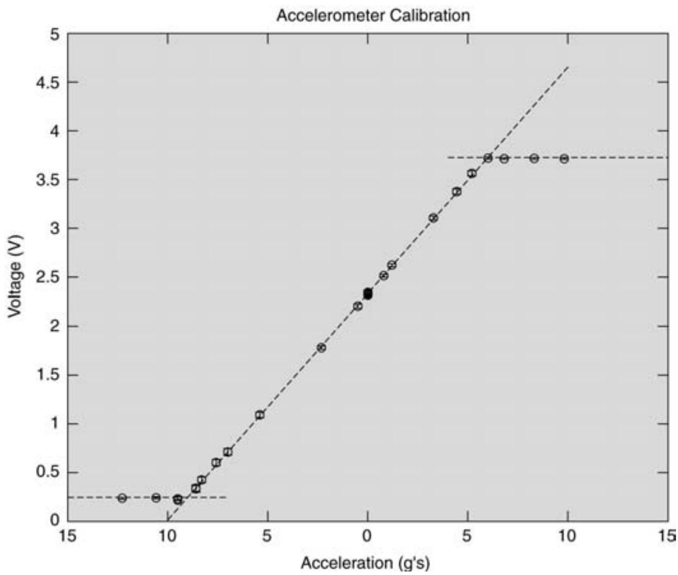

FIGURE 3.21
Calibration of the accelerometer.

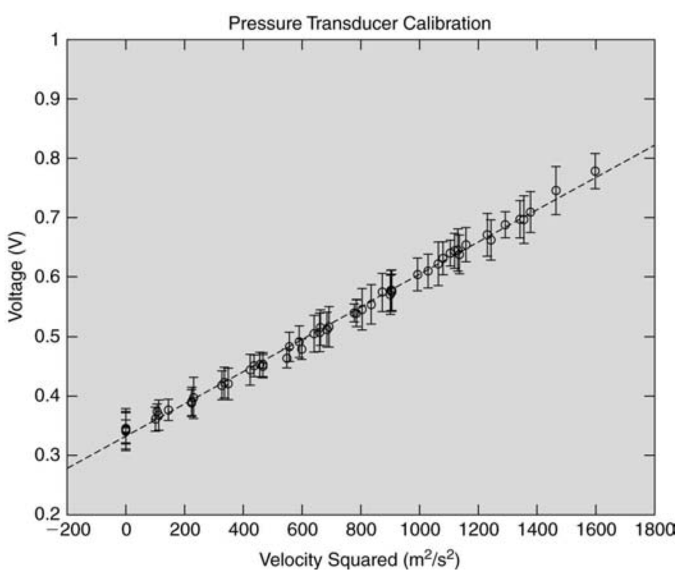

FIGURE 3.22
Calibration of the pressure transducer.

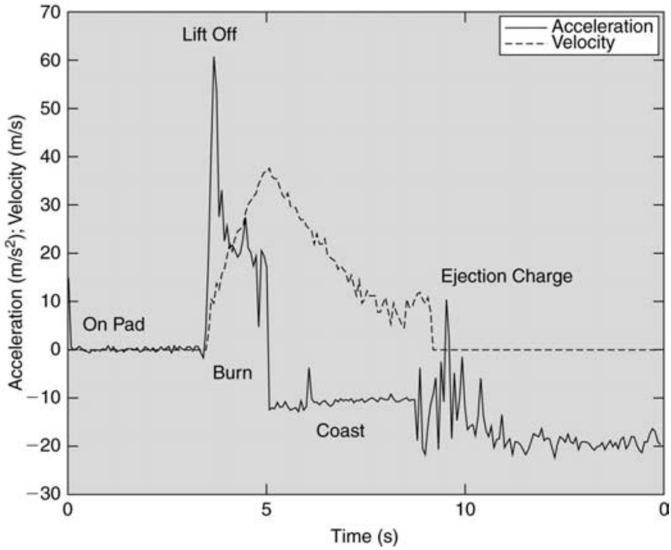

FIGURE 3.23
Example rocket velocity and acceleration data.

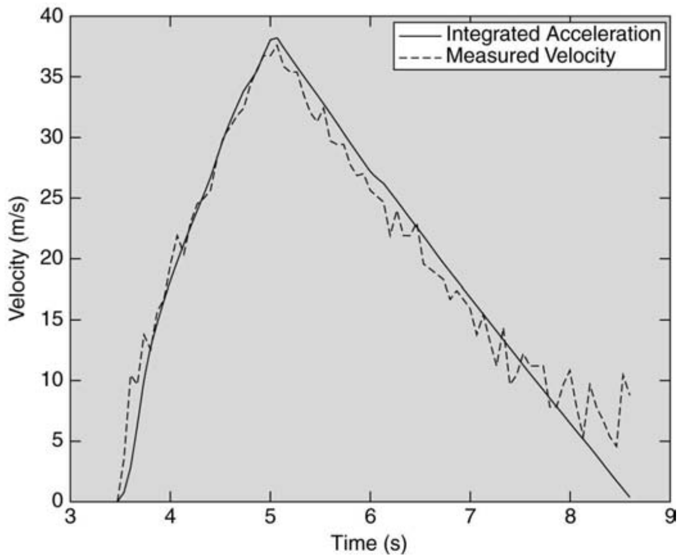

FIGURE 3.24
Integrated acceleration and velocity comparison.

3.8 Problem Topic Summary

Topic	Review Problems	Homework Problems
Components	1, 2, 4, 5, 7	1, 2, 4, 5, 8, 9, 12, 13
Systems	3, 6, 8	3, 6, 7, 10, 11, 14, 15

TABLE 3.5
Chapter 3 Problem Summary

3.9 Review Problems

1. Modern automobiles are equipped with a system to measure the temperature of the radiator fluid and output this temperature to a computer monitoring system. A thermistor is manufactured into the car radiator. A conducting cable leads from the thermistor and connects the thermistor to one arm of a Wheatstone bridge. The voltage output from the Wheatstone bridge is input into the car computer that digitally samples the signal 10 times each second. If the radiator fluid temperature exceeds an acceptable limit, the computer sends a signal to light a warning indicator to alert the driver. Match the following components of the fluid temperature measurement system (radiator fluid temperature, thermistor, Wheatstone bridge, and car computer) with their function in terms of a generalized measurement system (sensor, physical variable, transducer, and signal processor).

2. Which of the following instruments is used to interface analog systems to digital ones? (a) A/C converter, (b) D/C converter, (c) A/D converter, (d) AC/DC converter.

3. A metallic wire embedded in a strain gage is 4.2 cm long with a diameter of 0.07 mm. The gage is mounted on the upper surface of a cantilever beam to sense strain. Before strain is applied, the initial resistance of the wire is 64 Ω. Strain is applied to the beam, stretching the wire 0.1 mm, and changing its electrical resistivity by 2×10^{-8} Ωm. If Poisson's ratio for the wire is 0.342, find the change in resistance in the wire due to the strain to the nearest hundredth ohm.

4. What is the time constant (in seconds) of a single-pole, low-pass, passive filter having a resistance of 2 kΩ and a capacitance of 30 μF?

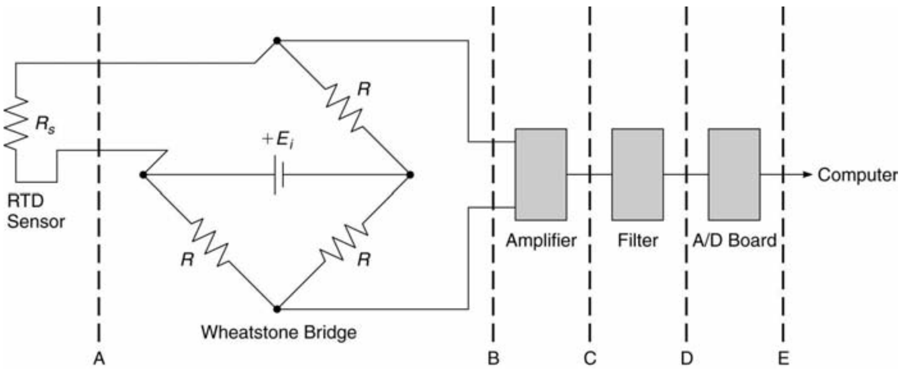

FIGURE 3.25
An example temperature measurement system configuration.

5. A single-stage, low-pass RC filter with a resistance of 93 Ω is designed to have a cut-off frequency of 50 Hz. Determine the capacitance of the filter in units of μF.

6. Two resistors, R_A and R_B, arranged in parallel, serve as the resistance, R_1, in the leg of a Wheatstone bridge where $R_2 = R_3 = R_4 = 200\ \Omega$ and the excitation voltage is 5.0 V. If $R_A = 1000\ \Omega$, what value of R_B is required to give a bridge output of 1.0 V?

7. The number of bits of a 0 V-to-5 V A/D board having a quantization error of 0.61 mV is (a) 4, (b) 8, (c) 12, (d) 16, or (e) 20.

8. Determine the output voltage, in V, of a Wheatstone bridge having resistors with resistances of 100 Ω and an input voltage of 5 V.

3.10 Homework Problems

1. Consider the amplifier between stations B and C of the temperature measurement system shown in Figure 3.25. (a) Determine the minimum input impedance of the amplifier (in Ω) required to keep the amplifier's voltage measurement loading error, e_V, less than 1 mV for the case when the bridge's output impedance equals 30 Ω and its output voltage equals 0.2 V. (b) Based upon the answer in part (a), if an operational amplifier were used, would it satisfy the requirement of e_V less than 1 mV (Hint: Compare the input impedance obtained in part (a) to that of a typical operational amplifier)? Answer yes or no and explain why or why not.

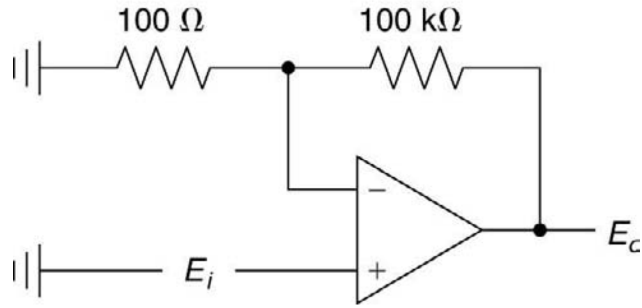

FIGURE 3.26
A closed-loop operational amplifier configuration.

(c) What would be the gain, G, required to have the amplifier's output equal to 9 V when $T = 72$ °F?

2. Consider the A/D board between stations D and E of the temperature measurement system shown in Figure 3.25. Determine how many bits ($M = 4, 8, 12$, or 16) would be required to have less than ±0.5 % quantization error for the input voltage of 9 V with $E_{FSR} = 10$ V.

3. The voltage from a 0 kg-to-5 kg strain gage balance scale has a corresponding output voltage range of 0 V to 3.50 mV. The signal is recorded using a new 16 bit A/D converter having a unipolar range of 0 V to 10 V, with the resulting weight displayed on a computer screen. An intelligent aerospace engineering student decides to place an amplifier between the strain gage balance output and the A/D converter such that 1 % of the balance's full scale output will be equal to the resolution of 1 bit of the converter. Determine (a) the resolution (in mV/bit) of the converter and (b) the gain of the amplifier.

4. The operational amplifier shown in Figure 3.26 has an *open-loop* gain of 10^5 and an output resistance of 50 Ω. Determine the *effective* output resistance (in Ω) of the op amp for the given configuration.

5. A single-stage, passive, low-pass (RC) filter is designed to have a cut-off frequency, f_c, of 100 Hz. Its resistance equals 100 Ω. Determine the filter's (a) magnitude ratio at $f = 1$ kHz, (b) time constant (in ms), and (c) capacitance (in μF).

6. A voltage-sensitive Wheatstone bridge (refer to Figure 3.27) is used in conjunction with a hot-wire sensor to measure the temperature within a jet of hot gas. The resistance of the sensor (in Ω) is $R_1 = R_o[1 + \alpha(T - T_o)]$, where $R_o = 50$ Ω is the resistance at $T_o = 0$ °C and $\alpha = 0.00395$/°C. For $E_i = 10$ V and $R_3 = R_4 = 500$ Ω, determine (a) the value of R_2 (in Ω) required to balance the bridge at $T = 0$ °C. Using

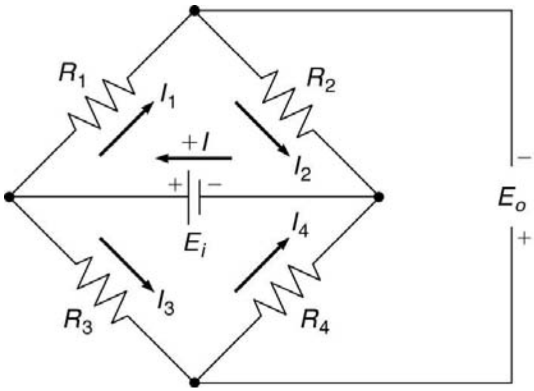

FIGURE 3.27
The Wheatstone bridge configuration.

this as a fixed R_2 resistance, further determine (b) the value of R_1 (in Ω) at $T = 50$ °C, and (c) the value of E_o (in V) at $T = 50$ °C. Next, a voltmeter having an input impedance of 1000 Ω is connected across the bridge to measure E_o. Determine (d) the percentage loading error in the measured bridge output voltage. Finally, (e) state what other electrical component, and in what specific configuration, could be added between the bridge and the voltmeter to reduce the loading error to a negligible value.

7. An engineer is asked to specify several components of a temperature measurement system. The output voltages from a Type J thermocouple referenced to 0 °C vary linearly from 2.585 mV to 3.649 mV over the temperature range from 50 °C to 70 °C. The thermocouple output is to be connected directly to an A/D converter having a range from −5 V to +5 V. For both a 12-bit and a 16-bit A/D converter determine (a) the quantization error (in mV), (b) the percentage error at $T = 50$ °C, and (c) the percentage error at $T = 70$ °C. Now if an amplifier is installed in between the thermocouple and the A/D converter, determine (d) the amplifier's gain to yield a quantization error of 5 % or less.

8. Consider the filter between stations C and D of the temperature measurement system shown in Figure 3.25. Assume that the temperature varies in time with frequencies as high as 15 Hz. For this condition, determine (a) the filter's cut-off frequency (in Hz) and (b) the filter's time constant (in ms). Next, find (c) the filter's output voltage (peak-to-peak) when the amplifier's output voltage (peak-to-peak) is 8 V and the temperature varies with a frequency of 10 Hz and (d) the signal's phase lag through the filter (in ms) for this condition.

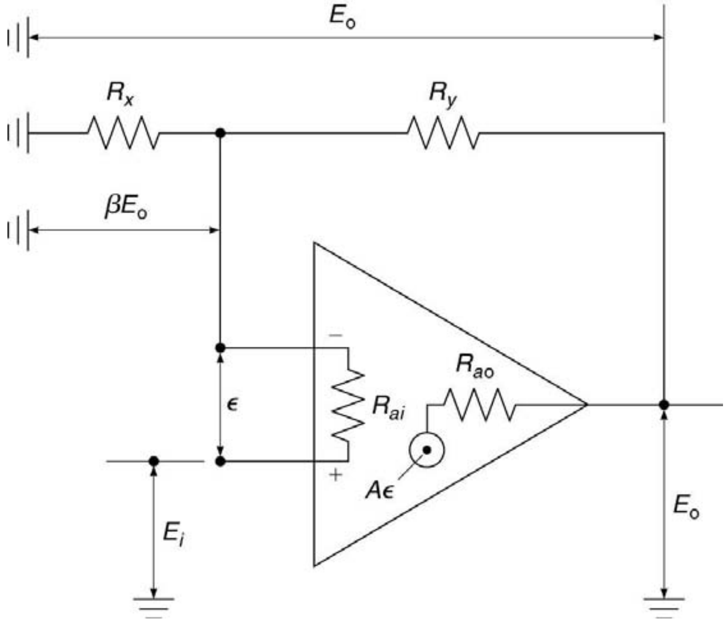

FIGURE 3.28
The operational amplifier in the voltage-follower configuration.

9. An op amp in the negative-feedback, voltage-follower configuration is shown in Figure 3.28. In this configuration, a voltage difference, ϵ, between the op amp's positive and negative inputs results in a voltage output of $A\epsilon$, where A is the open-loop gain. The op amp's input and output impedances are R_{ai} and R_{ao}, respectively. E_i is its input voltage and E_o its output voltage. Assuming that there is negligible current flow into the negative input, determine (a) the value of β, and (b) the closed-loop gain, G, in terms of β and A. Finally, recognizing that A is very large ($\sim 10^5$ to 10^6), (c) derive an expression for E_o as a function of E_i, R_x, and R_y.

10. Refer to the information given previously for the configuration shown in Figure 3.28. When E_i is applied to the op amp's positive input, a current I_{in} flows through the input resistance, R_{ai}. The op amp's effective input resistance, R_{ci}, which is the resistance that would be measured between the op amp's positive and negative inputs by an ideal ohmmeter, is defined as E_i/I_{in}. (a) Derive an expression for R_{ci} as a function of R_{ai}, β, and A. Using this expression, (b) show that this is a very high value.

11. Refer to the information given previously for the configuration shown in Figure 3.28. The op amp's output voltage for this configuration is $E_o = A(E_i - \beta E_o)$. Now assume that there is a load connected to the op amp's

output that results in a current flow, I_{out}, across the op amp's output resistance, R_{ao}. This effectively reduces the op amp's output voltage by $I_{out}R_{ao}$. For the equivalent circuit, the Thévenin output voltage is E_o, as given in the above expression, and the Thévenin output impedance is R_{co}. (a) Derive an expression for R_{co} as a function of R_{ao}, β, and A. Using this expression, (b) show that this is a very low value.

12. A standard RC circuit might be used as a low-pass filter. If the output voltage is to be attenuated 3 dB at 100 Hz, what should the time constant, τ, be of the RC circuit to accomplish this?

13. Design an op amp circuit such that the output voltage, E_o, is the sum of two different input voltages, E_1 and E_2.

14. A pitot-static tube is used in a wind tunnel to determine the tunnel's flow velocity, as shown in Figure 3.18. Determine the following: (a) the flow velocity (in m/s) if the measured pressure difference equals 58 Pa, (b) the value of R_x (in Ω) to have $E_o = 0$ V, assuming $R = 100\ \Omega$ and $R_s = 200\ \Omega$ at a zero flow velocity, with $E_i = 5.0$ V, (c) the value of E_o (in V) at the highest flow velocity, at which the parallel combination of R_x and R_s increases by 20 %, (d) the amplifier gain to achieve 80 % of the full-scale range of the A/D board at the highest flow velocity, (e) the values of the resistances if the amplifier is a non-inverting operational amplifier, and (f) the number of bits of the A/D board such that there is less than 0.2 % error in the voltage reading at the highest flow velocity.

15. A force-balance system comprised of a cantilever beam with four strain gages has output voltages of 0 mV for 0 N and 3.06 mV for 10 N. The signal is recorded using a 16-bit A/D converter having a unipolar range of 0 V to 10 V, with the resulting voltage being displayed on a computer monitor. A student decides to modify the system to get better force resolution by installing an amplifier between the force-balance output and the A/D converter such that 0.2 % of the balance's output for 10 N of force will be equal to the resolution of 1 bit of the converter. Determine (a) the resolution (in mV/bit) of the converter, (b) the gain that the amplifier must have in the modified system, and (c) the force (in N) that corresponds to a 5 V reading displayed on the monitor when using the modified system.

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4

Calibration and Response

CONTENTS

A single number has more genuine and permanent value than an expensive library full of hypotheses.
Robert J. Mayer, c. 1840.

Measures are more than a creation of society, they **create** *society.*
Ken Alder. 2002. *The Measure of All Things*. London: Little, Brown.

It is easier to get two philosophers to agree than two clocks.
Lucius Annaeus Seneca, c. 40.

4.1 Chapter Overview

In this chapter, the performance of a measurement system is investigated. Calibration methods are presented that assure recorded values are accurate indicators of the variables sensed. Both the static and the dynamic response characteristics of linear measurement systems are examined. First-order and second-order systems are considered in detail, including how their output can lag in time the changes that occur in the experiment's environment. With this information, approaches to data acquisition and signal processing, which are the subjects of subsequent chapters, then can be considered.

4.2 Static Response Characterization

Measurement systems and their instruments are used in experiments to obtain measurand values that usually are either steady or varying in time. For both situations, errors arise in the measurand values simply because the instruments are not perfect; their outputs do not precisely follow their inputs. These errors can be quantified through the process of **calibration**.

In a calibration, a known input value (called the **standard**) is applied to the system and then its output is measured. Calibrations can either be **static** (not a function of time) or **dynamic** (both the **magnitude** and the **frequency** of the input signal can be a function of time). Calibrations can be performed in either **sequential** or **random** steps. In a sequential calibration the input is increased systematically and then decreased. Usually this is done by starting at the lowest input value and calibrating at every other input value up to the highest input value. Then the calibration is continued back down to the lowest input value by covering the alternate input values that were skipped during the upscale calibration. This helps to identify any unanticipated variations that could be present during calibration. In a random calibration, the input is changed from one value to another in no particular order.

From a calibration experiment, a **calibration curve** is established. A generic static calibration curve is shown in Figure 4.1. This curve has several characteristics. The **static sensitivity** refers to the slope of the calibration curve at a particular input value, x_1. This is denoted by K, where $K = K(x_1) = (dy/dx)_{x=x_1}$. Unless the curve is linear, K will not be a constant. More generally, sensitivity refers to the smallest change in a quantity that an instrument can detect, which can be determined knowing the value of K and the smallest indicated output of the instrument. There are two **ranges** of the calibration, the input range, $x_{max} - x_{min}$, and the output range, $y_{max} - y_{min}$.

Calibration accuracy refers to how close the measured value of a calibration is to the **true value**. Typically, this is quantified through the **absolute error**, e_{abs}, where

$$e_{abs} = |\text{true value} - \text{indicated value}|. \tag{4.1}$$

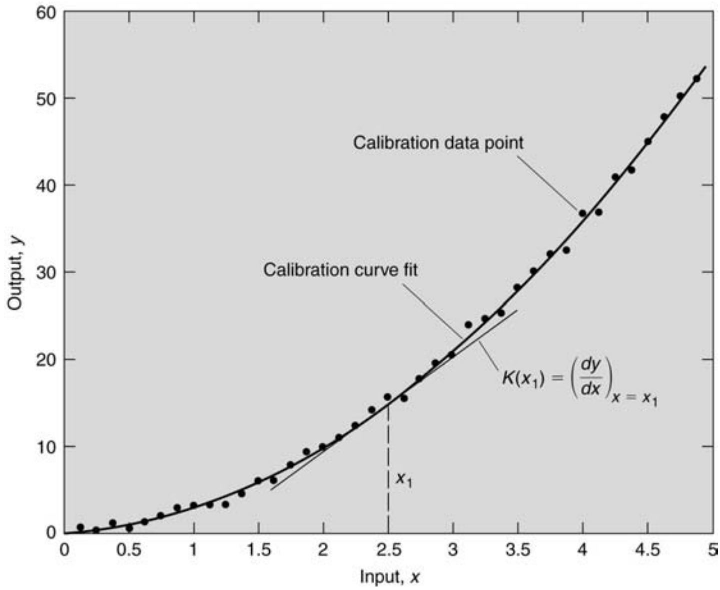

FIGURE 4.1
Typical static calibration curve.

The **relative error**, e_{rel}, is

$$e_{rel} = e_{abs}/|\text{true value}|. \tag{4.2}$$

The **accuracy** of the calibration, a_{cal}, is related to the absolute error by

$$a_{cal} = 1 - e_{rel}. \tag{4.3}$$

Calibration precision refers to how well a particular value is indicated upon repeated but independent applications of a specific input value. An expression for the precision in a measurement and the uncertainties that arise during calibration are presented in Chapter 7.

Example Problem 4.1

Statement: A hot-wire anemometer system was calibrated in a wind tunnel using a pitot-static tube. The data obtained is presented in Table 4.1. Using this data, a linear calibration curve-fit was made, which yielded

$$E^2 = 10.2 + 3.28\sqrt{U}.$$

Determine the following for the curve-fit: [a] the sensitivity, [b] the maximum absolute error, [c] the maximum relative error, and [d] the accuracy at the point of maximum relative error.

Solution: Because the curve-fit is linear, the sensitivity of the curve-fit is its slope, which equals 3.284 $\text{V}^2/\sqrt{m/s}$. The calculated voltages, E_c, from the curve-fit expression are given in Table 4.1. Inspection of the results reveals that the maximum difference

Velocity U (m/s)	Measured voltage E_m (V)	Calculated voltage E_c (V)
0.00	3.19	3.19
3.05	3.99	3.99
6.10	4.30	4.28
9.14	4.48	4.49
12.20	4.65	4.66

TABLE 4.1
Hot-wire anemometer system calibration data.

between the measured and calculated voltages is 0.02 V, which occurs at a velocity of 6.10 m/s. Thus, the maximum absolute error, e_{abs}, is 0.02 V, as defined by Equation 4.1. The relative error, e_{rel}, is defined by Equation 4.2. This also occurs at a velocity of 6.10 m/s, although maximum relative error does not always occur at the same calibration point as the maximum absolute error. Here, $e_{rel} = 0.02/4.30 = 0.01$, rounded to the correct number of significant figures. Consequently, by Equation 4.3, the accuracy at the point of maximum relative error is $1 - 0.01 = 0.99$, or 99 %.

4.3 Dynamic Response Characterization

In reality, almost every measurement system does not respond instantaneously to an input that varies in time. Often there is a time delay and amplitude difference between the system's input and output signals. This obviously creates a measurement problem. If these effects are not accounted for, **dynamic errors** will be introduced into the results.

To properly assess and quantify these effects, an understanding of how measurement systems respond to transient input signals must be gained. The ultimate goal would be to determine the output (response) of a measurement system for all conceivable inputs. The dynamic error in the measurement can be related to the difference between the input and output at a given time. In this chapter, only the basics of this subject will be covered. The response characteristics of several specific systems (zero, first, and second-order) to specific transient inputs (step and sinusoidal) will be studied. Hopefully, this brief foray into dynamic system response will give an appreciation for the problems that can arise when measuring time-varying phenomena.

First, examine the general formulation of the problem. The output signal, $y(t)$, in response to the input forcing function of a *linear* system, $F(t)$, can be modeled by a *linear* ordinary differential equation with *constant* coefficients $(a_0, ..., a_n)$ of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + ... + a_1 \frac{dy}{dt} + a_0 y = F(t). \tag{4.4}$$

In this equation n represents the **order** of the system. The input forcing function can be written as

$$F(t) = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + ... + b_0 x, \;\; m \leq n, \tag{4.5}$$

where $b_0, ..., b_n$ are *constant* coefficients, $x = x(t)$ is the forcing function, and m represents its order, where m must always be less than or equal to n to avoid having an over-deterministic system. By writing $F(t)$ as a polynomial, the ability to describe almost any shape of forcing function is retained.

The output response, $y(t)$, actually represents a physical variable followed in time. For example, it could be the displacement of the mass of an accelerometer positioned on a fluttering aircraft wing or the temperature of a thermocouple positioned in the wake of a heat exchanger. The exact ordinary differential equation governing each circumstance is derived from a conservation law, for example, from Newton's second law for the accelerometer or from the first law of thermodynamics for the thermocouple.

To solve for the output response, the exact form of the input forcing function, $F(t)$, must be specified. This is done by choosing values for the $b_0, ..., b_n$ coefficients and m. Then Equation 4.4 must be integrated subject to the initial conditions.

In this chapter, two types of input forcing functions, step and sinusoidal, are considered for linear, first-order, and second-order systems. There are analytical solutions for these situations. Further, as will be shown in Chapter 9, almost all types of functions can be described through Fourier analysis in terms of the sums of sine and cosine functions. So, if a linear system's response for sinusoidal-input forcing is determined, then its response to more complicated input forcing can be described. This is done by linearly superimposing the outputs determined for each of the sinusoidal-input forcing components that were identified by Fourier analysis. Finally, note that many measurement systems are linear, but not all. In either case, the response of the system almost always can be determined numerically. Numerical solution methods for such a model will be discussed in Section 4.8.

Now consider some particular systems by first specifying the order of the systems. This is done by substituting a particular value for n into Equation 4.4.

- For $n = 0$, a zero-order system is specified by

$$a_0 y = F(t). \tag{4.6}$$

Instruments that behave like zero-order systems are those whose output is directly coupled to its input. An electrical-resistance strain gage in itself is an excellent example of a zero-order system, where an input strain directly causes a change in the gage resistance. However, dynamic effects can occur when a strain gage is attached to a flexible structure. In this case the response must be modeled as a higher-order system.

- For $n = 1$, a first-order system is given by

$$a_1 \dot{y} + a_0 y = F(t). \tag{4.7}$$

Instruments whose responses fall into the category of first-order systems include thermometers, thermocouples, and other similar simple systems that produce a time lag between the input and output due to the *capacity* of the instrument. For thermal devices, the heat transfer between the environment and the instrument coupled with the thermal capacitance of the instrument produces this time lag.

- For $n = 2$, a second-order system is specified by

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = F(t). \tag{4.8}$$

Examples of second-order instruments include diaphragm-type pressure transducers, U-tube manometers, and accelerometers. This type of system is characterized by its *inertia*. In the U-tube manometer, for example, the fluid having inertia is moved by a pressure difference.

The responses of each of these systems are examined in the following sections.

4.4 Zero-Order System Dynamic Response

For a zero-order system the equation is

$$y = \left(\frac{1}{a_0}\right) F(t) = KF(t), \tag{4.9}$$

where K is called the static sensitivity or steady-state gain. It can be seen that the output, $y(t)$, exactly follows the input forcing function, $F(t)$, in time and that $y(t)$ is amplified by a factor, K. Hence, for a zero-order system, a plot of the output signal values (on the ordinate) versus the input signal values (on the abscissa) should yield a straight line of slope, K. In fact, instrument manufacturers often provide values for the steady-state gains of their instruments. These values are obtained by performing *static* calibration experiments.

4.5 First-Order System Dynamic Response

First-order systems are slightly more complicated. Their governing equation is

$$\tau \dot{y} + y = KF(t), \tag{4.10}$$

where τ is the **time constant** of the system $= a_1/a_0$. When the time constant is small, the derivative term in Equation 4.10 becomes negligible and the equation reduces to that of a zero-order system. That is, the smaller the time constant, the more instantaneous is the response of the system.

Now digress for a moment and examine the origin of a first-order-system equation. Consider a standard glass bulb thermometer initially at room temperature that is immersed into hot water. The thermometer takes a while to read the correct temperature. However, it is necessary to obtain an equation of the thermometer's temperature as a function of time after it is immersed in the hot water in order to be more specific.

Start by considering the liquid inside the bulb as a fixed mass into which heat can be transferred. When the thermometer is immersed into the hot water, heat (a form of energy) will be transferred from the hotter body (the water) to the cooler body (the thermometer's liquid). This leads to an increase in the total energy of the liquid. This energy transfer is governed by the first law of thermodynamics (conservation of energy), which is

$$\frac{dE}{dt} = \frac{dQ}{dt}, \tag{4.11}$$

in which E is the total energy of the thermometer's liquid, Q is the heat transferred from the hot water to the thermometer's liquid, and t is time. The rate at which the heat is transferred into the thermometer's liquid depends upon the physical characteristics of the interface between the outside of the thermometer and the hot water. The heat is transferred convectively to the glass from the hot water and is described by

$$\frac{dQ}{dt} = hA[T_{hw} - T], \tag{4.12}$$

where h is the convective heat transfer coefficient, A is the surface area over which the heat is transferred, and T_{hw} is the temperature of the hot water. Here it is assumed implicitly that there are no conductive heat transfer losses in the glass. All of the heat transferred from the hot water through the glass reaches the thermometer's liquid. Now as the energy is stored within the liquid, its temperature increases. For energy within the liquid to be conserved, it must be that

$$\frac{dE}{dt} = mC_v \frac{dT}{dt}, \tag{4.13}$$

where T is the liquid's temperature, m is its mass, and C_v is its specific heat at constant volume.

Thus, upon substitution of Equations 4.12 and 4.13 into Equation 4.11,

$$mC_v \frac{dT}{dt} = hA[T_{hw} - T]. \tag{4.14}$$

Rearranging,

$$\frac{mC_v}{hA} \frac{dT}{dt} + T = T_{hw}. \tag{4.15}$$

Comparing this equation to Equation 4.10 it can be seen that $y = T$, $\tau = mC_v/hA$, and $T_{hw} = F(t)$, with $K = 1$. This is the linear, first-order differential equation with constant coefficients that relates the time rate of change in the thermometer's liquid temperature to its temperature at any instance of time and the conditions of the situation. Equation 4.15 must be integrated to obtain the desired equation of the thermometer's temperature as a function of time after it is immersed in the hot water.

Another example of a first-order system is an electrical circuit comprised of a resistor of resistance, R, a capacitor of capacitance, C, both in series with a voltage source with voltage, $E_i(t)$. The voltage differences, ΔV, across each component in the circuit are $\Delta V = RI$ for the resistor and $\Delta V = Q/C$ for the capacitor, where the current, I, is related to the charge, Q, by $I = dQ/dt$. Application of Kirchhoff's voltage law to the circuit gives

$$RC\frac{dV}{dt} + V = E_i(t). \tag{4.16}$$

Comparing this equation to Equation 4.10 gives $\tau = RC$ and $K = 1$.

Now proceed to solve a first-order system equation to determine the response of the system subject to either a step change in conditions (by assuming a step-input forcing function) or a periodic change in conditions (by assuming a sinusoidal-input forcing function). The former, for example, could be the temperature of a thermometer as a function of time after it is exposed to a sudden change in temperature, as was examined above. The latter, for example, could be the temperature of a thermocouple in the wake behind a heated cylinder.

4.5.1 Response to Step-Input Forcing

Start by considering the governing equation for a first-order system

$$\tau\dot{y} + y = KF(t), \tag{4.17}$$

where the step-input forcing function, $F(t)$, is defined as A for $t > 0$ and the initial condition $y(0) = y_0$. Equation 4.17 is a linear, first-order ordinary differential equation. Its general solution (see [24]) is of the form

$$y(t) = c_0 + c_1 e^{-\frac{t}{\tau}}. \tag{4.18}$$

Substitution of this expression for y and the expression for its derivative $\dot{y}$ into Equation 4.17 yields $c_0 = KA$. Subsequently, applying the initial condition to Equation 4.18 gives $c_1 = y_0 - KA$. Thus, the specific solution can be written as

$$y(t) = KA + (y_0 - KA)\, e^{-\frac{t}{\tau}}. \tag{4.19}$$

Now examine this equation. When the time equals zero, the exponential term is unity, which gives $y(0) = y_0$. Also, when time becomes very large with respect to τ, the exponential term tends to zero, which gives an output equal to KA. Hence, the output rises exponentially from its initial value of y_o at $t = 0$ to its final value of KA at $t >> \tau$. This is what is seen in the solution, as shown in the left graph of Figure 4.2. Note that at the dimensionless time $t/\tau = 1$, the value the signal reaches approximately two-thirds (actually $1 - \frac{1}{e}$ or 0.6321) of its final value. The time that it takes the system to reach 90 % of its final value (which occurs at $t/\tau = 2.303$) is called the **rise time** of a first-order system. At $t/\tau = 5$ the signal has reached greater than 99 % of its final value.

The term y_0 can be subtracted from both sides of Equation 4.19 and then rearranged to yield

$$M(t) \equiv \frac{y(t) - y_0}{y_\infty - y_0} = 1 - e^{-\frac{t}{\tau}}, \tag{4.20}$$

noting that $y_\infty = KA$. $M(t)$ is called the **magnitude ratio** and is a dimensionless variable that represents the change in y at any time t from its initial value divided by its maximum possible change. When y reaches its final value, $M(t)$ is unity. The right side of Equation 4.20 is a dimensionless time, t/τ. Equation 4.20 is valid for *all* first-order systems responding to step-input forcing because the equation is dimensionless.

Alternatively, Equation 4.19 can be rearranged directly to give

$$\frac{y(t) - y_\infty}{y_0 - y_\infty} = e^{-\frac{t}{\tau}} \equiv \delta(t). \tag{4.21}$$

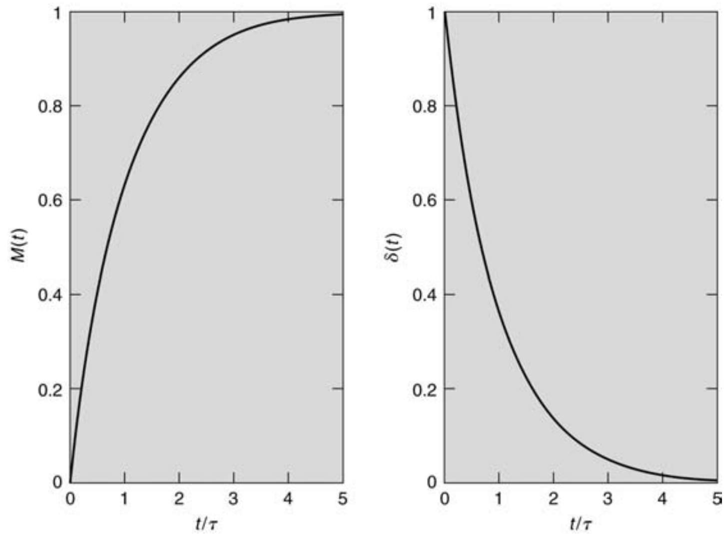

FIGURE 4.2
Response of a first-order system to step-input forcing.

In this equation $\delta(t)$ represents the fractional difference of y from its final value. This can be interpreted as the fractional **dynamic error** in y. From Equations 4.20 and 4.21,

$$\delta(t) = 1 - M(t). \tag{4.22}$$

This result is plotted in the right graph of Figure 4.2. At the dimensionless time $t/\tau = 1$, δ equals $0.3678 = 1/e$. Further, at $t/\tau = 5$, the dynamic error is essentially zero ($= 0.007$). That means for a first-order system subjected to a step change in input it takes approximately five time constants for the output to reach the input value. For perfect measurement system there would be no dynamic error ($\delta(t) = 0$) and the output would always follow the input $[M(t) = 1]$.

4.5.2 Response to Sinusoidal-Input Forcing

Now consider a first-order system that is subjected to an input that varies sinusoidally in time. The governing equation is

$$\tau \dot{y} + y = KF(t) = KA \sin(\omega t), \tag{4.23}$$

where K and A are arbitrary constants. The units of K would be those of y divided by those of A. The general solution is

$$y(t) = y_h + y_p = c_0 e^{-\frac{t}{\tau}} + c_1 + c_2 \sin(\omega t) + c_3 \cos(\omega t), \tag{4.24}$$

in which c_0 through c_3 are constants, where the first term on the right side of this equation is the homogeneous solution, y_h, and the remaining terms constitute the particular solution, y_p.

The constants c_1 through c_3 can be found by substituting the expressions for $y(t)$ and its derivative into Equation 4.23. By comparing like terms in the resulting equation,

$$c_1 = 0, \tag{4.25}$$

$$c_2 = \frac{KA}{\omega^2\tau^2 + 1}, \tag{4.26}$$

and

$$c_3 = -\omega\tau C_2 = \frac{-\omega\tau KA}{\omega^2\tau^2 + 1}. \tag{4.27}$$

The constant c_0 can be found by applying the initial condition $y(0) = y_0$ to Equation 4.24, where

$$c_0 = y + 0 - c_3 = \frac{\omega\tau KA}{\omega^2\tau^2 + 1}. \tag{4.28}$$

Thus, the final solution becomes

$$y(t) = (y_0 + \omega\tau D)e^{-\frac{t}{\tau}} + D\sin(\omega t) - \omega\tau D\cos(\omega t), \tag{4.29}$$

where

$$D = \frac{KA}{\omega^2\tau^2 + 1}. \tag{4.30}$$

Now Equation 4.29 can be simplified further. The sine and cosine terms can be combined in Equation 4.29 into a single sine term using the trigonometric identity

$$\alpha\cos(\omega t) + \beta\sin(\omega t) = \sqrt{\alpha^2 + \beta^2}\sin(\omega t + \phi), \tag{4.31}$$

where

$$\phi = \tan^{-1}(\alpha/\beta). \tag{4.32}$$

Equating this expression with the sine and cosine terms in Equation 4.29 gives $\alpha = -\omega\tau D$ and $\beta = D$. Thus,

$$D\sin(\omega t) - \omega\tau D\cos(\omega t) = D\sqrt{\omega^2\tau^2 + 1} = \frac{KA}{\sqrt{\omega^2\tau^2 + 1}} \tag{4.33}$$

and

$$\phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}(\omega\tau), \tag{4.34}$$

or, in units of degrees,

$$\phi^\circ = -(180/\pi)\tan^{-1}(\omega\tau). \tag{4.35}$$

The minus sign is present in Equations 4.34 and 4.35 by convention to denote that the output lags behind the input.

The final solution is

$$y(t) = y_0 + (\frac{\omega\tau KA}{\omega^2\tau^2 + 1})e^{-\frac{t}{\tau}} + \frac{KA}{\sqrt{\omega^2\tau^2 + 1}}\sin(\omega t + \phi). \tag{4.36}$$

The first term on the right side represents the **transient response** while the second term is the **steady-state response**. For $\omega\tau << 1$, the transient term becomes very small and the output follows the input. For $\omega\tau >> 1$, the output is droplets attenuated and its phase is *shifted* from the input by ϕ radians. The **phase lag** in seconds (lag time), β, is given by

$$\beta = \phi/\omega. \tag{4.37}$$

Examine this response further in a dimensionless sense. The magnitude ratio for this input-forcing situation is the ratio of the magnitude of the steady-state output to that of the input. Thus,

$$M(\omega) = \frac{KA/\sqrt{\omega^2\tau^2 + 1}}{KA} = \frac{1}{\sqrt{\omega^2\tau^2 + 1}}. \tag{4.38}$$

The dynamic error, using its definition in Equation 4.22 and Equation 4.38, becomes

$$\delta(\omega) = 1 - \frac{1}{\sqrt{\omega^2\tau^2 + 1}}. \tag{4.39}$$

Shown in Figures 4.3 and 4.4, respectively, are the magnitude ratio and the phase shift plotted versus the product $\omega\tau$. First examine Figure 4.3. For values of $\tau\omega$ less than approximately 0.1, the magnitude ratio is very close to unity. This implies that the system's output closely follows its input in this range. At $\omega\tau$ equal to unity, the magnitude ratio equals 0.707, that is, the output amplitude is approximately 71 % of its input. Here, the dynamic error would be $1 - 0.707 = 0.293$ or approximately 29 %. Now look at Figure 4.4. When $\omega\tau$ is unity, the phase shift equals $-45°$. That is, the output signal lags the input signal by 45° or 1/8th of a cycle.

The magnitude ratio often is expressed in units of decibels, abbreviated as dB. The decibel's origin began with the introduction of the Bel, defined in terms of the ratio of output power, P_2, to the input power, P_1, as

$$\text{Bel} = \log_{10}(P_2/P_1). \tag{4.40}$$

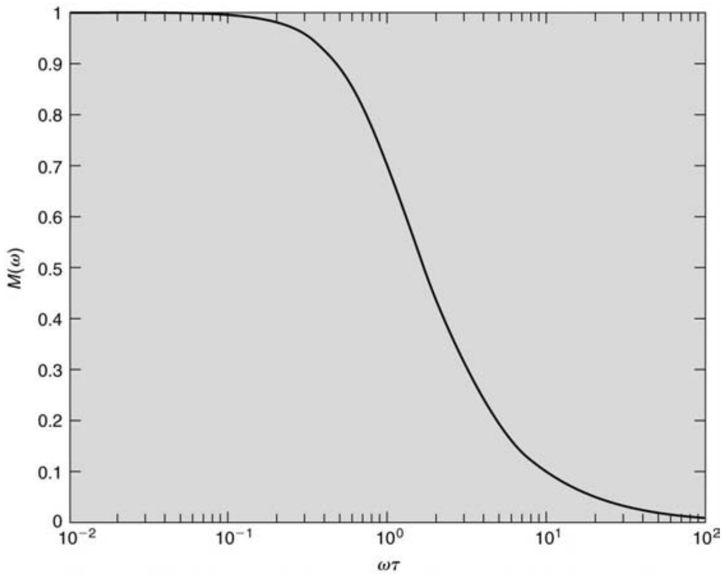

FIGURE 4.3
The magnitude ratio of a first-order system responding to sinusoidal-input forcing.

To accommodate the large power gains (output/input) that many systems had, the decibel (equal to 10 Bels) was defined as

$$\text{Decibel} = 10 \log_{10}(P_2/P_1). \tag{4.41}$$

Equation 4.41 is used to express sound intensity levels, where P_2 corresponds to the sound intensity and P_1 to the reference intensity, 10^{-12} W/m^2, which is the lowest intensity that humans can hear. The Saturn V on launch has a sound intensity of 172 dB; human hearing pain occurs at 130 dB; a soft whisper at a distance of 5 m is 30 dB.

There is one further refinement in this expression. Power is a squared quantity, $P_2 = {Q_2}^2$ and $P_1 = {Q_1}^2$, where Q_2 and Q_1 are the base measurands, such as volts for an electrical system. With this in mind, Equation 4.41 becomes

$$\text{Decibel} = 10 \log_{10}(Q_2/Q_1)^2 = 20 \log_{10}(Q_2/Q_1). \tag{4.42}$$

Equation 4.42 is the basic definition of the decibel as used in measurement engineering. Finally, Equation 4.42 can be written in terms of the magnitude ratio

$$\text{dB} = 20 \log_{10} M(\omega). \tag{4.43}$$

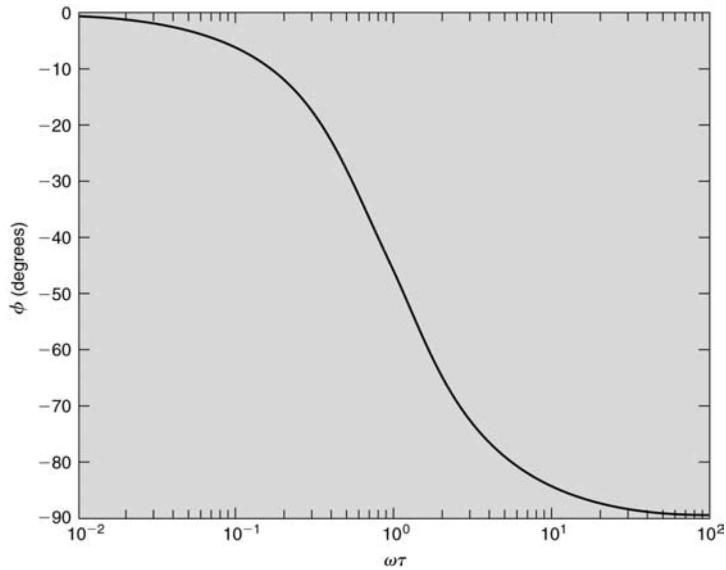

FIGURE 4.4
The phase shift of a first-order system responding to sinusoidal-input forcing.

The point $M(\omega) = 0.707$, which is a decrease in the system's amplitude by a factor of $1/\sqrt{2}$, corresponds to an attenuation of the system's input by -3 dB. Sometimes, this is called the droplets half-power point because, at this point, the power is one-half the original power.

Example Problem 4.2

Statement: Convert the sound intensity level of 30 dB to $\log_e M(\omega)$.
Solution: The relationship between logarithms of bases a and b is

$$\log_b x = \log_a x / \log_a b.$$

For this problem the bases are e and 10. So,

$$\log_e M(\omega) = \log_{10} M(\omega) / \log_{10} e.$$

Now, $\log_{10} e = 0.434\ 294$. Also, $\log_e 10 = 2.302\ 585$ and $e = 2.718\ 282$. Using Equation 4.43, $\log_{10} M(\omega)$ at 30 dB equals 1.500. Thus

$$\log_e M(\omega) = 1.500/0.434 = 3.456.$$

The relationship between logarithms of two bases is used often when converting back and forth between base 10 and base e systems.

Systems often are characterized by their bandwidth and center frequency. **Bandwidth** is the range of frequencies over which the output amplitude of a system remains above 70.7 % of its input amplitude. Over this range,

$M\omega) \geq 0.707$ or -3 dB. The lower frequency at which $M\omega) < 0.707$ is called the **low cut-off frequency**. The higher frequency at which $M\omega) < 0.707$ is called the **high cut-off frequency**. The **center frequency** is the frequency equal to one-half the sum of the low and high cut-off frequencies. Thus, the bandwidth is the difference between the high and low cut-off frequencies. Sometimes bandwidth is defined as the range of frequencies that contain most of the system's energy or over which the system's gain is almost constant. However, the above quantitative definition is preferred and used most frequently.

Example Problem 4.3

Statement: Determine the low and high cut-off frequencies, center frequency, and the bandwidth in units of hertz of a first-order system having a time constant of 0.1 s that is subjected to sinusoidal-input forcing.

Solution: For a first-order system, $M(\omega) \geq 0.707$ from $\omega\tau = 0$ to $\omega\tau = 1$. Thus, the low cut-off frequency is 0 Hz and the high cut-off frequency is (1 rad/s s)/[(0.1 s)(2π rad/cycle)] = $5/\pi$ Hz. The bandwidth equals $5/\pi$ Hz − 0 Hz = $5/\pi$ Hz. The center frequency is $5/2\pi$.

The following example illustrates how the time constant of a thermocouple affects its output.

Example Problem 4.4

Statement: Consider an experiment in which a thermocouple that is immersed in a fluid and connected to a reference junction/linearizer/amplifier micro-chip with a static sensitivity of 5 mv/°C. Its output is $E(t)$ in millivolts. The fluid temperature varies sinusoidally in degrees Celsius as $115 + 12\sin(2t)$. The time constant τ of the thermocouple is 0.15 s. Determine $E(t)$, the dynamic error $\delta(\omega)$ and the time delay $\beta(\omega)$ for $\omega = 2$. Assume that this system behaves as a first-order system.

Solution: It is known that

$$\tau\dot{E} + E = KF(t).$$

Substitution of the given values yields

$$0.15\dot{E} + E = 5[115 + 12\sin 2t] \tag{4.44}$$

with the initial condition of $E(0) = (5 \text{ mv/°C})(115 \text{ °C}) = 575$ mV.

To solve this linear, first-order differential equation with constant coefficients, a solution of the form $E(t) = E_h + E_p$ is assumed, where $E_h = C_0 e^{-t/\tau}$ and $E_p = c_1 + c_2\sin 2t + c_3\cos 2t$. Substitution of this expression for $E(t)$ into the left side and grouping like terms gives

$$c_1 = 575, \; c_2 = 55.1, \text{ and } c_3 = -16.5.$$

Equation 4.44 then can be rewritten as

$$E(t) = k_0 e^{-t/0.15} + 575 + 55.1\sin 2t - 16.5\cos 2t.$$

Using the initial condition,

$$c_0 = 16.5.$$

Thus, the final solution for E(t) is

$$E(t) = 575 + 16.5e^{-t/0.15} + 55.1\sin 2t - 16.5\cos 2t$$

or, in units of °C temperature

$$T(t) = 115 + 3.3e^{-t/0.15} + 11.0\sin 2t - 3.3\cos 2t.$$

The output (measured) temperature is plotted in Figure 4.5 along with the input (actual) temperature. A careful comparison of the two signals reveals that the output lags the input in time and has a slightly attenuated amplitude. At $t = 2$ s, the actual temperature is ∼106 °C, which is less than the measured temperature of ∼109 °C. Whereas, at $t = 3$ s, the actual temperature is ∼112 °C, which is greater than the measured temperature of ∼109 °C. So, for this type of forcing, the measured temperature can be greater or less than the actual temperature, depending upon the time at which the measurement is made.

The time lag and the percent reduction in magnitude can be found as follows. The dynamic error is

$$\delta(\omega = 2) = 1 - M(\omega = 2) = 1 - \frac{1}{[1 + (2 \times 0.15)^2]^{1/2}} = 0.04,$$

which is a 4 % reduction in magnitude. The time lag is

$$\beta(\omega = 2) = \frac{\phi(\omega = 2)}{\omega} = \frac{-\tan^{-1}\omega\tau}{\omega} = \frac{(-16.7\,°)(\pi \text{ rad}/180\,°)}{2 \text{ rad/s}} = -0.15 \text{ s},$$

which implies that the output signal lags the input signal by 0.15 s. The last two terms in the temperature expression can be combined using a trigonometric identity (see Chapter 9), as

$$11.0\text{sin}2t - 3.3\text{cos}2t = 11.48\text{sin}(2t - 0.29), \tag{4.45}$$

where 0.29 rad = 16.7° is the phase lag found before.

4.6 Second-Order System Dynamic Response

The response behavior of second-order systems is more complex than first-order systems. Their behavior is governed by the equation

$$\frac{1}{{\omega_n}^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = KF(t), \tag{4.46}$$

where $\omega_n = \sqrt{a_0/a_2}$ denotes the natural frequency and $\zeta = a_1/2\sqrt{a_0 a_2}$ damping ratio of the system. Note that when $2\zeta >> 1/\omega_n$, the second derivative term in Equation 4.46 becomes negligible with respect to the other terms, and the system behavior approaches that of a first-order system with a system time constant equal to $2\zeta/\omega_n$.

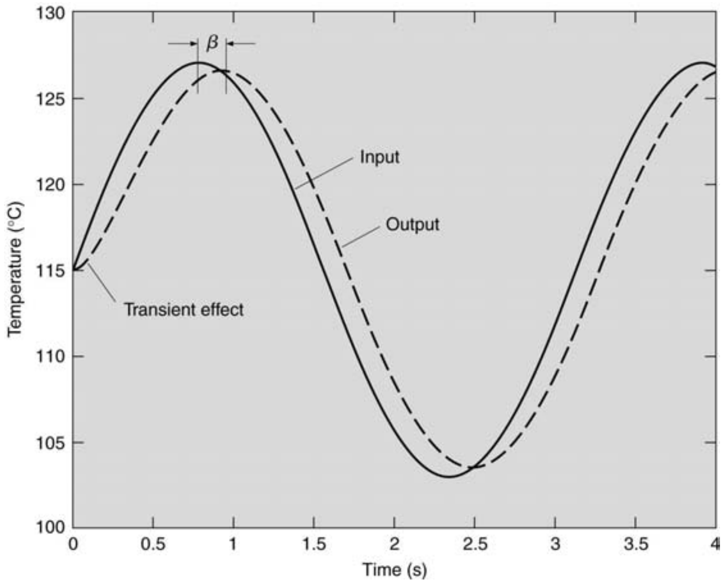

FIGURE 4.5
The time history of the thermocouple system.

Equation 4.46 could represent, among other things, a mechanical spring-mass-damper system or an electrical capacitor-inductor-resistor circuit, both with forcing. The solution to this type of equation is rather lengthy and is described in detail in many applied mathematics texts (see [24]). Now examine where such an equation would come from by considering the following example.

A familiar situation occurs when a bump in the road is encountered by a car. If the car has a good suspension system it will absorb the effect of the bump. The bump hardly will be felt. On the other hand, if the suspension system is old, an up-and-down motion is present that may take several seconds to attenuate. This is the response of a linear, second-order system (the car with its suspension system) to an input forcing (the bump).

The car with its suspension system can be modeled as a mass (the body of the car and its passengers) supported by a spring (the suspension coil) and a damper (the shock absorber) in parallel (usually there are four sets of spring-dampers, one for each wheel). Newton's second law can be applied, which states that the mass times the acceleration of a system is equal to the sum of the forces acting on the system. This becomes

$$m\frac{d^2y}{dt^2} = \sum_i F_i = F_g + F_s(t) + F_d(t) + F(t), \tag{4.47}$$

in which y is the vertical displacement, F_g is the gravitational force ($= mg$),

$F_s(t)$ is the spring force $(= -k[L^* + y])$, where k is the spring constant and L^* the initial compressed length of the spring, $F_d(t)$ is the damping force $(= -\gamma dy/dt)$, where γ is the damping coefficient, and $F(t)$ is the forcing function. Note that the spring and damping forces are negative because they are *opposite* to the direction of motion. The height of the bump as a function of time as dictated by the speed of the car would determine the exact shape of $F(t)$. Now when there is no vertical displacement, which is the case just before the bump is encountered, the system is in equilibrium and y does not change in time. Equation 4.47 reduces to

$$0 = mg - kL^*. \tag{4.48}$$

This equation can be used to replace L^* in Equation 4.47 to arrive at

$$\frac{m}{k}\frac{d^2y}{dt^2} + \frac{\gamma}{k}\frac{dy}{dt} + y = \frac{1}{k}F(t). \tag{4.49}$$

Comparing this equation to Equation 4.46 yields $\omega_n = \sqrt{k/m}$, $\zeta = \gamma/\sqrt{4km}$, and $K = 1/k$.

Another example of a second-order system is an electrical circuit comprised of a resistor, R, a capacitor, C, and an inductor, L, in series with a voltage source with voltage, $E_i(t)$, that completes a closed circuit. The voltage differences, ΔV, across each component in the circuit are $\Delta V = RI$ for the resistor, $\Delta V = LdI/dt$ for the inductor, and $\Delta V = Q/C$ for the capacitor, where the current, I, is related to the charge, Q, by $I = dQ/dt$. Application of Kirchhoff's voltage law to the circuit's closed loop gives

$$LC\frac{d^2I}{dt^2} + RC\frac{dI}{dt} + I = C\frac{dE_i(t)}{dt}. \tag{4.50}$$

Comparing this equation to Equation 4.46 gives $\omega_n = \sqrt{1/LC}$, $\zeta = R/\sqrt{4L/C}$, and $K = C$.

The approach to solving a nonhomogeneous, linear, second-order, ordinary differential equation with constant coefficients of the form of Equation 4.46 involves finding the homogeneous, $y_h(t)$, and particular, $y_p(t)$, solutions and then linearly superimposing them to form the complete solution, $y(t) = y_h(t) + y_p(t)$. The values of the arbitrary coefficients in the $y_h(t)$ solution are determined by applying the specified initial conditions, which are of the form $y(0) = y_o$ and $\dot{y}(0) = \dot{y}_o$. The values of the arbitrary coefficients in the $y_p(t)$ solution are found through substitution of the general form of the $y_p(t)$ solution into the differential equation and then equating like terms.

The form of the homogeneous solution to Equation 4.46 depends upon roots of its corresponding characteristic equation

$$\frac{1}{{\omega_n}^2}r^2 + \frac{2\zeta}{\omega_n}r + 1 = 0, \tag{4.51}$$

which are

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (4.52)$$

Depending upon the value of the discriminant $\sqrt{\zeta^2 - 1}$, there are three possible families of solutions (see the text web site for the step-by-step solutions):

- $\zeta^2 - 1 > 0$: the roots are real, negative, and distinct. The general form of the solution is

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \quad (4.53)$$

- $\zeta^2 - 1 = 0$: the roots are real, negative, and equal to $-\omega_n$. The general form of the solution is

$$y_h(t) = c_1 e^{rt} + c_2 t e^{rt}. \quad (4.54)$$

- $\zeta^2 - 1 < 0$: the roots are complex and distinct. The general form of the solution is

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t), \quad (4.55)$$

  using Euler's formula $e^{it} = \cos t + i \sin t$ and noting that

$$r_{1,2} = \lambda \pm i\mu, \quad (4.56)$$

  with $\lambda = -\zeta\omega_n$ and $\mu = \omega_n\sqrt{1 - \zeta^2}$.

All three general forms of solutions have exponential terms that decay in time. Thus, as time increases, all homogeneous solutions tend toward a value of zero. Such solutions often are termed **transient solutions**. When $0 < \zeta < 1$ (when $\sqrt{\zeta^2 - 1} < 0$) the system is called **under-damped**; when $\zeta = 1$ (when $\sqrt{\zeta^2 - 1} = 0$) it is called **critically damped**; when $\zeta > 1$ (when $\sqrt{\zeta^2 - 1} > 0$) it is called **over-damped**. The reasons for these names will be obvious later. Now examine how a second-order system responds to step and sinusoidal inputs.

4.6.1 Response to Step-Input Forcing

The responses of a second-order system to a step input having $F(t) = A$ for $t > 0$ with the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$ are as follows:

- For the under-damped case ($0 < \zeta < 1$)

$$y(t) = KA\left\{1 - e^{-\zeta\omega_n t}\left[\frac{1}{\sqrt{1-\zeta^2}}\sin(\omega_n t\sqrt{1-\zeta^2}+\phi)\right]\right\} \quad (4.57)$$

where

$$\phi = \sin^{-1}(\sqrt{1-\zeta^2}). \quad (4.58)$$

As shown by Equation 4.57, the output initially overshoots the input, lags it in time, and is oscillatory. As time continues, the oscillations damp out and the output approaches, and eventually reaches, the input value. A special situation arises for the no-damping case when $\zeta = 0$. For this situation the output lags the input and repeatedly overshoots and undershoots it forever.

- For the critically damped case ($\zeta = 1$),

$$y(t) = KA\left\{1 - e^{-\omega_n t}(1 + \omega_n t)\right\}. \quad (4.59)$$

No oscillation is present in the output. Rather, the output slowly and monotonically approaches the input, eventually reaching it.

- For the over-damped case ($\zeta > 1$),

$$y(t) = KA \cdot$$
$$\{1 - e^{-\zeta\omega_n t}[\cosh(\omega_n t\sqrt{\zeta^2-1}) + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh(\omega_n t\sqrt{\zeta^2-1})]\}. \quad (4.60)$$

The behavior is similar to the $\zeta = 1$ case. Here the larger the value of ζ, the longer it takes for the output to reach the value of the input signal.

Note that in the equations of all three cases the quantity $\zeta\omega_n$ in the exponential terms multiplies the time. Hence, the quantity $1/\zeta\omega_n$ represents the time constant of the system. The larger the value of the time constant, the longer it takes the response to approach steady state. Further, because the magnitude of the step-input forcing equals KA, the magnitude ratio, $M(t)$, for all three cases is obtained simply by dividing the right sides of Equations 4.57, 4.59, and 4.60 by KA.

Equations 4.57 through 4.60 appear rather intimidating. It is helpful to plot these equations rewritten in terms of their magnitude ratios and examine their form. The system response to step-input forcing is shown in Figure 4.6 for various values of ζ. The quickest response to steady state is when $\zeta = 0$ (that is when the time constant $1/\zeta\omega_n$ is minimum). However, such a value of ζ clearly is not optimum for a measurement system because the amplitude ratio overshoots, then undershoots, and continues to oscillate

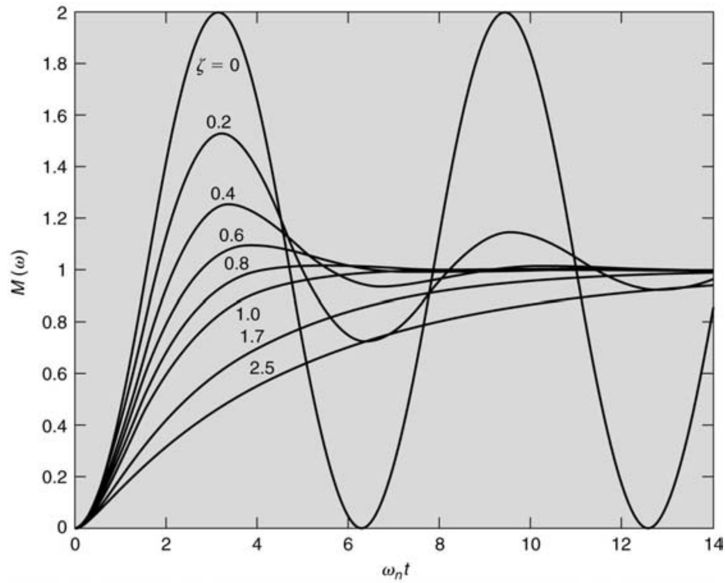

FIGURE 4.6
The magnitude ratio of a second-order system responding to step-input forcing.

about a value of $M(\omega) = 1$ forever. The oscillatory behavior is known as *ringing* and occurs for all values of $\zeta < 1$.

Shown in Figure 4.7 is the response of a second-order system having a value of $\zeta = 0.2$ to step-input forcing. Note the oscillation in the response about an amplitude ratio of unity. In general, this oscillation is characterized by a period T_d, where $T_d = 2\pi/\omega_d$, with the **ringing frequency** $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The **rise time** for a second-order system is the time required for the system to initially reach 90 % of its steady-state value. The **settling time** is the time beyond which the response remains within ± 10 % of its steady-state value.

A value of $\zeta = 0.707$ quickly achieves a steady-state response. Most second-order instruments are designed for this value of ζ. When $\zeta = 0.707$, the response overshoot is within 5 % of $M(t) = 1$ within about one-half of the time required for a $\zeta = 1$ system to achieve steady state. For values of $\zeta > 1$, the system eventually reaches a steady-state value, taking longer times for larger values of ζ.

4.6.2 Response to Sinusoidal-Input Forcing

The response of a second-order system to a sinusoidal input having $F(t) = KA\sin(\omega t)$ with the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$ is

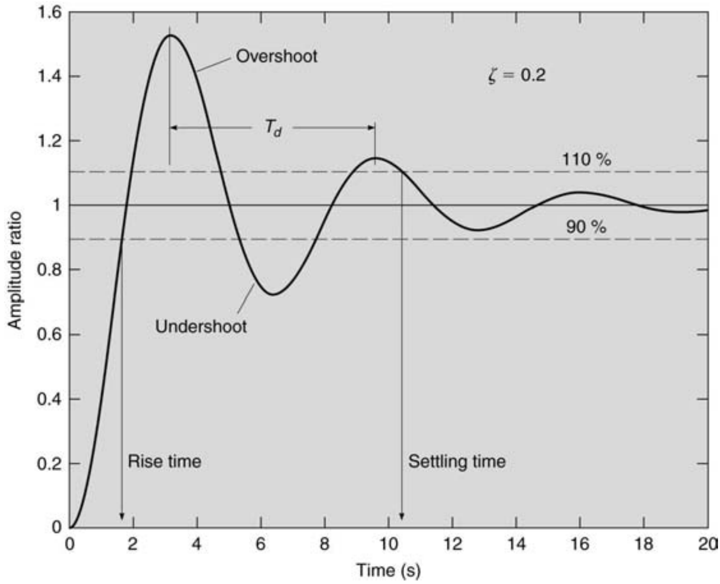

FIGURE 4.7
The temporal response of a second-order system with $\zeta = 0.2$ to step-input forcing.

$$y_p(t) = \frac{KA\sin[\omega t + \phi(\omega)]}{\{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2\}^{1/2}}, \tag{4.61}$$

where the phase lag in units of radians is

$$\phi(\omega) = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \text{ for } \frac{\omega}{\omega_n} \leq 1, \tag{4.62}$$

or

$$\phi(\omega) = -\pi - \tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \text{ for } \frac{\omega}{\omega_n} > 1. \tag{4.63}$$

Note that Equation 4.61 is the particular solution, which also is the steady-state solution. This is because the homogeneous solutions for all ζ are transient and tend toward a value of zero as time increases. Hence, the steady-state magnitude ratio based upon the input $KA\sin(\omega t)$, Equation 4.61 becomes

$$M(\omega) = \frac{1}{\{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2\}^{1/2}}. \tag{4.64}$$

These equations show that the system response will contain both magnitude and phase errors. The magnitude and phase responses for different values of ζ are shown in Figures 4.8 and 4.9, respectively. Note that the magnitude ratio is a function of frequency, ω, for the sinusoidal-input forcing case, whereas it is a function of time, t, for the step-input forcing case.

First examine the magnitude response shown in Figure 4.8. For low values of ζ, approximately 0.6 or less, and $\omega/\omega_n \leq 1$, the magnitude ratio exceeds unity. The maximum magnitude ratio occurs at the value of $\omega/\omega_n = \sqrt{1-2\zeta^2}$. For $\omega/\omega_n \geq\sim 1.5$, the magnitude ratio is less than unity and decreases with increasing values of ω/ω_n.

Typically, magnitude attenuation is given in units of dB/decade or dB/octave. A **decade** is defined as a 10-fold increase in frequency (any 10:1 frequency range). An **octave** is defined as a doubling in frequency (any 2:1 frequency range). For example, using the information in Figure 4.8, there would be an attenuation of approximately -8 dB/octave [$= 20\log(0.2) - 20\log(0.5)$] in the frequency range $1 \leq \omega/\omega_n \leq 2$ when $\zeta = 1$.

Now examine the phase response shown in Figure 4.9. As ω/ω_n increases, the phase angle becomes more negative. That is, the output signal begins to lag the input signal in time, with this lag time increasing with ω/ω_n. For values of $\omega/\omega_n < 1$, this lag is greater for greater values of ζ. At $\omega/\omega_n = 1$, all second-order systems having any value of ζ have a phase lag of $-90°$ or 1/4 of a cycle. For $\omega/\omega_n > 1$, the increase in lag is less for systems with greater values of ζ.

4.7 Higher-Order System Dynamic Response

As seen in this chapter, the responses of linear, first, and second-order systems to simple step and sinusoidal inputs are rather complex. Most experiments involve more than one instrument. Thus, the responses of most experimental measurement systems will be even more complex than the simple cases examined here.

When each instrument in a measurement system is linear, as described in Chapter 2, the total measurement system response can be calculated easily. For the overall system, [a] the static sensitivity is the *product* of all of the static sensitivities, [b] the magnitude ratio is the *product* of all of the magnitude ratios, and [c] the phase shift is the *sum* of all of the phase shifts.

In the end, the most appropriate way to determine the dynamic response characteristics of a measurement system is through dynamic calibration. This can be accomplished by subjecting the system to a range of either step or sinusoidal inputs of amplitudes and frequencies that span the entire

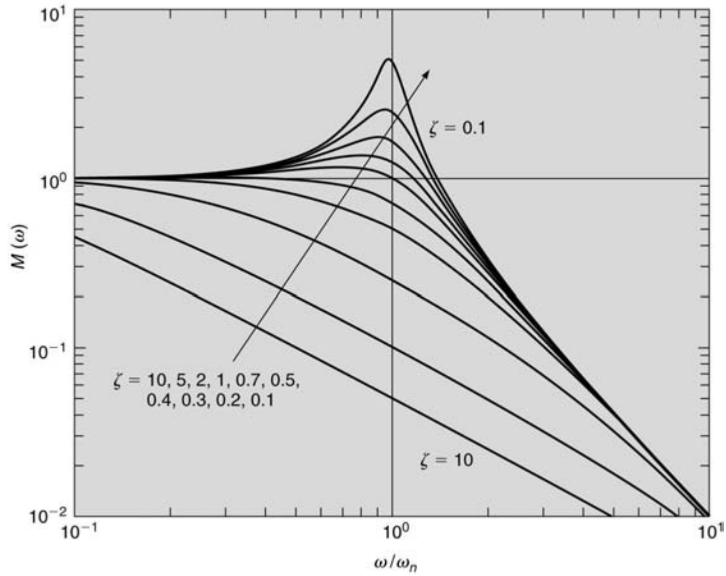

FIGURE 4.8
The magnitude ratio of a second-order system responding to sinusoidal-input forcing.

range of those that would be encountered in an actual experiment. With this approach, the system's dynamic errors can be quantified accurately.

Example Problem 4.5

Statement: A pressure transducer is connected through flexible tubing to a static pressure port on the surface of a cylinder that is mounted inside a wind tunnel. The structure of the flow local to the port is such that the static pressure, $p(t)$, varies as

$$p(t) = 15\text{sin}2t,$$

in which t is time. Both the tubing and the pressure transducer behave as second-order systems. The natural frequencies of the transducer, $\omega_{n,trans}$, and the tubing, $\omega_{n,tube}$, are 2000 rad/s and 4 rad/s, respectively. Their damping ratios are $\zeta_{trans} = 0.7$ and $\zeta_{tube} = 0.2$, respectively. Find the magnitude attenuation and phase lag of the pressure signal, as determined from the output of the pressure transducer, and then write the expression for this signal.

Solution: Because this measurement system is linear, the system's magnitude ratio, $M_s(\omega)$, is the product of the components' magnitude ratios, and the phase lag, $\phi_s(\omega)$, is the sum of the components' phase lags, where ω the circular frequency of the pressure. Thus,

$$M_s(\omega) = M_{tube}(\omega) \times M_{trans}(\omega)$$

and

$$\phi_s(\omega) = \phi_{tube}(\omega) + \phi_{trans}(\omega).$$

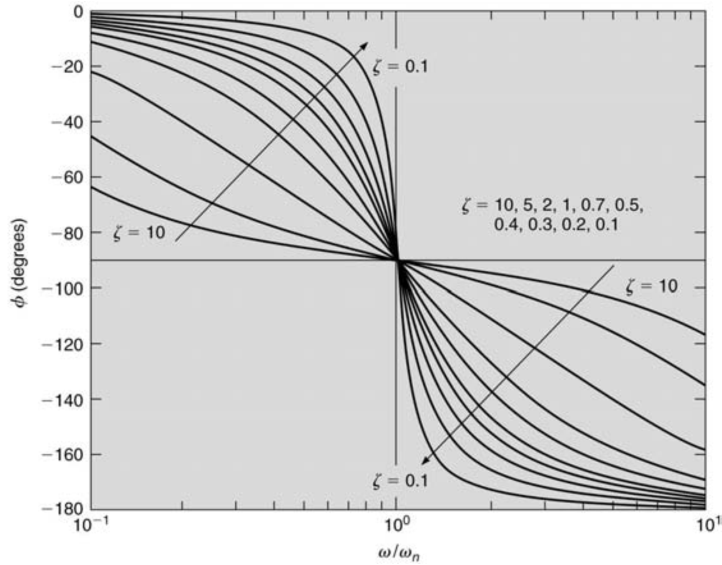

FIGURE 4.9
The phase shift of a second-order system responding to a sinusoidal-input forcing.

Also, $\omega/\omega_{tube} = 2/4 = 0.5$ and $\omega/\omega_{trans} = 2/2000 = 0.001$. Application of Equations 4.62 and 4.64, noting $\zeta_{trans} = 0.7$ and $\zeta_{tube} = 0.2$, yields $\phi_{tube} = -14.9°$, $\phi_{trans} = -0.1°$, $M_{tube} = 1.29$, and $M_{trans} = 1.00$. Thus, $\phi_s(2) = -14.9° + -0.1° = -15.0°$ and $M_s(2) = (1.29)(1.00) = 1.29$. The pressure signal, as determined from the output of the transducer, is $p_s(t) = (15)(1.29)\sin[2t - (15.0)(\pi/180)] = 19.4\sin(2t - 0.26)$. Thus, the magnitude of the pressure signal at the output of the measurement system will appear 129 % greater than the actual pressure signal and be delayed in time by 0.13 s [(0.26 s)/(2 rad/s)].

4.8 *Numerical Solution Methods

Differential equations governing a system's response to input forcing may be nonlinear and not have exact solutions. Fortunately, methods are available to numerically integrate most ordinary differential equations and obtain the system response [24]. The basic solution approach is to reduce any higher-order ordinary differential equations to a system of coupled, first-order ordinary differential equations. Then, the first-order equations are solved using finite-difference methods. For example, the second-order ordinary differential equation

$$\frac{d^2y(t)}{dt^2} - cy(t) = d \tag{4.65}$$

can be reduced to two first-order ordinary differential equations by using the substitution $dy/dt = z(t)$, which yields the system of equations

$$\begin{aligned} \frac{dy(t)}{dt} &= z(t) \text{ and} \\ \frac{dz(t)}{dt} &= cy(t) + d. \end{aligned} \tag{4.66}$$

Two initial conditions are needed to obtain a specific solution.

The numerical solution of a first-order ordinary differential equation can be obtained using various finite-difference methods [3]. The exact differential, $dy(t)/dt = f(y,t)$, is approximated by a finite difference. There are many ways to approximate $dy(t)/dt$. The choice depends upon the required accuracy and computation time. A straightforward finite-difference approximation for $dy(t)/dt$ is the forward Euler expression

$$f(y_n, t_n) \approx \frac{y_{n+1} - y_n}{\Delta t}, \tag{4.67}$$

where n and $n+1$ denote the n-th and $n+1$-th points. Equation 4.67 leads directly to

$$y_{n+1} = y_n + \Delta t \; f(y_n, t_n). \tag{4.68}$$

The expression for $f(y,t)$ is obtained from the governing first-order ordinary differential equation. An initial condition, $y(0)$, also is specified. This permits the value of y_{n+1} to be computed for a fixed Δt from Equation 4.68. This algorithm is applied successively up to the desired final time.

Other methods can be used to determine an expression analogous to Equation 4.67. All these methods are easy to implement. The following more commonly used methods replace $f(y_n, t_n)$ in Equation 4.68 by

$$f(y_{n+1}, t_{n+1}) \tag{4.69}$$

for the backward Euler method, and

$$\frac{f(y_n, t_n) + f(y_{n+1}, t_{n+1})}{2} \tag{4.70}$$

for the improved Euler method. The improved Euler method is more accurate than the forward and backward Euler methods. The fourth-order Runge-Kutta method replacement for Equation 4.67 is

$$\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}, \tag{4.71}$$

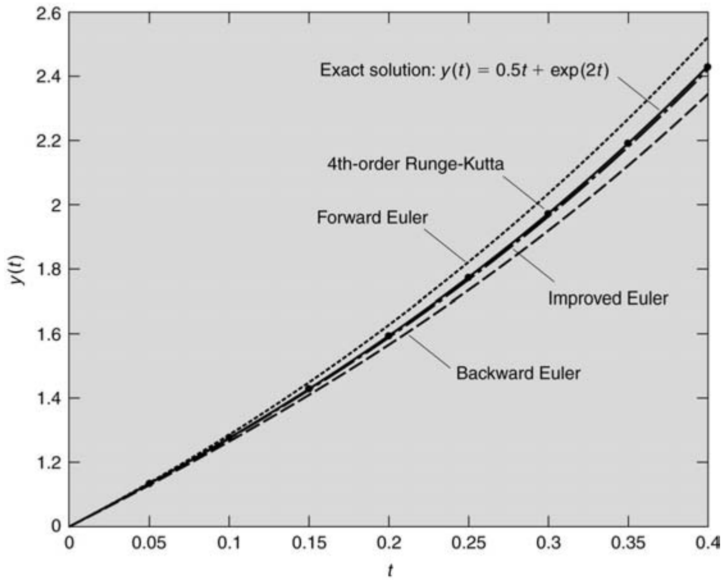

FIGURE 4.10
The response of the system $\frac{dy(t)}{dt} - 2y(t) = F(t) = 0.5 - t$ to forcing as determined by the MATLAB M-file `odeint.m`.

where

$$\begin{aligned} k_1 &= f(y_n, t_n), \\ k_2 &= f(y_n + \frac{\Delta t}{2} k_1, t_n + \frac{\Delta t}{2}), \\ k_3 &= f(y_n + \frac{\Delta t}{2} k_2, t_n + \frac{\Delta t}{2}), \text{ and} \\ k_4 &= f(y_n + \Delta t\, k_3, t_n + \Delta t). \end{aligned} \tag{4.72}$$

The fourth-order Runge-Kutta method is more accurate than any Euler method. It is the method used most frequently and is quite sufficient for most numerical integrations [3].

Example Problem 4.6

Statement: A first-order system is described by the equation

$$\frac{dy(t)}{dt} - 2y(t) = F(t) = 0.5 - t,$$

with the initial condition $y(0) = 1$. Solve this differential equation numerically and analytically. Use four numerical methods: [1] forward Euler, [2] backward Euler, [3] improved Euler, and [4] fourth-order Runge-Kutta. Use a step size of 0.05 s. Plot all the results for comparison.

Solution: The MATLAB M-file `odeint.m` can be used for this purpose. The results are presented in Figure 4.10. The fourth-order Runge-Kutta and improved Euler solutions follow the exact solution closely. The backward Euler method underestimates the response. The forward Euler method overestimates the response.

4.9 Problem Topic Summary

Topic	Review Problems	Homework Problems
System Basics	1, 3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 19, 21, 22, 23	
First-Order	2, 8, 11, 12, 18, 20	1, 2, 3, 4, 5, 8, 11, 13, 17
Second-Order	9, 10	6, 7, 9, 10, 12, 14, 15, 16, 18, 19, 20

TABLE 4.2
Chapter 4 Problem Summary

4.10 Review Problems

1. Does a smaller diameter thermocouple or a larger diameter thermocouple have the large time constant?

2. The dynamic error in a temperature measurement using a thermocouple is 70 % at 3 s after an input step change in temperature. Determine the magnitude ratio of the thermocouple's response at 1 s.

3. Determine the % dynamic error of a measurement system that has an output of $3\sin(200t)$ for an input of $4\sin(200t)$.

4. Determine the attenuation (reduction) in units of dB/decade for a measurement system that has an output of $3\sin(200t)$ for an input of $4\sin(200t)$ and an output of $\sin(2000t)$ for an input of $4\sin(2000t)$.

5. Is a strain gage in itself classified as a zero, first, second, or higher-order system?

6. Determine the damping ratio of a RLC circuit with $LC = 1$ s^2 that has a magnitude ratio of 8 when subjected to a sine wave input with a frequency of 1 rad/s.

7. Determine the phase lag in degrees for a simple RC filter with $RC = 5$ s when its input signal has a frequency of $1/\pi$ Hz.

8. A first-order system is subjected to a step input of magnitude B. The time constant in terms of B equals (a) $0.707B$, (b) $0.5B$, (c) $(1 - \frac{1}{e})B$, or (d) B/e.

9. A second-order system with $\zeta = 0.5$ and $\omega_n = 2$ rad/s is subjected to a step input of magnitude B. The system's time constant equals (a) 0.707 s, (b) 1.0 s, (c) $(1 - \frac{1}{e})$ s, or (d) not enough information.

10. A second-order system with $\zeta = 0.5$ and $\omega_n = 2$ rad/s is subjected to a sinusoidal input of magnitude $B\sin(4t)$. The phase lag of the output signal in units of degrees is (a) -3, (b) -146, (c) -34, or (d) -180.

11. A first-order system is subjected to an input of $B\sin(10t)$. The system's time constant is 1 s. The amplitude of the system's output is approximately (a) $0.707B$, (b) $0.98B$, (c) $(1 - \frac{1}{e})B$, or (d) $0.1B$.

12. A first-order system is subjected to an input of $B\sin(10t)$. The system's time constant is 1 s. The time lag of the system's output is (a) -0.15 s, (b) -0.632 s, (c) $-\pi$ s, or (d) -84.3 s.

13. What is the static sensitivity of the calibration curve $F = 250W + 125$ at $W = 2$?

14. The magnitude of the static sensitivity of the calibration curve $V = 3 + 8\sqrt{F}$ at $F = 16$ is (a) 0, (b) 1, (c) 3, (d) 4, or (e) 8.

15. What is the order of each of the following systems? (a) Strain gage, (b) pressure transducer, (c) accelerometer, (d) RC circuit, (e) thermocouple, (f) pitot-static tube.

16. What is the magnitude ratio that corresponds to -6 dB?

17. What is the condition for an RLC circuit to be underdamped, critically damped, or overdamped?

18. A large thermocouple has a time constant of 10 s. It is subjected to a sinusoidal variation in temperature at a cyclic frequency of $1/(2\pi)$ Hz. The phase lag, in °, is approximately (a) -0.707, (b) -3, (c) -45, or (d) -85.

19. What is the sensitivity of the linear calibration curve at $E = 0.5\exp(10/T)$ at (a) $T = 283$ K, (b) $T = 300$ K, and (c) $T = 350$ K. (d) What type of temperature sensor might result in such an exponential calibration curve?

20. Consider a first-order system where the frequency of the sinusoidal forcing function is 10 Hz and the system response lags by 90°. What is the phase lag in seconds?

Time (ms)	Temperature (°C)
0	24.8
40	22.4
120	19.1
200	15.5
240	13.1
400	9.76
520	8.15
800	6.95
970	6.55
1100	6.15
1400	5.75
1800	5.30
2000	5.20
2200	5.00
3000	4.95
4000	4.95
5000	4.95
6000	4.95
7000	4.95

TABLE 4.3
Thermocouple Response Data

21. The signal $10\sin(2\pi t)$ passes through a filter whose magnitude ratio is 0.8 and then through a linear amplifier. What must be the gain of the amplifier for the amplifier's output signal to have an amplitude of 16?

22. An electronic manometer is calibrated using a fluid based manometer as the calibration standard. The resulting calibration curve fit is given by the equation $V = 1.1897P - 0.0002$, where the unit of P is inches of H_20 and V is volts. The static sensitivity (in V/in. H_20) is (a) 0.0002, (b) $1.1897P^2$ - $0.0002P$, (c) 1.1897, or (d) -0.0002.

23. Determine the static sensitivity at $x = 2.00$ for a calibration curve having $y = 0.8 + 33.72x + 3.9086x^2$. Express the result with the correct number of significant figures.

4.11 Homework Problems

1. A first-order system has $M(f = 200 \text{ Hz}) = 0.707$. Determine (a) its time constant (in milliseconds) and (b) its phase shift (in degrees).

2. A thermocouple held in room-temperature air is suddenly immersed into a beaker of cold water. Its temperature as a function of time is recorded. Determine the thermocouple's time constant by plotting the data listed in Table 4.3, assuming that the thermocouple behaves as a first-order system. A more accurate method of determining the time constant is by performing a least-squares *linear* regression analysis (see Chapter 8) after transforming the temperatures into their appropriate nondimensional variables.

3. A first-order system with a time constant equal to 10 ms is subjected to a sinusoidal forcing with an input amplitude equal to 8.00 V. When the input forcing frequency equals 100 rad/s, the output amplitude is 5.66 V; when the input forcing frequency equals 1000 rad/s, the output amplitude is 0.80 V. Determine (a) the magnitude ratio for the 100 rad/s forcing case, (b) the roll-off slope (in units of dB/decade) for the $\omega\tau = 1$ to $\omega\tau = 10$ decade, and (c) the phase lag (in degrees) for the 100 rad/s forcing case.

4. The dynamic error in a temperature measurement using a thermometer is 70 % at 3 s after an input step change in temperature. Determine (a) the magnitude ratio at 3 s, (b) the thermometer's time constant (in seconds), and (c) the magnitude ratio at 1 s.

5. A thermocouple is immersed in a liquid to monitor its temperature fluctuations. Assume the thermocouple acts as a first-order system. The temperature fluctuations (in degrees Celsius) vary in time as $T(t) = 50 + 25\cos(4t)$. The output of the thermocouple transducer system (in V) is linearly proportional to temperature and has a static sensitivity of 2 mV/°C. A step-input calibration of the system reveals that its rise time is 4.6 s. Determine the system's (a) time constant (in seconds), (b) output $E(t)$ (in millivolts), and (c) time lag (in seconds) at $\omega = 0.2$ rad/s.

6. A knowledgeable aerospace student selects a pressure transducer (with $\omega_n = 6284$ rad/s and $\zeta = 2.0$) to investigate the pressure fluctuations within a laminar separation bubble on the suction side of an airfoil. Assume that the transducer behaves as an over-damped second-order system. If the experiment requires that the transducer response has $M(\omega) \geq 0.707$ and $|\phi(\omega)| \leq 20°$, determine the maximum frequency (in hertz) that the transducer can follow and accurately meet the two criteria.

7. A strain gage system is mounted on an airplane wing to measure wing oscillations and strain during wind gusts. The system is second order, having a 90 % rise time of 100 ms, a *ringing* frequency of 1200 Hz, and a damping ratio of 0.8. Determine (a) the dynamic error when subjected to a 1 Hz oscillation and (b) the time lag (in seconds).

8. In a planned experiment a thermocouple is to be exposed to a step change in temperature. The response characteristics of the thermocouple must be such that the thermocouple's output reaches 98 % of the final temperature within 5 s. Assume that the thermocouple's bead (its sensing element) is spherical with a density equal to 8000 kg/m^3, a specific heat at constant volume equal to 380 J/(kg·K), and a convective heat transfer coefficient equal to 210 W/(m^2·K). Determine the *maximum* diameter that the thermocouple can have and still meet the desired response characteristics.

9. Determine by calculation the damping ratio value of a second-order system that would be required to achieve a magnitude ratio of unity when the sinusoidal-input forcing frequency equals the natural frequency of the system.

10. The pressure tap on the surface of a heat exchanger tube is connected via flexible tubing to a pressure transducer. *Both* the tubing and the transducer behave as second-order systems. The natural frequencies are 30 rad/s for the tubing and 6280 rad/s for the transducer. The damping ratios are 0.45 for the tubing and 0.70 for the transducer. Determine the magnitude ratio and the phase lag for the system when subjected to a sinusoidal forcing having a 100 Hz frequency. What, if anything, is the problem in using this system for this application?

11. Determine the percent dynamic error in the temperature measured by a thermocouple having a 3 ms time constant when subjected to a temperature that varies sinusoidally in time with a frequency of 531 Hz.

12. The output of an under-damped second-order system with $\zeta = 0.1$ subjected to step-input forcing initially oscillates with a period equal to 1 s until the oscillation dissipates. The same system then is subjected to sinusoidal-input forcing with a frequency equal to 12.62 rad/s. Determine the phase lag (in degrees) at this frequency.

13. A thermocouple is at room temperature (70 °F) and at equilibrium when it is plunged into a water bath at a temperature of 170 °F. It takes the thermocouple 1 s to read a temperature indication of 120 °F. What is the time constant of the thermocouple-fluid system? This same thermocouple is used to measure a sinusoidally varying temperature. The variation in degrees Fahrenheit is given by the equation

$$T = 100 + 200\sin(10t).$$

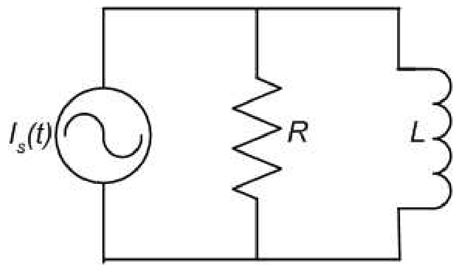

FIGURE 4.11
Current-pulse RL circuit.

What temperature does the thermocouple indicate after steady state conditions are reached?

14. A pressure transducer that behaves as a second-order system is supposed to have a damping ratio of 0.7, but some of the damping fluid has leaked out, leaving an unknown damping ratio. When the transducer is subjected to a harmonic input of 1850 Hz, the phase angle between the input and the output is 45°. The manufacturer states that the natural frequency of the transducer is 18 500 rad/s. (a) What is the dynamic error in the transducer output for a harmonic pressure signal of 1200 Hz? (b) If the transducer indicates a pressure amplitude of 50 psi, what is the true amplitude of the pressure?

15. The output of an under-damped second-order system with $\zeta = 0.1$ subjected to step-input forcing initially oscillates with a period equal to 1 s until the oscillation dissipates. The same system then is subjected to sinusoidal-input forcing with a frequency equal to 12.62 rad/s. Determine the phase lag (in degrees) at this frequency.

16. Consider the RL circuit shown in the Figure 4.11, where the source is the current pulse $I_s(t) = 6\,[u(t) - u(t-1)]$ A, $R = 5\ \Omega$, and $L = 5$ H. What is the current response of the circuit, $i(t)$?

17. For an RC circuit ($R = 2\ \Omega$; $C = 0.5$ F) with step-input forcing from 0 V to 1 V, determine (a) the voltage of the circuit at 1 s, (b) the voltage of the circuit at 5 s, and (c) the dynamic error at 1 s.

18. For an RLC circuit ($R = 2\ \Omega$; $C = 0.5$ F; $L = 0.5$ H) with sinusoidal-input forcing of the form $F(t) = 2\sin(2t)$, determine (a) the phase lag in degrees, (b) the phase lag in seconds, and (c) the magnitude ratio.

19. For an RLC circuit, (a) what are the mathematical relationships involving R, L, and C for the system to be under-damped, critically damped, or over-damped? (b) What is the equivalent time constant of this system?

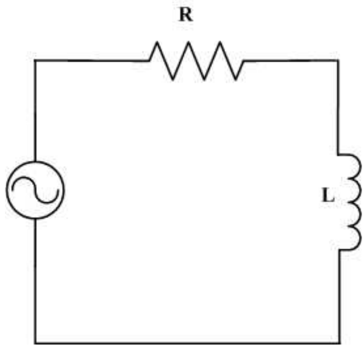

FIGURE 4.12
Simple *RL* circuit.

20. Consider the simple *RL* circuit shown in Figure 4.12 in which $R = 10$ Ω and $L = 5$ H. (a) What is the governing equation for the current in this circuit? Is it first order or second order? (b) What is the time constant for this system? (c) If the voltage source has a sinusoidal input of $5\sin(10t)$ V, what is the solution to the governing equation? What is the magnitude ratio? What is the phase lag (in seconds)? (d) Plot the current response versus time.

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5

Probability

CONTENTS

When you deal in large numbers, probabilities are the same as certainties. I wouldn't bet my life on the toss of a single coin, but I would, with great confidence, bet on heads appearing between 49 % and 51 % of the throws of a coin if the number of tosses was 1 billion.

Brian L. Silver. 1998. *The Ascent of Science*. Oxford: Oxford University Press.

It is the nature of probability that improbable things will happen.

Aristotle, c. 350 B.C.

... it is a very certain truth that, whenever we are unable to identify the most true options, we should follow the most probable...

René Descartes. 1637. *Discourse on Method.*

Chance favours only the prepared mind.

Louis Pasteur, 1879.

5.1 Chapter Overview

Probability underlies all of our lives. How often has one heard that the chance of rain tomorrow will be 50 % or that there is a good chance to be a winner in today's lottery? It is hard to avoid its mention in an electronically connected society. But what science is behind such statements? Similar questions can be asked in relation to experiments, such as the probability that a pressure will exceed a certain limit.

In this chapter we will study some of the tools of probability. We will start by examining the differences between a population and a sample when using statistics. Next we will find out how to calculate and present the statistical information about a population or a sample. Then, we will explore the concept of the probability density function and its integral, the probability distribution function. The chapter concludes with a review of basic probability concepts. After finishing with this chapter, you will have studied most of the basic concepts of probability. This will prepare you to begin the study of statistics, which is the subject of Chapter 6.

5.2 Relation to Measurements

Probability and statistics are two distinct but closely related fields of science. Probability deals with the likelihood of events. The mathematics of probability shows us how to calculate the likelihood or chance of an event based on theoretical populations. Statistics involve the collection, presentation, and interpretation of data, usually for the purpose of making inferences about the behavior of an underlying population or for testing theory. Both fields can be used to answer many practical questions that arise when performing an experiment, such as the following:

- How frequently does this event occur?
- What are the chances of rejecting a correct theory?
- How repeatable are the results?
- What confidence is there in the results?
- How can the fluctuations and drift in the data be characterized?
- How much data is necessary for an adequate sample?

Armed with a good grasp of probability and statistics, all of these questions can be answered quantitatively.

5.3 Sample versus Population

Quantitative information about a process or population usually is gathered through an experiment. From this information, certain characteristics of the process or population can be estimated. This approach is illustrated schematically in Figure 5.1. The **population** refers to the complete collection of all members relevant to a particular issue and the **sample** to a subset of that population. Some populations are finite, such as the number of students in a measurements class. Some populations essentially are infinite, such as the number of molecules in the earth's atmosphere. Many populations are finite but are so large that they are considered infinite, such as the number of domestic pets in the U.S. The sample is drawn randomly and independently from the population. **Statistics** of the sample, such as its **sample mean value**, $\bar{x}$, and its **sample variance**, S_x^2, can be computed. From these statistics, the population's **parameters**, which literally are *almost measurements*, such as its **true mean value**, x', and its **true variance**, σ^2, can be estimated using methods of **statistical inference**.

The term *statistic* was defined by R.A. Fisher, the renowned statistician, as the number that is derived from observed measurements and that estimates a parameter of a distribution [1]. Other useful information also can be obtained using statistics, such as the probability that a future measurand will have a certain value. The interval within which the true mean and true variance are contained also can be ascertained assuming a certain level of confidence and the distribution of all possible values of the population.

The process of sampling implicitly involves random selection. When a measurement is made during an experiment, a value is selected randomly from an infinite number of possible values in the measurand population. That is, the process of selecting the next measurand value does not depend upon any previously acquired measurand values. The specific value of the selected measurand is a **random variable**, which is a real number between $-\infty$ and $+\infty$ that can be associated with each possible measurand value. So, the term *random* refers to the selection process and *not* to the often-misinterpreted meaning that the acquired measurand values form a set of random numbers. If the selection process is not truly random, then erroneous conclusions about the population may be made from the sample.

5.4 Plotting Statistical Information

Usually the first thing done after an experiment is to plot the data and to observe its trends. This data typically is a set of measurand values acquired with respect to time or space. The representation of the variation in a mea-

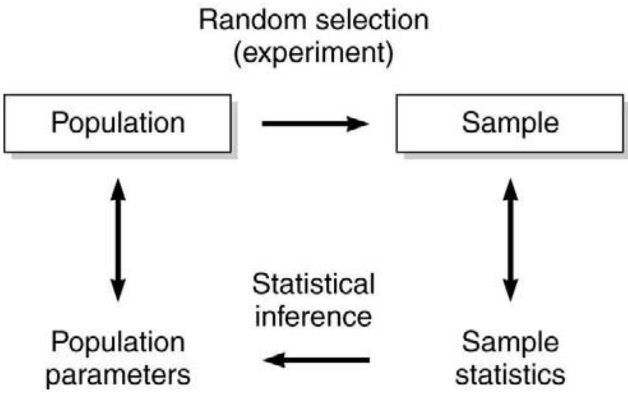

FIGURE 5.1
The finite sample versus the infinite population.

surand's magnitude with respect to time or space is called its **signal**. This signal usually is one of three possible representations: **analog** (continuous in both magnitude and time or space), **discrete** (continuous in magnitude but at specific, fixed-interval values in time or space), and **digital** (having specific, fixed-interval values in both magnitude and time or space). These three representations of the same signal are illustrated in Figure 5.2 and discussed further in Chapter 9. In that figure the analog signal is denoted by the solid curve, the digital signal by open circles, and the discrete signal by solid circles. The discrete representation sometimes is called a **scattergram**.

A cursory examination of the continuous signal displayed in Figure 5.2 shows that the signal has **variability** (its magnitude varies in time) and exhibits a **central tendency** (its magnitude varies about a mean value). How can this information be quantified? What other ways are there to view the sample such that more can be understood about the underlying physical process?

One way to view the central tendency of the signal and the frequency of occurrence of the signal's values is with a **histogram**, which literally is a *picture of cells*. Galileo may have used a frequency diagram to summarize some astronomical observations in 1632 [8]. John Graunt probably was the first to invent the histogram in 1662 in order to present the mortality rates of the black plague [9].

Consider the *digital* representation of the signal shown in Figure 5.2. There are 10 values in units of volts (1.5, 1.0, 2.5, 4.0, 3.5, 2.0, 2.5, 3.0, 2.5, and 0.5). The resolution of the digitization process for this case is 0.5 V. A histogram of this signal is formed by simply counting the number of times that each value occurs and then plotting the count for each value on the ordinate axis versus the value on the abscissa axis. The histogram is

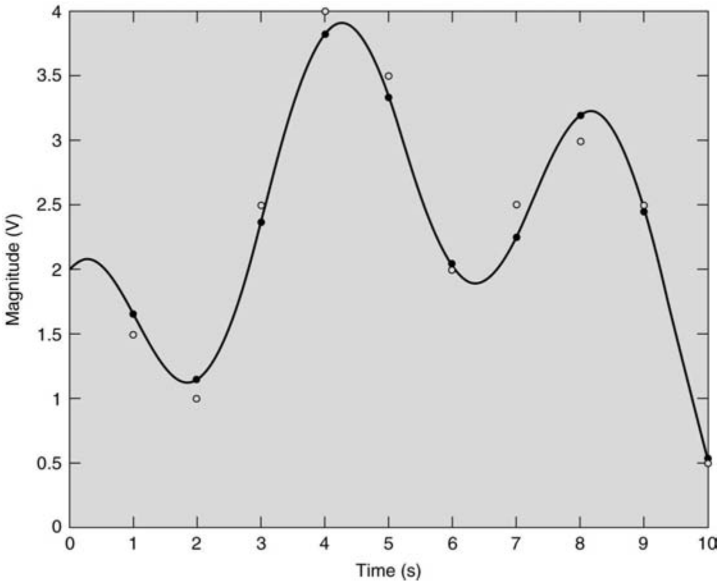

FIGURE 5.2
Various representations of the same signal.

shown in Figure 5.3. Several features are immediately evident. The most frequently occurring value is 2.5 V. This value occurs three out of ten times, so it comprises 30 % of the signal's values. The range of values is from 0.5 V to 4.0 V. The average value of the signal appears to be between 2.0 V and 2.5 V (its actual value is 2.3 V).

What are the mechanics and rules behind constructing a histogram? In practice there are two types of histograms. Equal-probability interval histograms have **class intervals** (bins) of variable width, each containing the same number of occurrences. Equal-width interval histograms have class intervals of fixed width, each possibly containing a different number of occurrences. The latter is used most frequently. It is more informative because it clearly shows both the frequency and the distribution of occurrences. The number of intervals, hence the interval width, and the interval origin must be determined first before constructing an equal-width interval histogram. There are many subtleties involved in choosing the optimum interval width and interval origin. The reader is referred to Scott [9] for a thorough presentation.

But how is the number of intervals chosen? Too few or too many intervals yield histograms that are not informative and do not reflect the distribution of the population. At one extreme, all the occurrences can be contained in one interval; at the other extreme, each occurrence can be in its own interval. Clearly, there must be an optimum number of intervals that yields the most

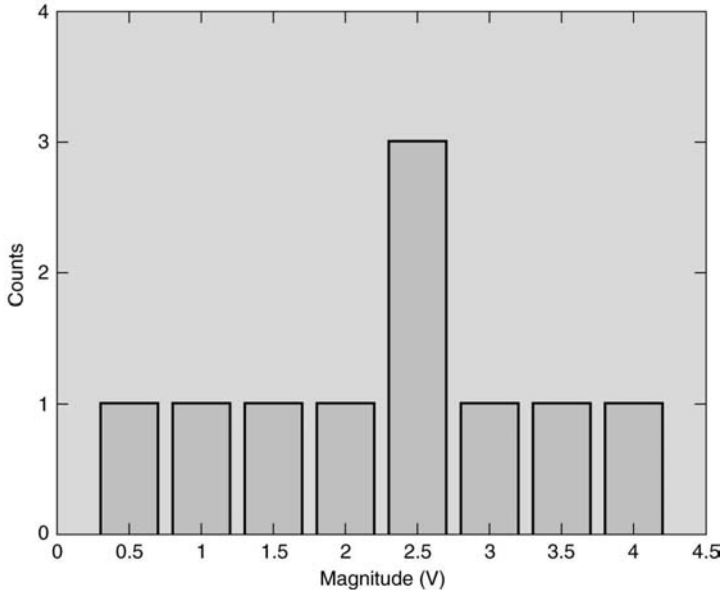

FIGURE 5.3
The histogram of a digital signal representation.

representative histogram. An example of how the choice of the number of intervals affects the histogram's fidelity is presented in Figure 5.4. In that figure, the theoretical values of the population are represented by black dots in the left and center histograms and by a white curve in the right histogram. The data used for the histogram consisted of 5000 values drawn randomly from a normally distributed population (this type of distribution is discussed in Chapter 6). In the left histogram, too few intervals are chosen and the population is over-estimated in the left and right bins and under-estimated in the center bin. In the right histogram, too many intervals are chosen and the population is consistently over-estimated in almost all bins. In the middle histogram, the optimum number of class intervals is chosen and excellent agreement between the observed and expected values is achieved.

For equal-probability interval histograms, the intervals have different widths. The widths typically are determined such that the probability of an interval equals $1/K$, where K denotes the number of intervals. Bendat and Piersol [17] present a formula for K that was developed originally for continuous distributions by Mann and Wald [11] and modified by Williams [12]. It is valid strictly for $N \geq 450$ at the 95 % confidence level, although Mann and Wald [11] state that it is probably valid for $N \geq 200$ or even lower N. The exact expression given by Mann and Wald is $K = 2[2(N-1)^2/c^2]^{0.2}$, where $c = 1.645$ for 95 % confidence. Various spreadsheets as well as Montgomery and Runger [6] suggest the formula $K = \sqrt{N}$, which agrees

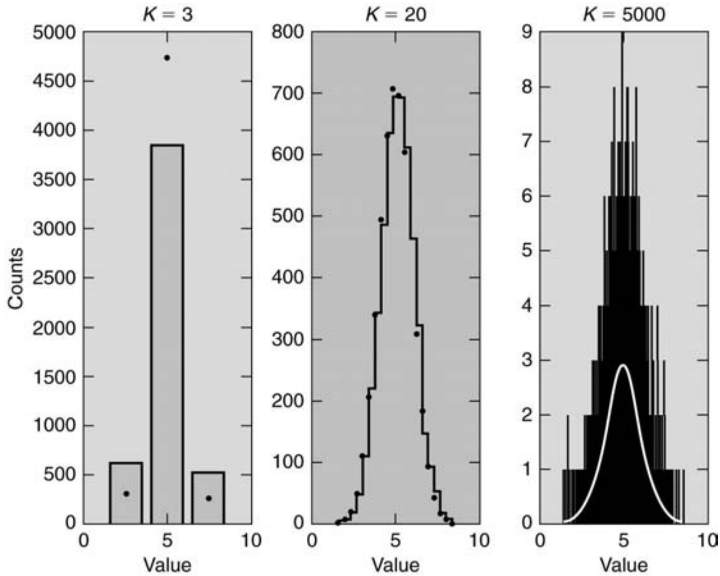

FIGURE 5.4
Histograms with different numbers of intervals for the same data.

within 10 % with the modified Mann and Wald formula up to approximately $N = 1000$.

For equal-width interval histograms, the interval width is constant and equal to the range of values (maximum minus minimum values) divided by the number of intervals. Sturgis' formula [10] determines K from the number of binomial coefficients needed to have a sum equal to N and can be used for values of N as low as approximately 30. Based upon this formula, various authors, such as Rosenkrantz [5], suggest using values of K between 5 and 20 (this would cover between approximately $N = 2^5 = 32$ to $N = 2^{20} \simeq 10^6$). Scott's formula for K ([13] and [14]), valid for $N \geq 25$, was developed to minimize the integrated mean square error that yields the best fit between Gaussian data and its parent distribution. The exact expression is $\Delta x = 3.49\sigma N^{-1/3}$, where Δx is the interval width and σ the standard deviation. Using this expression and assuming a range of values based upon a certain percent coverage, an expression for K can be derived.

The formulas for K for both types of histograms are presented in Table 5.1 and displayed in Figure 5.5. The number of intervals for equal-probability interval histograms at a given value of N is approximately two to three times greater than the corresponding number for equal-width interval histograms. More intervals are required for the former case because the center intervals must be narrow and numerous to maintain the same probability as those intervals on the tails of the distribution. Also, because of the

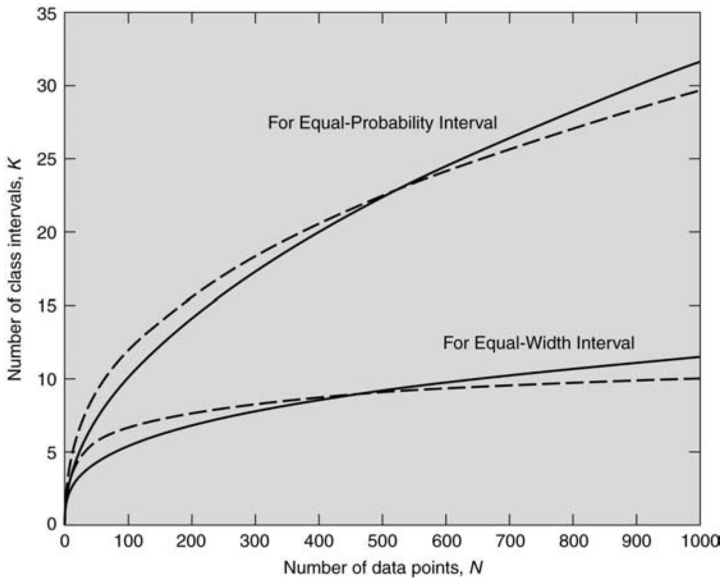

FIGURE 5.5
$K = f(N)$ formulas for equal-width and equal-probability histograms. Equal-probability interval symbols: modified Mann and Wald formula (dash) and square-root formula (solid). Equal-width interval symbols: Sturgis formula (dash) and Scott formula (solid).

low probabilities of occurrence at the tails of the distribution, some intervals for equal-width histograms may need to be combined to achieve greater than five occurrences in an interval. This condition (see [14]) is necessary for proper comparison between theoretical and experimental distributions. However, for small samples it may not be possible to meet this condition. Another very important condition that must be followed is that $\Delta x \geq u_x$, where u_x is the uncertainty of the measurement of x. That is, the interval width should *never* be smaller than the uncertainty of the measurand.

To construct equal-width histograms, these steps must be followed:

1. Identify the minimum and maximum values of the measurand x, x_{min}, and x_{max}, thereby finding its range, $x_{range} = x_{max} - x_{min}$.

2. Determine the number of class intervals, K, using the appropriate formula for equal-width histograms. Preferably, this should be Scott's formula.

3. Calculate the width of each interval, where $\Delta x = x_{range}/K$.

Interval Type	Formula	Reference
Equal-probability	$K = 1.87(N-1)^{0.40}$	[11], [12], [17]
Equal-probability	$K = \sqrt{N}$	[6], [15]
Equal-width	$K = 3.322 \log_{10} N$	[10]
Equal-width	$K = 1.15N^{1/3}$	[13], [14]

TABLE 5.1
Formulas for the number of histogram intervals with 95 % confidence.

4. Count the number of occurrences, n_j ($j = 1$ to K), in each Δx interval. Check that the sum of all the n_j's equals N, the total number of data points.

5. Check that the conditions for $n_j > 5$ (if possible) and $\Delta x \geq u_x$ (definitely) are met.

6. Plot n_j vs xm_j, where xm_j is discretized as the mid-point value of each interval.

Instead of examining the distribution of the *number* of occurrences of various magnitudes of a signal, the *frequency* of occurrences can be determined and plotted. The plot of $n_j/N = f_j$ versus xm_j is known as the **frequency distribution** (sometimes called the relative frequency distribution). The area bounded laterally by any two x values in a frequency distribution equals the frequency of occurrence of that range of x values or the probability that x will assume values in that range. Also, the sum of all the f_j's equals 1. The frequency distribution often is preferred over the histogram because it corresponds to the probabilities of occurrence. Further, as the sample size becomes large, the sample's frequency distribution becomes similar to the distribution of the population's probabilities, which is called the probability density function.

The distribution of all of the values of the infinitely large population is given by its probability density function, $p(x)$. This will be defined in the next section. Typically $p(x)$ is normalized such that the integral of $p(x)$ over all x equals unity. This effectively sets the sum of the probabilities of all the values between $-\infty$ and $+\infty$ to be unity or 100 %. Similar to the frequency distribution, the area under the portion of the probability density function over a given measurand range equals the percent probability that the measurands will have values in that range.

To properly compare a frequency distribution with an assumed probability density function on the same graph, the frequency distribution first must be converted into a **frequency density distribution**. The frequency density is denoted by f_j^*, where $f_j^* = f_j/\Delta x$. This is because the probability density function is related to the frequency distribution by the expression

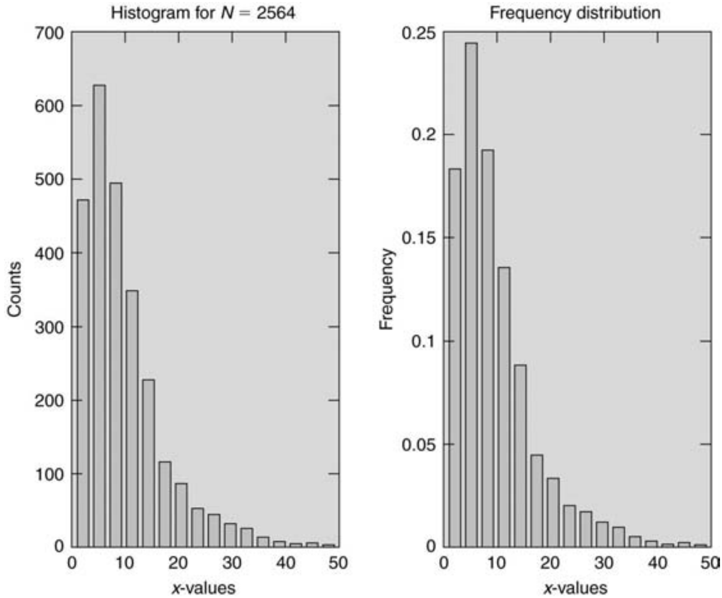

FIGURE 5.6
Sample histogram and frequency distributions of the same data.

$$p(x) = \lim_{N \to \infty, \Delta x \to 0} \sum_{j=1}^{K} f_j / \Delta x = \lim_{N \to \infty, \Delta x \to 0} \sum_{j=1}^{K} f_j{}^*. \quad (5.1)$$

The N required for this comparative limit to be attained within a certain confidence can be determined using the law of large numbers [5], which is

$$N \geq \frac{1}{4\epsilon^2(1 - P_o)}. \quad (5.2)$$

This law, derived by Jacob Bernoulli (1654-1703), considered the father of the quantification of uncertainty, was published posthumously in 1713 [1]. The law determines the N required to have a probability of at least P_o that $f_j^*(x)$ differs from $p(x)$ by less than ϵ.

Example Problem 5.1

Statement: One would like to determine whether or not a coin used in a coin toss is fair. How many tosses would have to be made to assess this?

Solution: Assume one wants to be at least 68 % confident ($P_o = 0.68$) that the coin's fairness is assessed to within 5 % ($\epsilon = 0.05$). Then, according to Equation 5.2, the coin must be tossed at least 310 times ($N \geq 310$).

Often the frequency distribution of a finite sample is used to identify the probability density function of its population. The probability density function's shape tells much about the physical process governing the population. Once the probability density function is identified, much more information about the process can be obtained.

5.5 Probability Density Function

Consider the time history record of a random variable, $x(t)$. Its probability density function, $p(x)$, reveals how the values of $x(t)$ are distributed over the entire range of $x(t)$ values. The probability that $x(t)$ will be between x^* and $x^* + \Delta x$ is given by

$$p(x) = \lim_{\Delta x \to 0} \frac{Pr[x^* < x(t) \leq x^* + \Delta x]}{\Delta x}. \tag{5.3}$$

The probability that $x(t)$ is in the range x to $x + \Delta x$ over a total time period T also can be determined. Assume that T is large enough such that the statistical properties are truly representative of that time history. Further assume that a single time history is sufficient to fully characterize the underlying process. These assumptions are discussed further in Chapter 9. For the time history depicted in Figure 5.7, the total amount of time during the time period T that the signal is between x and $x + \Delta x$ is given by T_x, where

$$T_x = \sum_{j=1}^{m} \Delta t_j \tag{5.4}$$

for m occurrences. In other words,

$$Pr[x < x(t) \leq x + \Delta x] = \lim_{T \to \infty} \left[\frac{T_x}{T}\right] = \lim_{T \to \infty} \frac{1}{T} \sum_{j=1}^{m} \Delta t_j. \tag{5.5}$$

This implies that

$$p(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\lim_{T \to \infty} \frac{1}{T} \sum_{j=1}^{m} \Delta t_j\right] = \lim_{\Delta x \to 0, T \to \infty} \left[\frac{T_x/T}{\Delta x}\right]. \tag{5.6}$$

Likewise, x could be the number of occurrences of a variable with a Δx interval, n_j, where the total number of occurrences is N. Here N is like T and n_j is like T_x, so

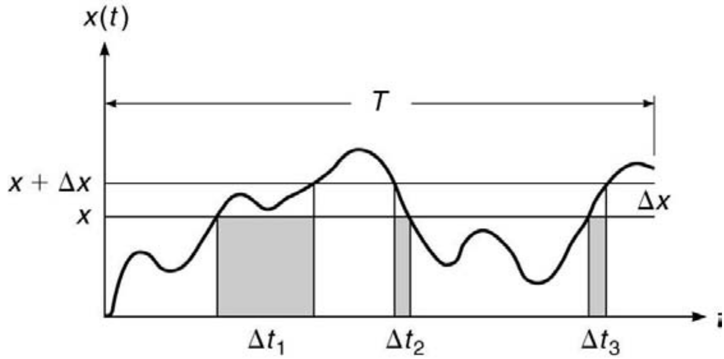

FIGURE 5.7
A time history record.

$$p(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\lim_{N \to \infty} \sum_{j=1}^{m} \frac{n_j}{N} \right] = \lim_{\Delta x \to 0, N \to \infty} \sum_{j=1}^{m} \left[\frac{n_j/N}{\Delta x} \right]. \tag{5.7}$$

Equations 5.6 and 5.7 show that the limit of the frequency density distribution is the probability density function.

The probability density function of a signal that repeats itself in time can be found by applying the aforementioned concepts. To determine the probability density function of this type of signal, the signal only needs to be examined over one period T. Equation 5.6 then becomes

$$p(x) = \frac{1}{T} \lim_{\Delta x \to 0} \frac{1}{\Delta x} \sum_{j=1}^{m} \Delta t_j. \tag{5.8}$$

Now as $\Delta x \to 0$, $\Delta t_j \to \Delta x \cdot |dt/dx|_j$. Thus, in the limit as $\Delta x \to 0$, Equation 5.8 becomes

$$p(x) = \frac{1}{T} \sum_{j=1}^{m} |dt/dx|_j, \tag{5.9}$$

noting that m is the number of times the signal is between x and $x + \Delta x$.

Example Problem 5.2

Statement: Determine the probability density function of the periodic signal $x(t) = x_o \sin(\omega t)$ with $\omega = 2\pi/T$.

Solution: Differentiation of the signal with respect to time yields

$$dx = x_o \omega \cos(\omega t) dt \;\text{ or }\; dt = dx/[x_o \omega \cos(\omega t)]. \tag{5.10}$$

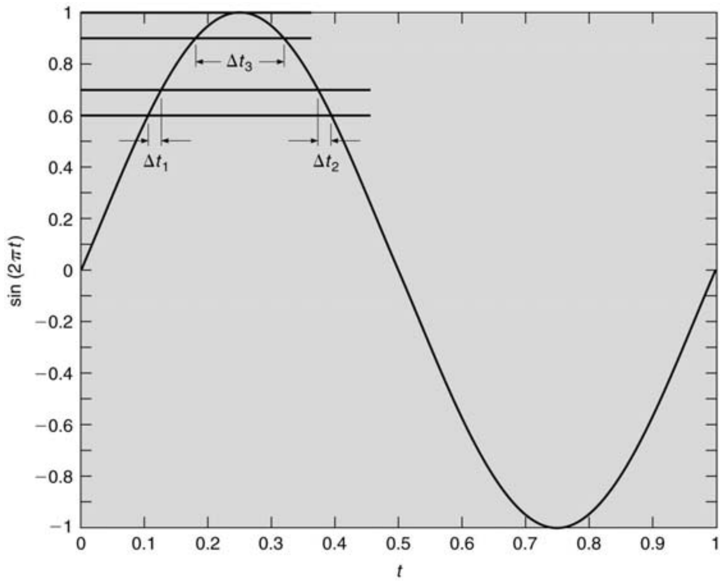

FIGURE 5.8
Constructing $p(x)$ from the time history record.

Now, the number of times that this particular signal resides during one period in the x to $x + \Delta x$ interval is 2. Thus, for this signal, using Equations 5.9 and 5.10, the probability density function becomes

$$p(x) = \frac{\omega}{2\pi} 2 | \frac{1}{x_o \omega \cos(\omega t)} | = | \frac{1}{\pi x_o \cos(\omega t)} |. \tag{5.11}$$

The probability density functions of other deterministic, continuous functions of time can be found using the same approach.

The probability density function also can be determined graphically by analyzing the time history of a signal in the following manner:

1. Given $x(t)$ and the sample period, T, choose an amplitude resolution Δx.
2. Determine T_x, then $T_x/(T\Delta x)$, noting also the mid-point value of x for each Δx interval.
3. Construct the probability density function by plotting $T_x/(T_x\Delta x)$ for each interval on the ordinate (y-axis) versus the mid-point value of x for that interval on the abscissa (x-axis).

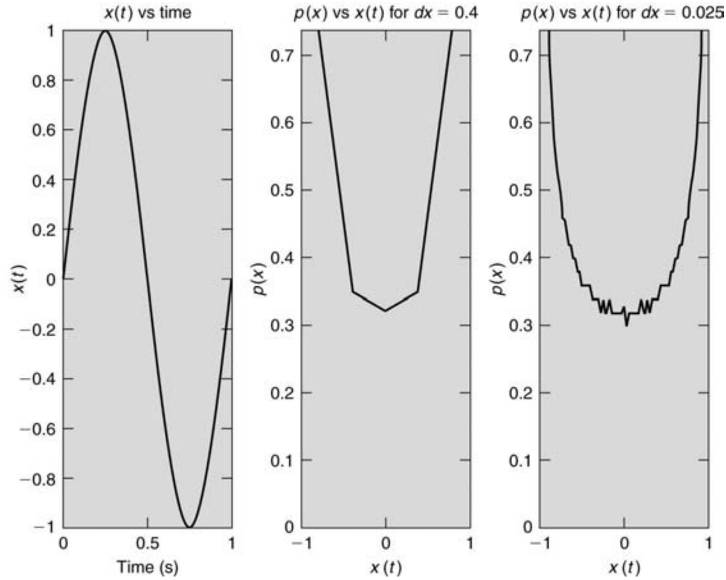

FIGURE 5.9
Approximations of $p(x)$ from $x(t) = \sin(2\pi t)$.

Note that the same procedure can be applied to examining the time history of a signal that does not repeat itself in time. By graphically determining the probability density function of a known periodic signal, it is easy to observe the effect of the amplitude resolution Δx on the resulting probability density function in relation to the exact probability density function.

Example Problem 5.3

Statement: Consider the signal $x(t) = x_o \sin(2\pi t/T)$. For simplicity, let $x_o = 1$ and $T = 1$, so $x = \sin(2\pi t)$. Describe how the probability density function could be determined graphically.

Solution: First choose $\Delta x = 0.10$. As illustrated in Figure 5.8, for the interval $0.60 < \sin(2\pi t) \leq 0.70$, $T_x = 0.020 + 0.020 = 0.040$, which yields $T_x/(T_x\Delta x) = 0.40$ for the mid-point value of $x = 0.65$. Likewise, for the interval $0.90 < \sin(2\pi t) \leq 1.00$, $T_x = 0.14$, which yields $T_x/(T\Delta x) = 1.40$ for the mid-point value of $x = 0.95$. Using this information gathered for all Δx intervals, an estimate of the probability density function can be made by plotting $T_x/(T/\Delta x)$ versus x.

One simple way of interpreting the underlying probability density function of a signal is to consider it as the projection of the density of the signal's amplitudes, as illustrated below in Figure 5.10 for the signal $x = \sin(2\pi t)$. The more frequently occurring values appear more dense as viewed along the horizontal axis.

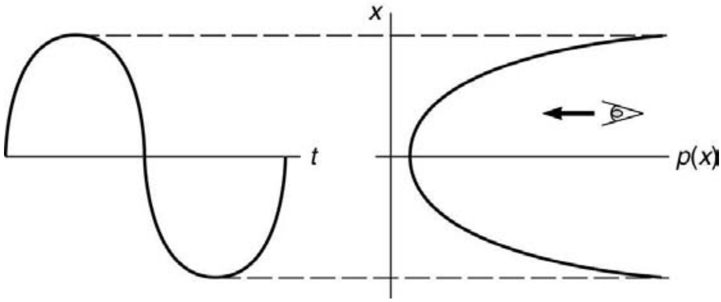

FIGURE 5.10
Projection of signal's amplitude densities.

In some situations, particularly when examining the time history of a random signal, the aforementioned procedure of measuring the time spent at each Δx interval to determine $p(x)$ becomes quite laborious. Recall that if $p(x)dx$ is known, the probability of occurrence of x for any range of x is given by

$$p(x)dx = \lim_{\Delta x \to 0} p(x)\Delta x = \lim_{T \to \infty} \left[\frac{T_x}{T} \right]. \tag{5.12}$$

Recognizing this, an alternative approach can be used to determine the quantity $p(x)dx$ by choosing a very small Δx, such as the thickness of a pencil line. If a horizontal line is moved along the amplitude axis at constant-amplitude increments and the number of times that the line crosses the signal is determined for each amplitude increment, C_x, then

$$p(x)dx = \lim_{C \to \infty} \left[\frac{C_x}{C} \right], \tag{5.13}$$

where C is a very large number. Note that $p(x)dx$ was determined by this approach, *not* $p(x)$, as was done previously.

5.6 Various Probability Density Functions

The concept of the probability density function was introduced in Chapter 5. There are many specific probability density functions. Each represents a different population that is characteristic of some physical process. In the following, a few of the more common ones will be examined.

Some of the probability density functions are for discrete processes (those having only discrete outcomes), such as the **binomial** probability density

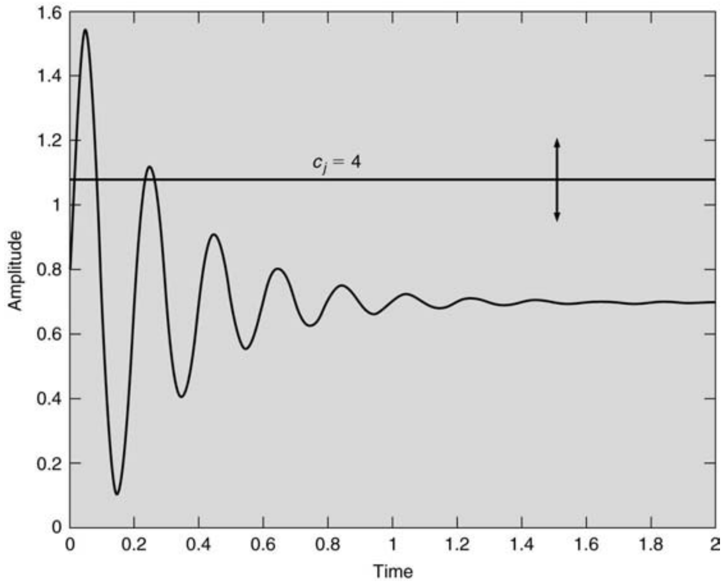

FIGURE 5.11
Graphical approach to determine $p(x)dx$.

function. This describes the probability of the number of successful outcomes, n, in N repeated trials, given that only either success (with probability P) or failure (with probability $Q = 1 - P$) is possible. The binomial probability density function, for example, describes the probability of obtaining a certain sum of the numbers on a pair of dice when tossed or the probability of getting a particular number of heads and tails for a series of coin tosses. The **Poisson** probability density function models the probability of rarely occurring events. It can be derived from the binomial probability density function. Two examples of processes that can be modeled by the Poisson probability density function are number of disintegrative emissions from an isotope and the number of micrometeoroid impacts on a spacecraft. Although the outcomes of these processes are discrete whole numbers, the process is considered continuous because of the very large number of events considered. This essentially amounts to possible outcomes that span a large, continuous range of whole numbers.

Other probability density functions are for continuous processes. The most common one is the **normal** (**Gaussian**) probability density function. Many situations closely follow a normal distribution, such as the times of runners finishing a marathon, the scores on an exam for a very large class, and the IQs of everyone without a college degree (or with one). The **Weibull** probability density function is used to determine the probability of fatigue-induced failure times for components. The **lognormal** probability density function is similar to the normal probability density function but considers its variable to be related to the logarithm of another variable. The diameters of raindrops are lognormally distributed, as are the populations of various biological systems. Most recently, scientists have suggested a new probability density function that can be used quite successfully to model the occurrence of clear-air turbulence and earthquakes. This probability density function is similar to the normal probability density function but is skewed to the left and has a larger tail to the right to account for the observed higher frequency of more rarely occurring events.

5.6.1 Binomial Distribution

Consider first the binomial distribution. In a *repeated trials* experiment consisting of N *independent* trials with a probability of success, P, for an individual trial, the probability of getting exactly n successes (for $n \leq N$) is given by the binomial probability density function

$$p(n) = \left[\frac{N!}{(N-n)!n!}\right] P^n (1-P)^{N-n}. \tag{5.14}$$

The mean, $\bar{n}$, and the variance, σ^2, are NP and NPQ, respectively, where Q is the probability of failure, which equals $1 - P$. The higher-order central moments of the skewness and kurtosis are $(Q-P)/(NPQ)^{0.5}$ and $3 + [(1 - 6PQ)/NPQ]$, respectively.

As shown in Figure 5.12, for a fixed N, as P becomes larger, the probability density function becomes skewed more to the right. For a fixed P, as N becomes larger, the probability density function becomes more symmetric. Tending to the limit of large N and small but finite P, the probability density function approaches a normal one. The MATLAB M-file `bipdfs.m` was used to generate this figure based upon the `binopdf(n,N,P)` command.

Example Problem 5.4

Statement: Suppose five students are taking a probability course. Typically, only 75 % of the students pass this course. Determine the probabilities that exactly 0, 1, 2, 3, 4, or 5 students will pass the course.

Solution: These probabilities are calculated using Equation 5.14, where $N = 5$, $P = 0.75$, and $n = 0$, 1, 2, 3, 4, and 5. They are displayed immediately below and plotted in Figure 5.13.

$p(0)$	=	$1 \times 0.75^0 \times 0.25^5$	=	0.0010
$p(1)$	=	$5 \times 0.75^1 \times 0.25^4$	=	0.0146
$p(2)$	=	$10 \times 0.75^2 \times 0.25^3$	=	0.0879
$p(3)$	=	$10 \times 0.75^3 \times 0.25^2$	=	0.2637
$p(4)$	=	$5 \times 0.75^4 \times 0.25^1$	=	0.3955
$p(5)$	=	$1 \times 0.75^5 \times 0.25^0$	=	0.2373
		sum	=	1.0000

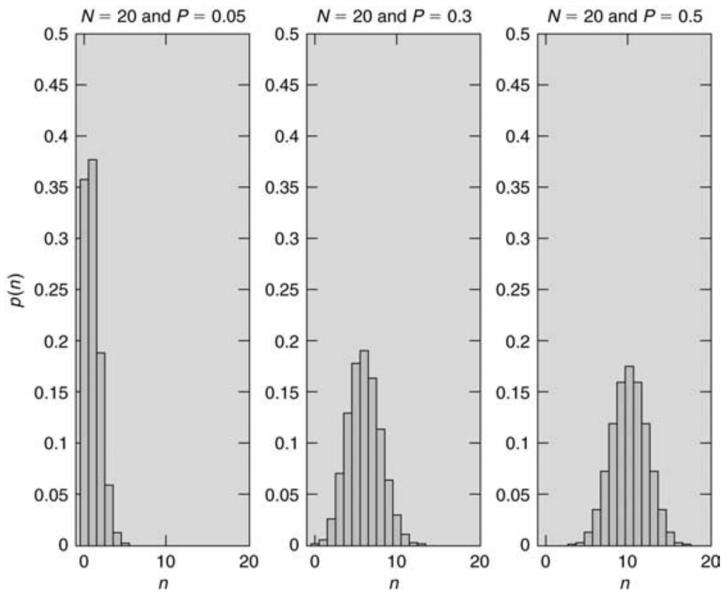

FIGURE 5.12
Binomial probability density functions for various N and P.

5.6.2 Poisson Distribution

Next consider the Poisson distribution. In the limit when N becomes very large and P becomes very small (close to zero, which implies a rare event) in such a way that the mean ($= NP$) remains finite, the binomial probability density function very closely approximates the Poisson probability density function.

For these conditions, the Poisson probability density function allows us to determine the probability of n rare event successes (occurrences) out of a large number of N repeated trial experiments (during a series of N time intervals) with the probability P of an event success (during a time interval) as given by the probability density function

$$p(n) = \frac{(NP)^n}{n!} e^{-NP}. \tag{5.15}$$

The MATLAB command `poisspdf(n,N*P)` can be used to calculate the probabilities given by Equation 5.15. The mean and variance both equal NP, noting $(1 - P) \approx 1$. The skewness and kurtosis are $(NP)^{-0.5}$ and $3 + 1/NP$, respectively. As NP is increased, the Poisson probability density function approaches a normal probability density function.

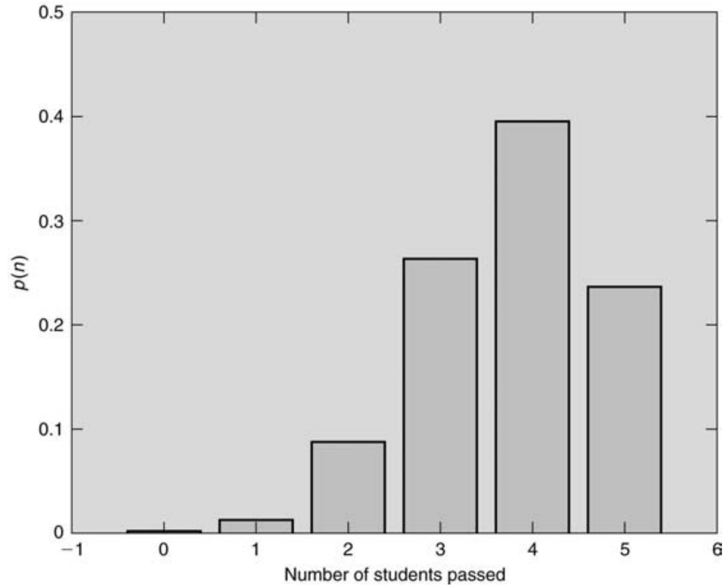

FIGURE 5.13
Probabilities for various numbers of students passed.

Example Problem 5.5

Statement: There are 2×10^{-20} α particles per second emitted from the nucleus of an isotope. This implies that the probability for an emission from a nucleus to occur in one second is 2×10^{-20}. Assume that the total material to be observed is comprised of 10^{20} atoms. Emissions from the material are observed at one-second intervals. Determine the resulting probabilities that a total of $0, 1, 2, ..., 8$ emissions occur in the interval.

Solution: The probabilities are calculated using Equation 5.15, where $N = 10^{20}$, $P = 2 \times 10^{-20}$ and $n = 0$ through 8. The results are

$p(0)$	=	0.135
$p(1)$	=	0.271
$p(2)$	=	0.271
$p(3)$	=	0.180
$p(4)$	=	0.090
$p(5)$	=	0.036
$p(6)$	=	0.012
$p(7)$	=	0.003
$p(8)$	=	0.001
sum	=	0.999

These results are displayed in Figure 5.14.

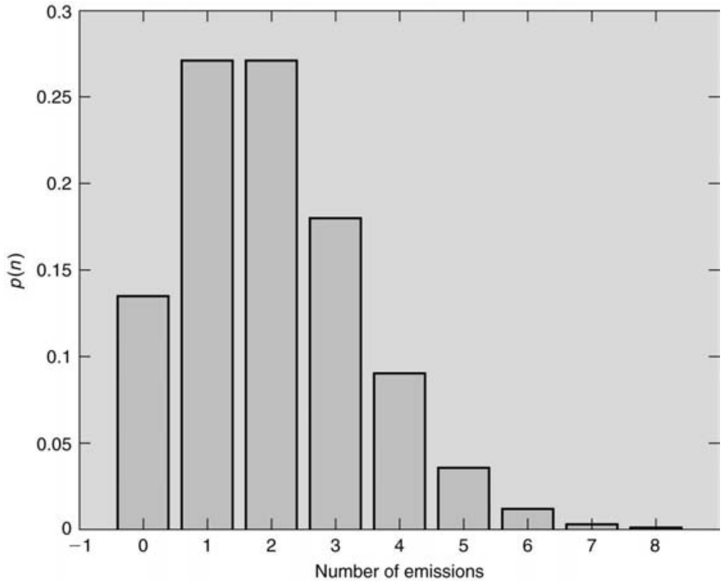

FIGURE 5.14
Poisson example of isotope emissions.

5.7 Central Moments

Once the probability density function of a signal has been determined, this information can be used to determine the values of various parameters. These parameters can be found by computing the **central moments** of the probability density function. Computations of statistical moments are similar to those performed to determine mechanical moments, such as the moment of inertia of an object. The term *central* refers to the fact that the various statistical moments are computed with respect to the centroid or mean of the probability density function of the population. The **m-th central moment** is defined as

$$\langle (x - x')^m \rangle = E\left[(x - x')^m\right] = \mu_m \equiv \int_{-\infty}^{+\infty} (x - x')^m p(x)dx. \quad (5.16)$$

Either $\langle\ \rangle$ or $E[\,]$ denotes the **expected value** or **expectation** of the quantity inside the brackets. This is the value that is expected (in the probabilistic sense) if the integral is performed.

When the centroid or mean, x', equals 0, the central moments are known as moments about the origin. Equation 5.16 becomes

$$\langle x^m \rangle \equiv \int_{-\infty}^{+\infty} x^m p(x) dx = \mu'_m. \tag{5.17}$$

Further, the central moment can be related to the moment about the origin by the transformation

$$\mu_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \mu_1^i \mu'_{m-i} \text{ where } \binom{m}{i} = \frac{m!}{i!(m-i)!}. \tag{5.18}$$

The **zeroth central moment**, μ_o, is an identity

$$\mu_o = \int_{-\infty}^{+\infty} p(x) dx = 1. \tag{5.19}$$

Having $\mu_o = 1$ assures that $p(x)$ is normalized correctly.

The **first central moment**, μ_1, leads to the definition of the **mean** value (the centroid of the distribution). For $m = 1$,

$$\left\langle (x - x')^1 \right\rangle = \int_{-\infty}^{+\infty} (x - x') p(x) dx. \tag{5.20}$$

Expanding the left side of Equation 5.20 yields

$$\langle (x - x') \rangle = \langle x \rangle - \langle x' \rangle = \langle x \rangle - x' = 0, \tag{5.21}$$

because the expectation of x, $\langle x \rangle$, is the true mean value of the population, x'. Hence, $\mu_1 = 0$. Now expanding the right side of Equation 5.20 reveals that

$$\int_{-\infty}^{+\infty} (x - x') p(x) dx = \int_{-\infty}^{+\infty} x p(x) dx - x' \int_{-\infty}^{+\infty} p(x) dx = \int_{-\infty}^{+\infty} x p(x) dx - x'. \tag{5.22}$$

Because the right side of the equation must equal zero, it follows that

$$x' = \int_{-\infty}^{+\infty} x p(x) dx. \tag{5.23}$$

Equation 5.23 is used to compute the mean value of a distribution given its probability density function.

The **second central moment**, μ_2, defines the **variance**, σ^2, as

$$\mu_2 = \int_{-\infty}^{+\infty} (x - x')^2 p(x) dx = \sigma^2, \tag{5.24}$$

which has units of x^2. The **standard deviation**, σ, is the square root of the variance. It describes the width of the probability density function.

The variance of x can be expressed in terms of the expectation of x^2, $E[x^2]$, and the square of the mean of x, x'^2. Equation 5.16 for this case becomes

$$\begin{aligned} \sigma^2 &= E[(x - x')^2] \\ &= E[x^2 - 2xx' + x'^2] \\ &= E[x^2] - 2x'E[x] + x'^2 \\ &= E[x^2] - 2x'x' + x'^2 \\ &= E[x^2] - x'^2. \end{aligned} \tag{5.25}$$

So, when the mean of x equals 0, the variance of x equals the expectation of x^2. Further, if the mean and variance of x are known, then $E[x^2]$ can be computed directly from Equation 5.25.

Example Problem 5.6

Statement: Determine the *mean* power dissipated by a 2 Ω resistor in a circuit when the current flowing through the resistor has a mean value of 3 A and a variance of 0.4 A^2.

Solution: The power dissipated by the resistor is given by $P = I^2R$, where I is the current and R is the resistance. The mean power dissipated is expressed as $E[P]$. Assuming that R is constant, $E[P] = E[I^2R] = RE[I^2]$. Further, using Equation 5.25, $E[I^2] = \sigma_I^2 + I'^2$. So, $E[P] = R(\sigma_I^2 + I'^2) = 2(0.4 + 3^2) = 18.8$ W. Expressed with the correct number of significant figures (one), the answer is 20 W.

The **third central moment**, μ_3, is used in the definition of the **skewness**, Sk, where

$$Sk = \frac{\mu_3}{\sigma^3} = \frac{1}{\sigma^3}\int_{-\infty}^{+\infty} (x - x')^3 p(x)dx. \tag{5.26}$$

Defined in this manner, the skewness has no units. It describes the symmetry of the probability density function, where a positive skewness implies that the distribution is skewed or stretched to the right. This is shown in Figure 5.15. When the distribution has positive skewness, the probability density function's mean is greater than its mode, where the **mode** is the most frequently occurring value. A negative skewness implies the opposite (stretched to the left with mean $<$ mode). The sign of the mean minus the mode is the sign of the skewness. For the normal distribution, $Sk = 0$ because the mean equals the mode.

The **fourth central moment**, μ_4, is used in the definition of the **kurtosis**, Ku, where

$$Ku = \frac{\mu_4}{\sigma^4} = \frac{1}{\sigma^4}\int_{-\infty}^{+\infty} (x - x')^4 p(x)dx, \tag{5.27}$$

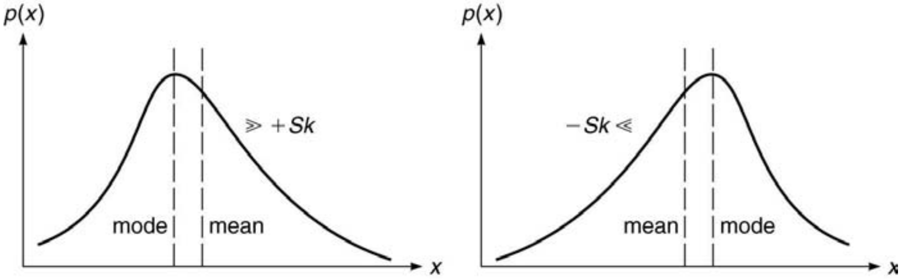

FIGURE 5.15
Distributions with positive and negative skewness.

which has no units. The kurtosis describes the peakedness of the probability density function. A *leptokurtic* probability density function has a slender peak, a *mesokurtic* one a middle peak, and a *platykurtic* one a flat peak.

For the normal distribution, $Ku = 3$. Sometimes, another expression is used for the kurtosis, where $Ku^* = Ku - 3$ such that $Ku^* < 0$ implies a probability density function that is flatter than the normal probability density function, and $Ku^* > 0$ implies one that is more peaked than the normal probability density function.

For the special case in which x is a normally distributed random variable, the m-th central moments can be written in terms of the standard deviation, where $\mu_m = 0$ when m is odd and > 1, and $\mu_m = 1 \cdot 3 \cdot 5 \cdots (m-1)\sigma^m$ when m is even and > 1. This formulation obviously is useful in determining higher-order central moments of a normally distributed variable when the standard deviation is known.

5.8 Probability Distribution Function

The probability that a value x is less than or equal to some value of x^* is defined by the **probability distribution function**, $P(x)$. Sometimes this also is referred to as the cumulative probability distribution function. The probability distribution function is expressed in terms of the integral of the probability density function

$$P(x^*) = Pr[x \leq x^*] = \int_{-\infty}^{x^*} p(x)dx. \tag{5.28}$$

From this, the probability that a value of x will be between the values of x_1^* and x_2^* becomes

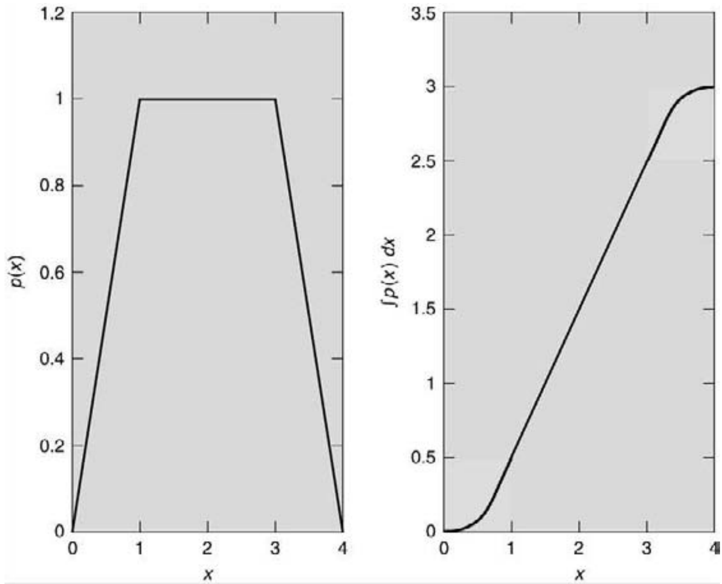

FIGURE 5.16
Example probability density (left) and probability distribution (right) functions.

$$Pr[x_1^* \leq x \leq x_2^*] = P(x_2^*) - P(x_1^*) = \int_{x_1^*}^{x_2^*} p(x)dx. \tag{5.29}$$

Note that $p(x)$ is dimensionless. The units of $P(x)$ are those of x.

Example plots of $p(x)$ and $P(x)$ are shown in Figure 5.16. The probability density function is in the left figure and the probability distribution function in the right figure. The $P(x)$ values for each x value are determined simply by finding the area under the $p(x)$ curve up to each x value. For example, the area at the value of $x = 1$ is a triangular area equal to $0.5 \times 1.0 \times 1.0 = 0.5$. This is the corresponding value of the probability distribution function at $x = 1$. Note that the probability density in this example is not normalized. The final value of the probability distribution function does not equal unity. What does the maximum value of the probability density function have to be for the probability density function to be normalized correctly, as described in Section 5.7? The answer is unity. So, to normalize $p(x)$ correctly, all values of $p(x)$ should be divided by 1/3 in this example.

5.9 *Probability Concepts

The concept of the probability of an occurrence or outcome is intuitive. Consider the toss of a fair die. The probability of getting any one of the six possible numbers is 1/6. Formally this is written as $Pr[A] = 1/6$ or approximately 17 %, where the A denotes the occurrence of any one specific number. The **probability** of an occurrence can be defined as the number of times of the occurrence divided by the total number of times considered (the times of the occurrence plus the times of no occurrence). If the probabilities of getting 1 or 2 or 3 or 4 or 5 or 6 on a single toss are added, the result is $Pr[1]+Pr[2]+Pr[3]+Pr[4]+Pr[5]+Pr[6] = 6(1/6) = 1$. That is, the sum of all of the possible probabilities is unity. A probability of 1 implies absolute certainty; a probability of 0 implies absolute uncertainty or impossibility.

Now consider several tosses of a die and, based upon these results, determine the probability of getting a specific number. Each toss results in an **outcome**. The tosses when the specific number occurred comprise the **set** of occurrences for the **event** of getting that specific number. The tosses in which the specific number did *not* occur comprise the **null set** or **complement** of that event. Remember, the event for this situation is not a single die toss, but rather all the tosses in which the specific number occurred. Suppose, for example, the die was tossed eight times, obtaining eight outcomes: 1, 3, 1, 5, 5, 4, 6, and 2. The probability of getting a 5 based upon these outcomes would be $Pr[5] = 2/8 = 1/4$. That is, two of the eight possible outcomes comprise the set of events where 5 is obtained. The probability of the event of getting a 3 would be $Pr[3] = 1/8$. These results do not imply necessarily that the die is unfair, rather, that the die has not been tossed enough times to assess its fairness. This subject was considered briefly (see Equation 5.2). It addresses the question of how many measurements need to be taken to achieve a certain level of confidence in an experiment.

5.9.1 *Union and Intersection of Sets

Computing probabilities in the manner just described is correct provided the one event that is considered has nothing in common with the other events. Continuing to use the previous example of eight die tosses, determine the probability that either an even number or the numbers 3 or 4 occur. There is $Pr[\text{even}] = 3/8$ (from 4, 6, and 2) and $Pr[3] = 1/8$ and $Pr[4] = 1/8$. Adding the probabilities, the sum equals 5/8. Inspection of the results, however, shows that the probability is 1/2 (from 3, 4, 6, and 2). Clearly the method of simply adding these probabilities for this type of situation is not correct.

To handle the more complex situation when events have members in common, the union of the various sets of events must be considered, as illustrated in Figure 5.17. The lined triangular region marks the set of events

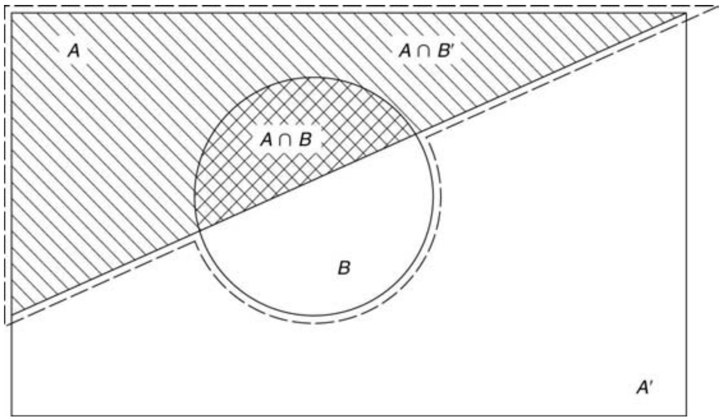

FIGURE 5.17
The union and the intersection of the sets A and B.

A and the circular region the set of events B. The complement of A is denoted by A'. The sample space is **exhaustive** because A and A' comprise the entire sample space. For two sets A and B, the **union** is the set of all members of A *or* B *or* both, as denoted by the region bordered by the dashed line. This is written as $Pr[A \cup B]$.

If the sets of A and B are **mutually exclusive** where they do not share any common members, then

$$Pr[A \cup B] = Pr[A] + Pr[B]. \tag{5.30}$$

This would be the case if the sets A and B did not overlap in the figure (if the circular region was outside the triangular region). Thus, the probability of getting 3 or 4 in the eight-toss experiment is 1/4 (from 3 and 4).

If the sets do overlap and have common members, as shown by the cross-hatched region in the figure, then

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B], \tag{5.31}$$

where $Pr[A \cap B]$ is the probability of the intersection of A and B. The **intersection** is the set of all members in *both* A and B. So, the correct way to compute the desired probability is $Pr[\text{even}\cup 3\cup 4] = Pr[\text{even}] + Pr[3] + Pr[4] - Pr[\text{even}\cap 3\cap 4] = 3/8 + 1/8 + 1/8 - 1/8 = 1/2$. $Pr[\text{even}\cap 3\cap 4] = 1/8$, because only one common member, 4, occurred during the eight tosses.

5.9.2 *Conditional Probability

The moment questions are asked such as, "What is the chance of getting a 4 on the second toss of a die given either a 1 or a 3 was rolled on the first toss?", more thought is required to answer them. This is a problem in **conditional probability**, the probability of an event given that specified events have occurred in the past. This concept can be formalized by determining the probability that event B occurs given that event A occurred previously. This is written as $Pr[B \mid A]$, where

$$Pr[B \mid A] \equiv \frac{Pr[B \cap A]}{Pr[A]}. \tag{5.32}$$

Rearranging this definition gives

$$Pr[B \cap A] = Pr[B \mid A]Pr[A]. \tag{5.33}$$

This is known as the *multiplication rule* of conditional probability. Further, because $Pr[A \cap B] = Pr[B \cap A]$, Equation 5.33 implies that

$$Pr[B \mid A]Pr[A] = Pr[A \mid B]Pr[B]. \tag{5.34}$$

Examination of Figure 5.17 further reveals that

$$Pr[A] = Pr[A \cap B] + Pr[A \cap B'], \tag{5.35}$$

where $A \cap B'$ is shown as the lined region and B' means *not* B. Using the converse of Equation 5.33, Equation 5.35 becomes

$$Pr[A] = Pr[A \mid B]Pr[B] + Pr[A \mid B']Pr[B']. \tag{5.36}$$

Equation 5.36 is known as the *total probability rule* of conditional probability. It can be extended to represent more than two mutually exclusive and exhaustive events [6].

When events A and B are mutually exclusive, then $Pr[B \mid A] = 0$, which leads to $Pr[B \cap A] = 0$. When event B is **independent** of event A, the outcome of event A has no influence on the outcome of event B. Then

$$Pr[B \mid A] = Pr[B]. \tag{5.37}$$

Thus, for independent events, it follows from Equations 5.33 and 5.37 that

$$Pr[B \cap A] = Pr[B]Pr[A]. \tag{5.38}$$

That is, the conditional probability of a series of independent events is the *product* of the individual probabilities of each of the events. Hence, to answer the question posed at the beginning of this section, $Pr[B \cap A] = Pr[A]$ $Pr[B] = (1/6)(2/6) = 2/36$. That is, there is approximately a 6 % chance that either an even number or the numbers 3 or 4 occurred.

The chance of winning the lottery can be determined easily using this information. The results suggest not to bet in lotteries. Assume that four balls are drawn from a bin containing white balls numbered 1 through 49, and then a fifth ball is drawn from another bin containing red balls with the same numbering scheme. Because each number selection is independent of the other number selections, the probability of guessing the numbers on the four white balls correctly would be $Pr[1]\ Pr[2]\ Pr[3]\ Pr[4] = (1/49)(1/48)(1/47)(1/46) = 1/(5\ 085\ 024)$. The probability of guessing the numbers on all five balls correctly would be $Pr[1]\ Pr[2]\ Pr[3]\ Pr[4]\ Pr[5] = [1/(5\ 085\ 024)](1/49) = 1/(249\ 166\ 176)$ or about 1 chance in 250 million! That chance is about equivalent to tossing a coin and getting 28 heads in a row. Recognizing that the probability that an event will *not* occur equals one minus the probability that it will occur, the chance of *not* winning the lottery is 99.999 999 6 %. So, it is very close to impossible to win the lottery.

Other useful conditional probability relations can be developed. Using Equation 5.34 and its converse, noting that $Pr[A \cap B] = Pr[B \cap A]$,

$$Pr[B \mid A] = \frac{Pr[B]Pr[A \mid B]}{Pr[A]}. \tag{5.39}$$

This relation allows us to determine the probability of event B occurring given that event A has occurred from the probability of event A occurring given that event B has occurred.

Example Problem 5.7

Statement: Determine the probability that the temperature, T, will exceed 100 °F in a storage tank whenever the pressure, p, exceeds 2 atmospheres. Assume the probability that the pressure exceeds 2 atmospheres whenever the temperature exceeds 100 °F is 0.30, the probability of the pressure exceeding 2 atmospheres is 0.10 and the probability of the temperature exceeding 100 °F is 0.25.

Solution: Formally, $Pr[p > 2 \mid T > 100] = 0.30$, $Pr[p¿2] = 0.10$ and $Pr[T > 100] = 0.25$. Using Equation 5.39, $Pr[T > 100 \mid p > 2] = (0.25)(0.30)/(0.10) = 0.75$. That is, whenever the pressure exceeds 2 atmospheres there is a 75 % chance that the temperature will exceed 100 °F.

Equation 5.36 can be substituted into Equation 5.39 to yield

$$Pr[B \mid A] = \frac{Pr[A \mid B]Pr[B]}{Pr[A \mid B]Pr[B] + Pr[A \mid B']Pr[B']}. \tag{5.40}$$

This equation expresses what is known as *Bayes' rule*, which is valid for mutually exclusive and exhaustive events. It was discovered accidentally by the Reverend Thomas Bayes (1701-1761) while manipulating formulas for conditional probability [1]. Its power lies in the fact that it allows one to calculate probabilities inversely, to determine the probability of a *before* event conditional upon an *after* event. This equation can be extended to represent more than two events [6]. Bayes' rule can be used to solve many practical problems in conditional probability. For example, the probability that a component identified as defective by a test of known accuracy actually is defective can be determined knowing the percentage of defective components in the population.

Before determining this probability specifically, examine these types of probabilities in a more general sense. Assume that the probability of an event occurring is p_1. A test can be performed to determine whether or not the event has occurred. This test has an accuracy of $100p_2$ percent. That is, it is correct $100p_2$ percent of the time and incorrect $100(1-p_2)$ percent of the time. What is the percent probability that the test can predict an actual event? There are four possible situations that can arise: (1) the test indicates that the event has occurred *and* the event actually has occurred (a true positive), (2) the test indicates that the event has occurred *and* the event actually has not occurred (a false positive), (3) the test indicates that the event has not occurred *and* the event actually has occurred (a false negative), and (4) the test indicates that the event has not occurred *and* the event actually has not occurred (a true negative). Here, the terms positive and negative refer to the *indicated* occurrence of the event and the terms true and false refer to whether or not the indicated occurrence agrees with the *actual* occurrence of the event. So, the probabilities of the four possible combinations of events are p_2p_1 for true positive, $(1-p_2)(1-p_1)$ for false positive, $(1-p_2)p_1$ for false negative, and $p_2(1-p_1)$ for true negative. Now the probability that an event actually occurred given that a test indicated that it had occurred would be the ratio of the actual probability of occurrence (the true positive probability) to the sum of all indicated positive occurrences (the true positive plus the false positive probabilities). That is,

$$Pr[A \mid IA] = \frac{p_2p_1}{p_2p_1 + (1-p_2)(1-p_1)}, \tag{5.41}$$

where IA denotes the event of an *indicated* occurrence of A and A symbolizes the event of an *actual* occurrence.

Alternatively, Equation 5.41 can be derived directly using Bayes' rule. Here,

$$Pr[A \mid IA] = \frac{Pr[IA \mid A]Pr[A]}{Pr[IA \mid A]Pr[A] + Pr[IA \mid A']Pr[A']}, \tag{5.42}$$

which is identical to Equation 5.41 because $\Pr[IA \mid A] = p_2$, $\Pr[A] = p_1$, $\Pr[IA \mid A'] = (1 - p_2)$ and $\Pr[A'] = (1 - p_1)$.

Example Problem 5.8

Statement: An experimental technique is being developed to detect the removal of a microparticle from the surface of a wind tunnel wall as the result of a turbulent sweep event. Assume that a sweep event occurs 14 % of the time when the detection scheme is operated. The experimental technique can detect a sweep event correctly 73 % of the time. [a] What is the probability that a sweep event will be detected during the time period of operation? [b] What is the probability that a sweep event will be detected if the experimental technique is correct 90 % of the time?

Solution: The desired probability is the ratio of true positive identifications to true positive plus false positive identifications. Let p_1 be the probability of an actual sweep event occurrence during the time period of operation and p_2 be the experimental technique's reliability (the probability to identify correctly). For this problem, $p_1 = 0.14$ and $p_2 = 0.73$ for part [a] and $p_2 = 0.90$ for part [b]. Substitution of these values into Equation 5.41 yields $P = 0.31$ for part [a] and $P = 0.59$ for part [b]. First, note that the probability that a sweep event will be detected is only 31 % with a 73 % reliability of the experimental technique. This in part is because of the relatively low percentage of sweep event occurrences during the period of operation. Second, an increase in the technique's reliability from 73 % to 90 %, or by 17 %, increases the probability from 31 % to 59 %, or by 28 %. An increase in technique reliability increases the probability of correct detection relatively by a greater amount.

Example Problem 5.9

Statement: Suppose that 4 % of all transistors manufactured at a certain plant are defective. A test to identify a defective transistor is 97 % accurate. What is the probability that a transistor identified as defective actually is defective?

Solution: Let event A denote that the transistor actually is defective and event B that the transistor is indicated as defective. What is $Pr[A \mid B]$? It is known that $Pr[A] = 0.04$ and $Pr[B \mid A] = 0.97$. It follows that $Pr[A'] = 1 - Pr[A] = 0.96$ and $\Pr[B \mid A'] = 1 - Pr[B \mid A] = 0.03$ because the set of all possible events are mutually exclusive and exhaustive. Direct application of Bayes' rule gives

$$Pr[A \mid B] = \frac{(0.97)(0.04)}{(0.97)(0.04) + (0.03)(0.96)} = 0.57.$$

So, there is a 57 % chance that a transistor identified as defective actually is defective. At first glance, this percentage seems low. Intuitively, the value would be expected to be closer to the accuracy of the test (97 %). However, this is not the case. In fact, to achieve a 99 % chance of correctly identifying a defective transistor, the test would have to be 99.96 % accurate!

It is important to note that the way statistics are presented, either in the form of probabilities, percentages, or absolute frequencies, makes a noticeable difference to some people in arriving at the correct result. Studies [6] have shown that when statistics are expressed as frequencies, a far greater number of people arrive at the correct result. The previous problem can be solved again by using an alternative approach [6].

Example Problem 5.10

Statement: Suppose that 4 % of all transistors manufactured at a certain plant are defective. A test to identify a defective transistor is 97 % accurate. What is the probability that a transistor identified as defective actually is defective?

Solution:

- Step 1: Determine the base rate of the population, which is the fraction of defective transistors at the plant (0.04).
- Step 2: Using the test's accuracy and the results of the first step, determine the fraction of defective transistors that are identified by the test to be defective ($0.04 \times 0.97 = 0.04$).
- Step 3: Using the fraction of good transistors in the population and the test's false-positive rate ($1 - 0.97 = 0.03$), determine the fraction of good transistors that are identified by the test to be defective ($0.96 \times 0.03 = 0.03$).

- Step 4: Determine the desired probability, which is 100 times the fraction in step 2 divided by the sum of the fractions in steps 2 and 3 (0.04 / [0.04 + 0.03] = 0.57) or 57 %.

Which approach is easier to understand?

5.9.3 *Coincidences

Conditional probability can be used to explain what appear to be rare coincidences. My wife and I took a tour of Scotland in 1999 along with five other people whom we had never met before. When we boarded the tour bus in Scotland we were astounded to find out that five out of the seven of us lived in Indiana! Our reaction was common – what a rare coincidence, especially because only about one out of every 1000 people in the world (approximately 0.1 %) live in Indiana and about 3/4 of us on the bus were from Indiana. But then I started to think and ask questions. It turns out that the United Kingdom is a very popular vacation spot for people from the midwestern and eastern United States and that a commercial airline was having a special offer that included a flight and tour of Scotland for those flying out of Chicago's O'Hare and New York City's Kennedy airports (all of those on the bus lived near Chicago or New York City). Granted these conditions do not explain why five people from Indiana versus another midwestern or eastern state were there, but they do make this coincidence much more probable and certainly not rare.

As remarked by Stewart [2], "Because we notice coincidences and ignore noncoincidences, we make coincidences seem more significant than they really are." In fact, even today many people still attribute the occurrences of apparently rarely occurring events to mysterious causes. Perhaps it is easier to believe in an inexplicable cause than to identify the conditions under which the event occurred. Most likely, such occurrences are not so rare after all.

5.9.4 *Permutations and Combinations

The probability of an event can be determined knowing the number of occurrences of the event and the total number of occurrences of all possible events. Finding the number of all possible events can sometimes be confusing. Consider an experiment in which there are three possible occurrences denoted by a, b, and c, each of which can occur only once without replacement. What is the probability of getting c, then a, then b, which is $Pr[cab]$? To determine this probability, the total number of ways that a, b, and c can be arranged *respective* of their order must be known. That is, the number of permutations of a, b, and c must be determined. The number of **permutations** of n objects is

$$n! = n(n-1)(n-2)...1, \tag{5.43}$$

where $n!$ is called n **factorial**. Stirling's formula is sometimes useful, where $n! \simeq \sqrt{2\pi n}(n/\exp(n))^n$, which agrees with Equation 5.43 within 1 % for $n > 9$. So, there are six possible ways (*abc*, *acb*, *bca*, *bac*, *cab*, and *cba*) to arrange a, b, and c. Thus, $Pr[cab] = 1/6$.

Now what if the experiment had four possible occurrences, a, b, c, and d, and $Pr[cab]$ needs to be determined? The number of permutations of n objects taken m at a time, P^n_m, is

$$P^n_m = \frac{n!}{(n-m)!} = n(n-1)(n-2)...(n-m+1). \tag{5.44}$$

So, there are 4!/(4-3)! or 24 possible ways to arrange three of the four possible occurrences. Thus, Pr[*cab*] = 1/24. The probability of getting c, then a, then b is reduced from 1/6 to 1/24 when the possibility of a fourth occurrence is introduced. Often it is easy to calculate the number of permutations using a spread sheet program. For example, using Microsoft EXCEL, the value of Equation 5.44 is given by the command `PERMUT(n,m)`.

Further consider the same experiment, but where the number of possible combinations of c, a, and b are determined *irrespective* of the order. That is, the number of **combinations** of n objects taken m at a time, C^n_m, given by

$$C^n_m = \frac{n!}{m!(n-m)!} = \frac{n(n-1)(n-2)...(n-m+1)}{m!}, \tag{5.45}$$

must be found. There are only 4!/(3!1!) or four possible combinations of three out of four possible occurrences (*abc*, *abd*, *cdb*, and *cda*). So, the $Pr[cab]$ = 1/4 for this case. That is, there is a ten-time greater chance (25 % versus 2.5 %) of getting a, b, and c in any order versus getting the particular order of c, a, and b. Using Microsoft EXCEL, the value of Equation 5.45 is given by the command `COMBIN(n,m)`.

Finally, if there is repetition with replacement, then the number of possible combinations of n objects taken m at a time with **replacement**, $C^n_{m(r)}$, is

$$C^n_{m(r)} = \frac{(m+n-1)!}{(m!)(n-1)!}. \tag{5.46}$$

For our experiment, Equation 5.46 gives 6!/(3!3!) or 20 possible combinations of three out of four possible occurrences. This leads to $Pr[cab] = 1/20$. Clearly, when there is repetition, the number of possible combinations increases.

5.9.5 *Birthday Problems

There are two classic birthday problems that challenge one's ability to compute the probability of an occurrence [1]. The first is to determine the probability of at least two out of n people having the same birth date of the year (same day and month but not necessarily the same year). The second is to determine the probability of at least two out of n people having a *specific* birth date, such as January 18th. For simplicity, assume that there are 365 days per year and that the probability of having a birthday on any day of the year is the same (both assumptions are, in fact, not true).

Consider the first problem. Often it is easier first to compute the probability that an event will *not* occur and then subtract that probability from unity to obtain the probability that the event will occur. For the *second* person there is a probability of 364/365 (= $[366 - n]/365$ where $n = 2$) of *not* having the same birth date of the year as the first person. For the third person it is (364/365)(363/365) of *not* having the same birth date of the year as the first person *or* the second person. Here each event is independent of the other, so the joint probability is the product of the two. Continuing this logic, the probability, Q, of n people *not* having the same birth date of the year is

$$Q = \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{(366 - n)}{365}. \tag{5.47}$$

With a little algebra and using the definition of the factorial, Equation 5.47 can be rewritten as

$$Q = \frac{1}{365^n} \frac{365!}{(365 - n)!} = \frac{1}{365^n} P_n^{365}. \tag{5.48}$$

Thus, the probability, P, of at least two of n people having the same birth date of the year is

$$P = 1 - Q = 1 - \frac{1}{365^n} P_n^{365}. \tag{5.49}$$

For $n = 10$, Equation 5.49 gives $P = 11.69$ % and for $n = 50$, $P = 97.04$ %. How many people have to be in a room to have a greater than 50 % chance of at least two people having the same birth date of the year? The answer is at least 23 people ($P = 50.73$ %).

Now examine the second problem, which differs from the first problem in that a specific birth date is specified. The probability of the second of n people not having that specific birth date is (364/365) and the probability of the third of n people not having that specific birth date is the same, and so on. Thus, the probability of n people *not* having a specific birth date is

$$Q = \left[\frac{364}{365}\right]^{n-1}. \tag{5.50}$$

The probability of at least two of n people having a *specific* birth date of the year is

$$P = 1 - Q = 1 - \left[\frac{364}{365}\right]^{n-1}. \tag{5.51}$$

For $n = 10$, Equation 5.51 gives $P = 2.44$ % and for $n = 50$, $P = 12.58$ %. So how many people have to be in a room to have a greater than 50 % chance of at least two people having a specific birth date of the year? The answer is at least 254 people ($P = 50.05$ %).

5.10 Problem Topic Summary

Topic	Review Problems	Homework Problems
Basic Probability	1, 2, 3, 11, 14, 15	1
Conditional Probability	10, 16, 17, 18	2, 3, 10
Moments	6, 7, 8, 9, 13	7, 9
Displaying Probabilities	4, 5, 12	4, 5, 6, 8

TABLE 5.2
Chapter 5 Problem Summary

5.11 Review Problems

1. Assuming equal probability of being born any day of the year, match each of the following birthday occurrence possibilities (for one person) with its correct probability given in Table 5.3.

2. One of each US coin currencies is placed into a container (a penny, a nickel, a dime, and a quarter). Given that the withdrawal of a coin from the container is random, find the correct value for each of the following described quantities: (a) If two coins are drawn, find the probability of any one permutation occurring. (b) If three coins are drawn without replacement, find the probability of the total being the maximum possible monetary value. (c) If three coins are drawn with replacement, find the probability of the total being the maximum possible monetary value. (d) If two coins are drawn with replacement, find the probability of the total number of cents being even.

Possibility of Occurrence	Probability
on the 31st of a month	0.0849
in August	0.329
on Feb. 29, 1979 (for one born in that year)	0
in a month with 30 days	0.0192

TABLE 5.3
Birthday occurrences and probabilities.

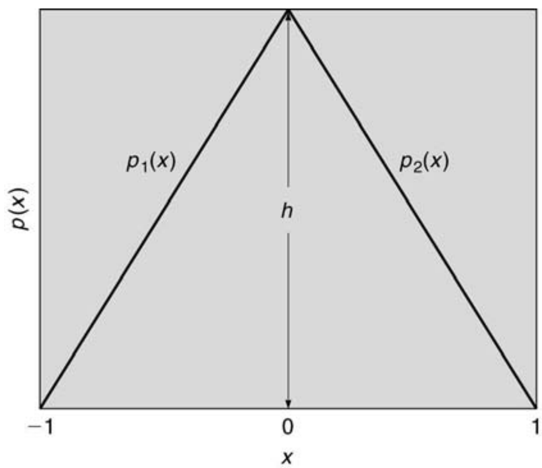

FIGURE 5.18
A triangular probability density function.

3. A sports bar hosts a gaming night where students play casino games using play money. A business major has \$1500 in play money and decides to test a strategy on the roulette wheel. The minimum bet is \$100 with no maximum. He decides to bet that the ball will land on red each time the wheel is spun. On the first bet, he bets the minimum. For each consecutive spin of the wheel, he doubles his previous bet. He decides beforehand that he will play roulette the exact number of times that his cash stock would allow if he lost each time consecutively. What is the probability that he will run out of money before leaving the table?

4. An engineering student samples the wall pressure exerted by a steady-state flow through a pipe 1233 times using an analog-to-digital converter. Using the recommendations made in this chapter, how many equal-interval bins should the student use to create a histogram of the measurements? Respond to the nearest whole bin.

5. Given the probability density function pictured in Figure 5.18, compute the height, h, that conserves the zeroth central moment.

6. Compute the first central moment from the probability density function pictured in Figure 5.18.

7. Compute the kurtosis of the probability density function pictured in Figure 5.18.

8. Compute the skewness of the probability density function pictured in Figure 5.18.

9. Compute the standard deviation of the probability density function pictured in Figure 5.18.

10. A diagnostic test is designed to detect a cancer precursor enzyme that exists in 1 of every 1000 people. The test falsely identifies the presence of the enzyme in 20 out of 1000 people who actually do not have the enzyme. What is the percent chance that a person identified as having the enzyme actually does have the enzyme?

11. What is the chance that you will throw either a 3 or a 5 on the toss of a fair die? (a) 1/12, (b) 1/6, (c) 1/3, (d) 1/2, (e) 1/250.

12. A pressure transducer's output is in units of volts. N samples of its signal are taken each second. The frequency density distribution of the sampled data has what units? (a) 1/volts, (b) volts times seconds, (c) volts/N, (d) none; it is nondimensional, (e) seconds.

13. What is the kurtosis? (a) Bad breath, (b) the fourth central moment, (c) the mean minus the mode of a distribution, (d) the name of a new, secret football play that hopefully will make a difference next season, (e) the square of the standard deviation.

14. How many license plates showing five symbols, specifically, two letters followed by three digits, could be made?

15. A box contains ten screws, and three of them are defective. Two screws are drawn at random. Find the probability that neither of the two screws is defective. Determine the probability with and without replacement.

16. The actual probability of a college student having bronchitis is 50 %. The student health center's diagnostic test for bronchitis has an accuracy of 80 %. Determine the percent probability that a student who has tested positive for bronchitis actually has bronchitis.

17. A newly developed diagnostic test indicates that 80 % of students predicted to score 100 % on an exam actually do. The diagnostic test has an accuracy of 90 %. Determine the actual probability of a student scoring 100 % on an exam.

18. Four strain gages are place on a beam to determine an unknown force, and they are arranged in a Wheatstone bridge configuration so that the output signal is in millivolts. If N samples are recorded each second, what are the units of the corresponding frequency density distribution?

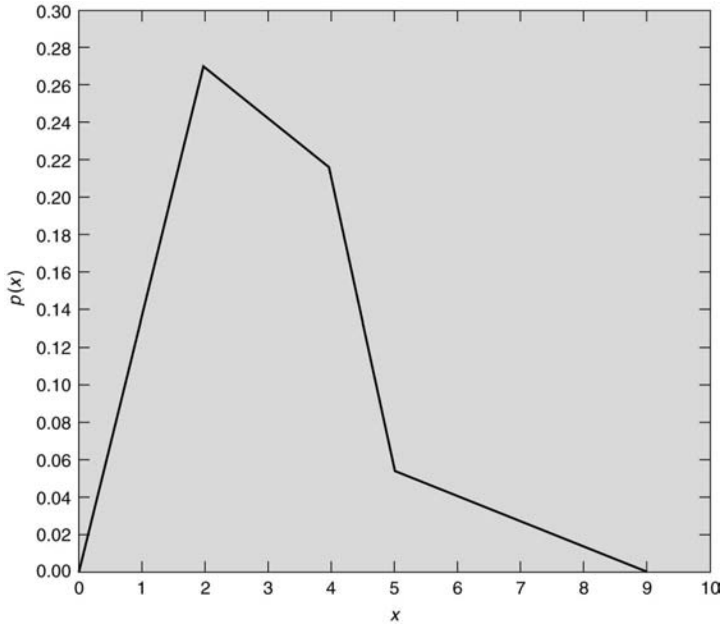

FIGURE 5.19
A probability density function.

5.12 Homework Problems

1. Determine (a) the percent probability that at least 2 out of 19 students in a classroom will have a birthday on the *same* birth date of the year, (b) how many people would have to be in the room in order to have a greater-than-50 % chance to have a birthday on the *same* birth date of the year, and (c) the percent probability that at least 2 of the 19 students will have a birthday on a *specific* birth date of the year.

2. A cab was involved in a hit-and-run accident during the night near a famous mid-western university. Two cab companies, the Blue and the Gold, operate in the city near the campus. There are two facts: (1) 85 % of the cabs in the city are Gold and 15 % are Blue, and (2) a witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed the night of the accident and concluded that the witness correctly identified each of the two colors 80 % of the time and failed to do so 20 % of the time. Determine the percent probability that the cab involved in the accident was Blue.

3. A diagnostic test is designed to detect a bad aircraft component whose prevalence is one in a thousand. The test has a false positive rate of 5 %, where it identifies a *good* component as *bad* 5 % of the time. What is the percent chance that a component identified as *bad* really is bad?

4. Use the data file `diam.dat`. This text file contains two columns of time (s) and diameter (μm) data in approximately 2500 rows. For the diameter data only (column 2), using MATLAB, plot (a) its histogram and (b) its frequency distribution. Use Sturgis's formula for the number of bins, K, as related to the number of data points, N: $K = 1 + 3.322 \log_{10} N$. HINT: MATLAB's function `hist(x,k)` plots the histogram of x with k bins. The statement `[a,b] = hist(x,k)` produces the column matrices a and b, where a contains the counts in each bin and b is the center location coordinate of each bin. MATLAB's function `bar(b,a/N)` will plot the frequency distribution, where N is the total number of x values.

5. Using the graph of the probability density function of an underlying population presented in Figure 5.19, determine (a) the percent probability that one randomly selected value from this population will be between the values of 2 and 5. If a sample of 20 values are drawn randomly from this population, determine (b) how many will have values greater than 2 and (c) how many will have values greater than 5.

6. Let ζ be described by the probability density function $p(x) = 0.75(1 - x^2)$, if $(-1 \leq x \leq 1)$ and zero otherwise. Find (a) the probability distribution function, $P(x)$, (b) the probability $Pr(-1/2 \leq \zeta \leq 1/2)$ and $Pr(1/4 \leq \zeta \leq 2)$, and (c) the value of x such that $Pr(\zeta \leq x) = 0.95$.

7. For the measurand values of 7, 3, 1, 5, and 4, determine (a) the sample mean, (b) the sample variance, and (c) the sample skewness.

8. For the probability density function of *something*, $p(x)$, shown in Figure 5.20, determine (a) $Pr[2 \leq x \leq 3]$, (b) $Pr[x \leq 3]$, and (c) $Pr[x \leq 7]$.

9. When a very high voltage is applied between two electrodes in a gas, the gas will break down and sparks will form (much like lightning). The voltage at which these sparks form depends on a number of variables including gas composition, pressure, temperature, humidity, and the surface of the electrodes. In an experiment, the breakdown voltage was measured 10 times in atmospheric air and the breakdown voltages in units of volts were 2305, 2438, 2715, 2354, 2301, 2435, 2512, 2621, 2139, and 2239. From these measured voltages, determine the (a) mean, (b) variance, and (c) standard deviation.

10. Assume that 14 % of the handguns manufactured throughout the world are 8-mm handguns. A witness at the scene of a robbery states the perpetrator was using an 8-mm handgun. The court performs a weapon identification test on the witness and finds that she can identify the

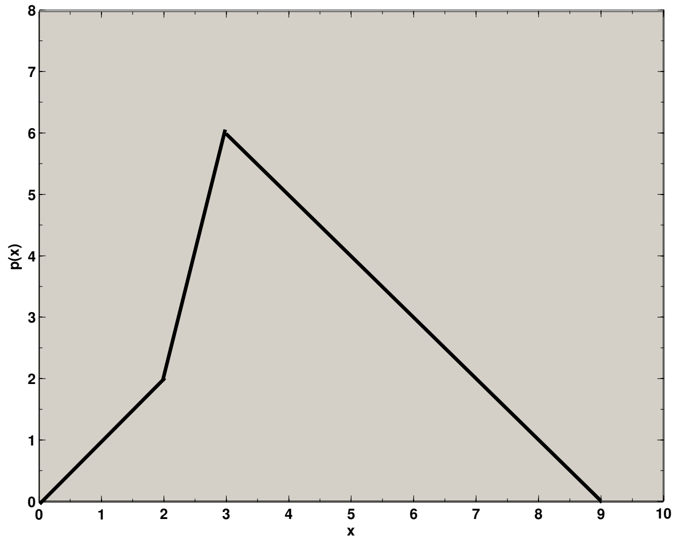

FIGURE 5.20
A probability density function of *something*.

weapon correctly 73 % of the time. (a) What is the probability that an 8-mm handgun was used in the crime? (b) What is the probability if the witness was able to identify an 8-mm handgun correctly 90 % of the time?

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6

Statistics

CONTENTS

That is not exactly true, but it's probably more true than false.

Murray Winn, St. Joseph County Republican Chairman, cited in *The South Bend Tribune*, South Bend, IN, November 6, 2003.

...very many have striven to discover the cause of this direction ... but they wasted oil and labor, because, not being practical in the research of objects in nature, being acquainted only with books, ..., they constructed certain ratiocinations on a basis of mere opinions, and old-womanishly dreamt the things that were not.

William Gilbert, 1600, cited in *De Magnete*. 1991. New York: Dover Press.

6.1 Chapter Overview

Statistics are at the heart of many claims. How many times have you heard that one candidate is ahead of another by a certain percentage in the latest poll or that it is safer to fly than to drive? How confident can we be in

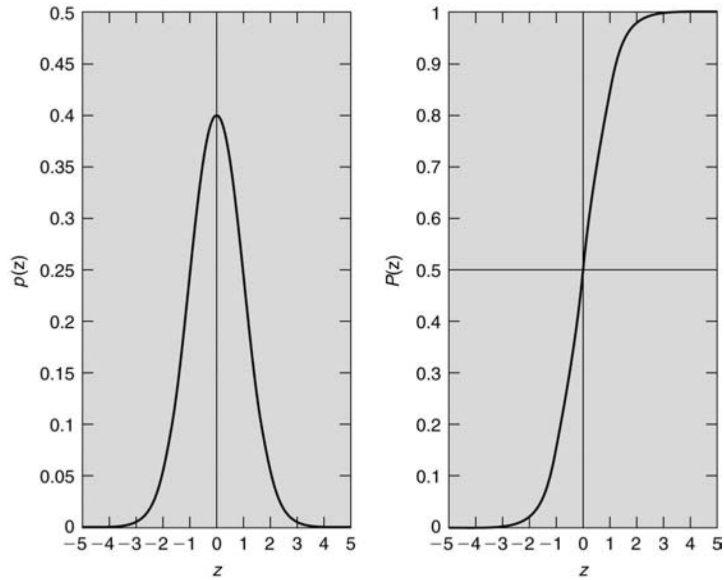

FIGURE 6.1
The probability density and distribution functions for the normal distribution.

such statements? Similar questions arise when interpreting the results of experiments.

In this chapter we will study statistics. We start by examining some frequently used distributions, including the normal, Student's t, and χ^2. We will learn how to use them to determine the probabilities of events and various statistical quantities. We will examine statistical inference and learn how to estimate the characteristics of a population from finite information. Finally, we will investigate how experiments are planned efficiently using methods of statistics. After finishing with this chapter, you will have most of the tools necessary to perform an experiment and to interpret its results correctly.

6.2 Normal Distribution

Now, consider the normal distribution in more detail. In the limit when N becomes very large and P is finite, assuming that the variance remains constant, the binomial probability density function becomes the normal probability density function.

Consider a random error to be comprised of a large number of N elementary errors of equal and infinitesimally small magnitude, e, with an equally likely chance of being either positive or negative, where $P = 1/2$. The normal distribution allows us to find the probability of occurrence of any error in the range from $-Ne$ to $+Ne$, where the probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi NP(1-P)}} \exp\left[\frac{-(x-NP)^2}{2NP(1-P)}\right]. \tag{6.1}$$

The mean and variance are the same as the binomial distribution, NP and NPQ, respectively, where $Q = 1 - P$. The higher-order central moments of the skewness and kurtosis are 0 and 3, respectively.

Utilizing expressions for the mean, x', and the variance, σ^2, in Equation 6.1, the probability density function assumes the more familiar form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-x')^2\right]. \tag{6.2}$$

The normal probability density function is shown in the left plot in Figure 6.1, in which $p(x)$ is plotted versus the nondimensional variable $z = (x - x')/\sigma$. Its maximum value equals 0.3989 at $z = 0$.

The normal probability density function is very significant. Many probability density functions tend to the normal probability density function when the sample size is large. This is supported by the central limit and related theorems. The central limit theorem can be described loosely as follows [10]. Given a population of values with finite variance, if independent samples are taken from this population, all of size N, then the new population formed by the averages of these samples will tend to be governed by the normal probability density function, *regardless* of what distribution governed the original population. Alternatively, the central limit theorem states that whatever the distribution of the independent variables, subject to certain conditions, the probability density function of their sum approaches the normal probability density function (with a mean equal to the sum of their means and a variance equal to the sum of their variances) as N approaches infinity. The conditions are that (1) the variables are expressed in a standardized, nondimensional format, (2) no single variate dominates, and (3) the sum of the variances tends to infinity as N tends to infinity. The central limit theorem also holds for certain classes of dependent random variables.

The normal probability density function describes well those situations in which the departure of a measurand from its central tendency is brought about by *a very large number* of *small* random effects. This is most appropriate for experiments in which all systematic errors have been removed and a large number of values of a measurand are acquired. This probability density function consequently has been found to be the appropriate probability density function for many types of physical measurements. In essence, a measurement subject to many small random errors will be distributed normally. Further, the mean values of finite samples drawn from a distribution other than normal will most likely be distributed normally, as assured by the central limit theorem.

Francis Galton (1822-1911) devised a mechanical system called a *quincunx* to demonstrate how the normal probability density function results from a very large number of small effects with each effect having the same probability of success or failure. This is illustrated in Figure 6.2. As a ball enters the quincunx and encounters the first effect, it falls a lateral distance e to either the right or the left. This event has caused it to depart slightly from its true center. After it encounters the second event, it can either return to the center or depart a distance of $2e$ from it. This process continues for a very large number, N, of events, resulting in a continuum of possible outcomes ranging from a value of $\bar{x} - Ne$ to a value of $\bar{x} + Ne$. The key, of course, to arrive at such a continuum of normally distributed values is to have e small and N large. This illustrates why many phenomena are normally distributed. In many situations there are a number of very small, uncontrollable effects always present that lead to this distribution.

6.3 Normalized Variables

For convenience in performing statistical calculations, the statistical variable often is nondimensionalized. For any statistical variable x, its **standardized normal variate**, β, is defined by

$$\beta = (x - x')/\sigma, \tag{6.3}$$

in which x' is the mean value of the population and σ its standard deviation. In essence, the dimensionless variable β signifies how many standard deviations that x is from its mean value. When a *specific* value of x, say x_1, is considered, the standardized normal variate is called the **normalized** z **variable**, z_1, as defined by

$$z_1 = (x_1 - x')/\sigma. \tag{6.4}$$

These definitions can be incorporated into the probability expression of a dimensional variable to yield the corresponding expression in terms of the nondimensional variable. The probability that the dimensional variable x will be in the interval $x' \pm \delta x$ can be written as

$$P(x' - \delta x \leq x \leq x' + \delta x) = \int_{x'-\delta x}^{x'+\delta x} p(x)dx. \tag{6.5}$$

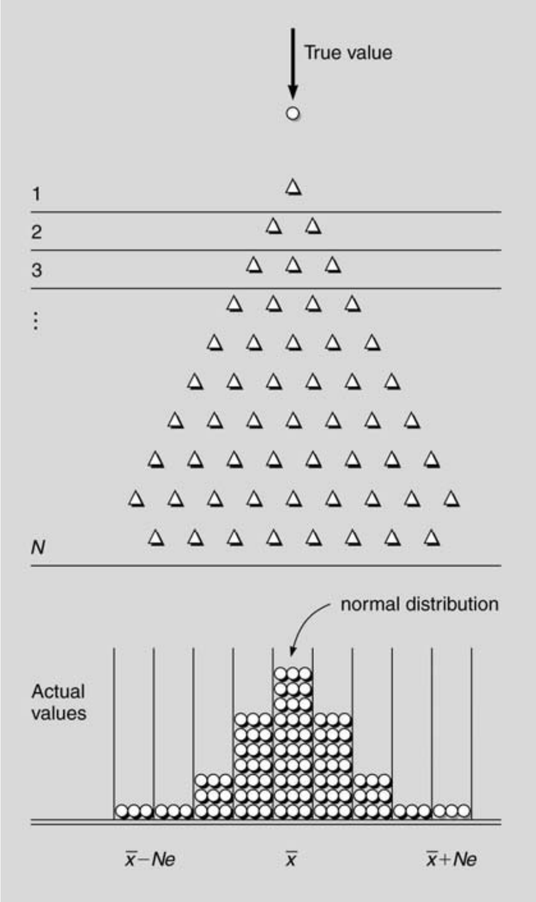

FIGURE 6.2
Galton's quincunx.

Note that the width of the interval is $2\delta x$, which in some previous expressions was written as Δx. Likewise,

$$P(-x_1 \leq x \leq +x_1) = \int_{-x_1}^{+x_1} p(x)dx. \tag{6.6}$$

This general expression can be written specifically for a normally distributed variable as

$$P(-x_1 \leq x \leq +x_1) = \int_{-x_1}^{+x_1} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x - x')^2\right] dx. \tag{6.7}$$

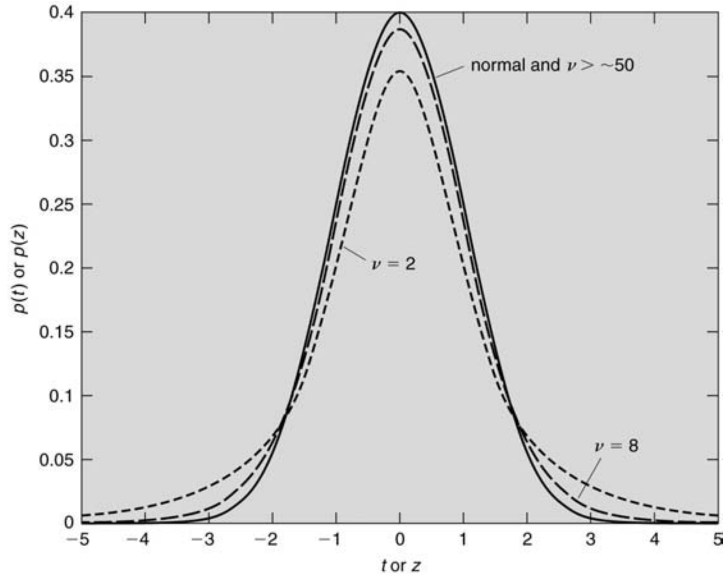

FIGURE 6.3
The normal and Student's t probability density functions.

Using Equations 6.3 and 6.4 and noting that $dx = \sigma d\beta$, Equation 6.7 becomes

$$\begin{aligned} P(-z_1 \leq \beta \leq +z_1) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-z_1}^{+z_1} \exp\left[\frac{-\beta^2}{2}\right] \sigma d\beta, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-z_1}^{+z_1} \exp\left[\frac{-\beta^2}{2}\right] d\beta, \\ &= 2\left\{\frac{1}{\sqrt{2\pi}} \int_{0}^{+z_1} \exp\left[\frac{-\beta^2}{2}\right] d\beta\right\}. \end{aligned} \quad (6.8)$$

The factor of 2 in the last equation reflects the symmetry of the normal probability density function, which is shown in Figure 6.3, in which $p(z)$ is plotted as a function of the normalized-z variable. The term in the { } brackets is called the **normal error function**, denoted as $p(z_1)$. That is

$$p(z_1) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+z_1} \exp\left[\frac{-\beta^2}{2}\right] d\beta. \quad (6.9)$$

The values of $p(z_1)$ are presented in Table 6.2 for various values of z_1. For example, there is a 34.13 % probability that a normally distributed variable z_1 will be within the range from $x_1 - x' = 0$ to $x_1 - x' = \sigma$ [$p(z_1) = 0.3413$]. In other words, there is a 34.13 % probability that a normally distributed

z_P	% P
1	68.27
1.645	90.00
1.960	95.00
2	95.45
2.576	99.00
3	99.73
4	99.99

TABLE 6.1
Probabilities of some common z_P values.

variable will be within one standard deviation *above* the mean. Note that the normal error function is one-sided because it represents the integral from 0 to $+z_1$. Some normal error function tables are two-sided and represent the integral from $-z_1$ to $+z_1$. Always check to see whether such tables are either one-sided or two-sided.

Using the definition of z_1, the probability that a normally distributed variable, x_1, will have a value within the range $x' \pm z_1\sigma$ is

$$2p(z_1) = \frac{2}{\sqrt{2\pi}} \int_0^{+z_1} \exp\left[-\frac{\beta^2}{2}\right] d\beta = \frac{\%\,P}{100}. \tag{6.10}$$

In other words, there is P percent probability that the normally distributed variable, x_i, will be within $\pm z_P$ standard deviations of the mean. This can be expressed formally as

$$x_i = x' \ \pm \ z_P\sigma \ \ (\%\,P). \tag{6.11}$$

The percent probabilities for the z_P values from 1 to 4 are presented in Table 6.1. As shown in the table, there is a 68.27 % chance that a normally distributed variable will be within ± one standard deviation of the mean and a 99.73 % chance that it will be within ± three standard deviations of the mean.

Example Problem 6.1

Statement: Consider the situation in which a large number of voltage measurements are made. From this data, the mean value of the voltage is 8.5 V and that its variance is 2.25 V^2. Determine the probability that a single voltage measurement will fall in the interval between 10 V and 11.5 V. That is, determine $Pr[10.0 \leq x \leq 11.5]$.

Solution: Using the definition of the probability distribution function, $Pr[10.0 \leq x \leq 11.5] = Pr[8.5 \leq x \leq 11.5] - Pr[8.5 \leq x \leq 10.0]$. The two probabilities on the right side of this equation are found by determining their corresponding normalized z-variable values and then using Table 6.2.

First, $Pr[8.5 \leq x \leq 10.0]$:

$$z = \frac{x - x'}{\sigma} = \frac{10 - 8.5}{1.5} = 1 \ \Rightarrow \ P(8.5 \leq x \leq 10.0) = \frac{.6827}{2} = 0.3413.$$

Then, $Pr[8.5 \leq x \leq 11.5]$:

$$z = \frac{11.5 - 8.5}{1.5} = 2 \Rightarrow P(8.5 \leq x \leq 11.5) = \frac{.9545}{2} = 0.4772.$$

Thus, $Pr[10.0 \leq x \leq 11.5]$ = 0.4772 - 0.3413 = 0.1359 or 13.59 %. Likewise, the probability that a single voltage measurement will fall in the interval between 10 V and 13 V is 15.74 %.

Example Problem 6.2

Problem Statement: Based upon a large data base, the State Highway Patrol has determined that the average speed of Friday-afternoon drivers on an interstate is 67 mph with a standard deviation of 4 mph. How many drivers out of 1000 travelling on that interstate on Friday afternoon will be travelling in excess of 72 mph?

Problem Solution: Assume that the speeds of the drivers follow a normal distribution. The 72 mph speed first converted into its corresponding z-variable value is

$$z = \frac{72 - 67}{4} = 1.2. \tag{6.12}$$

Thus, we need to determine

$$Pr[z > 1.2] = 1 - Pr[z \leq 1.2] = 1 - (Pr[-\infty \leq z \leq 0] + Pr[0 \leq z \leq 1.2]). \tag{6.13}$$

From the one-sided z-variable probability table

$$Pr[0 \leq z \leq 1.2] = 0.3849. \tag{6.14}$$

Also, because the normal probability distribution is symmetric about its mean

$$Pr[-\infty \leq z \leq 0] = 0.5000. \tag{6.15}$$

Thus,

$$Pr[z > 1.2] = 1 - (0.5000 + 0.3849) = 0.1151. \tag{6.16}$$

This means that approximately 115 of the 1000 drivers will be travelling in excess of 72 mph on that Friday afternoon.

6.4 Student's t Distribution

It was about 100 years ago that William Gosset, a statistician working for the Guinness brewery, recognized a problem in using the normal distribution to describe the distribution of a small sample. As a consequence of his observations it was recognized that the normal probability density function *overestimated* the probabilities of the small-sample members near its mean and *underestimated* the probabilities far away from its mean. Using his data as a guide and working with the ratios of sample estimates, Gosset was able to develop a new distribution that better described how the members of a small sample drawn from a normal population were actually distributed.

z_P	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4758	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4799	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4988	.4989	.4989	.4989	.4990
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
4.0	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

TABLE 6.2
Values of the normal error function.

Because his employer would not allow him to publish his findings, he published them under the pseudonym "Student"[2]. His distribution was named the **Student's t distribution**. "Student" continued to publish significant works for over 30 years. Mr. Gosset did so well at Guinness that he eventually was put in charge of its entire Greater London operations [1].

The essence of what Gosset found is illustrated in Figure 6.3. The solid curve indicates the normal probability density function values for various z. It also represents the Student's t probability density function for various t and a large sample size ($N > 100$). The dashed curve shows the Student's t probability density function values for various t for a sample consisting of 9 members ($\nu = 8$), and the dotted curve for a sample of 3 members ($\nu = 2$). It is clear that as the sample size becomes smaller, the normal probability density function near its mean (where $z = 0$) overestimates the sample probabilities and, near its extremes (where $z > \sim 2$ and $z < \sim -2$), underestimates the sample probabilities. These differences can be quantified easily using the expressions for the probability density functions.

The probability density function of Student's t distribution is

$$p(t, \nu) = \frac{\Gamma[(\nu+1)/2]}{\sqrt{\pi\nu}\Gamma(\nu/2)} \left[1 + \frac{t^2}{\nu}\right]^{-(\nu+1)/2}, \tag{6.17}$$

where ν denotes the degrees of freedom and Γ is the gamma function, which has these properties:

$$\begin{aligned}
\Gamma(n) &= (n-1)! \text{ for } n = \text{ whole integer} \\
\Gamma(m) &= (m-1)(m-2)...(3/2)(1/2)\sqrt{\pi} \text{ for } m = \text{ half} - \text{integer} \\
\Gamma(1/2) &= \sqrt{\pi}
\end{aligned}$$

Note in particular that $p = p(t, \nu)$ and, consequently, that there are an *infinite* number of Student's t probability density functions, one for each value of ν. This was suggested already in Figure 6.3 in which there were different curves for each value of N.

The statistical concept of **degrees of freedom** was introduced by R.A. Fisher in 1924 [1]. The number of degrees of freedom, ν, at any stage in a statistical calculation equals the number of recorded data, N, minus the number of different, independent restrictions (constraints), c, used for the required calculations. That is, $\nu = N - c$. For example, when computing the sample mean, there are no constraints ($c = 0$). This is because only the actual sample values are required (hence, no constraints) to determine the sample mean. So for this case, $\nu = N$. However, when either the sample standard deviation or the sample variance is computed, the value of the sample mean value is required (one constraint). Hence, for this case, $\nu = N - 1$. Because both the sample mean and sample variance are contained implicitly in t in Equation 6.17, $\nu = N - 1$. Usually, whenever a probability density function expression is used, values of the mean and the variance are required. Thus, $\nu = N - 1$ for these types of statistical calculations.

The expressions for the mean and standard deviation were developed in Chapter 5 for a **continuous** random variable. Analogous expressions can be developed for a **discrete** random variable. When N is very large,

$$x' = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i \tag{6.18}$$

and

$$\sigma^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - x')^2. \tag{6.19}$$

When N is small,

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{6.20}$$

and

$$S_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2. \tag{6.21}$$

Here $\bar{x}$ denotes the **sample mean**, whose value can (and usually does) vary from that of the **true mean**, x'. Likewise, S_x^2 denotes the **sample variance** in contrast to the **true variance** σ^2. The factor $N-1$ occurs in Equation 6.21 as opposed to N to account for loosing one degree of freedom ($\bar{x}$ is needed to calculate S_x^2).

Example Problem 6.3

Statement: Consider an experiment in which a finite sample of 19 values of a pressure are recorded. These are in units of kPa: 4.97, 4.92, 4.93, 5.00, 4.98, 4.92, 4.91, 5.06, 5.01, 4.98, 4.97, 5.02, 4.92, 4.94, 4.98, 4.99, 4.92, 5.04, and 5.00. Estimate the range of pressure within which another pressure measurement would be at P = 95 % given the recorded values.

Solution: From this data, using the equations for the sample mean and the sample variance for small N,

$$\bar{p} = \frac{1}{19} \sum_{i=1}^{19} p_i = 4.97 \text{ and } S_p = \sqrt{\frac{1}{19-1} \sum_{i=1}^{19} (p_i - \bar{p})^2} = 0.046.$$

Now $\nu = N - 1 = 18$, which gives $t_{\nu,P} = t_{18,95} = 2.101$ using Table 6.4. So,

$$p_i = \bar{p} \pm t_{\nu,P} S_p \ (\%\ P) \ \Rightarrow \ p_i = 4.97 \pm 0.10 \ (95\ \%)$$

.

Thus, the next pressure measurement is estimated to be within the range of 4.87 kPa to 5.07 kPa at 95 % confidence.

Now, what if the sample had the same mean and standard deviation values but they were determined from only five measurements? Then

$$t_{\nu,P} = t_{4,95} = 2.770 \ \Rightarrow \ p_i = 4.97 \pm 0.13 \ (95\ \%)$$

$t_{\nu,P}$	$\%P_{\nu=2}$	$\%P_{\nu=8}$	$\%P_{\nu=100}$
1	57.74	65.34	68.03
2	81.65	91.95	95.18
3	90.45	98.29	99.66
4	94.28	99.61	99.99

TABLE 6.3
Probabilities for some typical $t_{\nu,P}$ values.

.

For this case the next pressure measurement is estimated to be within the range of 4.84 kPa to 5.10 kPa at 95 % confidence. So, for the same confidence, a smaller sample size implies a broader range of uncertainty.

Further, what if the original sample size was used but only 50 % confidence was required in the estimate? Then

$$t_{\nu,P} = t_{18,50} = 0.668 \Rightarrow p_i = 4.97 \pm 0.03 \ (50\ \%)$$

.

For this case the next pressure measurement is estimated to be within the range of 4.94 kPa to 5.00 kPa at 50 % confidence. Thus, for the same sample size but a lower required confidence, the uncertainty range is narrower. On the contrary, if 100 % confidence was required in the estimate, the range would have to extend over *all* possible values.

In a manner analogous to the method for the normalized z variable in Equation 6.4, Student's t variable is defined as

$$t_1 = (x_1 - \bar{x})/S_x. \tag{6.22}$$

It follows that the normally distributed variable x_i in a small sample will be within $\pm t_{\nu,P}$ sample standard deviations from the sample mean with % P confidence. This can be expressed formally as

$$x_i = \bar{x} \pm t_{\nu,P} S_x \ \ (\%\ P). \tag{6.23}$$

The interval $\pm t_{\nu,P} S_x$ is called the **precision interval**. The percentage probabilities for the $t_{\nu,P}$ values of 1, 2, 3, and 4 for three different values of ν are shown in Table 6.3. Thus, in a sample of nine ($\nu = N - 1 = 8$), there is a 65.34 % chance that a normally distributed variable will be within ± one sample standard deviation from the sample mean and a 99.61 % chance that it will be within ± four sample standard deviations from the sample mean. Also, as the sample size becomes smaller, the percent P that a sample value will be within ± a certain number of sample standard deviations becomes less. This is because Student's t probability density function is slightly broader than the normal probability density function and extends out to larger values of t from the mean for smaller values of ν, as shown in Figure 6.3.

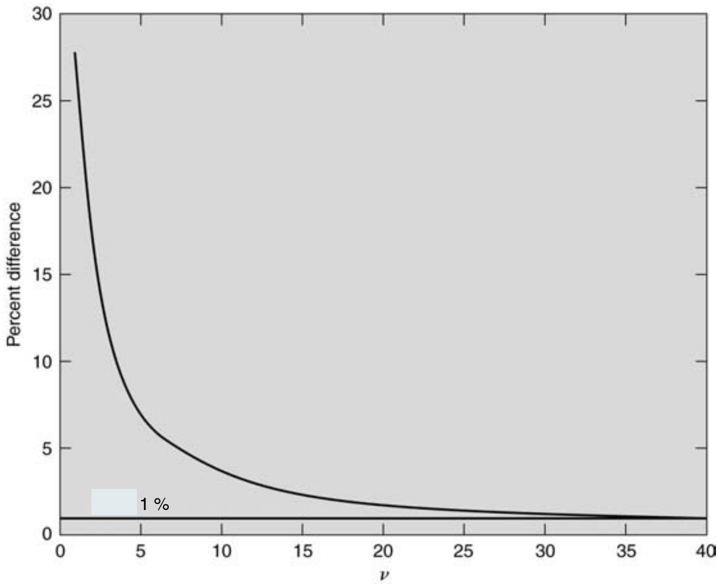

FIGURE 6.4
Comparison of Student's t and normal probabilities.

Another way to compare the Student's t distribution with the normal distribution is to examine the percent difference in the areas underneath their probability density functions for the same range of z and t values. This implicitly compares their probabilities, which can be done for various degrees of freedom. The results of such a comparison are shown in Figure 6.4. The probabilities are compared between t and z equal to 0 up to t and z equal to 5. It can be seen that the percent difference decreases as the number of degrees of freedom increases. At $\nu = 40$, the difference is less than 1 %. That is, the areas under their probability density functions over the specified range differ by less than 1 % when the number of measurements are approximately greater than 40.

The values for $t_{\nu,P}$ are given in Table 6.4. Using this table, for $\nu = 8$ there is a 95 % probability that a sample value will be within ± 2.306 sample standard deviations of the sample mean. Likewise, for $\nu = 40$, there is a 95 % probability that a sample value will be within ± 2.021 sample standard deviations of the sample mean.

A relationship between Student's t variable and the normalized-z variable can be found directly by equating the $x_i's$ of Equations 6.11 and 6.23.

ν	$t_{\nu,P=50\ \%}$	$t_{\nu,P=90\ \%}$	$t_{\nu,P=95\ \%}$	$t_{\nu,P=99\ \%}$
1	1.000	6.341	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.192	5.841
4	0.741	2.132	2.770	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
30	0.683	1.697	2.042	2.750
40	0.681	1.684	2.021	2.704
50	0.680	1.679	2.010	2.679
60	0.679	1.671	2.000	2.660
120	0.677	1.658	1.980	2.617
∞	0.674	1.645	1.960	2.576

TABLE 6.4
Student's t variable values for different P and ν.

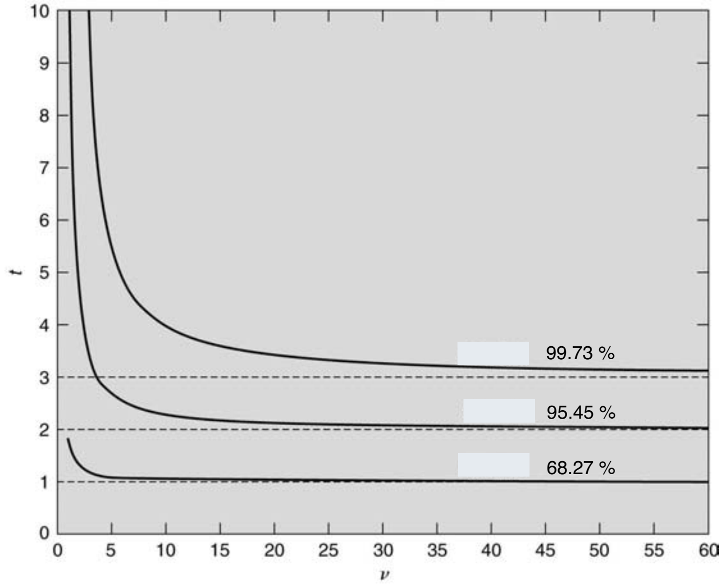

FIGURE 6.5
Student's t values for various degrees of freedom and percent probabilities.

This is

$$t_{\nu,P} = \pm \left\{ \frac{(x' - \bar{x}) \pm z_P \sigma}{S_x} \right\} \quad (\% \ P). \tag{6.24}$$

Now, in the limit as $N \to \infty$, the sample mean, $\bar{x}$, tends to the true mean, x', and the sample standard deviation, S_x, tends to the true standard deviation σ. It follows from Equation 6.24 that $t_{\nu,P}$ tends to z_P. This is illustrated in Figure 6.5, in which the $t_{\nu,P}$ values for $P = 68.27$ %, 95.45 % and 99.73 % are plotted versus ν. This figure was constructed using the MATLAB M-file `tnuP.m`. As shown in the figure, for increasing values of ν, the $t_{\nu,P}$ values for $P = 68.27$ %, 95.45 % and 99.73 % approach the z_P values of 1, 2, and 3, respectively. In other words, Student's t distribution approaches the normal distribution as N tends to infinity.

6.5 Standard Deviation of the Means

Consider a sample of N measurands. From its sample mean, $\bar{x}$, and its sample standard deviation, S_x, the region within which the true mean of the underlying population, x', can be inferred. This is done by statistically

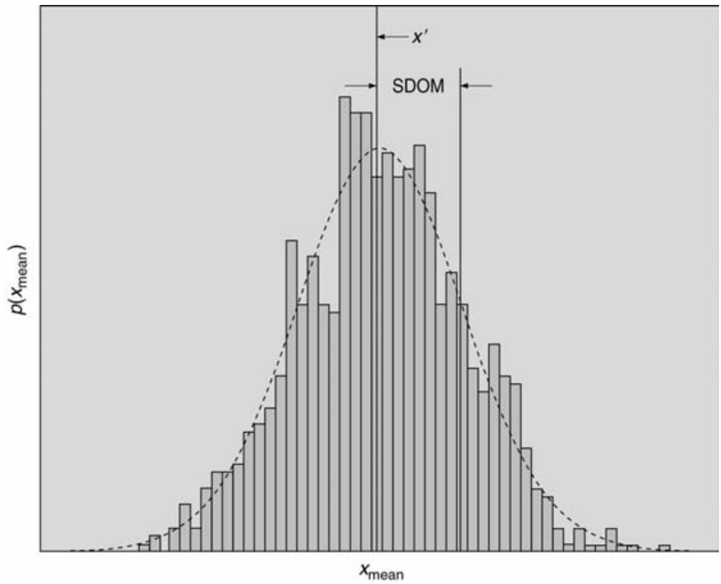

FIGURE 6.6
The probability density function of the mean values of x.

relating the sample to the population through the **standard deviation of the means** (SDOM).

Assume that there are M sets (samples), each comprised of N measurands. A specific measurand value is denoted by x_{ij}, where $i = 1$ to N refers to the specific number within a set and $j = 1$ to M refers to the particular set. Each set will have a mean value, $\bar{x}_j$, where

$$\bar{x}_j = \frac{1}{N} \sum_{i=1}^{N} x_{ij}, \tag{6.25}$$

and a sample standard deviation, S_{x_j}, where

$$S_{x_j} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_{ij} - \bar{x}_j)^2}. \tag{6.26}$$

Now each $\bar{x}_j$ is a random variable. The central limit theorem assures that the $\bar{x}_j$ values will be normally distributed about their mean value (the mean of the mean values), $\bar{\bar{x}}$, where

$$\bar{\bar{x}} = \frac{1}{M} \sum_{j=1}^{M} \bar{x}_j. \tag{6.27}$$

This is illustrated in Figure 6.6.

The standard deviation of the mean values (termed the *standard deviation of the means*), then will be

$$S_{\bar{x}} = \left[\frac{1}{M-1} \sum_{j=1}^{M} (\bar{x} - \bar{\bar{x}})^2 \right]^{1/2}. \tag{6.28}$$

It can be proven using Equations 6.26 and 6.28 [4] that

$$S_{\bar{x}} = S_x / \sqrt{N}. \tag{6.29}$$

This deceptively simple formula allows us to determine from the values of only *one* finite set the range of values that contains the true mean value of the entire population. Formally

$$x' = \bar{x} \pm t_{\nu,P} S_{\bar{x}} = \bar{x} \pm t_{\nu,P} \frac{S_x}{\sqrt{N}} \; (\% \; P). \tag{6.30}$$

This formula implies that the bounds within which x' is contained can be reduced, which means that the estimate of x' can be made more precise, by increasing N or by decreasing the value of S_x. There is a moral here. It is better to carefully plan an experiment to minimize the number of random effects beforehand and hence to reduce S_x, rather than to spend the time acquiring more data to achieve the same bounds on x'.

The interval $\pm t_{\nu,P} S_{\bar{x}}$ is called the **precision interval of the true mean**. As N becomes large, from Equation 6.29 it follows that the SDOM becomes small and the sample mean value tends toward the true mean value. In this light, the precision interval of the true mean value can be viewed as a measure of the *uncertainty* in determining x'.

Example Problem 6.4

Statement: Consider the differential pressure transducer measurements in the previous example. What is the range within which the true mean value of the differential pressure, p', is contained?

Solution: Equation 6.30 reveals that

$$p' = \bar{p} \pm t_{\nu,P} S_{\bar{p}} = \bar{p} \pm t_{\nu,P} \frac{S_p}{\sqrt{N}} \; (\% \; P)$$

$$= 4.97 \pm \frac{(2.101)(0.046)}{\sqrt{19}} = 4.97 \pm 0.02 \; (95 \; \%)$$

.

Thus, the true mean value is estimated at 95 % confidence to be within the range from 4.95 kPa to 4.99 kPa.

Finally, it is very important to note that although Equations 6.23 and 6.30 appear to be similar, they are uniquely different. Equation 6.23 is used to estimate the range within which another x_i value will be with a given confidence, whereas Equation 6.30 is used to estimate the range that contains the true mean value for a given confidence. Both equations use the values of the sample mean, standard deviation, and number of measurands in making these estimates.

6.6 Chi-Square Distribution

The range that contains the true mean of a population can be estimated using the values from only a single sample of N measurands and Equation 6.30. Likewise, there is an analogous way of estimating the range that contains the true variance of a population using the values from only one sample of N measurands. The estimate involves using one more probability distribution, the chi-square distribution.

The chi-square distribution is used in many statistical calculations. For example, it can be used to determine the precision interval of the true variance, to quantify how well a sample matches an assumed parent distribution, and to compare two samples of same or different size with one another. The statistical variable, χ^2, represents the sum of the squares of the differences between the measured and expected values normalized by their variance. Thus, the value of χ^2 is dependent upon the number of measurements, N, at which the comparison is made, and, hence, the number of degrees of freedom, $\nu = N - 1$. From this definition it follows that χ^2 is related to the standardized variable, $z_i = (x_i - x')/\sigma$, and the number of measurements by

$$\chi^2 = \sum_{i=1}^{N} z_i^2 = \sum_{i=1}^{N} \frac{(x_i - x')^2}{\sigma^2}. \tag{6.31}$$

χ^2 can be viewed as a quantitative measure of the total deviation of all x_i values from their population's true mean value with respect to their population's standard deviation. This concept can be used, for example, to compare the χ^2 value of a sample with the value that would be expected for a sample of the same size drawn from a normally distributed population. Using the definition of the sample variance given in Equation 6.21, this expression becomes

$$\chi^2 = \nu S_x^2/\sigma^2. \tag{6.32}$$

So, in the limit as $N \to \infty$, $\chi^2 \to \nu$.

The probability density function of χ^2 (for $\chi^2 \geq 0$) is

$$p(\chi^2, \nu) = [2^{\nu/2}\Gamma(\nu/2)]^{-1}(\chi^2)^{(\nu/2)-1}\exp(-\chi^2/2), \tag{6.33}$$

where Γ denotes the gamma function given by

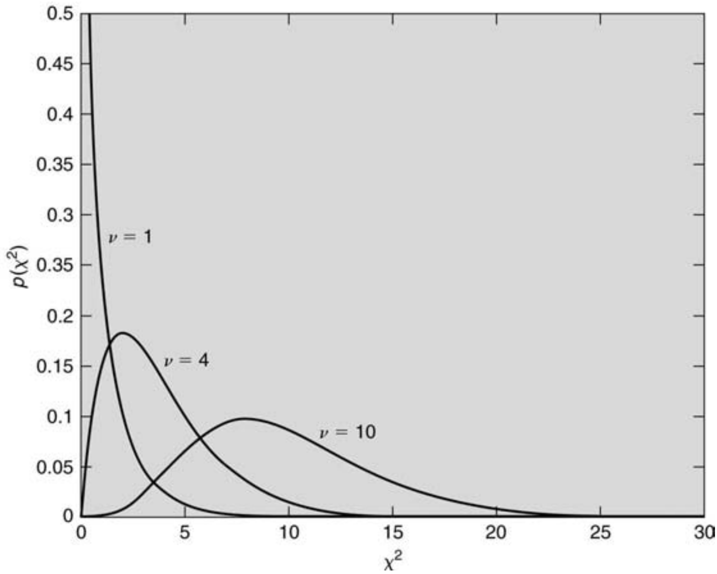

FIGURE 6.7
Three χ^2 probability density functions.

$$\Gamma(\nu/2) = \int_0^\infty x^{(\nu/2)-1} \exp(-x) dx = (\frac{\nu}{2} - 1)! \qquad (6.34)$$

and the mean and the variance of $p(\chi^2, \nu)$ are ν and 2ν, respectively. Sometimes values of χ^2 are normalized by the expected value, ν. The appropriate parameter then becomes the **reduced chi-square variable**, which is defined as $\tilde{\chi}^2 \equiv \chi^2/\nu$. The mean value of the reduced chi-square variable then equals unity. Finally, note that there is a different probability density function of χ^2 for each value of ν.

The χ^2 probability density functions for three different values of ν are plotted versus χ^2 in Figure 6.7. The MATLAB M-file `chipdf.m` was used to construct this figure. The value of $p(\chi^2 = 10,\ \nu = 10)$ is 0.0877, whereas the value of $p(\chi^2 = 1,\ \nu = 1)$ is 0.2420. For a sample of only $N = 2$ ($\nu = N - 1 = 1$), there is almost a 100 % chance that the value of χ^2 will be less than approximately 9. However, if $N = 11$ there is approximately a 50 % chance that the value of χ^2 will be less than approximately 9.

The corresponding probability distribution function, given by the integral of the probability density function from 0 to a specific value of χ^2, denoted by χ^2_α, is called the chi-square distribution with ν degrees of freedom. It denotes the probability $P(\chi^2_\alpha) = 1 - \alpha$ that $\chi^2 \leq \chi^2_\alpha$, where α denotes the **level of significance**. In other words, the area under a specific χ^2 probability *density* function curve from 0 to χ^2_α equals $P(\chi^2_\alpha)$ and

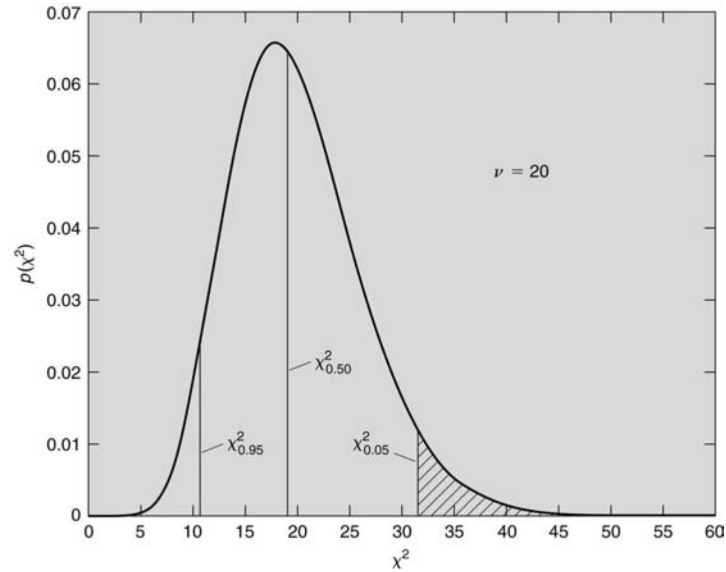

FIGURE 6.8
The χ^2 probability density function for $\nu = 20$.

the area from χ^2_α to ∞ equals α. The χ^2 probability density function for $\nu = 20$ is plotted in Figure 6.8. The MATLAB M-file `chicdf.m` was used to construct this figure. Three χ^2_α values (for $\alpha = 0.05$, 0.50, and 0.95) are indicated by vertical lines. The lined area represents 5 % of the total area under the probability density curve, corresponding to $\alpha = 0.05$. The χ^2_α values for various ν and α are presented in Table 6.5. Using this table, for $\nu = 20$, $\chi^2_{0.95} = 10.9$, $\chi^2_{0.50} = 19.3$, and $\chi^2_{0.05} = 31.4$. That is, when $\nu = 20$, 50 % of the area beneath the curve is between $\chi^2 = 0$ and $\chi^2 = 19.3$.

The χ^2 probability distribution functions for the same values of ν used in Figure 6.7 are plotted versus χ^2 in Figure 6.9. For $N = 2$, there is a 99.73 % chance that the value of χ^2 will be less than 9. For $N = 11$, there is a 46.79 % chance that the value of χ^2 will be less than 9. Finally, for $\nu = 4$, as already determined using Table 6.5, a value of $\chi^2 = 3.36$ yields $P(\chi^2_\alpha) = 0.50$.

6.6.1 Estimating the True Variance

Consider a finite set of x_i values drawn randomly from a normal distribution having a true mean value x' and a true variance σ^2. It follows directly from Equation 6.32 and the definition of χ^2_α that there is a probability of $1-\frac{\alpha}{2}$ that $\nu S_x^2/\sigma^2 \leq \chi^2_{\alpha/2}$ or that $\nu S_x^2/\chi^2_{\alpha/2} \leq \sigma^2$. Conversely, there is a probability of $1-(1-\frac{\alpha}{2}) = \alpha/2$ that $\sigma^2 \leq \nu S_x^2/\chi^2_{\alpha/2}$. Likewise, there is a probability of

ν	$\chi^2_{0.99}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.90}$	$\chi^2_{0.50}$	$\chi^2_{0.05}$	$\chi^2_{0.025}$	$\chi^2_{0.01}$
1	0.000	0.000	0.000	0.016	0.455	3.84	5.02	6.63
2	0.020	0.051	0.103	0.211	1.39	5.99	7.38	9.21
3	0.115	0.216	0.352	0.584	2.37	7.81	9.35	11.3
4	0.297	0.484	0.711	1.06	3.36	9.49	11.1	13.3
5	0.554	0.831	1.15	1.61	4.35	11.1	12.8	15.1
6	0.872	1.24	1.64	2.20	5.35	12.6	14.4	16.8
7	1.24	1.69	2.17	2.83	6.35	14.1	16.0	18.5
8	1.65	2.18	2.73	3.49	7.34	15.5	17.5	20.1
9	2.09	2.70	3.33	4.17	8.34	16.9	19.0	21.7
10	2.56	3.25	3.94	4.78	9.34	18.3	20.5	23.2
11	3.05	3.82	4.57	5.58	10.3	19.7	21.9	24.7
12	3.57	4.40	5.23	6.30	11.3	21.0	23.3	26.2
13	4.11	5.01	5.89	7.04	12.3	22.4	24.7	27.7
14	4.66	5.63	6.57	7.79	13.3	23.7	26.1	29.1
15	5.23	6.26	7.26	8.55	14.3	25.0	27.5	30.6
16	5.81	6.91	7.96	9.31	15.3	26.3	28.8	32.0
17	6.41	7.56	8.67	10.1	16.3	27.6	30.2	33.4
18	7.01	8.23	9.39	10.9	17.3	28.9	31.5	34.8
19	7.63	8.91	10.1	11.7	18.3	30.1	32.9	36.2
20	8.26	9.59	10.9	12.4	19.3	31.4	34.2	37.6
30	15.0	16.8	18.5	20.6	29.3	43.8	47.0	50.9
40	22.2	24.4	26.5	29.1	39.3	55.8	59.3	63.7
50	29.7	32.4	34.8	37.7	49.3	67.5	71.4	76.2
60	37.5	40.5	43.2	46.5	59.3	79.1	83.3	88.4
70	45.4	48.8	51.7	55.3	69.3	90.5	95.0	100.4
80	53.5	57.2	60.4	64.3	79.3	101.9	106.6	112.3
90	61.8	65.6	69.1	73.3	89.3	113.1	118.1	124.1
100	70.1	74.2	77.9	82.4	99.3	124.3	129.6	135.8

TABLE 6.5
χ^2_α values for various ν and α.

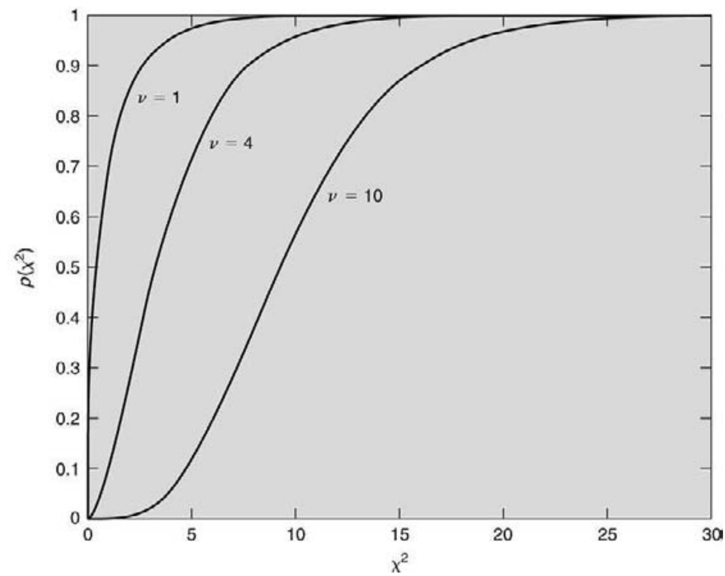

FIGURE 6.9
Three χ^2 probability distribution functions.

$\alpha/2$ that $\nu S_x^2/\sigma^2 \leq \chi^2_{1-\frac{\alpha}{2}}$ or that $\nu S_x^2/\chi^2_{1-\frac{\alpha}{2}} \leq \sigma^2$. Thus, the true variance of the underlying population, σ^2, is *within* the **sample variance precision interval**, from $\nu S_x^2/\chi^2_{\alpha/2}$ to $\nu S_x^2/\chi^2_{1-\frac{\alpha}{2}}$ with a probability $1-(\alpha/2)-(\alpha/2) = P$. Also there is a probability $\alpha/2$ that it is *below* the lower bound of the precision interval, a probability $\alpha/2$ that it is *above* the upper bound of the precision interval, and a probability α that it is outside of the bounds of the precision interval. Formally, this is

$$\frac{\nu S_x^2}{\chi^2_{\alpha/2}} \leq \sigma^2 \leq \frac{\nu S_x^2}{\chi^2_{1-\alpha/2}} \quad (\%\ P). \tag{6.35}$$

The width of the precision interval of the true variance in relation to the probability P can be examined further. First consider the two extreme cases. If $P = 1$ (100 %), then $\alpha = 0$ which implies that $\chi_0^2 = \infty$ and $\chi_1^2 = 0$. Thus, the sample variance precision interval is from 0 to ∞ according to Equation 6.35. That is, there is a 100 % chance that σ^2 will have a value between 0 and ∞. If $P = 0$ (0 %), then $\alpha = 1$ which implies that the sample variance precision intervals are the same. That is, there is a 0 % chance the σ^2 will exactly equal one specific value out of an infinite number of possible values (when $\alpha = 1$ and $\nu >> 1$, that unique value would be S_x^2). These two extreme-case examples illustrate the upper and lower limits of the sample variance precision interval and its relation to P and α. As α varies from 0 to 1 (hence, P varies from 1 to 0), the precision interval width decreases from

∞ to 0. In other words, the probability, α, that the true variance is *outside* of the sample variance precision interval increases as the precision interval width decreases.

6.6.2 Establishing a Rejection Criterion

The relation between the probability of occurrence of a χ^2 value being less than a specified χ^2 value can be utilized to ascertain whether or not effects other than random ones are present in an experiment or process. This is particularly relevant, for example, in establishing a rejection criterion for a manufacturing process or in an experiment. If the sample's χ^2 value exceeds the value of χ^2_α based upon the probability of occurrence $P = 1 - \alpha$, it is likely that systematic effects (biases) are present. In other words, the level of significance α also can be used as a chance indicator of random effects. A low value of α implies that there is very little chance that the noted difference is due to random effects and, thus, that a systematic effect is the cause for the discrepancy. In essence, a low value of α corresponds to a relatively high value of χ^2, which, of course, has little chance to occur randomly. It does, however, have *some* chance to occur randomly, which leads to the possibility of falsely identifying a random effect as being systematic, which is a Type II error (see Section 6.8). For example, a batch sample yielding a low value of α implies that the group from which it was drawn is suspect and probably (but not definitely) should be rejected. A high value of α implies the opposite, that the group probably (but not definitely) should be accepted.

Example Problem 6.5

Statement: This problem is adapted from [18]. A manufacturer of bearings has compiled statistical information that shows the true variance in the diameter of "good" bearings is 3.15 μm^2. The manufacturer wishes to establish a batch rejection criterion such that only small samples need to be taken and assessed to check whether or not there is a flaw in the manufacturing process that day. The criterion states that when a batch sample of 20 manufactured bearings has a sample variance > 5.00 μm^2 the batch is to be rejected. This is because, most likely, there is a flaw in the manufacturing process. What is the probability that a batch sample will be rejected even though the true variance of the population from which it was drawn was within the tolerance limits or, in other words, of making a Type II error?

Solution: From Equation 6.32,

$$\chi^2_\alpha(\nu) \quad = \quad \nu S_x^2/\sigma^2 = \frac{(20-1)(5.00)}{(3.15)} = 30.16.$$

For this value of χ^2 and $\nu = 19$, $\alpha \cong 0.05$ using Table 6.5. So, there is approximately a 5 % chance that the discrepancy is due to random effects (that a new batch will be rejected even though its true variance is within the tolerance limits), or a 95 % chance that it is not. Thus, the standard for rejection is good. That is, the manufacturer should reject any sample that has $S_x^2 > 5$ μm^2. In doing so, he risks only a 5 % chance of falsely identifying a good batch as bad. If the χ^2 value equaled 11.7 instead, then there would be a 90 % chance that the discrepancy is due to random effects.

Now what if the size of the batch sample was reduced to $N = 10$? For this case, $\alpha = 0.0004$. So, there is a 0.04 % chance that the discrepancy is due to random effects.

In other words, getting a χ^2 value of 30.16 with a batch sample of 10 instead of 20 gives us even more assurance that the criterion is a good one.

6.6.3 Comparing Observed and Expected Distributions

In some situations, a sample distribution should be compared with an expected distribution to determine whether or not the expected distribution actually governs the underlying process. When comparing two distributions using a χ^2 analysis,

$$\chi^2 \approx \sum_{j=1}^{K} \frac{(O_j - E_j)^2}{E_j}, \tag{6.36}$$

with O_j and E_j the number of observed and expected occurrences in the j-th bin, respectively. The expected occurrence for the j-th bin is the product of the total number of occurrences, N, and the probability of occurrence, P_j. The probability of occurrence is the difference of the probability *distribution* function's values at the j-th bin's two end points. It also can be approximated by the product of the bin width and the probability *density* function value at the bin's mid-point value. Equation 6.36 follows from Equation 6.31 by noting that $\sigma^2 \sim \nu \sim E$. Strictly speaking, this expression is an *approximation* for χ^2 and is subject to the additional constraint that $E_j \geq 5$ [7]. The number of degrees of freedom, ν, are given by $\nu = K - (L + n)$, where K is the number of bins, preferably using Scott's formula for equal-width intervals that was described in Section 5.4. Here, $n = 2$ because two values are needed to compute the expected probabilities from the assumed distribution (one for the mean and one for the variance). There is an additional constraint ($L = 1$), because the number of expected values must be determined. Thus, whenever χ^2 analysis of this type is performed, $\nu = K - 3$.

From this type of analysis, agreement between observed and expected distributions can be ascertained with a certain confidence. The percent probability that the expected distribution is the correct one is specified by α. By convention, when $\alpha < 0.05$, the disagreement between the sample and expected distributions is *significant* or the agreement is *unlikely*. When $\alpha < 0.01$, the disagreement between the sample and expected distributions is *highly significant* or the agreement is *highly unlikely*.

Example Problem 6.6

Statement: Consider a study conducted by a professor who wishes to determine whether or not the 300 undergraduate engineering students in his department are normal with respect to their heights. He determines this by comparing the distribution of their heights to that expected for a normally distributed student population. His height data are presented in Table 6.6. Are their heights normal?

Bin number k	Heights in bin	Observed number, O_k	Expected number, E_k
1	less than $X - 1.5\sigma$	19	20.1
2	between $X - 1.5\sigma$ and $X - \sigma$	25	27.5
3	between $X - \sigma$ and $X - 0.5\sigma$	44	45.0
4	between $X - 0.5\sigma$ and X	59	57.5
5	between X and $X + 0.5\sigma$	60	57.5
6	between $X + 0.5\sigma$ and $X + \sigma$	45	45.0
7	between $X + \sigma$ and $X + 1.5\sigma$	30	27.5
8	above $X + 1.5\sigma$	18	20.1

TABLE 6.6
Observed and expected heights

Solution: For this case, $\nu = 8 - 3 = 5$, where $K = 8$ was determined using Scott's formula (actually, $K = 7.7$, which is rounded up). The expected values are calculated for each bin by noting that $E_k = NP_k$ where $N = 300$. For example, for bin 2 where $-1.5\sigma \leq x \leq -\sigma$, $P_k = Pr(-1.5\sigma \leq x \leq -\sigma) = Pr(z_1 = -1.5) - Pr(z_1 = -1) = 0.4332 - 0.3413$ (using Table 6.2) $= 0.0919$. So, the expected number in bin 2 is $(0.0919)(300) = 27.5$. The results for every bin are shown in Table 6.6.

Substitution of these results into Equation 6.36 yields $\chi^2 = 0.904$. For the values of $\chi^2_\alpha = 0.904$ and $\nu = 5$, from Table 6.5, $\alpha \simeq 0.97$. That is, the probability of obtaining this χ^2 value or less is ~97 %, under the assumption that the expected distribution is correct. Thus, agreement with the assumed normal distribution is *significant*.

6.7 *Pooling Samples

In some situations it may be necessary to combine the data gathered from M replicate experiments, each comprised of N measurands. The measurands can be *pooled* to form one set of MN measurands [18].

For the j-th experiment

$$\bar{x}_j = \frac{1}{N}\sum_{i=1}^{N} x_{ij} \text{ and } S^2_{x_j} = \frac{1}{N-1}\sum_{i=1}^{N}(x_{ij} - \bar{x}_j)^2. \quad (6.37)$$

From these expressions the following expressions can be developed. The mean of all $\bar{x}_j$'s, called the **pooled mean** of x, $\{\bar{x}\}$, the mean of the means, then becomes

$$\bar{\bar{x}} = \{\bar{x}\} = \frac{1}{M}\sum_{j=1}^{M}\bar{x}_j = \frac{1}{MN}\sum_{j=1}^{M}\sum_{i=1}^{N} x_{ij}. \quad (6.38)$$

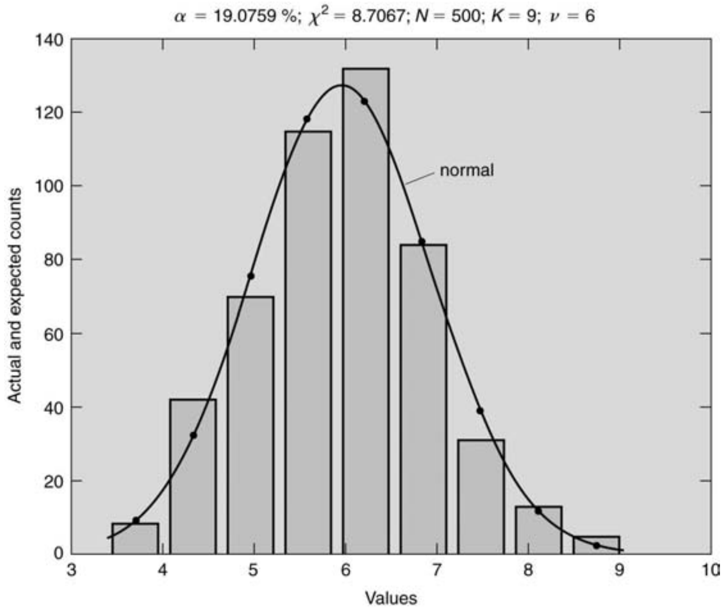

FIGURE 6.10
Analysis of 500 test scores using `chinormchk.m`.

The **pooled variance** of x, $\{S_x^2\}$, is actually the average of the variances of the M experiments where $\{S_x^2\}$ is treated as a random variable and is given by

$$\{S_x^2\} = \frac{1}{M}\sum_{j=1}^{M} S_{x_j}^2 = \frac{1}{M(N-1)}\sum_{j=1}^{M}\sum_{i=1}^{N}(x_{ij} - \bar{x}_j)^2. \tag{6.39}$$

The **pooled standard deviation**, $\{S_x\}$, is the positive square root of the pooled variance. The **pooled standard deviation of the means**, $\{S_{\bar{x}}\}$, is

$$\{S_{\bar{x}}\} = \frac{\{S_x\}}{\sqrt{MN}}. \tag{6.40}$$

Now consider when the number of measurands varies in each experiment, when N is not constant. There are N_j measurands for the j-th experiment. The resulting pooled statistical properties must be weighted by N_j. The **pooled weighted mean**, $\{\bar{x}\}_w$, is

$$\{\bar{x}\}_w = \frac{\sum_{j=1}^{M} N_j\bar{x}_j}{\sum_{j=1}^{M} N_j}. \tag{6.41}$$

	accept H_0	reject H_0
H_0 true	correct (1-α)	type I error (α)
H_0 false	type II error (β)	correct (1-β)

TABLE 6.7
Null hypothesis decisions and their associated probabilities and errors.

The **pooled weighted standard deviation**, $\{S_x\}_w$, is

$$\{S_x\}_w = \sqrt{\frac{\nu_1 S_{x_1}^2 + \nu_2 S_{x_2}^2 + ... + \nu_M S_{xM}^2}{\nu}}, \tag{6.42}$$

where

$$\nu = \sum_{j=1}^{M} \nu_j = \sum_{j=1}^{M} (N_j - 1). \tag{6.43}$$

The **pooled weighted standard deviation of the means**, $\{S_{\bar{x}}\}_w$, is

$$\{S_{\bar{x}}\}_w = \frac{\{S_x\}_w}{\left[\sum_{j=1}^{M} N_j\right]^{1/2}}. \tag{6.44}$$

6.8 *Hypothesis Testing

Hypothesis testing [6] incorporates the tools of statistics into a decision-making process. In the terminology of statistics, a null hypothesis is indicated by H_0 and an alternative hypothesis by H_1. The alternative hypothesis is considered to be the complement of the null hypothesis. There is the possibility that H_0 could be rejected, that is, considered false, when it is actually true. This is called a *Type I error*. Conversely, H_0 could be accepted, that is, considered true, when it is actually false. This is termed a *Type II error*. Type II errors are of particular concern in engineering. Sound engineering decisions should be based upon the assurance that Type II error is minimal. For example, if H_0 states that a structure will not fail when its load is less than a particular safety-limit load, then it is important to assess the probability that the structure can fail *below* the safety-limit load. This can be quantified by the power of the test, where the power is defined as 1 − probability of Type II error. For a fixed level of significance (see Section 6.6), the power increases as the sample size increases. Large values of power signify better precision. Null hypothesis decisions are summarized in Table 6.7.

Consider the rationale behind using statistical analysis to determine whether or not the mean of a population, x', will have a particular value, x_o. In an experiment, each measurand value will be subject to small, random variations because of minor, uncontrolled variables. The null hypothesis would be $H_0 : x' = x_o$ and the alternative hypothesis $H_1 : x' \neq x_o$. Because the alternative hypothesis would be true if either $x' < x_o$ or $x' > x_o$, the appropriate hypothesis test would be a *two-sided* t-test. If the the null hypothesis were either $H_0 : x' \leq x_o$ or $H_0 : x' \geq x_o$, then the appropriate hypothesis test would be a *one-sided* t-test. The modifier t implies that Student's t variable is used to assess the hypothesis. These tests implicitly require that all measurand values are provided such that their sample mean and sample standard deviation can be determined.

Decision of either hypothesis acceptance or rejection is made using Student's t distribution. For a one-sided t-test, if $H_0 : x' \leq x_o$, then its associated probability, $\Pr[X \leq t]$, must be determined. X represents the value of a single sample that is drawn randomly from a t-distribution with $\nu = N-1$ degrees of freedom. Likewise, if $H_0 : x' \geq x_o$, then its associated probability, $\Pr[X \geq t]$ must be found. For a two-sided t-test, the sum of the probabilities $\Pr[X \leq t]$ and $\Pr[X \geq t]$ must be determined. This sum equals $2\Pr[X \geq |t|]$ because of the symmetry of Student's t distribution. These probabilities are determined through Student's t value. For hypothesis testing, the particular t value, termed the *t-statistic*, is based upon the sample standard deviation of the means, where $t = (\bar{x} - x_o)/(S_x/\sqrt{N})$.

A p-value, sometimes referred to as the *observed* level of significance, is defined for the null hypothesis of a set of measurands as the probability of obtaining the measurand set or a set having less agreement with the hypothesis. The p-value is proportional to the plausibility of the null hypothesis. The criteria for accepting or rejecting the null hypothesis are the following:

- $p < 0.01$ indicates non-credible H_0, so reject H_0 and accept H_1.
- $0.01 \leq p \leq 0.10$ is inconclusive, so acquire more data.
- $p > 0.10$ indicates plausible H_0, so accept H_0 and reject H_1.

Sometimes, $p = 0.05$ is used as a decision value in order to avoid an inconclusive result, where $p < 0.05$ implies plausibility and $p > 0.05$ signifies non-credibility. Keep in mind that only the plausibility, not the exact truth, of a null hypothesis can be ascertained. Rejecting the null hypothesis of a two-sided test means $x' \neq x_o$. Accepting the null hypothesis implies that x_o is a plausible value of x', but not necessarily that $x_o = x'$. So, rejecting a null hypothesis is more exact statistically than accepting a null hypothesis. Rejecting the null hypothesis $H_0 : x' \leq x_o$ means $x' \geq x_o$. Accepting the null hypothesis indicates that, plausibly, $x' \leq x_o$. Again, it is more exact statistically to reject the null hypothesis or, conversely, to accept the alternative hypothesis. Hence, it is better to pose the null hypothesis such that its alternative hypothesis most likely will be accepted.

p-value	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
$p \geq 0.10$	accept	accept	accept
$0.05 < p < 0.10$	accept	accept	reject
$0.01 < p < 0.05$	accept	reject	reject
$p < 0.01$	reject	reject	reject

TABLE 6.8
Null hypothesis decisions and associated p and α values.

Stated differently, if x' is on the side of x_o that favors the null hypothesis, then the hypothesis should be accepted. If it is not, then the plausibility of the hypothesis must be ascertained, based upon the aforementioned p-value criteria. If the hypothesis test is one-sided, then $x' \leq x_o$ means $t \leq 0$ and $p > 0.50$, which indicates acceptance. Also, $x' >> x_o$ means $t > 0$ and $p \sim 0$, which implies rejection. Further, x' slightly greater than x_o means t is slightly greater than 0 and p is non-zero and finite, signifying plausibility. If the hypothesis test is two-sided, the larger the value of $|t|$, the farther away x' is from x_0. The p-value is calculated with the probability that a measurand set with $x' = x_o$ has a t-statistic with an absolute value greater than $|t|$. Any measurand set with a t-statistic that is greater than $|t|$ or less than -$|t|$ has less agreement with the null hypothesis. The acceptance or rejection of the null hypothesis is based upon the the same aforementioned p-value criteria. The following serves to illustrate how the p-values specifically are determined for one-sided and two-sided hypothesis tests.

Consider the one-sided test where $H_0 : x' \leq 10$. For this example, $\bar{x} = 12$, $S_x = 3$ and $N = 20$. Thus, the t-statistic value equals $(12 - 10)/(3\sqrt{20})$ = 2.981. For $\nu = 19$, the corresponding p-value equals $1 - \Pr[t \leq 2.981] = 1 - 0.9962 = 0.0038$. Thus, the null hypothesis is rejected and the alternative hypothesis, that x' is greater than 10, is accepted.

Next examine the two-sided test where $H_0 : x' = 10$. Using the same statistical parameters as in the previous example, where the t-statistic value is 2.981, the p-value equals $2\Pr[t \geq |2.981|] = (2)(0.0038) = 0.0076$. Here the absolute value of the t-statistic, 2.981, is greater than the p-value, 0.0076. So, the measurand set with $x' = 10$ has less agreement with the null hypothesis. In fact, because $p < 0.01$, the null hypothesis is not credible.

More specificity about accepting or rejecting a null hypothesis can be obtained by associating this decision with a level of significance, α. Here, the null hypothesis is accepted if the p-value is larger than α and rejected if the p-value is less than α. The level of significance is the probability of a Type I error. The relationships between null hypothesis acceptance or rejection and their associated p and α values are presented in Table 6.8.

Test no.	Set A δ_A (mm)	Set B δ_B (mm)	$\delta_A - \delta_B$ (mm)
1	3.0806	2.9820	0.0986
2	3.0232	2.9902	0.0330
3	2.9010	3.0728	-0.1718
4	3.1340	2.9107	0.2233
5	3.0290	2.9775	0.0514
6	3.1479	2.9348	0.2131
7	3.1138	2.9881	0.1257
8	2.9316	3.2303	-0.2987
9	2.8708	2.9090	-0.0382
10	2.9927	2.7979	0.1948

TABLE 6.9
Boundary-layer thickness measurements.

Example Problem 6.7

Statement: A test is conducted to assess the reliability of a transducer designed to indicate when the pressure in a vessel is 120 psi. The vessel pressure is recorded each time the sensor gives an indication. The test is repeated 20 times, resulting in a mean pressure at detection of 121 psi with a standard deviation of 3 psi. Determine the reliability of the transducer based upon a 5 % level of significance.

Solution: It is best statistically to test the null hypothesis that the transducer's detection level is 120 psi. The alternative hypothesis would be that the detection level is either less than or greater than 120 psi. This appropriate test is a two-sided t-test. For this case, the value of the t-statistic is

$$t = \frac{\bar{x} - x_o}{S_x/\sqrt{N}} = \frac{121 - 120}{3/\sqrt{20}} = 1.491.$$

For $\nu = 19$, the corresponding p-value equals $2P[|t| \geq 1.491] = (2)(0.0762) = 0.1524$. According to Table 6.8, the null hypothesis is acceptable. Thus, the transducer can be considered reliable. At this level of significance, the p-value would have to be less than 0.05 before the null hypothesis could be rejected and the transducer considered unreliable.

This type of analysis also can be employed to test the hypothesis that two measurand sets with *paired* samples (each having the same number of samples) come from the same population. This is illustrated by the following problem.

Example Problem 6.8

Statement: Ten thermal boundary-layer thickness measurements were made at a specific location along the length of a heat-exchanger plate. Ten other thickness measurements were made after the surface of the heat-exchanger plate was modified to improve its heat transfer. The results are shown in Table 6.9. Determine the percent confidence that the plate surface modification has no effect on the thermal boundary-layer thickness.

Solution: Assume that the thicknesses follow a t-distribution. This implies that the differences of the thicknesses for each set, $\delta_{A-B} = \delta_A - \delta_B$, also follow a t-distribution. The mean and the standard deviation of the differences can be computed from the sample data. They are 0.043 mm and 0.171 mm, respectively. Now if both samples come from the same population (here, this would imply that the surface modification had no detectable effect on the boundary-layer thickness), then the difference of their true mean values must be zero. Thus, the problem can be rephrased as follows. What is the confidence that a parameter with a mean value of 0.043 mm, $\bar{\delta}$, and a standard deviation of 0.171 mm, $S_{\bar{\delta}}$, determined from 10 samples, actually comes from a population whose mean value is zero?

This involves a two-sided hypothesis test. The null hypothesis is that the mean value of the differences is zero (that the surface modification has no detectable effect) and the alternative hypothesis is that the mean value of the differences is not zero (that the surface modification has a detectable effect). The t-statistic value for this case is

$$t = \frac{\sqrt{N}(\bar{\delta} - 0)}{S_{\bar{\delta}}} = \frac{\sqrt{10}(0.043)}{0.171} = 0.795.$$

For $\nu = 9$, the p-value equals $2P[t \geq |0.795|] = (2)(0.2235) = 0.4470$. Thus, there is approximately a 45 % chance that the surface modification has a detectable effect and a 55 % chance that it does not. So the present experiment gives an ambiguous result. If the mean of the thickness difference was smaller, say 0.020 mm, given everything else the same, then the p-value would be 0.7200, based upon $t = 0.37$. Now there is more confidence in the hypothesis that the surface modification has no detectable effect. However, this still is not significant enough. In fact, to have 95 % confidence in the hypothesis, the mean of the thickness difference would have to be 0.004 mm, given everything else is the same. This type of analysis can be extended further to experiments involving unpaired samples with or without equal variances [6].

6.9 *Design of Experiments

Statistical tools can be used in experimental planning. The method of **design of experiments** (DOE) provides an assessment of an experiment's output sensitivity to its independent variables. In DOE terminology, this method assesses the sensitivity of the *result* (the measurand or dependent variable) to various *factors* (independent variables) that comprise the *process* (experiment). The significance of DOE is that it can be carried out *before* an experiment is conducted. DOE, for example, can be used to identify the variables that most significantly affect the output. In essence, DOE provides an efficient way to plan and conduct experiments.

Methods of DOE have been known for many years. According to Hald [14], fundamental work on DOE was carried out by R. A. Fisher and published in 1935 [8]. DOE, and the related topic, Taguchi methods, have become popular in recent years because of the quest through experimentation for improved quality in consumer and industrial products (for example, see

	low	high
A(mm)	5	10
B(ms)	20	50

TABLE 6.10
Factors and their levels for the detector experiment.

[10], [11], and [12]). These methods also can be applied to forensic science [13].

The main objective of DOE is to determine how the factors influence the result of a process. The approach is to determine this by running trials (actual and/or computer trials), and measuring the process response for planned, controlled changes in the factors. A feature of DOE is that it provides a ranking of the sensitivity of the result to each factor; it ranks the factors in terms of their effects. It provides the direction of the sensitivity, whether a factor change increases or decreases the result. A major and important feature of DOE is that it provides this information with a minimum number of measurements or calculations. An additional advantage of DOE is that knowledge of the statistical nature of the input is unnecessary. If statistical information is available, DOE can lead directly to an analysis of variance (ANOVA) (for example, see [14]) and hypothesis tests for the factors and their interactions. If the statistical nature of some or all of the factors is unavailable, methods still exist to examine and rank the sensitivity.

The basics of DOE can be illustrated readily. Consider a hypothetical experiment with a result, y_j, that depends on a number of factors designated by A, B, C, ... Trials of the experiment are conducted for different, predetermined values of the factors, where the j-th run produces the j-th value of the result. The primary function of DOE is to determine a quantitative measure of the effect of each of the factor changes on the result. The output must be quantitative or measurable. The factors can be *quantitative*. They also can be *attribute* variables, such as hot or cold, fast or slow, and so forth. In the coverage here, factors will be allowed to take on only two values called a low level, indicated with a minus sign, and a high level, indicated with a plus sign. The high and low levels of the factors are selected by the experimenter to represent a practical range of values large enough to have an influence, yet small enough to determine the local behavior of the process. It is not uncommon to carry out one exploratory DOE to establish ranges of variables, and then perform another DOE, based on the results of the first, for a more refined analysis. In the case of attribute variables, such as fast and slow, the choice of high and low can be purely arbitrary. In the case of quantitative variables, the choice usually is intuitive, but it still remains arbitrary because the signs of the results reverse if the levels are reversed.

	A: low	A: high
B: low	40.9	47.8
B: high	42.4	50.2

TABLE 6.11
Percentage changes for four trials.

To illustrate the method of DOE, consider a hypothetical experiment in which an experimentalist wishes to assess the sensitivity of a light-level detector to two factors, the position of the detector from the surface of an object (factor A) and the time response of the detector (factor B). The percentage change in the amplitude of the detector's output from a reference amplitude is chosen as the result. Table 6.10 lists the factors and their levels.

Four trials are carried out. This provides a *complete* experiment, in which all four possible combinations of factor type and level are considered. In one trial, for example, the detector with the shortest time response, 20 ms (the low value of B), is placed near the surface, at 5 mm (the low value of A). The result for this case is an increase in the amplitude of 40.9 %, as displayed in Table 6.11. Examination of all of the results reveals that the greatest output is achieved by placing a 50-ms detector 10 mm from the surface of the object. DOE can be extended readily to consider more than two factors. To achieve a *complete* experiment, 2^k trials are required, where k is the number of factors, with two levels for each factor.

6.10 *Factorial Design

The method of DOE suggests a manner in which the contribution of each factor on an experimental result can be assessed. This is the method of **factorial design**. Because this method identifies the effect of each factor, it can be used to organize and minimize the number of experimental trials.

When dealing with the effects of changes in two factors, two levels and four runs, a measure of the effect, or *main effect*, ME, resulting from changing a factor from its low to high value can be estimated using the average of the two observed changes in the response. So for factor A,

$$ME_A = \frac{1}{2}\left[(y_2 - y_1) + (y_4 - y_3)\right]. \tag{6.45}$$

Similarly, a measure of the effect of changing factor B from its low to high level can be estimated by averaging two corresponding changes as

$$ME_B = \frac{1}{2}\left[(y_3 - y_1) + (y_4 - y_2)\right]. \tag{6.46}$$

Trial	A	B	AB	M	Response Total
1	-	-	+	+	T_1
2	+	-	-	+	T_2
3	-	+	-	+	T_3
4	+	+	+	+	T_4

TABLE 6.12
Sign pattern for two factors, A and B.

These average effects from four runs have significantly greater reliability than changes computed from three runs. In addition, an interaction may exist between the factors. An effect of the interaction can be estimated from the four runs by taking the differences between the diagonal averages, where

$$ME_{AB} = \frac{1}{2}(y_1 + y_4) - \frac{1}{2}(y_2 + y_3) = \frac{1}{2}\left[(y_1 - y_2) + (y_4 - y_3)\right]. \tag{6.47}$$

Finally, a measure of the overall level of the process can be based on the average as

$$ME = \frac{1}{4}\left[y_1 + y_2 + y_3 + y_4\right]. \tag{6.48}$$

Equations 6.45 through 6.48 provide the basic structure of a factorial design with two levels per factor and four runs. Examine the forms of the above main effect equations.

If the parentheses in Equations 6.45 through 6.47 are dropped and the responses are placed in the order of their subscripts, these equations become

$$ME_A = \frac{1}{2}\left[-y_1 + y_2 - y_3 + y_4\right], \tag{6.49}$$

$$ME_B = \frac{1}{2}\left[-y_1 - y_2 + y_3 + y_4\right], \tag{6.50}$$

and

$$ME_{AB} = \frac{1}{2}\left[+y_1 - y_2 - y_3 + y_4\right]. \tag{6.51}$$

A certain pattern of plus and minus signs from left to right appears in each equation. Table 6.12 lists the pattern of plus and minus signs in columns under each factor, A and B, the interaction, AB, and the overall gain, M. Note that the signs of the interaction are products of the signs of the factors.

A full set of trials often is repeated to permit estimation of the effects of the influence of uncontrolled variables. The above equations can be expressed more conveniently in terms of the sums or totals of the responses rather than the responses themselves. That is, for r runs or full sets of trials, the total, T_j, for the responses, y_{ji}, is given by adding the r responses as

$$T_j = \sum_{i=1}^{r} y_{ji}. \tag{6.52}$$

Experiments organized as above are referred to as 2^k designs, which yield four trials for two levels and two factors. A general form of Equations 6.49 through 6.51 that provides the estimates of the effects for k factors is [14]

$$ME_j = \frac{1}{r2^{k-1}} \left[\sum_{i=1}^{2^k} \pm T_i \right], \tag{6.53}$$

where $j = A, B, AB, \ldots$. For two factors, the proper signs for each term from left to right in Equation 6.53 are those signs in the column under the j-th factor of Table 6.12. For example, for $r = 1$, $y_j = T_j$, if the main effect of factor B is to be estimated, then Equation 6.53 gives

$$ME_B = \frac{1}{2} \left[-T_1 - T_2 + T_3 + T_4 \right], \tag{6.54}$$

where the signs are those in the column under B in Table 6.12. For values of $k \geq 2$, a listing of the sequence of signs is given in statistics texts (for example, see [15]). Note that Equation 6.53 must be modified to calculate the overall mean, M, of the responses, where,

$$M = \frac{1}{r2^{k}} \left[\sum_{i=1}^{2^k} +T_i \right], \tag{6.55}$$

in which $j = A, B, AB, \ldots$, and all the signs are plus signs.

The sensitivity of a process to a factor level change generally differs from factor to factor. A small change in one factor may cause a large change in the response while another does not. Sensitivity is the slope of the response curve. In factorial design, the sensitivity, ζ, of the response to a certain factor is the main effect divided by the change in the factor. For example, in the case of factor B,

$$\zeta_B = \frac{ME_B}{B^+ - B^-}. \tag{6.56}$$

A problem can arise with the application of Equation 6.56 when the factors are attributes rather than numeric. As a result, sensitivity is usually viewed as being determined directly by the main effects themselves.

The example presented in Section 6.9 now can be analyzed using factorial analysis. Suppose that two runs were conducted, giving the results that are presented in Table 6.13. With this information, the main effects can be computed using Equation 6.53 and using the sign patterns in Table 6.12. For example, the main effect of factor A, detector response time, using Equation 6.53, is

	A: low	**A: high**
B: low	$y_{11} = 40.9$ $y_{12} = 41.6$ $T_1 = 82.5$	$y_{21} = 47.8$ $y_{22} = 39.9$ $T_2 = 87.7$
B: high	$y_{31} = 42.4$ $y_{32} = 42.0$ $T_3 = 84.4$	$y_{41} = 50.2$ $y_{42} = 46.5$ $T_4 = 96.7$

TABLE 6.13
Percentage changes for four trials, each conducted twice.

$$ME_A = \frac{1}{4}[-82.5 + 87.7 - 84.4 + 96.7] = 4.4. \tag{6.57}$$

Similarly, $ME_B = 2.7$ and $ME_{AB} = 1.8$. These results can be interpreted. When going from the low to high level of factor A, that is, switching the detector position from 5 mm to 10 mm, there is a 4.4 % increase in the detector's amplitude. Similarly, when going from the low to high level of factor B, that is, changing from a 20 ms response detector to a 50 ms response detector, there is a 2.7 % increase in amplitude. The interaction main effect implies that changing from the combination of a 5 mm position of the 20 ms response detector to a 10 mm position of the 50 ms response detector increases the amplitude by 1.8 %. Finally, the average amplitude increase, M, is found from Equation 6.55, for $r = 2$ and $k = 2$, to be 43.9 %.

An inherent feature of any 2^k factorial design is that it is presented in a form that can be analyzed easily using ANalysis Of VAriance (ANOVA) (for example, see [6], [15]), provided there are two or more full sets of runs, that is, $r > 1$. ANOVA yields an important piece of information. It determines whether the effects of changing the levels of factors are statistically insignificant. If this is so, it means that uncontrolled variables were present in the experiment and caused changes greater than the controlled factor changes. If only one run is made, a 2^k design provides no measure of uncontrolled variations, known as the statistical error, and methods other than ANOVA must be used to measure significance.

6.11 Problem Topic Summary

Topic	Review Problems	Homework Problems
Normal	1, 2, 3, 4, 6, 9, 12, 14, 16, 21, 25	1, 2, 3, 9, 15, 17, 22, 24, 25
Student's t	5, 7, 8, 10, 11, 13, 17, 18, 22, 23, 24	3, 4, 5, 6, 8, 11, 14, 18, 19, 20, 21, 23, 24
Chi-square	15, 19, 20	6, 7, 10, 11, 12, 13, 15, 16, 25

TABLE 6.14
Chapter 6 Problem Summary

6.12 Review Problems

1. Given 1233 instantaneous pressure measurements that are distributed normally about a mean of 20 psi with a standard deviation of 0.5 psi, what is the probability that a measured value will be between 19 psi and 21 psi?

2. What is the probability, in decimal form, that a normally distributed variable will be within 1.500 standard deviations of the mean?

3. A laser pinpoints the target for an advanced aircraft weapons system. In a system test, the aircraft simulates targeting a flight-test aircraft equipped with an optical receiver. Data recorders show that the standard deviation of the angle of the beam trajectory is 0.1400° with a mean of 0°. The uncertainty in the angle of the beam trajectory is caused by precision errors, and the angle is distributed normally. What is the probability, in decimal form, that the aircraft laser system will hit a target 10 cm wide at a range of 100 m?

4. The average age of the dining clientele at a restaurant is normally distributed about a mean of 52 with a standard deviation of 20. What is the probability, in decimal form, of finding someone between the ages of 18 and 21 in the restaurant?

5. Each of 10 engineering students measures the diameter of a spherical ball bearing using dial calipers. The values recorded in inches are 0.2503,

0.2502, 0.2501, 0.2497, 0.2504, 0.2496, 0.2500, 0.2501, 0.2494, and 0.2502. With knowledge that the diameters of the ball bearings are distributed normally, find the probability that the diameter of a bearing is within +0.0005 and −0.0005 of the mean based on the sample statistics.

6. A series of acceleration measurements is normally distributed about a mean of 5.000 m/s^2 with a standard deviation of 0.2000 m/s^2. Find the value such that the probability of any value occurring below that value is 95 %.

7. Measured accelerations (in m/s^2), which are normally distributed, are 9.81, 9.87, 9.85, 9.78, 9.76, 9.80, 9.89, 9.77, 9.78, and 9.85. Estimate the range of acceleration within which the next measured acceleration would be at 99 % confidence.

8. From the data given in the previous problem, what range contains the true mean value of the acceleration?

9. If normal distribution (population) statistics are used (incorrectly!) to compute the sample statistics, find the percent probability that the next measured acceleration will be within the sample precision interval computed in the previous accelerometer problem. Compute statistics as if they were population statistics.

10. A student determines, with 95 % confidence, that the true mean value based upon a set of 61 values equals 6. The sample mean equals 4. Determine the value of the sample standard deviation to the nearest hundredth.

11. The values of x measured in an experiment are 5, 1, 3, and 6. Determine with 95 % confidence the upper value of the range that will contain the next x value.

12. A student determines, with 95 % confidence, that the true mean value based upon a set of N values equals 8. The sample mean equals 7. Assuming that N is very large (say > 100), determine the value of the standard deviation of the means to the nearest hundredth. Remember that the standard deviation of the means has a positive value.

13. The mean and standard deviation of a normally distributed population are 105 and 2, respectively. Determine the percent probability that a member of the population will have a value between 101 and 104.

14. The scores of the students who took the SAT math exam were normally distributed with a mean of 580 and a standard deviation of 60. Determine the percentage of students who scored greater than 750 to the nearest hundredth of a percent.

15. The percent probability that systematic effects have resulted in a χ^2 value greater, equal to, or greater than 25 based upon 16 measurements is (a) 5, (b) 10, (c) 90, (d) 95, or (e) 97.5.

16. Determine the percent probability that a student will score between 60 and 90 on an exam, assuming that the scores are normally distributed with a mean of 60 and a standard deviation of 15.

17. Determine the range of scores on a test within which 95 % of 12 students who took an exam having a mean of 60 and a standard deviation of 15, in whole numbers.

18. Given a mean and a standard deviation of 15 and 2.0, respectively, for a sample of 11, determine the range that contains the true variance, estimated at 95 % confidence.

19. A pressure pipeline manufacturer has performed wall thickness measurements for many years and knows that the true variance of the wall thickness of a pipe is 0.500 mm^2. If the variance of a sample is 1.02 mm^2, find the percent probability that this sample variance results from only random events. Assume a sample size of 16.

20. Determine the sample skewness for the measurand values of 7, 3, 1, 5, and 4.

21. An engineer has performed wall thickness measurements many times and knows that the true variance of the wall thickness of a pipe is 0.500 mm^2. If the rejection criterion for sample variance is 0.7921 mm^2 for a single wall, find the probability that the rejection criterion is good for a sample size of 21.

22. What is the probability that a student will score between 75 and 90 on an exam, assuming that the scores are distributed normally with a mean of 60 and a standard deviation of 15?

23. What is the probability that a student will score between 75 and 90 on an exam, assuming that the scores are based on only three students, with a mean of 60 and a standard deviation of 15?

24. Determine the probability that a student will score between 75 and 90 on an examination, assuming that the scores are based upon nine students, with a mean of 60 and a standard deviation of 15.

25. It is known that the statistics of a well-defined voltage signal are given by a mean of 8.5 V and a variance of 2.25 V^2. If a single measurement of the voltage signal is made, determine the probability that the measured value will be between 10 V and 11.5 V.

26. What are the units of the standardized normal variate and the normalized z variable?

6.13 Homework Problems

1. A February 14, 1997 *Observer* article cited a NCAA report on a famous midwestern university's admission gap between all 1992-95 entering freshmen and the subset of entering freshman football team members. The article reported that the mean SAT scores were 1220 for all entering freshmen and 894 for the football team members. Assume that the standard deviations of the SAT scores were 80 and 135 for all freshmen and all football team members, respectively. Determine (a) the percentage of all freshmen who scored greater than 1300 on their SATs, (b) the percentage of football players who scored greater than 1300 on their SATs, and (c) the number of football players who scored greater than half of all of the freshman class, assuming that there were 385 football players. State all assumptions.

2. Assume that students who took the SAT math exam were normally distributed about a mean value of 580 with a standard deviation of 60. Determine what percentage of the students scored higher than 750 on the exam.

3. Using MATLAB, determine for a class of 69 the percent probability, to four significant figures, of getting a test score within ± 1.5 standard deviations of the mean, assuming that the test scores are distributed according to (a) Student's t distribution and (b) the normal distribution.

4. During an experiment, an aerospace engineering student measures a wind tunnel's velocity N times. The student reports the following information, based on 90 % confidence, about the finite data set: mean velocity = 25.00 m/s, velocity standard deviation = 1.50 m/s, and uncertainty in velocity = ± 2.61 m/s. Determine (a) N, (b) the standard deviation of the means based upon this data set (in m/s), (c) the uncertainty, at 95 % confidence, in the estimate of the true mean value of the velocity (in m/s), and (d) the interval about the sample mean over which 50 % of the data in this set will be (in m/s).

5. An aerospace engineering student performs an experiment in a wind tunnel to determine the lift coefficient of an airfoil. The student takes 61 measurements of the vertical force using a force balance, yielding a sample mean value of 44.20 N and a sample variance of 4.00 N^2. Determine (a) the percent probability that an additional measurement will be between 45.56 N and 48.20 N, (b) the range (in N) over which the true mean value will be, assuming 90 % confidence, and (c) the range (in N^2) over which the true variance will be assuming 90 % confidence.

6. An airplane manufacturer intends to establish a component acceptance criterion that is based upon sound statistical methods. Preliminary tests on 61 acceptable components have determined that the mean load to produce component failure is 500 psi with a standard deviation of 25 psi. Based upon this information, provide (a) an estimate, with 99 % confidence, of the value of the next (the 62nd) measured load to produce failure, (b) an estimate, with 99 % confidence, of the true mean load to produce failure, and (c) an estimate, with 98 % confidence, of the true variance. Finally, the manufacturer wants to be 99 % confident that if the batch sample meets the acceptance criterion. (d) Determine the *range* of sample standard deviation values (in psi) that the batch sample can have and still meet the test criterion.

7. The sample mean of 21 golf ball weights equals 0.42 N, and the sample variance equals 0.04 N^2. Determine the range (in N^2) that contains the true variance, with 90 % confidence.

8. The values of $x = 5$, 3, 1, and 6 were measured in an experiment. Find the range within which will contain the next data point with 95 % confidence.

9. The mean and standard deviation of a normally distributed population of values x are $x^{'} = 105$ and $\sigma = 2$. Find the percent probability that a value of x will be in the range between 101 and 104.

E_k	6.4	13.6	13.6	6.4
O_k	8	10	16	6

TABLE 6.15
Expected and observed occurrences.

10. The expected number of occurrences, E_k, (assuming a normal distribution) and the observed number of occurrences, O_k, for 40 measurements are given in Table 6.15. Use the χ^2 test to determine the probability that the discrepancies between the observed and expected data are due to chance alone. The choices are (a) between 95 % and 90 %, (b) between 90 % and 50 %, (c) between 50 % and 5 %, and (d) between 5 % and 0 %.

11. For the data values of 1, 3, 5, 7, and 9, determine, with 95 % confidence, the values of the ranges that contain (a) the true mean, (b) the true variance, and (c) the next measured value if one more data point is taken.

12. A battery manufacturer guarantees that his batteries will last, on average, 3 years, with a standard deviation of 1 year. His claims are based

upon a very large population of his 'good' batteries. A consumer watch group decides to test his guarantee. Their small sample of his 'good' batteries indicates battery lifetimes (in years) of 1.9, 2.4, 3.0, 3.5, and 4.2. Determine (a) the percent confidence that the difference between the watch group's sample variance and manufacturer's true variance is due *solely* to random effects. Next, based upon the manufacturer's battery population average life time and standard deviation, (b) determine the probabilities that a battery lifetime will be less than 1.9 years and (c) between 3 and 4 years.

13. R measurand values have been obtained under steady-state operating conditions. An estimate of the value of the next measurand value, the $R + 1$ value, is between 2 and 12, and an estimate of the true mean value is between 2 and 4. Both estimates are made with 90 % confidence. Determine (a) the value of R and (b) the sample variance.

14. Given that the mean and standard deviation are 10 and 1.5, respectively, for a sample of 16, estimate with 95 % confidence, the ranges within which are (a) the true mean and (b) the true standard deviation.

15. The sample standard deviation of the length of 12 widgets taken off an assembly line is 0.20 mm. Determine the widgets population's standard deviation to support the conclusion that the probability is 50 % for any difference between the sample's and the population's standard deviations to be the result of random effects.

16. Determine the percent confidence that an experimenter should properly claim if the estimated true variance of a variable is between 6.16 and 14.6, based upon 31 measurements and a sample standard deviation of 3.

17. Assuming that the performance of a class is normally distributed (which in most cases it is not), (a) what is the probability that a student will score above a 99 % on the final exam if the mean is 76 % and the standard deviation is 11 %? (b) What if the mean is only 66 % but the standard deviation increases to 22 %?

18. The sample mean of 13 bowling balls measured from a manufacturing line is 10.12 lbf with a sample variance of 0.28 lbf^2. Determine the range (in N) that contains the true standard deviation of all the bowling balls assuming 90 % confidence.

19. A student, working on the development of an airship, wishes to determine the quality of his pressure transducer. During a controlled air-ship experiment, he measures the pressure (in psia) of 14.2, 14.2, 14.4, 14.8, and 14.5. Determine the upper value of the range within the which the next data point will be to the nearest hundredth at 95 % confidence.

20. Satisfied with the pressure transducer, an aviator takes his airship to an assumed altitude. However, he has no way of verifying the altitude. Therefore, he decides to measure air pressure at altitude and compare it to that in a table. He measures pressures (in psia) of 9.8, 9.9, 10.2, 9.0, 10.4, 10.1, 10.0, and 10.6. Determine the range that contains 95 % of the actual pressures.

21. A student determines that the true mean of a set of 31 values is 301.23 with 99 % confidence, while his sample mean equaled 299.89. What is the standard deviation of the sample? What is the standard deviation of the means?

22. Based on a large data base, the State Highway Patrol has determined that the average speed of Friday afternoon drivers on an interstate is 67 mph with a standard deviation of 4 mph. How many drivers of 1000 travelling on that interstate on Friday afternoon will be travelling in excess of 72 mph?

23. A small piece of cloth was found at the scene of a crime. One suspect was found wearing a sport coat having similar material. Ten fiber-diameter measurement tests were conducted on each of the two samples. The diameters (in mm) for cloth A were 3.0806, 3.0232, 2.9010, 3.1340, 3.0290, 3.1479, 3.1138, 2.9316, 2.8708, and 2.9927; for cloth B they were 2.9820, 2.9902, 3.0728, 2.9107, 2.9775, 2.9348, 2.9881, 3.2303, 2.9090, and 2.7979. What is the percent confidence that the crime-scene cloth was from the sport coat?

24. Using the data file `heights.txt` that contains the heights in centimeters of 500 college students, determine and plot the running sample mean and running sample standard deviation of the heights, that is, the sample mean and sample standard computed for each N (1 through 500). Also, provide the values for $N = 10$, $N = 100$, and $N = 500$.

25. The following problems use the data file `signal.dat` that contains two columns, each with 5000 rows of data (the first column is the measured velocity in m/s and the second column is the sample time in s). The velocities were measured behind an obstruction that contained several cables of different diameters. The data was taken over a period of 5 s at a sample rate of 1 kHz (1000 samples/s). Assume that the sample rate was fast enough such that the Nyquist sampling criterion was met. The two M-files `hf.m` and `chinormchk.m` may be useful. Do the following by using a given M-file, writing a program, or using a spreadsheet. (a) Plot the histogram of the velocities presented in the first column in the data file. Use Scott's formula to determine the required number of bins. (b) Plot the frequency distribution of the velocities. Use Scott's formula. (c) Plot the number of occurrences predicted by the normal distribution in histogram format along with the actual number of occurrences (as

done for the histogram above). This essentially amounts to overlaying the predicted values on the histogram constructed for the first problem. Use Scott's formula. Assume that the mean and the standard deviation of the normal distribution are the same as for the velocity data. (d) How well do the velocities compare with those predicted, assuming a normal distribution? What does it mean physically if the velocities are normally distributed for this situation?

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7

Uncertainty Analysis

CONTENTS

Whenever we choose to describe some device or process with mathematical equations based on physical principles, we always leave the real world behind, to a greater or lesser degree. ... These approximations may, in individual cases, be good, fair, or poor, but some discrepancy between modeled and real behavior always exists.

E.O. Doebelin. 1995. *Engineering Experimentation.* New York: McGraw-Hill.

A measurement result is complete only if it is accompanied by a quantitative expression of its uncertainty. The uncertainty is needed to judge whether the result is adequate for its intended purpose and whether it is consistent with other similar results.

Ferson, S., Kreinovich, V., Hajagos, J., Oberkampf, W. and Ginzburg, L. 2007. *Experimental Uncertainty Estimation and Statistics for Data Having Interval Uncertainty.* SAND2007-0939. Albuquerque: Sandia National Laboratories.

7.1 Chapter Overview

Uncertainty is one part of life that cannot be avoided. Its presence is a constant reminder of our limited knowledge and inability to control each and every factor that influences us. This especially holds true in the physical sciences. Whenever a process is quantified, by either modeling or experiments, uncertainty is present. In the beginning of this chapter the uncertainties present in modeling and experiments are identified. Agreement between modeling and experiment is described in the context of uncertainty. Measurement uncertainties are studied in detail. Conventional methods on how to characterize, quantify, and propagate them are presented. The generic cases of single or multiple measurements of a measurand or a result are considered. Then, numerical uncertainties associated with measurements are discussed. Finally, uncertainty that results from a lack of knowledge about a variable's specific value is considered.

7.2 Uncertainty

Aristotle first addressed uncertainty over 2300 years ago when he pondered the certainty of an outcome. However, it was not until the late 18th century that scientists considered the quantifiable effects of errors in measurements [1]. Continual progress on characterizing uncertainty has been made since then. Within the last 55 years, various methodologies for quantifying measurement uncertainty have been proposed [2]. In 1993, an international experimental uncertainty standard was developed by the International Organization for Standards (ISO) [3]. Its methodology now has been adopted by most of the international scientific community. In 1997 the National Conference of Standards Laboratories (NCSL) produced a U.S. guide almost identical to the ISO guide [4], "to promote consistent international methods in the expression of measurement uncertainty within U.S. standardization, calibration, laboratory accreditation, and metrology services." The American National Standards Institute with the American Society of Mechanical Engineers (ANSI/ASME) [5] and the American Institute of Aeronautics and Astronautics (AIAA) [6] also have new standards that follow the ISO guide. These new standards differ from the ISO guide only in that they use different names for the two categories of errors, random and systematic instead of Type A and Type B, respectively.

How is uncertainty categorized in the physical sciences? Whenever a physical process is quantified, uncertainties associated with modeling and computer simulation and/or with measurements can arise. Modeling and simulation uncertainties occur during the phases of "conceptual modeling of the physical system, mathematical modeling of the conceptual model, discretization and algorithm selection for the mathematical model, computer programming of the discrete model, numerical solution of the computer program model, and representation of the numerical solution" [7]. Such predictive uncertainties can be subdivided into modeling and numerical uncertainties [10]. Modeling uncertainties result from the assumptions and approximations made in mathematically describing the physical process. For example, modeling uncertainties occur when empirically based or simplified sub-models are used as part of the overall model. Modeling uncertainties perhaps are the most difficult to quantify, particularly those that arise during the conceptual modeling phase. Numerical uncertainties occur as a result of numerical solutions to mathematical equations. These include discretization, round-off, non-convergence, artificial dissipation, and related uncertainties. No standard for modeling and simulation uncertainty has been established internationally. Experimental or measurement uncertainties are inherent in the measurement stages of calibration and data acquisition. Numerical uncertainties also can occur in the analysis stage of the acquired data.

The terms *uncertainty* and *error* each have different meanings in modeling and experimental uncertainty analysis. **Modeling uncertainty** is defined as a *potential* deficiency due to a lack of knowledge and **modeling error** as a *recognizable* deficiency *not* due to a lack of knowledge [7]. According to Kline [8], **measurement error** is the difference between the true value and the measured value. It is a specific value. **Measurement uncertainty** is an estimate of the error in a measurement. It represents a range of possible values that the error might assume for a specific measurement. Additional uncertainty can arise because of a lack of knowledge of a *specific* measurand value within an interval of possible values, as described in Section 7.13.

By convention, the reported value of x is expressed with the same precision as its uncertainty, U_x, such as 1.25 $\pm$ 0.05. The magnitude of U_x depends upon the assumed confidence, the uncertainties that contribute to U_x, and how the contributing uncertainties are combined. The approach taken to determine U_x involves adopting an uncertainty standard, such as that presented in the ISO guide, identifying and categorizing all of the contributory uncertainties, assuming a confidence for the estimate, and then, finally, combining the contributory uncertainties to determine U_x. The types of error that contribute to measurement uncertainty must be identified first. The remainder of this chapter focuses primarily on measurement uncertainty analysis. Its associated numerical uncertainties are considered near the end of this chapter.

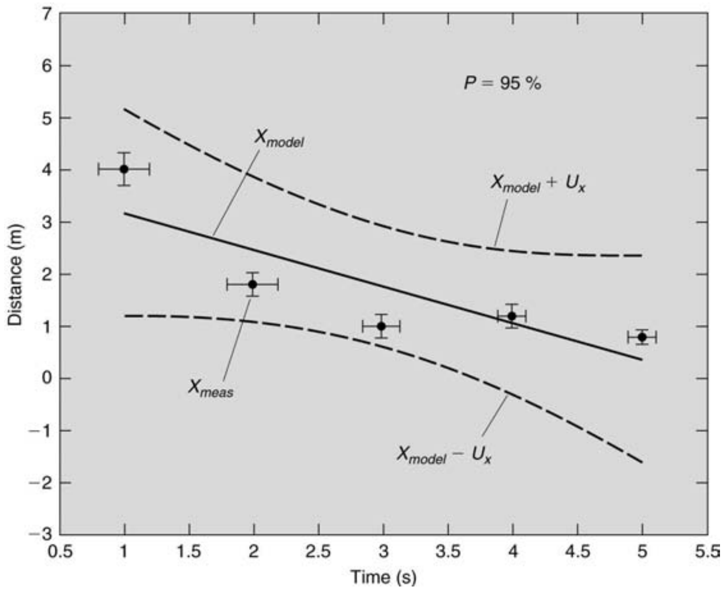

FIGURE 7.1
Graphical presentation of a comparison between predictive and experimental results.

7.3 Comparing Theory and Measurement

Given that both modeling and experimental uncertainties exist, what does agreement between the two mean? How should such a comparison be illustrated? Conventionally, the experimental uncertainty typically is denoted in a graphical presentation by error bars centered on measurement values and the modeling uncertainty by dashed or dotted curves on both sides of the theoretical curve. Both uncertainties should be estimated with the same statistical confidence. An example is shown in Figure 7.1. When all data points and their error bars are within the model's uncertainty curves, then the experiment and theory are said to agree *completely* within the assumed confidence. When some data points and either part or all of their error bars lay within the predictive uncertainty curves, then the experiment and theory are said to agree *partially* within the assumed confidence. There is *no* agreement when all of the data points and their error bars are outside of the predictive uncertainty curves.

Does agreement between experiment and theory imply that both correctly represent the process under investigation? Not all of the time. Caution must be exercised whenever such comparisons are made. Agreement between theory and experiment necessarily does not imply correctness. This issue has been addressed by Aharoni [11], who discusses the interplay between good and bad theory and experiments. There are several possibilities. A bad theory can agree with bad data. The wrong theory can agree with a good experiment by mere coincidence. A correct theory may disagree with the experiment simply because an unforeseen variable was not considered or controlled during the experiment. Therefore, agreement does not necessarily assure correctness.

Caution also must be exercised when arguments are made to support agreement between theory and experiment. Scientific data can be misused [12]. The data may have been presented selectively to fit a particular hypothesis while ignoring other hypotheses. Indicators may have been chosen with units to support an argument, such as the number of automobile deaths per journey instead of the number per kilometer travelled. Inappropriate scales may have been used to exaggerate an effect. Taking the logarithm of a variable appears graphically as having less scatter. This can be deceptive. Other illogical mistakes may have been made, such as confusing cause and effect, and implicitly using unproven assumptions. Important factors and details may have been neglected that could lead to a different conclusion. *Caveat emptor!*

Example Problem 7.1

Statement: Lemkowitz *et al.* [12] illustrate how different conclusions can be drawn from the same data. Consider the number of fatalities per 100 million passengers for two modes of transport given in the 1992 British fatality rates. For the automobile, there were 4.5 fatalities per journey and 0.4 fatalities per km. For the airplane, there were 55 fatalities per journey and 0.03 fatalities per km. Therefore, on a per km basis airplane travel had approximately 10 times fewer fatalities than the automobile. Yet, on a per journey basis, the auto had approximately 10 times fewer fatalities. Which of the two modes of travel is safer and why?

Solution: There is no unique answer to this question. In fact, on a per hour basis, the automobile and airplane have the the *same* fatality rate, which is 15 fatalities per 100 million passengers. Perhaps driving for shorter distances and flying for longer distances is safer than the converse.

7.4 Uncertainty as an Estimated Variance

When measurements are made under fixed conditions, the recorded values of a variable still will vary to an extent. This implicitly is caused by small variations in uncontrolled variables. The extent of these variations can be characterized by the variance of the values. Further, as the number of acquired values becomes large ($N \geq 30$), the sample variance approaches its true variance, σ^2, and the distribution evolves to a normal distribution.

Thus, if uncertainty is considered to represent the range within which an acquired value will occur, then uncertainty can be viewed as an estimate of the true variance of a normal distribution.

Consider the most general situation of a result, r, where $r = r(x_1, x_2, ...)$ and $x_1, x_2, ...,$ represent measured variables whose distributions are normal. In uncertainty analysis, a **result** is defined as a variable that is not measured but is related functionally to variables that are measured, which are called **measurands**. The result's variance is defined as

$$\sigma_r^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(r_i - r')^2\right], \tag{7.1}$$

where N is the number of determinations of r based upon the set of $x_1, x_2, ...$ measurands. The difference between a particular r_i value and its mean value, r', can be expressed in terms of a Taylor series expansion and the measurands' differences by

$$(r_i - r') \simeq (x_{1_i} - x_1')\left(\frac{\partial r}{\partial x_1}\right) + (x_{2_i} - x_2')\left(\frac{\partial r}{\partial x_2}\right) + ... \tag{7.2}$$

In this equation the higher order terms involving second derivatives and beyond are assumed to be negligible. Equation 7.2 can be substituted into Equation 7.1 to yield

$$\begin{aligned}\sigma_r^2 &\simeq \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left[(x_{1_i} - x_1')\left(\frac{\partial r}{\partial x_1}\right) + (x_{2_i} - x_2')\left(\frac{\partial r}{\partial x_2}\right) + ...\right]^2 \\ &\simeq \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}[(x_{1_i} - x_1')^2\left(\frac{\partial r}{\partial x_1}\right)^2 + (x_{2_i} - x_2')^2\left(\frac{\partial r}{\partial x_2}\right)^2 \\ &\quad +2(x_{1_i} - x_1')(x_{2_i} - x_2')\left(\frac{\partial r}{\partial x_1}\right)\left(\frac{\partial r}{\partial x_2}\right) + ...].\end{aligned} \tag{7.3}$$

The first two terms on the right side of Equation 7.3 are related to the variances of x_1 and x_2, where

$$\sigma_{x_1}^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(x_{1_i} - x_1')^2\right] \tag{7.4}$$

and

$$\sigma_{x_2}^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(x_{2_i} - x_2')^2\right]. \tag{7.5}$$

The third term is related to the **covariance** of x_1 and x_2, σ_{x_1,x_2}, where

$$\sigma_{x_1,x_2} = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(x_{1_i} - x_1')(x_{2_i} - x_2')\right]. \tag{7.6}$$

When x_1 and x_2 are statistically independent, $\sigma_{x_1x_2} = 0$. Substituting Equations 7.4, 7.5, and 7.6 into Equation 7.3 gives

$$\sigma_r^2 \simeq \sigma_{x_1}^2 \left(\frac{\partial r}{\partial x_1}\right)^2 + \sigma_{x_2}^2 \left(\frac{\partial r}{\partial x_2}\right)^2 + 2\sigma_{x_1x_2} \left(\frac{\partial r}{\partial x_1}\right)\left(\frac{\partial r}{\partial x_2}\right) + \dots . \quad (7.7)$$

Equation 7.7 can be extended to relate the uncertainty in a result as a function of measurand uncertainties. Defining the squared uncertainty u_i^2 as an estimate of the variance σ_i^2, Equation 7.7 becomes

$$u_c^2 \simeq u_{x_1}^2 \left(\frac{\partial r}{\partial x_1}\right)^2 + u_{x_2}^2 \left(\frac{\partial r}{\partial x_2}\right)^2 + 2u_{x_1x_2} \left(\frac{\partial r}{\partial x_1}\right)\left(\frac{\partial r}{\partial x_2}\right) + \dots . \quad (7.8)$$

This equation shows that the uncertainty in the result is a function of the estimated variances (uncertainties) u_{x_1} and u_{x_2} and their estimated covariance $u_{x_1x_2}$. It forms the basis for more detailed uncertainty expressions that are developed in the remainder of this chapter and used to estimate the overall uncertainty in a variable. u_c^2 is called the **combined estimated variance**. The **combined standard uncertainty** is u_c. This is denoted by u_r for a result and by u_m for a measurand. In order to determine the combined standard uncertainty, the types of errors that contribute to the uncertainty must be examined first.

7.5 Systematic and Random Errors

When a single measurement is performed, a number is assigned that represents the magnitude of the sensed physical variable. Because the measurement system used is not perfect, an error is associated with that number. If the system's components have been calibrated against more accurate standards, these standards have their own inaccuracies. The act of calibration itself introduces further uncertainty. All these factors contribute to the measurement uncertainty of a single measurement. Further, when the measurement is repeated, its value most likely will not be the same as it was the first time. This is because small, imperceptible changes in variables that affect the measurement have occurred in the interim, despite any attempts to perform a controlled experiment. Fortunately, almost all experimental uncertainties can be estimated, provided there is a consistent framework that identifies the types of uncertainties and establishes how to quantify them. The first step in this process is to identify the types of errors that give rise to measurement uncertainty.

Following the convention of the 1998 ANSI/ASME guidelines [5], the errors that arise in the measurement process can be categorized into either

systematic (bias) or **random** (precision) **errors**. Systematic errors sometimes can be difficult to detect and can be found and minimized through the process of **calibration**, which is the comparison with a true, known value. They determine the **accuracy** of the measurement. Further, they lack any statistical information. The systematic error of the experiment whose results are illustrated in Figure 7.2 is the difference between the true mean value and the sample mean value. In other words, if the estimate of a quantity does not equal the actual value of the quantity, then the quantity is biased. Random errors are related to the scatter in the data obtained under fixed conditions. They determine the **precision**, or repeatability, of the measurement. The random error of the experiment whose results are shown in Figure 7.2 is the difference between a confidence limit (either upper or lower) and the sample mean value. This confidence limit is determined from the standard deviation of the measured values, the number of measurements, and the assumed percent confidence. Random errors are statistically quantifiable. Therefore, an ideal experiment would be highly accurate *and* highly repeatable. High repeatability alone does not imply minimal error. An experiment could have hidden systematic errors and yet highly repeatable measurements, thereby always yielding approximately the same, yet inaccurate, values. Experiments having no bias but poor precision also are undesirable. In essence, systematic errors can be minimized through careful calibration. Random errors can be reduced by repeated measurements and the careful control of conditions.

Example Problem 7.2

Problem Statement: Some car rental agencies use an onboard global positioning system (GPS) to track an automobile. Assume that a typical GPS's precision is 2 % and its accuracy is 5 %. Determine the combined standard uncertainty in position indication that the agency would have if [1] it uses the GPS system as is, and [2] it recalibrates the GPS to within an accuracy of 1 %.

Problem Solution: Denote the precision uncertainty by u_p and the accuracy as u_a. The combined uncertainty, u_c, obtained by applying Equation 7.8, is

$$u_c = \sqrt{u_a^2 + u_p^2}. \tag{7.9}$$

For case [1], $u_c = \sqrt{29} = 5.39$. For case [2], $u_c = \sqrt{5} = 2.24$. So the re-calibration decreases the combined uncertainty by 3.15 %.

Examine two circumstances that help to clarify the difference between systematic and random errors. First, consider a stop watch used to measure the time that a runner crosses the finish line. If the physical reaction times of a timer who is using the watch are equally likely to be over or under by some average reaction time in starting and stopping the watch, the measurement of a runner's time will have a random error associated with it. This error could be assessed statistically by determining the variation in the recorded times of a same event whose time is known exactly by another method. If

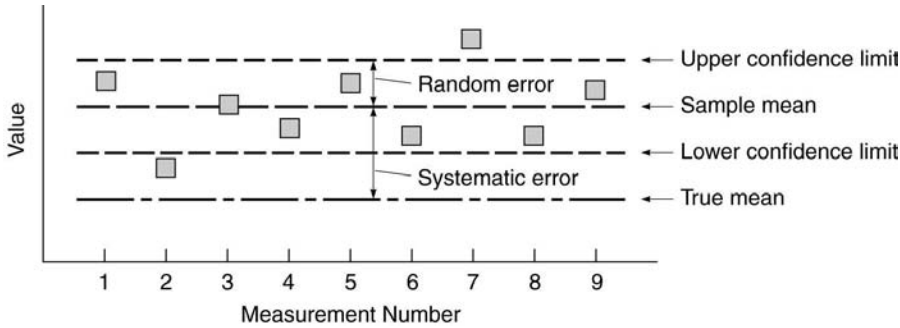

FIGURE 7.2
Nine recorded values of the same measurand.

the watch does not keep exact time, there is a systematic error in the time record itself. This error can be ascertained by comparing the mean value of recorded times of a same event with the exact time. Further, if the watch's inaccuracy is comparable to the timer's reaction time, the systematic error may be hidden within (or negligible with respect to) the random error. In this situation, the two types of errors are very hard to distinguish.

Next consider an analog panel meter that is used to record a signal. If the meter is read consistently from one side by an experimenter who records the reading, this method introduces a systematic error into the measurements. If the meter is read head-on, but not exactly so in every instance, then the experimenter introduces a random error into the recorded value.

For both examples, the systematic and random errors combine to determine the overall uncertainty. This is illustrated in Figure 7.2, which shows the results of an experiment. A **sample** of $N = 9$ recordings of the measurand, x, were made under fixed operating conditions. The *true* mean value of x, x_{true}, would be the sample mean value of the nine readings if no uncertainties were present. However, because of systematic and random errors, the two mean values differ. Fortunately, an estimate of the true mean value can be made to within certain confidence limits by using the sample mean value and methods of uncertainty analysis.

7.6 Measurement Process Errors

The experimental measurement process itself introduces systematic and random errors. Most experiments can be categorized either as **timewise** or as **sample-to-sample** experiments. In a timewise experiment, measurand values are recorded sequentially in time. In a sample-to-sample experiment, measurand values are recorded for multiple samples. The random error determined from a series of repeated measurements in a timewise experiment performed under steady conditions results from small, uncontrollable factors that vary during the experiment and influence the measurand. Some errors may not vary over short time periods but will over longer periods. So, the effect of the measurement time interval must be considered [5]. In the analogous sample-to-sample experiment, the random error arises from both sample-to-sample measurement system variability, and variations due to small, uncontrollable factors during the measurement process.

Errors that are not related directly to measurement system errors can be identified through repeating and replicating an experiment. In measurement uncertainty analysis, **repetition** implies that measurements are repeated in a particular experiment under the *same* operating conditions. **Replication** refers to the duplication of an experiment having *similar* experimental conditions, equipment, and facilities. The specific manner in which an experiment is replicated helps to identify various kinds of error. For example, replicating an experiment using a similar measurement system and the same conditions will identify the error resulting from using similar equipment. The definitions of repetition and replication differ from those commonly found in statistics texts (for example, see [14]), which consider an experiment repeated n times to be replicated $n+1$ times, with no changes in the fixed experimental conditions, equipment, or facility.

The various kinds of errors can be identified by viewing the experiment in the context of different **orders of replication levels**. At the **zeroth-order** replication level, only the errors inherent in the measurement system are present. This corresponds to either absolutely steady conditions in a timewise experiment or a single, fixed sample in a sample-to-sample experiment. This level identifies the smallest error that a given measurement system can have. At the **first-order** replication level, the additional random error introduced by small, uncontrolled factors that occur either timewise or from sample to sample are assessed. At the ***N*th-order** replication level, further systematic errors beyond the first-order level are considered. These, for example, could come from using different but similar equipment.

Measurement process errors originate during the calibration, measurement technique, data acquisition, and data reduction phases of an experiment [5]. Calibration errors can be systematic or random. Large systematic errors are reduced through calibration usually to the point where they are indistinguishable with inherent random errors. Uncertainty propagated from calibration against a more accurate standard still reflects the uncertainty of that standard. The order of standards in terms of increasing calibration errors proceeds from the primary standard through inter-laboratory, transfer, and working standards. Typically, the uncertainty of a standard used in a calibration is **fossilized**, that is, it is treated as a fixed systematic error in that calibration and in any further uncertainty calculations [13]. Data acquisition errors originate from the measurement system's components, including the sensor, transducer, signal conditioner, and signal processor. These errors are determined mainly from the elemental uncertainties of the instruments. Data reduction errors arise from computational methods, such as a regression analysis of the data, and from using finite differences to approximate derivatives and integrals. Other errors come from the techniques and methods used in the experiment. These can include uncertainties from uncontrolled environmental effects, inexact sensor placement, instrument disturbance of the process under investigation, operational variability, and so forth. All these errors can be classified as either systematic or random errors.

7.7 Quantifying Uncertainties

Before an overall uncertainty can be determined, analytical expressions for systematic and random errors must be developed. These expressions come from probabilistic considerations. Random error results from a large number of very small, uncontrollable effects that are independent of one another and individually influence the measurand. These effects yield measurand values that differ from one another, even under fixed operating conditions. It is reasonable to assume that the resulting random errors will follow a Gaussian distribution. This distribution is characterized through the mean and standard deviation of the random error. Because uncertainty is an estimate of the error in a measurement, it can be characterized by the standard deviation of the random error. Formally, the **random uncertainty** (precision limit) in the value of the measurand x is

$$P_x = t_{\nu_{P_x},C} \cdot S_{P_x}, \tag{7.10}$$

where S_{P_x} is the standard deviation of the random error, $t_{\nu_{P_x},C}$ is Student's t variable based upon ν_{P_x} degrees of freedom, and C is the confidence level. Likewise, the random uncertainty in the *average* value of the measurand x determined from N measurements is

$$P_{\bar{x}} = t_{\nu_{P_x},C} \cdot S_{P_{\bar{x}}} = t_{\nu_{P_x},C} \cdot S_{P_x}/\sqrt{N}. \tag{7.11}$$

Usually Equation 7.10 is used for experiments involving the single measurement of a measurand and Equation 7.11 for multiple measurements of a measurand. The degrees of freedom for the random uncertainty in x are

$$\nu_{P_x} = N - 1. \tag{7.12}$$

Systematic uncertainty can be treated in a similar manner. Although systematic errors are assumed to remain constant, their estimation involves the use of statistics. Following the ISO guidelines, systematic errors are assumed to follow a Gaussian distribution. The **systematic uncertainty** (bias limit) in the value of the measurand x is denoted by B_x. The value of B_x has a **reliability** of ΔB_x. Typically, a manufacturer provides a value for an instrument's accuracy. This number is assumed to be B_x. The value of the reliability is an estimate of the accuracy and is expressed in units of B_x. Hence, the lower its value, the more the confidence that is placed in the reported value of the accuracy. Formally, the systematic uncertainty in the value of x is

$$B_x = t_{\nu_{B_x},C} \cdot S_{B_x}, \tag{7.13}$$

where S_{B_x} is the standard deviation of the systematic error. Equation 7.13 can be rearranged to determine S_{B_x} for a given confidence and value of B_x. For example, $S_{B_x} \cong B_x/2$ for 95 % confidence. Finally, according to the ISO guidelines, the degrees of freedom for the systematic uncertainty in x are

$$\nu_{B_x} \cong \frac{1}{2}\left(\frac{\Delta B_x}{B_x}\right)^{-2} = \frac{1}{2}\left(\frac{\Delta S_{B_x}}{S_{B_x}}\right)^{-2}. \tag{7.14}$$

The quantity $\Delta B_x/B_x$ is termed the **relative systematic uncertainty** of B_x. More certainty in the estimate of B_x implies a smaller ΔB_x and, hence, a larger ν_{B_x}. One-hundred percent certainty corresponds to $\nu_{B_x} = \infty$. This effectively means that an infinite number of measurements are needed to assume 100 % certainty in the stated value of B_x.

Example Problem 7.3

Statement: A manufacturer states the the accuracy of a pressure transducer is 1 psi. Assuming that the reliability of this value is 0.5 psi, determine the relative systematic uncertainty in a pressure reading using this transducer and the degrees of freedom in the systematic uncertainty of the pressure.

Solution: According to the definition of the relative systematic uncertainty, $\Delta B_x/B_x = 0.5/1 = 0.5$ or 50 %. From Equation 7.14, the degrees of freedom for the systematic uncertainty in the pressure are $\frac{1}{2}(0.5)^{-2} = 2$. This is a relatively lower number, which reflects the high relative systematic uncertainty.

7.8 Measurement Uncertainty Analysis

Experimental uncertainty analysis involves the identification of errors that arise during all stages of an experiment and the propagation of these errors into the overall uncertainty of a desired result. The specific goal of uncertainty analysis is to obtain a value of the **overall uncertainty** (expanded uncertainty), U_x, of the variable, x, such that either the next-measured value, x_{next}, or the true value, x_{true}, can be estimated. The value of the next member of the sample of acquired data is represented by x_{next}. The value of the mean of the population from which the sample was drawn is given by x_{true}. For the case of a single measurement or result based on x, this is expressed as

$$x_{next} = x \ \pm U_x \ \ (\%C), \tag{7.15}$$

in which the estimate is obtained with $\%C$ confidence. For the case of a multiple measurement or result based on x, this becomes

$$x_{true} = \bar{x} \ \pm U_x \ \ (\%C), \tag{7.16}$$

in which $\bar{x}$ denotes the sample average of x. For either case, the overall uncertainty, U_x, can be expressed as

$$U_x = k \cdot u_c, \tag{7.17}$$

where k is the **coverage factor** and u_c the combined standard uncertainty. According to the ISO guidelines, the coverage factor is represented by the Student's t variable that is based upon the number of effective degrees of freedom. That is,

$$U_x = t_{\nu_{eff},C} \cdot u_c. \tag{7.18}$$

This assumes a normally distributed measurement error with zero mean and σ^2 variance. A zero mean implies that all significant systematic errors have been identified and removed from the measurement system prior to acquiring data. How these uncertainties contribute to the combined standard uncertainty and how they determine ν_{eff} depends upon the type of experimental situation encountered. The value of ν_{eff} is determined knowing the values of ν_{P_x} and ν_{B_x} given by Equations 7.12 and 7.14, respectively.

There are four situations that typify most experiments:

1. The single-measurement measurand experiment, in which the value of the measurand is based upon a single measurement (example: a measured temperature).

2. The single-measurement result experiment, in which the value of a result depends upon single values of one or more measurands (example: the volume of a cylinder based upon its length and diameter measurements).

3. The multiple-measurement measurand experiment, in which the mean value of a measurand is determined from a number of repeated measurements (example: the average temperature determined from a series of temperature measurements).

4. The multiple-measurement result experiment, in which the mean value of a result depends upon the values of one or more measurands, each determined from the same or a different number of measurements (example: the mean density of a perfect gas determined from N_1 temperature and N_2 pressure measurements).

Two standard types of uncertainty analysis can be used for experiments, general and detailed. Which type of uncertainty analysis applies to a particular experiment is not fixed. General uncertainty analysis usually applies to the first two situations and detailed uncertainty analysis to the latter two. Coleman and Steele [10] give a thorough presentation of both types.

General uncertainty analysis is a simplified approach that considers each measurand's overall uncertainty and its propagation into the final result. It does not consider the specific systematic and random errors that contribute to the overall uncertainty. This type of analysis typically is done during the planning stage of an experiment. It helps to identify sources of error and their contribution to the overall uncertainty. It also aids in determining whether or not a particular measurement system is appropriate for a planned experiment.

Detailed uncertainty analysis is a more thorough approach that identifies the systematic and random errors contributing to each measurand's overall uncertainty. The propagation of systematic and random errors into the final result is computed in parallel. The framework of detailed uncertainty analysis in this text is consistent with that presented in the ISO guide [3]. This type of uncertainty analysis usually is done for more-involved experimental designs and follows the calibration, data acquisition, and data reduction phases of an experiment.

In the following, each of the four common situations will be examined in more detail using the uncertainty analysis approach that is most appropriate.

7.9 General Uncertainty Analysis

General uncertainty analysis is most applicable to experimental situations involving either a single-measurement measurand or a single-measurement result. The uncertainty of a single-measurement measurand is related to its instrument uncertainty, which is determined from calibration, and to the resolution of instrument used to read the measurand value. The uncertainty of a single-measurement result comes directly from the uncertainties of its associated measurands. The expressions for these uncertainties follow directly from Equation 7.8.

For the case of J measurands, the combined standard uncertainty in a result becomes

$$u_r^2 \simeq \sum_{i=1}^{J} (\theta_i)^2 u_{x_i}^2 + 2 \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} (\theta_i)(\theta_j) u_{x_i,x_j}, \tag{7.19}$$

where

$$u_{x_i,x_j} = \sum_{k=1}^{L} (u_i)_k (u_j)_k, \tag{7.20}$$

with L being the number of elemental error sources that are common to measurands x_i and x_j, and $\theta_i = \partial r / \partial x_i$. θ_i is the **absolute sensitivity coefficient**. This coefficient should be evaluated at the expected value of x_i. Note that the covariances in Equation 7.19 should not be ignored simply for convenience when performing an uncertainty analysis. Variable interdependence should be assessed. This occurs through common factors, such as ambient temperature and pressure or a single instrument used for different measurands.

When the covariances are negligible, Equation 7.19 for J *independent* variables simplifies to

$$u_r^2 \simeq \sum_{i=1}^{J} \left(\theta_i u_{x_i}\right)^2, \tag{7.21}$$

where u_{x_i} is the **absolute uncertainty**. The values of the result's uncertainty will follow Student's t distribution [3], based upon the number of **effective degrees of freedom**, ν_{eff}, with

$$\nu_{eff} = \nu_r = \frac{u_r^4}{\sum_{i=1}^{J} (\theta_i^4 u_{x_i}^4)/\nu_i} = \frac{[\sum_{i=1}^{J} \theta_i^2 u_{x_i}^2]^2}{\sum_{i=1}^{J} (\theta_i^4 u_{x_i}^4)/\nu_i}, \tag{7.22}$$

where $\nu_i = N_i - 1$ is the number of degrees of freedom for u_{x_i} and $\nu_{eff} \leq \sum_{i=1}^{J} \nu_i$. This equation, known as the Welch-Satterthwaite formula, was presented originally by Welch [9]. The value of ν_{eff} obtained from the formula is rounded to the nearest whole number.

Equation 7.19 can be applied to estimate the uncertainty in a measurand. In this case, all the absolute sensitivity coefficients equal unity, and Equation 7.19 reduces to

$$u_m^2 \simeq \sum_{i=1}^{J} u_{x_i}^2 + 2 \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} u_{x_i,x_j}, \tag{7.23}$$

where u_m is the measurand uncertainty. Further, when the covariances are negligible, Equation 7.23 simplifies to

$$u_m^2 \simeq \sum_{i=1}^{J} u_{x_i}^2. \tag{7.24}$$

The corresponding number of effective degrees of freedom becomes

$$\nu_{eff} = \nu_m = \frac{u_m^4}{\sum_{i=1}^{J} (u_{x_i}^4/\nu_i)} = \frac{[\sum_{i=1}^{J} u_{x_i}^2]^2}{\sum_{i=1}^{J} (u_{x_i}^4/\nu_i)}. \tag{7.25}$$

Example Problem 7.4

Statement: Two pressure transducers are used to determine the pressure difference, $\Delta P = P_2 - P_1$, between a wind tunnel's reference pressure tap and a static pressure tap on the surface of an airfoil placed inside the wind tunnel. Both transducers are identical and have a reported accuracy of 1 %, as determined by the manufacturer's calibration. An experimenter decides to recalibrate these transducers against a laboratory standard that has 0.3 % accuracy. Further, the recalibration method itself introduces an additional 0.5 % uncertainty. Determine the combined standard uncertainty in ΔP for two situations: one when the experimenter does not recalibrate the transducers (case A) and the other when she does (case B).

Solution: Both situations involve the determination of the uncertainty in a measurand. When the transducers are not recalibrated, each transducer has an uncertainty of 1 % and the uncertainties are independent. According to Equation 7.21, $u^2_{\Delta P_A} = u^2_{P_1} + u^2_{P_2}$. Thus, $u_{\Delta P_A} = \sqrt{0.01^2 + 0.01^2} \simeq 0.014$ or 1.4 %. When the transducers are recalibrated against the *same* standard, their uncertainties are correlated. Hence, the covariant term in Equation 7.19 must be considered. Thus, $u^2_{\Delta P_B} = u^2_{P_1} + u^2_{P_2} + 2\theta_1\theta_2 u_{P_1,P_2}$. Here $\theta_1 = \frac{\partial u_{P_1}}{\partial u_{\Delta P_B}} = -1$ and $\theta_2 = \frac{\partial u_{P_2}}{\partial u_{\Delta P_B}} = 1$. The uncertainty resulting from the recalibration according to Equation 7.20 is

$$u_{P_1,P_2} = \sum_{k=1}^{2} (u_i)_k (u_j)_k = (u_{P_1})_1 (u_{P_2})_1 + (u_{P_1})_2 (u_{P_2})_2 = (0.3)(0.3) + (0.5)(0.5) \simeq 0.3 \text{ \%}.$$

Thus, applying Equation 7.19, $u_{\Delta P_B} = \sqrt{0.01^2 + 0.01^2 - (2)(0.003)} \simeq 0.013$ or 1.3 %. So, the recalibration of the transducers reduces the uncertainty in the pressure difference from 1.4 % to 1.2 %.

Because this situation involves the uncertainty of a measured *difference*, one of the θ_i's is negative. This reduces the uncertainty in the difference *if* the instruments are calibrated against the *same* standard under the *same* conditions (time, place, and so forth). This reduction seems counterintuitive at first, that one can reduce the measurement uncertainty of a difference through recalibration, which itself introduces uncertainty. However, the uncertainty of a measured difference can be reduced because recalibration using the same standard and conditions introduces a *dependent* systematic error. This error effectively reduces the independent systematic errors of each instrument.

Each single-measurement situation is considered in the following.

7.9.1 Single-Measurement Measurand Experiment

Many times it is desirable to estimate the uncertainty of a single measurand taken using a certain instrument. Typically this is done before conducting an experiment. The contributory errors are considered fossilized, hence systematic. The expression for the combined standard uncertainty is Equation 7.23.

This particular type of uncertainty is known as the design-stage uncertainty, u_d, which is analogous to the combined standard uncertainty. Often it is used to choose an instrument that meets the accuracy required for a measurement. It is expressed as a function of the zero-order uncertainty of the instrument, u_0, and the instrument uncertainty, u_I, as

$$u_d = \sqrt{u_0^2 + u_I^2}, \tag{7.26}$$

which usually is computed at the 95 % confidence level.

Instruments have resolution, readability, and errors. The **resolution** of an instrument is the smallest *physically indicated* division that the instrument displays or is marked. The zero-order uncertainty of the instrument, u_0, is set arbitrarily to be equal to one-half the resolution, based upon 95 % confidence. Equation 7.26 shows that the design-stage uncertainty can never be less than u_0, which would occur when u_0 is much greater than u_I. In other words, even if the instrument is perfect and has no instrument errors, its output must be read with some finite resolution and, therefore, some uncertainty.

The **readability** of an instrument is the closeness with which the scale of the instrument is read by an experimenter. This is a subjective value. Readability does *not* enter into assessing the uncertainty of the instrument.

The instrument uncertainty usually is stated by the manufacturer and results from a number of possible elemental instrument uncertainties, e_i. Examples of e_i are hysteresis, linearity, sensitivity, zero-shift, repeatability, stability, and thermal-drift errors. Thus,

$$u_I = \sqrt{\sum_{i=1}^{N} e_i^2}. \tag{7.27}$$

Instrument errors (elemental errors) are identified through calibration. An elemental error is an error that can be associated with a *single* uncertainty source. Usually, it is related to the full-scale output (FSO) of the instrument, which is its maximum output value. The most common instrument errors are the following:

1. Hysteresis:

$$\tilde{e}_H = \left(\frac{e_{H,max}}{FSO}\right) = \left(\frac{|y_{up} - y_{down}|_{max}}{FSO}\right). \tag{7.28}$$

The hysteresis error is related to $e_{H,max}$, which is the greatest deviation between two output values for a given input value that occurs when performing an up-scale, down-scale calibration. This is a single calibration proceeding from the minimum to the maximum input values, then back to the minimum. Hysteresis error usually arises from having a physical change in part of the measurement system upon reversing the system's input. Examples include the mechanical sticking of a moving part of the system and the physical alteration of the environment local to the system, such as a region of recirculating flow called a separation bubble.

This region remains attached to an airfoil upon decreasing its angle of attack from the region of stall.

2. Linearity:
$$\tilde{e}_L = \left(\frac{e_{L,max}}{FSO}\right) = \left(\frac{|y - y_L|_{max}}{FSO}\right). \tag{7.29}$$
Linearity error is a measure of how linear is the best fit of the instrument's calibration data. It is defined in terms of its maximum deviation distance, $|y - y_L|_{max}$.

3. Sensitivity:
$$\tilde{e}_K = \left(\frac{e_{K,max}}{FSO}\right) = \left(\frac{|y - y_{nom}|_{max}}{FSO}\right). \tag{7.30}$$
Sensitivity error is characterized by the greatest change in the slope (static sensitivity) of the calibration fit.

4. Zero-shift:
$$\tilde{e}_Z = \left(\frac{e_{Z,max}}{FSO}\right) = \left(\frac{|y_{shift} - y_{nom}|_{max}}{FSO}\right). \tag{7.31}$$
Zero-shift error refers to the greatest possible shift that can occur in the intercept of the calibration fit.

5. Repeatability:
$$\tilde{e}_R = \left(\frac{2S_x}{FSO}\right). \tag{7.32}$$
Repeatability error is related to the precision of the calibration. This is determined by repeating the calibration many times for the same input values. The quantity $2S_x$ represents the precision interval of the data for a particular value of x.

6. Stability:
$$\tilde{e}_S = \left(\frac{e_{S,max} \cdot \Delta t}{FSO}\right). \tag{7.33}$$
Stability error is related to $e_{S,max}$, which is the greatest deviation in the output value for a fixed input value that could occur during operation. This deviation is expressed in units of $FSO/\Delta t$, with Δt denoting the time since instrument purchase or calibration. Stability error is a measure of how much the output can drift over a period of time for the same input.

7. Thermal-drift:
$$\tilde{e}_T = \left(\frac{e_{T,max}}{FSO}\right). \tag{7.34}$$
Thermal-drift error is characterized by the greatest deviation in the output value for a fixed input value, $e_{T,max}$, that could occur during

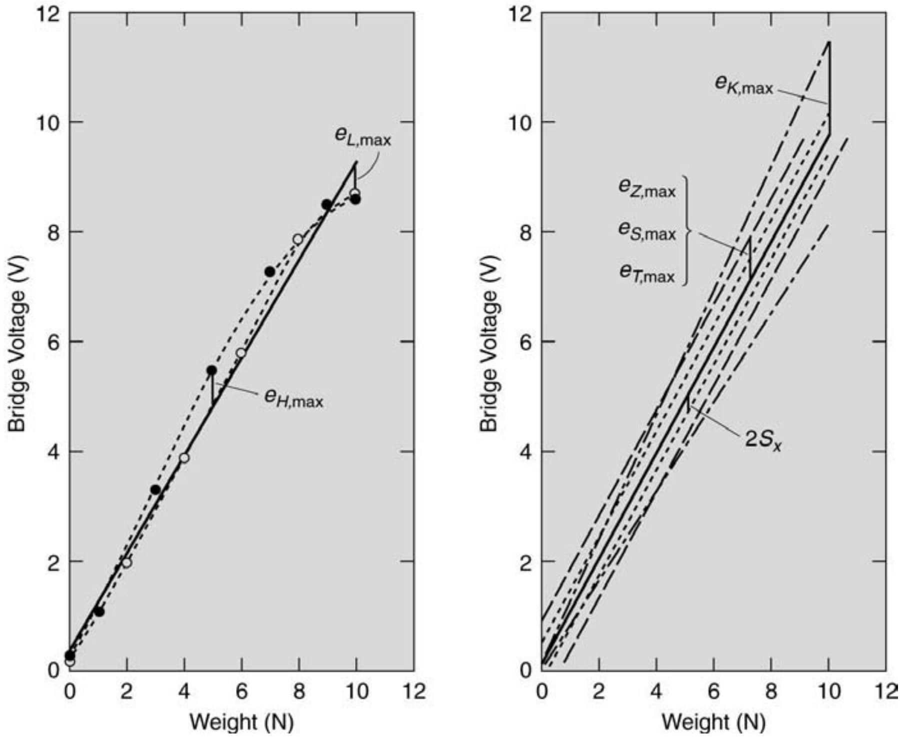

FIGURE 7.3
Elemental errors ascertained by calibration.

operation because of variations in the environmental temperature. Stability and thermal-drift errors are similar in behavior to the zero-shift error.

The instrument uncertainty, u_I, combines all the known instrument errors,

$$u_I = \sqrt{\sum e_i^2} = FSO \cdot \sqrt{\tilde{e}_H^2 + \tilde{e}_L^2 + \tilde{e}_K^2 + \tilde{e}_Z^2 + \tilde{e}_R^2 + \tilde{e}_S^2 + \tilde{e}_T^2 + \tilde{e}_{other}^2}, \quad (7.35)$$

where $\tilde{e}_{other}$ denotes any other instrument errors. All $\tilde{e}_i$'s expressed in Equation 7.35 are dimensionless.

How are these elemental errors actually assessed? Typically, hysteresis and linearity errors are determined by performing a *single* up-scale, down-scale calibration. The results of this type of calibration are displayed in the left graph of Figure 7.3. In that graph, the up-scale results are plotted as open circles and the down-scale results as solid circles. The dotted lines are linear interpolations between the data. Hysteresis is evident in this example by down-scale output values that are higher than their up-scale counterparts. The best-fit curve of the data is indicated by a solid line. Both the hysteresis and linearity errors are assessed with respect to the best-fit curve.

Sensitivity, repeatability, zero-shift, stability, and thermal-drift errors are ascertained by performing a *series* of calibrations and then determining each particular error by comparisons between the calibrations. The results of a series of calibrations are shown in the right graph of Figure 7.3. The solid curve represents the best-fit of the data from *all* the calibrations. The dotted curves indicate the limits within which a calibration is repeatable with 95 % confidence. The repeatability error is determined from the difference between either dotted curve and the best-fit curve. The dash-dot curves identify the calibration curves that have the maximum and minimum slopes. The sensitivity error is assessed in terms of the greatest difference between minimum or maximum sensitivity curve and the best-fit curve. The dashed curves denote shifts that can occur in the calibration because of zero-shift, stability, and thermal-drift errors. Each error can have a different value and is determined from the calibration curve having the greatest difference with calibration data that occurs with each effect, with respect to the best-fit curve.

The following two examples illustrate the effects of instrument errors on measurement uncertainty.

Example Problem 7.5

Statement: A pressure transducer is connected to a digital panel meter. The panel meter converts the pressure transducer's output in volts back to pressure in psi. The manufacturer provides the following information about the panel meter:

Resolution:	0.1 psi
Repeatability:	0.1 psi
Linearity:	within 0.1 % of reading
Drift:	less than 0.1 psi/6 months within the 32 °F to 90 °F range

The only information given about the pressure transducer is that it has "an accuracy of within 0.5 % of its reading".

Estimate the combined standard uncertainty in a measured pressure at a nominal value of 100 psi at 70 °F. Assume that the transducer's response is linear with an output of 1 V for every psi of input.

Solution: The uncertainty in the measured pressure, $(u_d)_{mp}$, is the combination of the uncertainties of the transducer, $(u_d)_t$, and the panel meter, $(u_d)_{pm}$. This can be expressed as

$$(u_d)_{mp} = \sqrt{[(u_d)_t]^2 + [(u_d)_{pm}]^2}.$$

For the transducer,

$$(u_d)_t = \sqrt{u_{I_t}^2 + u_{o_t}^2} = u_{I_t} = 0.005 \times 100 \text{ psi} = 0.50 \text{ psi}.$$

For the panel meter,

$$(u_d)_{pm} = \sqrt{u_{I_{pm}}^2 + u_{o_{pm}}^2}.$$

Now,

$$u_{o_{pm}} = 0.5 \text{ resolution} = 0.05 \text{ psi}, \tag{7.36}$$

$$u_{I_{pm}} = \sqrt{e_1^2 + e_2^2 + e_3^2},$$

where

$$\begin{aligned} e_1 \text{ (repeatability)} &= 0.1 \text{ psi} \\ e_2 \text{ (linearity)} &= 0.1 \text{ \% reading} = 0.001 \times 100\text{V}/(1\text{V/psi}) = 0.1 \text{ psi, and} \\ e_3 \text{ (drift)} &= 0.1 \text{ psi/6 months} \times 6 \text{ months} = 0.1 \text{ psi,} \end{aligned}$$

which implies that

$$\begin{aligned} u_{I_{pm}} &= 0.17 \text{ psi}, \\ (u_d)_{pm} &= 0.18 \text{ psi}, \\ (u_d)_{mp} &= \sqrt{0.50^2 + 0.18^2} = 0.53 \text{ psi}. \end{aligned}$$

Note that most of the combined standard uncertainty comes from the transducer. So, to improve the accuracy of the measurement system, a more accurate transducer is required.

Example Problem 7.6

Statement: An analog-to-digital (A/D) converter with the specifications listed below (see Chapter 3 for terminology) is to be used in an environment in which the A/D converter's temperature may change by ±10 °C. Estimate the contributions of conversion and quantization errors to the combined standard uncertainty in the digital representation of an analog voltage by the converter.

E_{FSR}	0 V to 10 V
M	12 bits
Linearity	±3 bits/E_{FSR}
Temperature drift	1 bit/5 °C

Solution: The instrument uncertainty is the combination of uncertainty due to quantization errors, e_Q, and to conversion errors, e_c,

$$(u_I)_E = \sqrt{e_Q^2 + e_I^2}.$$

The resolution of a 12-bit A/D converter with a full scale range of 0 V to 10 V is given by (see Chapter 2)

$$Q = \frac{E_{FSR}}{2^{12}} = \frac{10}{4096} = 2.4 \text{ mV/bit}.$$

The quantization error per bit is found to be

$$e_Q = 0.5Q = 1.2 \text{ mV}.$$

The conversion error is affected by two elements:

$$\begin{aligned} \text{linearity error} &= e_1 = 3 \text{ bits} \times 2.4 \text{ mV/bit} \\ &= 7.2 \text{ mV}. \\ \text{temperature error} &= e_2 = \frac{1\text{bit}}{5\ ^\circ\text{C}} \times 10\ ^\circ\text{C} \times 2.4 \text{ mV/bit} \\ &= 4.8 \text{ mV}. \end{aligned}$$

Thus, an estimate of the conversion error is

$$\begin{aligned} e_I &= \sqrt{e_1^2 + e_2^2} \\ &= \sqrt{(7.2 \text{ mV})^2 + (4.8 \text{ mV})^2} = 8.6 \text{ mV}. \end{aligned}$$

The combined standard uncertainty in the digital representation of the analog value due to the quantization and conversion errors becomes

$$\begin{aligned} (u_I)_E &= \sqrt{(1.2 \text{ mV})^2 + (8.6 \text{ mV})^2} \\ &= 8.7 \text{ mV}. \end{aligned}$$

Here the conversion errors dominate the uncertainty. So, a higher resolution converter is not necessary to reduce the uncertainty. A converter having smaller instrument errors is required.

These two examples illustrate the process of design-stage uncertainty estimation. Once the components of the measurement system have been chosen, uncertainty analysis can be extended to consider other types of errors that can effect the measurement, such as temporal variations in the system's output under fixed conditions. This involves multiple measurements, which are covered in Section 7.10.

7.9.2 Single-Measurement Result Experiment

The previous section considered estimating the uncertainties of a measurand. But what about the uncertainty in a result? This uncertainty was introduced beforehand and is given by Equation 7.21. From this equation, uncertainty expressions can be developed for specific analytical relations [4]:

1. If $r = Bx$, where B is a constant, then

$$u_r = |B| u_x. \tag{7.37}$$

   If r is directly proportional to a measurand through a constant of proportionality B, then the uncertainty in r is the product of the absolute value of the proportionality constant and the measurand uncertainty.

2. If $r = x + ... + z - (u + ... + w)$, then

$$u_r = \sqrt{(u_x)^2 + ... + (u_z)^2 + (u_u)^2 + ... + (u_w)^2}. \tag{7.38}$$

If r is related directly to all of the measurands, then the uncertainty in r is the combination in quadrature of the measurands' uncertainties.

3. If $r = (x...z)/(u...w)$, then

$$\frac{u_r}{|r|} = \sqrt{\left(\frac{u_x}{x}\right)^2 + ... + \left(\frac{u_z}{z}\right)^2 + \left(\frac{u_u}{u}\right)^2 + ... + \left(\frac{u_w}{w}\right)^2}. \tag{7.39}$$

The quantity $u_r/|r|$ is the **fractional uncertainty** of a result. If r is related directly and/or inversely to all of the measurands, then the fractional uncertainty in r is the combination in quadrature of the measurands' fractional uncertainties.

4. If $r = x^n$, then

$$\frac{u_r}{|r|} = |n|\frac{u_x}{|x|}. \tag{7.40}$$

This equation follows directly from Equation 7.39.

Estimation of the uncertainty in a result is shown in the following example.

Example Problem 7.7

Statement: The coefficient of restitution, e, of a ball can be determined by dropping the ball from a known height, h_a, onto a surface and then measuring its return height, h_b (as described in Chapter 11). For this experiment $e = \sqrt{h_b/h_a}$. If the uncertainty in the height measurement, u_h, is 1 mm, $h_a = 1.000$ m and $h_b = 0.800$ m, determine the combined standard uncertainty in e.

Solution: Direct application of Equation 7.21 yields

$$u_e = \sqrt{\left(\frac{\partial e}{\partial h_b} u_{h_b}\right)^2 + \left(\frac{\partial e}{\partial h_a} u_{h_a}\right)^2}.$$

Now, $\left(\frac{\partial e}{\partial h_b}\right) = \frac{1/h_a}{2\sqrt{h_b/h_a}}$ and $\left(\frac{\partial e}{\partial h_a}\right) = \frac{-h_b/h_a^2}{2\sqrt{h_b/h_a}}$. Substitution of the known values into the above equation gives $u_e = \sqrt{(5.59 \times 10^{-4})^2 + (4.47 \times 10^{-4})^2} = 7.16 \times 10^{-4} = 0.0007$.

Often there are experiments involving results that have angular dependencies. The values of these results can vary significantly with angle because of the presence of trigonometric functions in the denominators of their uncertainty expressions. The following two problems illustrate this point.

Example Problem 7.8

Statement: A radar gun determines the speed of a directly oncoming automobile within 4 %. However, if the gun is used off angle, another uncertainty arises. Determine

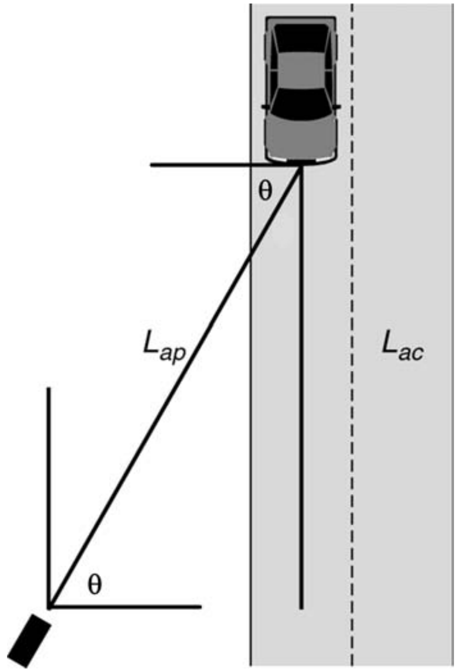

FIGURE 7.4
Radar detection of a car's speed.

the gun's off-angle uncertainty, u_{oa}, as a function of the angle at which the car is viewed. What is the combined uncertainty in the speed if the off-angle, θ_{oa}, equals 70°? Finally, what is the overall uncertainty in the speed assuming 95 % confidence?

Solution: A schematic of this problem is shown in Figure 7.4. Assume that the gun acquires a reading within a very short time period, Δt. The actual speed, s_{ac}, is the ratio of the actual highway distance travelled, L_{ac}, during the time period to Δt. Similarly, the apparent speed, s_{ap}, equals $L_{ap}/\Delta t$. From trigonometry,

$$L_{ac} = L_{ap} \times \sin(\theta). \tag{7.41}$$

Substitution of the speed definitions into this equation yields

$$s_{ac} = s_{ap} \sin(\theta). \tag{7.42}$$

The off-angle uncertainty can be defined as

$$u_{oa} = \frac{\mid s_{ac} - s_{ap} \mid}{s_{ac}} = \frac{\mid \sin(\theta) - 1 \mid}{\sin(\theta)}. \tag{7.43}$$

Note that when $\theta = 90°$, $\sin(90°) = 1$, which yields $u_{oa} = 0$. This is when the radar gun is pointed directly along the highway at the car. When $\theta = 70°$, $\sin(70°) = 0.940$, which yields $u_{oa} = (\mid 0.940 - 1 \mid)/0.940 = 0.064$ or 6.4 %. This uncertainty must be combined in quadrature with radar gun's instrument uncertainty, $u_I = 0.04$, to yield the combined uncertainty, u_c,

$$u_c = \sqrt{u_I^2 + u_{oa}^2} = \sqrt{0.04^2 + 0.064^2} = 0.075. \tag{7.44}$$

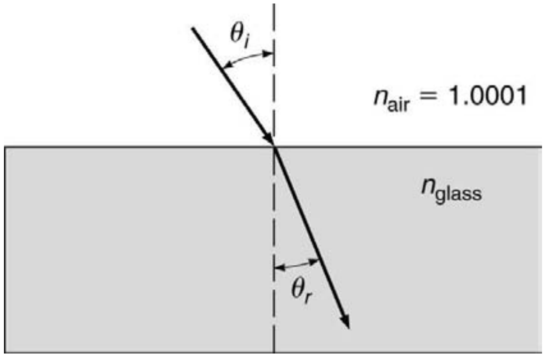

FIGURE 7.5
A light refraction experiment based on Snell's law.

So, the combined uncertainty is 7.5 % or almost twice the instrument uncertainty. This uncertainty increases as the off-angle increases. Assuming 95 % confidence, the overall uncertainty is approximately twice the combined uncertainty or 15 %. Thus, assuming that the indicated speed is 70 mph, the actual speed could be as low as approximately 60 mph (70 − 0.15 × 70) or as high as approximately 80 mph (70 + 0.15 × 70).

Example Problem 7.9

Statement: This problem is adapted from one in [4]. An experiment is constructed (see Figure 7.5) to determine the index of refraction of an unknown transparent glass. Find the fractional uncertainty, $\Delta n/n$, in the index of refraction, n, as determined using Snell's law, where $n = \sin\theta_i / \sin\theta_r$. Assume that the measurements of the angles are uncertain by $\pm 1^\circ$ or 0.02 rad.

Solution: It follows that

$$\frac{\Delta n}{n} = \sqrt{\left(\frac{\partial \sin\theta_i}{\sin\theta_i}\right)^2 + \left(\frac{\partial \sin\theta_r}{\sin\theta_r}\right)^2}.$$

Now,

$$\partial \sin\theta = \left|\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\right| \partial\theta = |\cos\theta|\partial\theta \ \text{(in rad)}.$$

So,

$$\frac{\partial \sin\theta}{|\sin\theta|} = |\cot\theta|\partial\theta \ \text{(in rad)}.$$

These considerations yield the following uncertainties:

$\theta_i \pm 1^\circ$	$\theta_r \pm 1^\circ$	$\sin\theta_i$	$\sin\theta_r$	n	$\frac{\partial \sin\theta_i}{\lvert\sin\theta_i\rvert}$	$\frac{\partial \sin\theta_r}{\lvert\sin\theta_r\rvert}$	$\frac{\Delta n}{n}$
20	13.0	0.342	0.225	1.52	5 %	8 %	9 %
40	23.5	0.643	0.399	1.61	2 %	4 %	5 %

Note that the percentage uncertainty in n decreases with increasing the angle of incidence. In fact, as the angle of incidence approaches that of normal incidence ($\theta_i = 0°$), the uncertainty tends to infinity.

Many times, experimental uncertainty analysis involves a series of uncertainty calculations that lead to the uncertainty of a desired result. In that situation, usually it is best to perform the analysis in steps, identifying the uncertainties in intermediate results. This not only helps to avoid mistakes made in calculations but also aids in identifying the variables that contribute significantly to the desired result's uncertainty. The following two examples illustrate this point.

Example Problem 7.10

Statement: Determine the combined standard uncertainty in the density of air, assuming $\rho = P/RT$. Assume negligible uncertainty in R ($R_{air} = 287.04$ J/kg·K). Let $T = 24$ °C $= 297$ K and $P = 760$ mm Hg.

Solution: The uncertainty in the density of air (a result) becomes

$$\begin{aligned} u_\rho &= \sqrt{\left(\frac{\partial \rho}{\partial T} u_T\right)^2 + \left(\frac{\partial \rho}{\partial P} u_P\right)^2} \\ &= \sqrt{\left(\frac{-P}{RT^2} u_T\right)^2 + \left(\frac{1}{RT} u_P\right)^2}, \end{aligned}$$

where

$$u_P = \frac{1}{2}(1 \text{ mm Hg}) = \frac{1}{2}\left(\frac{1.01 \times 10^5 \text{ Pa}}{760 \text{ mm Hg}} \times 1 \text{ mm Hg}\right) = \frac{1}{2}(133 \text{ Pa}) = 67 \text{ Pa},$$

and

$$u_T = 0.5(1°\text{C}) = 0.5(1 \text{ K}) = 0.5 \text{ K}.$$

Thus,

$$\begin{aligned} u_\rho &= \left[\left(\frac{101325}{(287.04)(297)^2}(0.5)\right)^2 + \left(\frac{1}{(287.04)(297)}(67)\right)^2\right]^{\frac{1}{2}} \\ &= [4.00 \times 10^{-6} + 0.62 \times 10^{-6}]^{\frac{1}{2}} \\ &= 2.15 \times 10^{-3} \text{ kg/m}^3. \end{aligned}$$

Finally,

$$\begin{aligned} \rho &= \frac{P}{RT} = \frac{101325}{(287.04)(297)} = 1.19 \text{ kg/m}^3 \\ \Rightarrow \frac{u_\rho}{\rho} &= \frac{2.15 \times 10^{-3}}{1.19} = 0.19 \ \%. \end{aligned}$$

This is a typical value for the combined standard uncertainty in the density as determined from pressure and temperature measurements in a contemporary laboratory.

Example Problem 7.11

Statement: Consider an experiment in which the static pressure distribution around the circumference of a cylinder in a cross flow in a wind tunnel is measured. Determine the combined standard uncertainty in the pressure coefficient, C_p, as defined by the equation

$$C_p \equiv \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2}. \tag{7.45}$$

Assume that the pressure difference $P - P_\infty$ is measured as Δp using an inclined manometer with

$$\begin{aligned} u_{\Delta p} &= 0.06 \text{ in. H}_2\text{O} = 15 \text{ N/m}^2 \quad (\Delta P = 996 \text{ N/m}^2), \\ u_\rho &= 2.15 \times 10^{-3} \text{ kg/m}^3 \quad (\rho = 1.19 \text{ kg/m}^3), \text{ and} \\ u_{V_\infty} &= 0.31 \text{ m/s} \quad (V_\infty = 40.9 \text{ m/s}). \end{aligned}$$

Solution: Now, as is clear from Equation 7.45, the pressure coefficient is a *result* and is a function of the density, the change in pressure, and the freestream velocity. In short,

$$C_p = f(\Delta p, \rho, V_\infty).$$

Therefore, applying Equation 7.21 yields

$$\begin{aligned} \Rightarrow u_{C_P} &= [(\frac{\partial C_p}{\partial \Delta p} u_{\Delta p})^2 + (\frac{\partial C_p}{\partial \rho} u_\rho)^2 + (\frac{\partial C_p}{\partial V_\infty} u_{V_\infty})^2]^{\frac{1}{2}} \\ &= [(\frac{2}{\rho V_\infty^2} u_{\Delta p})^2 + (-\frac{2\Delta p}{\rho^2 V_\infty^2} u_\rho)^2 + (-\frac{4\Delta p}{\rho V_\infty^3} u_{V_\infty})^2]^{\frac{1}{2}} \\ &= [(\frac{(2)(15)}{(1.19)(40.9)^2})^2 + (\frac{(2)(996)(2.15 \times 10^{-3})}{(1.19)^2)(40.9)^2})^2 + (\frac{(4)(996)(0.31)}{(1.19)(40.9)^3})^2]^{\frac{1}{2}} \\ &= [2.27 \times 10^{-4} + 3.27 \times 10^{-6} + 2.30 \times 10^{-4}]^{\frac{1}{2}} \\ &= 0.021. \end{aligned}$$

Alternatively, C_p as the ratio of two transducer differential pressures,

$$C_p \equiv \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2} = \frac{\Delta P}{\Delta P_{p-s}} \quad (\Delta p = 996 \text{ N/m}^2). \tag{7.46}$$

Now, assume that $u_{\Delta p} = u_{\Delta p_{p-s}} = 15 \text{ N/m}^2$.

The equation for the uncertainty in C_p when Equation 7.46 is used becomes

$$\begin{aligned} u_{C_p} &= [(\frac{\partial C_p}{\partial \Delta P_{cyl}} u_{\Delta P_{cyl}})^2 + (\frac{\partial C_P}{\partial \Delta P_{p-s}} u_{\Delta P_{p-s}})^2]^{\frac{1}{2}} \\ &= [(\frac{1}{\Delta P_{p-s}} u_{\Delta P})^2 + (-\frac{\Delta P_{cyl}}{\Delta P_{p-s}^2} u_{\Delta P_{p-s}})^2]^{\frac{1}{2}} \\ &= [2(\frac{15}{996})^2]^{\frac{1}{2}} \\ &= 0.021. \end{aligned}$$

The latter measurement approach is easier to determine C_p than the former. When designing an experimental procedure in which the pressure coefficient needs to be determined, it is preferable to ratio the two transducer differential pressures.

Finally, one may be interested in estimating the uncertainty of a result that can be found by different measurement approaches. This process is quite useful during the planning stage of an experiment. For example, take the simple case of an experiment designed to determine the volume of a cylinder, V. One approach would be to measure the cylinder's height, h, and diameter, d, and then compute its volume based upon the expression $V = \pi d^2 h/4$. An alternative approach would be to obtain its weight, W, and then compute its volume according to the expression $V = W/(\rho g)$, where ρ is the density of the cylinder and g the local gravitational acceleration. Which approach is chosen depends on the uncertainties in d, h, w, ρ, and g.

Example Problem 7.12

Statement: An experiment is being designed to determine the mass of a cube of Teflon. Two approaches are under consideration. Approach A involves determining the mass from a measurement of the cube's weight and approach B from measurements of the cube's length, width, and height (l, w, and h, respectively). For approach A, $m = W/g$; for approach B, $m = \rho V$, where m is the mass, W the weight, g the gravitational acceleration (9.81 m/s^2), ρ the density (2 200 kg/m^3), and V the volume (lwh). The fractional uncertainties in the measurements are W (2 %), g (0.1 %), ρ, (0.1 %), and l, w, and h (1 %). Determine which approach has the least uncertainty in the mass.

Solution: Because both equations for the mass involve products or quotients of the measurands, the fractional uncertainty of the mass can be computed using Equation 7.39. For approach A

$$\left(\frac{u_m}{|m|}\right)_A = \sqrt{\left(\frac{u_W}{W}\right)^2 + \left(\frac{u_g}{g}\right)^2} = \sqrt{(0.02)^2 + (0.001)^2} = 2.0\ \%.$$

For approach B, the fractional uncertainty must be determined in the volume. This is

$$\frac{u_V}{|V|} = \sqrt{\left(\frac{u_l}{l}\right)^2 + \left(\frac{u_w}{w}\right)^2 + \left(\frac{u_h}{h}\right)^2} = \sqrt{3 \times 0.01^2} = 1.7\ \%.$$

This result can be incorporated into the fractional uncertainty calculation for the mass

$$\left(\frac{u_m}{|m|}\right)_B = \sqrt{\left(\frac{u_\rho}{\rho}\right)^2 + \left(\frac{u_V}{V}\right)^2} = \sqrt{(0.001)^2 + (0.017)^2} = 0.017 = 1.7\ \%.$$

Thus, approach B has the least uncertainty in the mass. Note that the uncertainties in g and ρ both are negligible in these calculations.

7.10 Detailed Uncertainty Analysis

Detailed uncertainty analysis is appropriate for measurement situations that involve multiple measurements, either for a measurand or a result. Systematic and random errors are identified in this approach. Their contributions to the overall uncertainty are treated separately in the analysis until they are combined at the end. Detailed uncertainty analysis is performed *after* a statistically viable set of measurand values has been obtained under fixed operating conditions. Multiple measurements usually are made for one or both of two reasons. One is to assess the uncertainty present in an experiment due to uncontrollable variations in the measurands. The other is to obtain sufficient data such that the average value of a measurand can be estimated. Thus, detailed uncertainty calculations are more extensive than for the cases of a single-measurement measurand or result.

In general, for the situation in which both systematic and random errors are considered, application of Equation 7.19 for a result based upon J measurands leads directly to

$$u_r^2 = \sum_{i=1}^{J} {\theta_i}^2 S_{B_i}^2 + 2\sum_{i=1}^{J-1}\sum_{j=i+1}^{J} \theta_i\theta_j S_{B_i,B_j} + \sum_{i=1}^{J} {\theta_i}^2 S_{P_i}^2 + 2\sum_{i=1}^{J-1}\sum_{j=i+1}^{J} \theta_i\theta_j S_{P_i,P_j}, \tag{7.47}$$

where

$$S_{B_i}^2 = \sum_{k=1}^{M_B} (S_{B_i})_k^2, \tag{7.48}$$

$$S_{P_i}^2 = \sum_{k=1}^{M_P} (S_{P_i})_k^2, \tag{7.49}$$

$$S_{B_i,B_j} = \sum_{k=1}^{L_B} (S_{B_i})_k (S_{B_j})_k, \tag{7.50}$$

and

$$S_{P_i,P_j} = \sum_{k=1}^{L_P} (S_{P_i})_k (S_{P_j})_k. \tag{7.51}$$

M_B is the number of elemental systematic uncertainties, M_P is the number of elemental random uncertainties. $(S_{B_i})_k$ represents the k-th elemental error out of M_B elemental errors contributing to S_{B_i}, the estimate of the systematic error of the i-th measurand. Equation 7.48 is analogous to Equation 7.27 for the case of a single-measurement measurand. Further, L_B is the number of systematic errors that are common to B_i and B_j, and L_P the number of random errors that are common to P_i and P_j. $S_{B_i}^2$ and $S_{P_i}^2$ are estimates of the variances of the systematic and random errors of the i-th measurand, respectively. S_{B_i,B_j} and S_{P_i,P_j} are estimates of the covariances of the systematic and random errors of the i-th and j-th measurands, respectively.

The number of effective degrees of freedom of a multiple-measurement result, based upon the Welch-Satterthwaite formula, is

$$\nu_r = \frac{\{\sum_{i=1}^{J}[\theta_i^2 S_{B_i}^2 + \theta_i^2 S_{P_i}^2]\}^2}{\sum_{i=1}^{J}\left[(\theta_i^4 S_{P_i}^4)/\nu_{P_i}) + (\sum_{k=1}^{M_B}\theta_i^4 (S_{B_i})_k^4/\nu_{(S_{B_i})_k})\right]}, \tag{7.52}$$

where contributions are made from each i-th measurand. The random and systematic numbers of degrees of freedom are given by

$$\nu_{P_i} = N_i - 1, \tag{7.53}$$

where N_i denotes N measurements of the i-th measurand and

$$\nu_{(S_{B_i})_k} \cong \frac{1}{2}\left[\frac{\Delta(S_{B_i})_k}{(S_{B_i})_k}\right]^{-2}. \tag{7.54}$$

When S_{B_i} and S_{P_i} have similar values, the value of ν_r, as given by Equation 7.52, is approximately that of ν_{P_i} if $\nu_{B_i} >> \nu_{P_i}$. Conversely, the value of ν_r is approximately that of ν_{B_i} if $\nu_{P_i} >> \nu_{B_i}$. Further, if $S_{B_i} << S_{P_i}$, then the value of ν_r is approximately that of ν_{P_i}. The converse also is true.

Once ν_r is determined from Equation 7.52, the overall uncertainty in the result, U_r, can be found. This can be expressed using the definition of the overall uncertainty (Equation 7.18) and Equation 7.47 as

$$U_r^2 = t^2_{\nu_r,C}u_r^2 = B_r^2 + (t_{\nu_r,C}S_r)^2 = B_r^2 + P_r^2, \tag{7.55}$$

where

$$B_r = t_{\nu_r,C}\left[\sum_{i=1}^{J}{\theta_i}^2 S_{B_i}^2 + 2\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\theta_i\theta_j S_{B_i,B_j}\right]^{1/2} \tag{7.56}$$

and

$$P_r = t_{\nu_r,C}\left[\sum_{i=1}^{J}{\theta_i}^2 S_{P_i}^2 + 2\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\theta_i\theta_j S_{P_i,P_j}\right]^{1/2}. \tag{7.57}$$

The first term on the right side of Equation 7.56 is the sum of the estimated variances of of the systematic errors. The second term is the sum of the estimated covariances of the systematic errors. Likewise, for Equation 7.57, the first term is the estimated variance of the random errors and the second term is the estimated covariance of the random errors. The covariance of two systematic errors is zero if the errors are not correlated, or, in other words, they are independent of each other. Non-zero covariances arise when the error sources are common, such as when two pressure transducers are calibrated against the same standard. Correlated random errors can be identified by examining the amplitude variations in time of two measurands. When they follow the same trends, they may be correlated. Usually, however, the covariances of the systematic errors are assumed negligible in most uncertainty analysis.

When there are no correlated uncertainties,

$$u_r^2 = \sum_{i=1}^{J} {\theta_i}^2 S_{B_i}^2 + \sum_{i=1}^{J} {\theta_i}^2 S_{P_i}^2, \tag{7.58}$$

where

$$B_r = t_{\nu_r,C} \left[\sum_{i=1}^{J} {\theta_i}^2 S_{B_i}^2 \right]^{1/2} \tag{7.59}$$

and

$$P_r = t_{\nu_r,C} \left[\sum_{i=1}^{J} {\theta_i}^2 S_{P_i}^2 \right]^{1/2}. \tag{7.60}$$

For the situation when an experiment is replicated M times and a mean result is determined from the M individual results, the expression for S_r in Equation 7.55 is replaced by either

$$S_r = \sqrt{\frac{1}{M-1} \sum_{k=1}^{M} (r_k - \bar{r})^2} \tag{7.61}$$

or

$$S_{\bar{r}} = S_r / \sqrt{M}, \tag{7.62}$$

depending upon which outcome is desired (r or $\bar{r}$, respectively) [10]. The Welch-Satterthwaite formula is then

$$\nu_r = \frac{\{S_r^2 + \sum_{i=1}^{J} (\theta_i^2 S_{B_i}^2)\}^2}{(S_r^4/\nu_{S_r}) + \sum_{i=1}^{J} \sum_{k=1}^{M_B} \left[(\theta_i^4 (S_{B_i})_k^4)/\nu_{(S_{B_i})_k} \right]}, \tag{7.63}$$

with $\nu_{S_r} = M - 1$.

Equations 7.47 and 7.52 can be applied to estimate the uncertainty in a multiple-measurement measurand. For this case, these equations reduce to

$$u_m^2 = \sum_{i=1}^{J} S_{B_i}^2 + 2 \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} S_{B_i,B_j} + \sum_{i=1}^{J} S_{P_i}^2 + 2 \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} S_{P_i,P_j} \tag{7.64}$$

and

$$\nu_m = \frac{\{\sum_{i=1}^{J} [S_{B_i}^2 + S_{P_i}^2]\}^2}{\sum_{i=1}^{J} \left[(S_{P_i}^4/\nu_{P_i}) + (\sum_{k=1}^{M_B} (S_{B_i})_k^4/\nu_{(S_{B_i})_k}) \right]}. \tag{7.65}$$

Further simplification occurs where there are no correlated uncertainties. Equation 7.64 becomes

$$u_m^2 = \sum_{i=1}^{J} S_{B_i}^2 + \sum_{i=1}^{J} S_{P_i}^2. \tag{7.66}$$

When estimating the uncertainty in a *mean* value, S_{P_i} is replaced in Equation 7.66 by $S_{\bar{P}_i}$, where

$$S_{\bar{P}_i} = \sqrt{\sum_{i=1}^{M_P} \frac{S_{P_i}^2}{N_{P_i}}}. \tag{7.67}$$

For most engineering and scientific experiments, $\nu \geq 9$ [10]. When this is the case, it is reasonable to assume for 95 % confidence that $t_{\nu,95} \cong 2$ (when $\nu \geq 9$, $t_{\nu,95}$ is within 10 % of 2). This implies that

$$U_{x,95} \cong 2u_c = 2\sqrt{S_{B_i}^2 + S_{P_i}^2} = \sqrt{B_i^2 + P_i^2}, \tag{7.68}$$

using Equations 7.17, 7.10, and 7.13 where $S_{B_i}^2$ is shorthand notation for $\sum_{i=1}^{N} S_{B_i}^2$ and $S_{P_i}^2$ for $\sum_{i=1}^{N} S_{P_i}^2$. Equation 7.68 is known as the **large-scale approximation**. Its utility is that the overall uncertainty can be estimated directly from the systematic and random uncertainties. This is illustrated in the following example.

Example Problem 7.13

Statement: A load cell is used to measure the central load applied to a structure. The accuracy of the load cell as provided by the manufacturer is stated to be 2.3 N. The experimenter, based upon previous experience in using that manufacturer's load cells, estimates the relative systematic uncertainty of the load cell to be 10 %. A series of 11 measurements are made under fixed conditions, resulting in a standard deviation of the random error equal to 11.3 N. Determine the overall uncertainty in the load cell measurement at the 95 % confidence level.

Solution: The overall uncertainty can be determined using Equations 7.18 through 7.68. The standard deviations and the degrees of freedom for both the systematic and random errors need to be determined first. Here $\nu_{P_x} = N - 1 = 10$ and, from Equation 7.14, $\nu_{B_x} = \frac{1}{2}(0.10)^{-2} = 50$. Thus, at 95 % confidence, $t_{10,95} = 2.228$ and $t_{50,95} = 2.010$ for the random and systematic errors, respectively. Now $S_{P_x} = P_x/t_{10,95} = 11.3/2.228 = 5.072$ from Equation 7.10, and, from Equation 7.13, $S_{B_x} = B_x/t_{50,95} = 2.3/2.010 = 1.144$. Substitution of these values into Equation 7.65 yields

$$\nu_{eff} = \frac{(5.072^2 + 1.144^2)^2}{(5.072^4/10) + (1.144^4/50)} = 11.02 \cong 11.$$

Because $\nu_m \geq 9$, the large-scale approximation at 95 % confidence can be used, where

$$U_{x,95} \cong \sqrt{B_i^2 + P_i^2} = \sqrt{2.3^2 + 11.3^2} = 11.5 \text{ N}.$$

In the following two sections, the application of the above equations to multiple-measurement measurand and result experiments is presented.

7.10.1 Multiple-Measurement Measurand Experiment

Consider the uncertainty estimation of a measurand involving multiple measurements done to assess the contribution of temporal variability (temporal precision error) of a measurand under fixed conditions. For this situation the temporal variations of a measurand are treated as a random error and all other errors are considered to be systematic. The following example illustrates this process.

Example Problem 7.14

Statement: The supply pressure of a blow-down facility's plenum is to be maintained at set pressure for a series of tests. A compressor supplies air to the plenum through a regulating valve that controls the set pressure. A dial gauge (resolution: 1 psi; accuracy: 0.5 psi) is used to monitor pressure in the vessel. Thirty trials of pressurizing the vessel to a set pressure of 50 psi are attempted to estimate pressure controllability. The results show that the standard deviation in the set pressure is 2 psi. Estimate the combined standard uncertainty at 95 % confidence in the set pressure that would be expected during normal operation.

Solution: The uncertainty in the set pressure reading results from the instrument uncertainty, u_d, and the additional uncertainty arising from the temporal variability in the pressure under controlled conditions, u_1. That is,

$$u_p = \sqrt{u_d^2 + u_1^2},$$

where $u_1 = P_x = t_{\nu,P} S_x$ according to Equation 7.10. The instrument uncertainty is

$$u_d = \sqrt{u_o^2 + u_I^2},$$

where

$$\begin{aligned} u_o &= \frac{1}{2}(1.0 \text{ psi}) = 0.5 \text{ psi} \\ u_I &= 0.5 \text{ psi} \\ \Rightarrow u_d &= \sqrt{0.5^2 + 0.5^2} = 0.7 \text{ psi}. \end{aligned}$$

Also,

$$u_1 = t_{29,95} S_x = (2.047)(2) = 4.1 \text{ psi}$$

where $\nu = N - 1 = 29$ for this case. So,

$$u_p = \sqrt{0.7^2 + 4.1^2} = 4 \text{ psi}.$$

Note that the uncertainty primarily is the result of repeating the set pressure and not the resolution or accuracy of the gauge.

Next, consider the multiple-measurement situation of estimating the uncertainty in the range that contains the true value of a measurand. This is shown in the following example in which the contributory systematic and random errors are specified, having their elemental uncertainties already summed using Equations 7.48 and 7.49.

Example Problem 7.15

Statement: The stress on a loaded electric-powered remotely piloted vehicle (RPV) wing is measured using a system consisting of a strain gage, Wheatstone bridge, amplifier, and data acquisition system. The following systematic and random uncertainties arising from calibration, data acquisition, and data reduction are as follows:

Calibration:	$S_{B_1} = 1.0$ N/cm^2	$S_{P_1} = 4.6$ N/cm^2	$N_{P_1} = 15 \Rightarrow \nu_{P_1} = 14$
Data Acquisition:	$S_{B_2} = 2.1$ N/cm^2	$S_{P_2} = 10.3$ N/cm^2	$N_{P_2} = 38 \Rightarrow \nu_{P_2} = 37$
Data Reduction:	$S_{B_3} = 0.0$ N/cm^2	$S_{P_3} = 1.2$ N/cm^2	$N_{P_3} = 9 \Rightarrow \nu_{P_3} = 8$

Assume 100 % in the values of all systematic errors and that there are no correlated uncertainties. Determine for 95 % confidence the range that contains the true mean value of the stress, σ', given that the average value is $\bar{\sigma} = 223.4$ N/cm^2.

Solution: The systematic variances are

$$S_B^2 = S_{B_1}^2 + S_{B_2}^2 + S_{B_3}^2 = 1^2 + 2.1^2 + 0^2 = 5.4 \text{ N}^2/\text{cm}^4.$$

The random variances are

$$S_{\bar{P}}^2 = \frac{S_{P_1}}{N_{P_1}}^2 + \frac{S_{P_2}}{N_{P_2}}^2 + \frac{S_{P_3}}{N_{P_3}}^2 = \frac{4.6^2}{15} + \frac{10.3^2}{38} + \frac{1.2^2}{9} = 4.4 \text{ N}^2/\text{cm}^4.$$

It follows directly from Equation 7.66 that $u_\sigma^2 = 5.4 + 4.4 = 9.8$ N^2/cm^4. So, $u_\sigma = 3.1$ N/cm^2.

The number of effective degrees of freedom are determined using Equation 7.65, which becomes

$$\nu_\sigma = \frac{[1^2 + 2.1^2 + 0^2 + 4.6^2 + 10.3^2 + 1.2^2]^2}{[(4.6^4/14) + (10.3^4/37) + (1.2^4/8)]} = \frac{17982.8}{336.4} = 53.5 = 54,$$

noting that each $\nu_{B_i} = \infty$ because of its 100 % reliability. This yields $t_{54,95} \cong 2$.

Thus, the true mean value of the stress is

$$\sigma' = \bar{\sigma} \pm U_\sigma \text{ with } U_\sigma = t_{54,95} \times u_\sigma = 6.2 \text{ N/cm}^2 \ \ (95\ \%),$$

where the range is ± 6.2 N/cm^2. Thus,

$$\sigma' = 223.4 \pm 6.2 \text{ N/cm}^2 \ \ (95\ \%).$$

7.10.2 Multiple-Measurement Result Experiment

The uncertainty estimation of a result based upon multiple measurements of measurands can be made using Equations 7.47 through 7.54. This estimation is slightly more complicated than for the multiple-measurement measurand because it involves determinations of the absolute sensitivity coefficients. The following example illustrates the process in which the true mean value of a result is estimated.

Example Problem 7.16

Statement: This problem is adapted from [18]. An experiment is performed in which the density of a gas, assumed to be ideal, is determined from different numbers of separate pressure and temperature measurements. The gas is contained within a vessel of fixed volume. Pressure is measured within 1 % accuracy; temperature is measured to within 0.6 °R. Twenty measurements of pressure ($N_p = 20$) and ten measurements of temperature ($N_T = 10$) were performed, yielding

$\bar{p} = 2253.91$ psfa	$S_{P_p} = 167.21$ psfa
$\bar{T} = 560.4$ °R	$S_{P_T} = 3.0$ °R

Here, psfa denotes pound-force per square foot absolute. Estimate the true mean value of the density at 95 % confidence.

Solution: Assuming ideal gas behavior, the sample mean density, $\bar{\rho}$, becomes

$$\bar{\rho} = \frac{\bar{p}}{R\bar{T}} = 0.074 \text{ lbm/ft}^3.$$

There two sources of error in this experiment. One is from the variability in the pressure and temperature readings as signified by the values of S_{P_p} and S_{P_T} given above. These are random errors. The other is from the specified instrument inaccuracies. These are systematic errors. These errors are

$S_{B_p} = 1\ \% = 22.5$ psfa	$S_{B_T} = 0.6$ °R
$S_{P_{\bar{p}}} = \frac{S_{P_p}}{\sqrt{20}} = 37.4$ psfa	$S_{P_{\bar{T}}} = \frac{S_{P_T}}{\sqrt{10}} = 0.9$ °R

Here, the degrees of freedom are $\nu_P = 19$ and $\nu_T = 9$. Because the true mean value estimate is made from multiple measurements, the random uncertainties are based upon the standard deviations of their means. Thus, the combined random error becomes

$$S_{\bar{P}} = \sqrt{[\frac{\partial \rho}{\partial T} S_{P_{\bar{T}}}]^2 + [\frac{\partial \rho}{\partial P} S_{P_{\bar{p}}}]^2} = \sqrt{[\frac{-\bar{p}}{R\bar{T}^2} S_{P_{\bar{T}}}]^2 + [\frac{1}{R\bar{T}} S_{P_{\bar{p}}}]^2}$$
$$= \sqrt{[1.2 \times 10^{-4}]^2 + [1.2 \times 10^{-3}]^2} = 0.0012 \text{ lbm/ft}^3.$$

Likewise, the combined systematic error becomes 0.0007 lbm/ft^3. Using Equation 7.47, assuming no correlated uncertainties, yields

$$u_\rho = \sqrt{0.0012^2 + 0.0007^2} = 0.0014 \text{ lbm/ft}^3.$$

Further, assuming 100 % certainty in the stated accuracies of the instruments, the effective number of degrees of freedom as determined from Equation 7.52 is

$$\nu = \frac{\{[\frac{\partial \rho}{\partial T} S_{P_{\bar{T}}}]^2 + [\frac{\partial \rho}{\partial P} S_{P_{\bar{p}}}]^2\}^2}{\frac{[\frac{\partial \rho}{\partial T} S_{P_{\bar{T}}}]^4}{\nu_T} + \frac{[\frac{\partial \rho}{\partial P} S_{P_{\bar{p}}}]^4}{\nu_P}} = 23.$$

This yields $t_{23,95} = 2.06$.

Thus, the true mean value of the pressure is

$$\rho' = \bar{\rho} \pm U_\rho \text{ with } U_\rho = t_{23,95} \cdot u_\rho = 0.0029 \text{ lbm/ft}^3 \quad (95\ \%),$$

where the range is ± 0.0029 lbm/ft^3. Thus,

$$\rho' = 0.074 \pm 0.003 \text{ lbm/ft}^3 \quad (95\ \%),$$

which is an uncertainty of ±3.4 %.

7.11 Uncertainty Analysis Summary

Most uncertainty estimation situations involve a single-measurement measurand, a single-measurement result, a multiple-measurement measurand, or a multiple-measurement result. Expressions for the uncertainty in either a single- or multiple-measurement result contain absolute sensitivity coefficients. These coefficients are evaluated at typical measurand values. Those expressions for the uncertainty in either a single- or multiple-measurement measurand differ only in that the values of all absolute sensitivity coefficients become unity. When multiple measurements are considered, random uncertainties are expressed in terms of the standard deviations of the means of their random errors (Equation 7.11). This is the main difference between the single- and multiple-measurement cases.

The objective of any uncertainty analysis is to obtain an *estimate* of the overall uncertainty, U_x. The summary expression containing U_x involves either x_{next} or x_{true} (Equations 7.15 and 7.16). The overall uncertainty is expressed as the product of Student's t variable based upon the number of effective degrees of freedom (evaluated with $\%C$ confidence), $t_{\nu_{eff},C}$, and the combined standard uncertainty, u_c (Equation 7.18). The values of ν_{eff} and u_c depend upon the particular uncertainty estimation situation. When the effective number of degrees of freedom is greater than or equal to 9, the overall uncertainty can be estimated using the large-scale approximation (Equation 7.68) for 95 % confidence where $U_x = 2u_c$. This greatly simplifies the steps required to estimate U_x.

For single-measurement situations, generalized uncertainty analysis is the most appropriate (Section 7.9). No differentiation is made between systematic and random uncertainties. Expressions are available for the combined standard uncertainty of either a measurand or a result with (Equations 7.23 and 7.19) and without (Equations 7.24 and 7.21) correlated uncertainties. These expressions involve standard uncertainties for the measurands that originate primarily from instrument uncertainties determined from previous calibrations and assessments. There are associated expressions for the effective number of degrees of freedom (Equations 7.25 and 7.22 for a measurand and for a result, respectively).

The uncertainties for multiple-measurement situations are assessed best using detailed uncertainty analysis (Section 7.10). Errors are categorized as either systematic, S_{B_i}, or random, S_{P_i}. Expressions are developed for the combined standard uncertainty of either a measurand or a result with (Equations 7.64 and 7.47) and without (Equations 7.66 and 7.58) correlated uncertainties. There are associated expressions for the effective number of degrees of freedom (Equations 7.65 and 7.52 for a measurand and a result, respectively).

In summary, the overall uncertainty of either a measurand or a result can be estimated using the following steps:

1. *Determine which experimental situation applies.* Is the uncertainty estimate for a measurand or a result, and is it based upon single or multiple measurements?

2. *Identify all measurands and, if applicable, all results.*

3. *Identify all factors affecting the measurands and, if applicable, the results.* What instruments are used? What information is available about their calibrations? Are there any circumstances that lead to correlated uncertainties, such as the same instrument used for two different measurands?

4. *Define all functional relationships between the measurands and, if applicable, the results.* What are the nominal values of each measurand and result? Be sure to use the same system of units throughout all calculations.

5. *If the uncertainty in a result is estimated, determine the values of the absolute sensitivity coefficients from the functional relationships and nominal values.*

6. *Identify all uncertainties.* What are the instrument errors, systematic errors, and random errors? Are there temporal or spatial variations in the measurands that contribute to uncertainty?

7. *Calculate and propagate all uncertainties for the measurands and, if applicable, the results.* Proceed from estimates of the elemental uncertainties and calculate the systematic and random uncertainties.

8. *Propagate all uncertainties to obtain the combined standard uncertainty.*

9. *Determine the number of effective degrees of freedom.*

10. *Determine the value of the coverage factor based upon the assumed confidence and the number of effective degrees of freedom.* Use a value of two for the coverage factor if 95 % confidence is assumed and $\nu_{eff} \geq 9$.

11. *Determine the overall uncertainty.*

12. *Present your findings in the proper form.* $x \pm U_x$ $(\%C)$.

Example Problem 7.17

Statement: Very accurate weight standards were used to calibrate a force-measurement system. Nine calibration measurements were made, including repetition

Applied Weight, W (N)	Output Voltage, E (V)
1	1.0
2	3.0
3	5.0
4	7.0
5	9.1
5	8.9
5	8.9
5	9.1
5	9.0

TABLE 7.1
Force-measurement system calibration data.

of the measurement five times at 5 N of applied weight. The results are presented in Table 7.1. Determine (a) the static sensitivity of the calibration curve at 3.5 N, (b) the random uncertainty with 95 % confidence in the value of the output voltage based upon the data obtained for the 5 N application cases, (c) the range within which the true mean of the voltage is for the 5 N application cases at 90 % confidence, (d) the range within which the true variance of the voltage is for the 5 N application cases at 90 % confidence, (e) the standard error of the fit based upon all of the data, and (f) the design-stage uncertainty of the instrument at 95 % confidence assuming that 0.04 V instrument uncertainty was obtained through calibration.

Solution: The average value of the output voltage for the five values of the 5 N case is 9.0 V. Thus, the data can be fitted the best by the line $E = 2W + 1$. (a) The sensitivity of a linear fit is the slope, 2 V/N, which is the same for all applied-weight values. (b) The random uncertainty for the 5 N cases with 95 % uncertainty is $P_{5N} = t_{\nu,P=95\,\%}S_{P,5N}$. Here, $S_{P,5N} = 0.1$ V and $\nu = 4$ with $t_{4,95} = 2.770$. This implies $P_{5N} = 0.277$ V. (c) The range within which the true mean value is contained extends $\pm t_{\nu,P=90\,\%}S_{P,5N}/\sqrt{N} = 5$ from the sample mean value of 9 V. Here, $t_{4,90} = 2.132$. So the range is from $9 - (2.132)(0.1)/\sqrt{5}$ V to $9 + (2.132)(0.1)/\sqrt{5}$ V, or from 8.905 V to 9.095 V. (d) The range of the true variance is

$$\frac{\nu S^2_{P,5N}}{\chi^2_{\alpha/2}} \leq \sigma^2_{5N} \leq \frac{\nu S^2_{P,5N}}{\chi^2_{1-\alpha/2}}.$$

$P = 0.90$, which implies $\alpha = 0.10$. So, $\chi^2_{\alpha/2}$ for $\nu = 4$ equals 9.49 and $\chi^2_{1-\alpha/2}$ equals 0.711. Substitution of these values yields

$$\frac{(4)(0.01)}{9.49} \leq \sigma^2_{5N} \leq \frac{(4)(0.01)}{0.711},$$

or

$$4.22 \times 10^{-3}\ \text{V}^2 \leq \sigma^2_{5N} \leq 56.26 \times 10^{-3}\ \text{V}^2.$$

This also gives $0.065\ \text{V} \leq \sigma^2_{5N} \leq 0.237$ V. (e) The first four and one of the five 5 N applied-weight case values are on the best-fit line. Therefore, only four of the five 5 N case values contribute to the standard error of the fit. Thus

$$S_{yx} = \sqrt{(0.1)^2 + (0.1)^2 + (-0.1)^2 + (0.1)^2/(9-2)} = 0.0756 = 0.1\ \text{V}.$$

(f) The resolution of the voltage equals 0.1 V from inspection of the data. This implies that $u_o = 0.05$ V. This uncertainty is combined in quadrature with the instrument uncertainty of 0.04 V to yield a design-stage uncertainty equal to 0.064 = 0.06 V.

7.12 *Finite-Difference Uncertainties

There are additional uncertainties that need to be considered. These occur when experiments are conducted to determine a result that depends upon the integral or derivative of measurand values obtained at discrete locations or times. The actual derivative or integral only can be estimated from this discrete information. A discretization (truncation) error results. For example, consider an experiment in which the velocity of a moving object is determined from measurements of time as the object passes known spatial locations. The actual velocity may vary nonlinearly *between* the two locations, but it can only be approximated using the finite information available. Similarly, an actual velocity gradient in a flow only can be estimated from measured velocities at two adjacent spatial locations. Examples involving integral approximations are the lift and drag of an object, determined from a finite number of pressure measurements along the surface of the object, and the flow rate of a fluid through a duct, determined from a finite number of velocity measurements over a cross-section of the duct.

The discretization errors of integrals and derivatives can be estimated, as described in the following section. Numerical round-off errors also can occur in such determinations. For most experimental situations, however, discretization errors far exceed numerical round-off errors. When measurements are relatively few, the discretization error can be comparable to the measurement uncertainty. There are many excellent references that cover finite-difference approximation methods and their errors ([3], [22], [11], and [24]).

7.12.1 *Derivative Approximation

If the values of a measurand are known at several locations or times, its actual derivative can be approximated. This is accomplished by representing the actual derivative in terms of a finite-difference approximation based upon, most commonly, a Taylor series expansion. The amount of error is determined by the order of the expansion method used.

Suppose $f(x)$ is a continuous function with all of its derivatives defined at x. The next, or forward, value of $f(x + \Delta x)$ can be estimated using a Taylor series expansion of $f(x + \Delta x)$ about the point x. That is,

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x) + \frac{(\Delta x)^3}{6} f'''(x) + \ldots . \quad (7.69)$$

Equation 7.69 can be rearranged to solve for the derivative

$$f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} - \frac{(\Delta x)}{2} f''(x) - \frac{(\Delta x)^2}{6} f'''(x) + \ldots . \quad (7.70)$$

The first term on the right side is the finite-difference representation of $f'(x)$ and the subsequent terms define the discretization error. A finite-difference representation is termed n-th order when the leading term in the discretization error is proportional to $(\Delta x)^n$. Thus, Equation 7.70 is known as the first-order, forward-difference expression for $f'(x)$. If the actual $f(x)$ can be expressed as a polynomial of the first degree, then $f''(x) = 0$, the finite-difference representation of $f'(x)$ exactly equals the actual derivative and there is no discretization error. In general, an n-th order method is exact for polynomials of degree n.

Example Problem 7.18

Statement: The velocity profile of a fluid flowing between two parallel plates spaced a distance $2h$ apart is given by the expression $u(y) = u_o[1 - (y/h)^2]$, where y is the coordinate perpendicular to the plates. Determine the exact value of $u(0.2h)/u_o$ and compare it with the finite-difference values obtained from the Taylor series expansion that result as each term is included additionally in the series.

Solution: The exact value is found from direct substitution of $y = 0.2h$ into the velocity profile is $u(0.2h)/u_o|_{exact} = 0.96$. For the Taylor series given by Equation 7.69, the derivatives must be computed. The result is $u'(y) = -2u_o y/h^2$, $u''(y) = -2u_o/h^2$, and $u'''(y) = 0$. Noting that $0.2h = \Delta x$ for this case, substitutions into Equation 7.69 yield $u(0.2h)/u_o|_{series} = 1 - 0.08 + 0.04 + 0 + \ldots$. So, three terms are required in the series in this case to give the exact result; fewer terms result in a difference between the exact and series values.

Similarly, the first-order, backward-difference expression for $f'(x)$ is

$$f'(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x} + \frac{(\Delta x)}{2} f''(x) - \frac{(\Delta x)^2}{6} f'''(x) + \ldots . \quad (7.71)$$

Equation 7.70 can be added to Equation 7.71 to yield

$$f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} - \frac{(\Delta x)^2}{6} f'''(x) + \ldots , \quad (7.72)$$

resulting in a second-order, central-difference expression for $f'(x)$. Other expressions for second-order, central-difference, and central-mixed-difference second and higher derivatives can be obtained following a similar approach [22].

Usually second-order accuracy is sufficient for experimental analysis. Assuming this, the discretization error, e_d, of the first derivative approximated by a second-order central-difference estimate using values at two locations ($x - \Delta x$ and $x + \Delta x$) is

$$e_d \simeq f'''(x) \frac{(\Delta x)^2}{6}, \quad (7.73)$$

where $f'''(x)$ is evaluated somewhere in the interval, usually at its maximum value. A problem arises, however, because the value of $f'''(x)$ is not known.

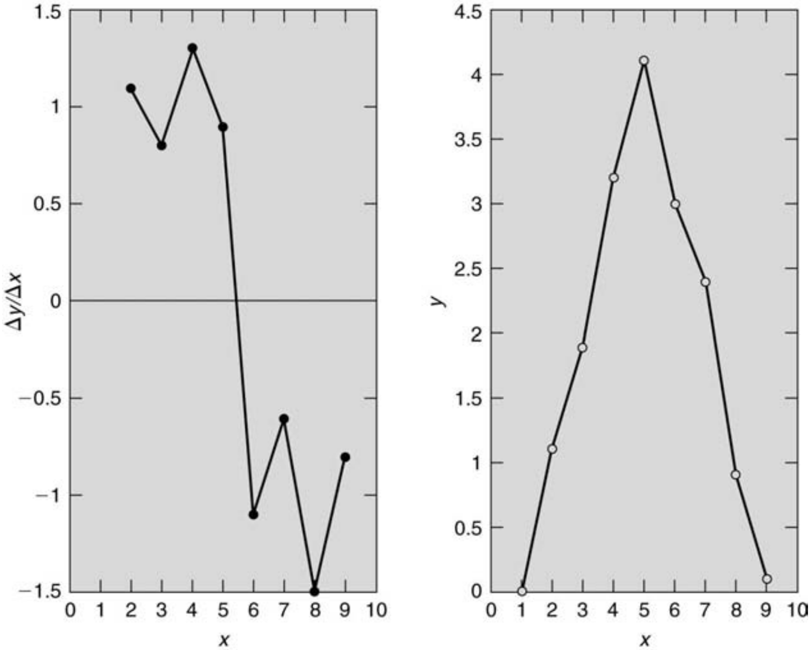

FIGURE 7.6
The output plot of `differ.m` for nine (x,y) data pairs of a triangular waveform.

So, only the order of magnitude of e_d can be estimated. Formally, the uncertainty in a first derivative approximated by a second-order central-difference method is

$$u_{f'(x)} \simeq C_{f'(x)}(\Delta x)^2, \tag{7.74}$$

where $C_{f'(x)}$ is a constant with same units as f'''. $C_{f'(x)}$ can be assumed to be of order one as a first approximation. The important point to note is that the discretization error is proportional to $(\Delta x)^2$. So, if Δx is reduced by 1/2, the discretization error is reduced by 1/4.

7.12.2 *Integral Approximation

Many different numerical methods can be used to determine the integral of a function. The method chosen depends upon the accuracy required, if the values of the function are known at its end points, if the numerical integration is done using equal-spaced intervals, and so forth. The trapezoidal rule is used most commonly for situations in which the intervals are equally spaced and the function's values are known at its end points.

The trapezoidal rule approximates the area under the curve $f(x)$ over the interval between a and b by the area of a trapezoid,

$$\int_a^b f(x) = \frac{(b-a)}{2}[f(b) + f(a)] + E, \tag{7.75}$$

where $E = (\Delta x)^3 f''(x)/24$, with $f''(x)$ evaluated somewhere in the interval from a to b. This rule can be extended to N points,

$$\begin{aligned}\int_{a=x_1}^{b=x_N} f(x)dx &= \Delta x[\frac{1}{2}f(x_1) + f(x_2) + \ldots + f(x_{N-1}) + \frac{1}{2}f(x_N)] + \sum_{i=1}^{N} E_i \\ &= \Delta x[\sum_{i=1}^{N} f(x_i) - (\frac{1}{2}f(x_1) + \frac{1}{2}f(x_N))] + \sum_{i=1}^{N} E_i, \end{aligned} \tag{7.76}$$

where $\Delta x = (b-a)/N$. The total discretization error, e_d, then becomes

$$e_d = \sum_{i=1}^{N} E_i \simeq \frac{1}{24}\sum_{i=1}^{N}(\Delta x)^3 f''(x) = \frac{N}{24}(\Delta x)^3 f''(x) = \frac{(b-a)^3}{24N^2} f''(x). \tag{7.77}$$

Thus, the uncertainty in applying the extended trapezoidal rule to approximate an integral is

$$u_{\int f(x)} \simeq C_{\int f(x)} N(\Delta x)^3, \tag{7.78}$$

where $C_{\int f(x)}$ is a constant with the same units as $f''(x)$. $C_{\int f(x)}$ can be assumed to be of order one as a first approximation.

A numerical estimate of $C_{\int f(x)}$ can be made if some expression for $f''(x)$ can be found. A second-order central second-difference approximation can be used, where

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2}. \tag{7.79}$$

This introduces an additional error of $\mathrm{O}[f''''(x_i)(\Delta x)^2]$. So, using Equation 7.79 does not improve the accuracy of the estimate; it simply provides a convenient way to estimate $f''(x_i)$. Now the discretization error also can be written alternatively as

$$e_d = \frac{(b-a)}{24N}\sum_{i=1}^{N} f''(x_i)(\Delta x)^2. \tag{7.80}$$

Substitution of Equation 7.79 into Equation 7.80 gives

$$u_{\int f(x)} \simeq \frac{(b-a)}{24N}\{\sum_{i=1}^{N}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})]^2\}^{1/2}. \tag{7.81}$$

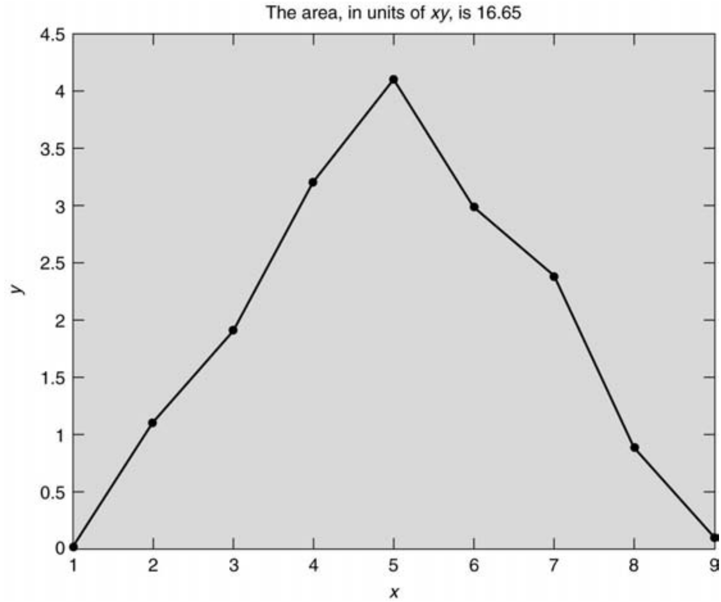

FIGURE 7.7
The output plot of `integ.m` for nine (x,y) data pairs of a triangular waveform.

Note that the terms with the brackets represent the discretization errors in the individual f'' estimates, which are combined in quadrature.

The following three problems illustrate how uncertainties arising from the finite-difference approximation of an integral factor into the uncertainty of a result.

Example Problem 7.19

Statement: Continuing with the experiment presented in the previous example, determine the uncertainties in the lift coefficient, C_L, and the drag coefficient, C_D, of the cylinder. The lift and drag coefficients are determined from 36 static pressure measurements around the cylinder's circumference done in 10° increments.

Solution: The lift coefficient is given by the equation

$$C_L = -\frac{1}{2} \int C_p(\theta) \sin(\theta) d\theta. \tag{7.82}$$

Because there are only 36 discrete measurements of the static pressure, the integral in Equation 7.82 must be approximated by a finite sum using the trapezoidal rule. The uncertainty that arises from this approximation is considered in a later example in this chapter. The general equation for the trapezoidal rule is

$$\int_a^b f(x)dx \cong \frac{(b-a)}{n}[\frac{f(a)}{2} + f(x_2) + \ldots + f(x_{n-1}) + \frac{f(b)}{2}].$$

Applying this formula to Equation 7.82 yields

$$\begin{aligned} C_L \cong -\frac{\pi}{72}[C_p(\theta = 0) &+ 2(C_p(\theta = \frac{10\pi}{180})\sin(\frac{10\pi}{180}) \\ &+ \ldots + C_p(\theta = \frac{350\pi}{180})\sin(\frac{350\pi}{180}) \\ &+ C_p(\theta = \frac{360\pi}{180})\sin(\frac{360\pi}{180})]\ (n = 36). \end{aligned}$$

Because the uncertainty in calculating C_L is an uncertainty in a result

$$u_{C_L} = \sqrt{(\frac{\partial C_L}{\partial C_p} u_{C_p})^2 + (\frac{\partial C_L}{\partial \sin\theta} u_{\sin\theta})^2}.$$

Now,

$$u_{\sin\theta} = \frac{\partial \sin\theta}{\partial \theta} u_\theta = u_\theta \cos\theta.$$

Therefore,

$$u_{C_L} = \sqrt{(u_{C_p} \sin\theta)^2 + (C_p(\theta) u_\theta \cos\theta)^2}.$$

This formulation must be applied to the finite-series approximation. Doing so leads to

$$\begin{aligned} u_{C_L} = \quad & \frac{\pi}{72}[(\sin(\theta = 0)u_{C_p})^2 + (C_p(\theta = 0)\cos(\theta = 0)u_\theta)^2 \\ & +(2\sin(\theta = \frac{10\pi}{180})u_{C_p})^2 + (2C_p(\theta = \frac{10\pi}{180})\cos(\frac{10\pi}{180})u_\theta)^2 \\ & +\ldots \\ & +(2\sin(\theta = \frac{350\pi}{180})u_{C_p})^2 + (2C_p(\theta = \frac{350\pi}{180})\cos(\frac{350\pi}{180})u_\theta)^2 \\ & +(\sin(\theta = \frac{360\pi}{180})u_{C_p})^2 + (C_p(\theta = \frac{360\pi}{180})\cos(\frac{360\pi}{180})u_\theta)^2]^{1/2}. \end{aligned}$$

This can be evaluated using a spreadsheet or MATLAB. For the case when $u_{C_p} = 0.021$ and $u_\theta = \pi/360$ ($\pm 0.5°$), $u_{C_L} = 0.0082$. Likewise, u_{C_D} can be evaluated. The expression is similar to u_{C_L} above but with cos and sin reversed. For $u_{C_p} = 0.021$ and $u_\theta = \pi/180$ ($\pm 1.0°$), $u_{C_D} = 0.0081$. Now assume that the experiment was performed within a Reynolds number range that yields $C_D \sim 1$. Thus, percent $u_{C_D} \cong 0.8$ %.

It is important to note that these u_{C_L} and u_{C_D} uncertainties do not include their finite series approximation uncertainties. These are determined in a following example.

Example Problem 7.20

Statement: Determine the uncertainty in the drag of the cylinder that was studied in the previous examples, where the drag, D, is defined as

$$D \equiv C_D \frac{1}{2}\rho V_\infty^2 A_{frontal}, \tag{7.83}$$

with $A_{frontal} = d \cdot L$.

Solution: In the experiment $\frac{1}{2}\rho V_\infty^2$ was measured as ΔP. Thus,

$$\begin{aligned} u_D &= [(\frac{\partial D}{\partial C_D} u_{C_D})^2 + (\frac{\partial D}{\partial \Delta P} u_{\Delta P})^2 + (\frac{\partial D}{\partial d} u_d)^2 + (\frac{\partial D}{\partial L} u_L)^2]^{1/2} \\ &= [(\Delta P d L u_{C_D})^2 + (C_D d L u_{\Delta P})^2 + (C_D \Delta P L u_d)^2 + (C_D \Delta P d u_L)^2]^{1/2}. \end{aligned}$$

Now, given that

$$\begin{aligned} \Delta P &= 4 \text{ in. } H_2O = 996 \text{ N/m}^2, u_{\Delta P} = 15 \text{ N/m}^2 \\ d &= 1.675 \text{ in.} = 0.0425 \text{ m}, u_d = 0.005 \text{ in.} = 1.27 \times 10^{-4} \text{ m} \\ L &= 16.750 \text{ in.} = 0.425\ 45 \text{ m}, u_L = 0.005 \text{ in.} = 1.27 \times 10^{-4} \text{ m} \\ C_D &\cong 1, u_{C_D} = 0.0092, \end{aligned}$$

then

$$\begin{aligned} u_D &= [(996)(0.0425)(0.425\ 45)(0.0092)^2 \\ &\quad (1)(0.0425)(0.425\ 45)(15)^2 \\ &\quad (1)(996)(0.425\ 45)(1.27 \times 10^{-4})^2 \\ &\quad (1)(996)(0.0425)(1.27 \times 10^{-4})^2]^{1/2} \\ &= [0.166^2 + 0.271^2 + 0.054^2 + 0.005^2]^{1/2} \\ &= 0.32. \end{aligned}$$

In order to get a percentage error, the nominal value of the drag is computed, where

$$D \cong (1)(996)(0.0425)(0.42545) = 18.0 \text{ N}.$$

Thus, the percentage error in the drag is 1.7 %, and $D = 18.0 \pm 0.3$ N.

Example Problem 7.21

Statement: Recall the experiment presented in a previous example in which 36 static pressure measurements were made around a cylinder's circumference. Determine the uncertainties in the lift and drag coefficients that arise when the extended trapezoidal rule is used to approximate the integrals involving the pressure coefficient, where

$$C_L = -\frac{1}{2} \int_0^{2\pi} C_P(\theta) \sin(\theta) d\theta \tag{7.84}$$

and

$$C_D = -\frac{1}{2} \int_0^{2\pi} C_P(\theta) \cos(\theta) d\theta.$$

Compare these numerical uncertainties to their respective measurement uncertainties which were obtained previously. Finally, determine the overall uncertainties in C_L and C_D.

Solution: Equation 7.81 implies

$$u_{\int f(x), C_L} \simeq \frac{2\pi}{24N} \{\sum_{i=1}^{N} [g(\theta_{i+1}) - 2g(\theta_i) + g(\theta_{i-1})]^2\}^{1/2} \tag{7.85}$$

and

$$u_{\int f(x),C_D} \simeq \frac{2\pi}{24N}\{\sum_{i=1}^{N}[h(\theta_{i+1}) - 2h(\theta_i) + h(\theta_{i-1})]^2\}^{1/2},$$

where

$$\begin{aligned} g(\theta) &= C_P(\theta)\sin(\theta), \\ h(\theta) &= C_P(\theta)\cos(\theta), \text{ and} \\ N &= 36. \end{aligned}$$

These uncertainties can be evaluated using a spreadsheet or MATLAB, yielding $u_{\int f(x),C_L} = 0.0054$ and $u_{\int f(x),C_D} = 0.0042$. These uncertainties are approximately one-half of the C_L and C_D measurement uncertainties.

Combining the C_L and C_D measurement and numerical approximation uncertainties gives

$$u_{C_L} = [0.0082^2 + 0.0054^2]^{1/2} = 0.0098 \text{ and}$$
$$u_{C_D} = [0.0081^2 + 0.0042^2]^{1/2} = 0.0092.$$

7.12.3 *Uncertainty Estimate Approximation

In some situations the direct approach to estimating the uncertainty in a result can be complicated if the mathematical expression relating the result to the measurands is algebraically complex. An alternative approach is to approximate numerically the partial derivatives in the uncertainty expression, thereby obtaining a more tractable expression that is amenable to spreadsheet or program analysis.

The partial derivative term, $\partial q/\partial x_i$, can be approximated numerically by the finite-difference expression

$$\frac{\partial q}{\partial x_i} \approx \frac{\Delta q}{\Delta x_i} = \frac{q|_{x_i+\Delta x_i} - q|_{x_i}}{\Delta x_i}. \tag{7.86}$$

This approximation is first-order accurate, as seen by examining Equation 7.70. Thus, its discretization error is of order $\Delta x f''(x)$. Use of Equation 7.86 yields the forward finite-difference approximation to Equation 7.21

$$u_q \approx \sqrt{(\frac{\Delta q}{\Delta x_1}u_{x_1})^2 + (\frac{\Delta q}{\Delta x_2}u_{x_2})^2 + ... + (\frac{\Delta q}{\Delta x_k}u_{x_k})^2}. \tag{7.87}$$

The value of Δx_i is chosen to be small enough such that the finite-difference expression closely approximates the actual derivative. Typically, $\Delta x_i = 0.01x_i$ is a good starting value. The value of Δx_i then should be decreased until appropriate convergence in the value of u_q is obtained.

Example Problem 7.22

Statement: In a previous example in this chapter, the uncertainty in the density for air was determined directly by the expression

$$u_\rho = \sqrt{\left(\frac{\partial \rho}{\partial T}u_T\right)^2 + \left(\frac{\partial \rho}{\partial P}u_P\right)^2} = 2.15 \times 10^{-3}\ \text{kg/m}^3,$$

where $u_P = 67$ Pa, $u_T = 0.5$ K, $\rho = 1.19$ kg/m^2, $T = 297$ K, $P = 101\ 325$ Pa, and air was assumed to be an ideal gas ($\rho = P/RT$). Determine the uncertainty in ρ by application of Equation 7.87.

Solution: The finite-difference expression for this case is

$$u_\rho \approx \sqrt{(\frac{\Delta \rho}{\Delta P}u_P)^2 + (\frac{\Delta \rho}{\Delta T}u_T)^2},$$

where $\frac{\Delta\rho}{\Delta P} = [\rho\ |_{P+\Delta P} - \rho_P]/\Delta P$ and $\frac{\Delta\rho}{\Delta T} = [\rho\ |_{T+\Delta T} - \rho_T]/\Delta T$. Letting $\Delta P = 0.01P$ and $\Delta T = 0.01T$ yields $\frac{\Delta\rho}{\Delta P} = [(P+\Delta P)/RT - P/RT]/\Delta P = 1/RT = \rho/P = 1.17 \times 10^{-5}$ and $\frac{\Delta\rho}{\Delta T} = [P/R(T-\Delta T) - P/RT]/\Delta T = -\rho/(T+\Delta T) = 3.97 \times 10^{-3}$.

Substitution of these values into the equation gives $u_\rho = \sqrt{0.62 \times 10^{-6} + 3.93 \times 10^{-6}} = 2.13 \times 10^{-3}$ kg/m^3. This agrees within 1 % of the value of 2.15×10^{-3} found using the direct method.

7.13 *Uncertainty Based upon Interval Statistics

In some measurements situations, specific values for part or all of the data may not be known. Rather, only a range or interval of possible values is known. This situation introduces another type of uncertainty, that due to a lack of knowledge about the specific values of the data, known as **epistemic** uncertainty. When all other experimental uncertainties are removed, epistemic uncertainty still remains and becomes the overall uncertainty. Thus, a different approach beyond the current ISO method, which assumes linearizably small and normally distributed measurement uncertainties, is required to handle this situation.

Ferson *et al.* [25] present such an approach that is based upon interval statistics. The ISO method assumes that measurement uncertainty results primarily from variability in the data that is caused by inherent randomness and/or finite sampling (termed **aleatory** uncertainty). Based upon this method, additional measurements will reduce the lower bound of the uncertainty estimate to the limit of design-stage uncertainties, which itself can be minimized significantly. However, when epistemic uncertainty is present, the situation can be different. The ISO method will *underestimate* the overall measurement uncertainty when epistemic uncertainty is present. Further, additional data will not necessarily lead to a reduced uncertainty. The reader is referred to the report by Ferson *et al.* [25] and the monograph by Salicone [26] for more detailed information.

Treating epistemic uncertainty, as explained in detail by Ferson *et al.* [25], can be illustrated by examining a data set consisting of N outcomes, each with a different range of possible values. The range for one outcome possibly may overlap that of another. How the values are distributed within a range is not specified. The outcomes can be represented as

$$\begin{bmatrix} y_{L_1}, y_{H_1} \\ y_{L_2}, y_{H_2} \\ \vdots \\ y_{L_N}, y_{H_N} \end{bmatrix},$$

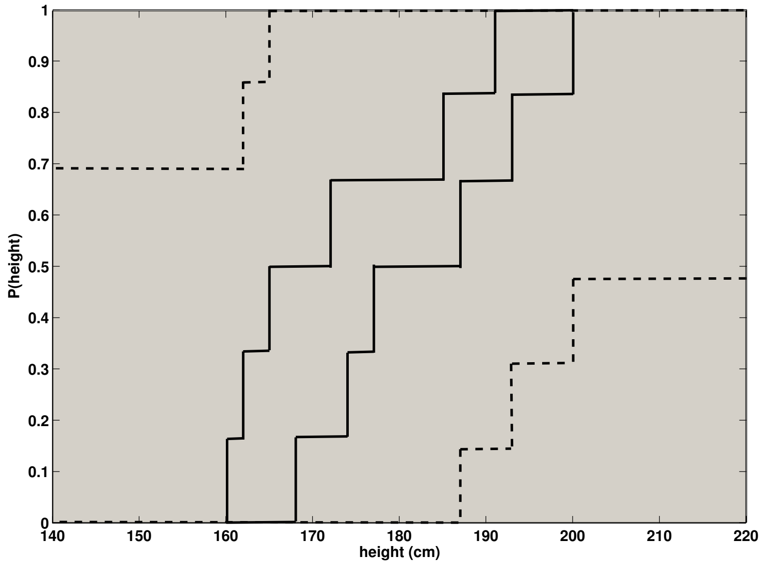

FIGURE 7.8
The p-box representation of the probability distribution function of the height estimates (solid) with 95 % confidence limits (dashed).

where y_{L_i} denotes the lower value of the range of the i-th estimate and y_{H_i} the upper value, with $i = 1, ..., N$.

Consider the example in which the heights of six different people are estimated visually by an observer as each person runs past a height scale located 10 m from the observer. The height estimates in centimeters made by the observer in ascending order are

$$\begin{bmatrix} 160, 168 \\ 162, 174 \\ 165, 177 \\ 172, 187 \\ 185, 193 \\ 191, 200 \end{bmatrix}.$$

These results are displayed graphically in a p-box representation in Figure 7.8. Here, y represents the measure (magnitude) of the height and $P(y)$ the empirically established probability distribution function of the height. The width of each box is the range for each height estimate and the height of each box is $1/N$ (here, 1/6), as indicated by the solid-line bounds in the figure. For example, based upon the observer's estimates, there is a 50.0 % probability that the observed heights were in the range from 160 cm (the minimum height observed) to a height of from 172 cm to 177 cm (172 cm if the lower bound at 50 % is used, and 177 cm if the upper bound at 50 % is used), and 83.3 % probability that they were in the range from 160 cm to a height of from 187 cm to 193 cm.

Confidence limits for $P(y)$, making no assumption of the specific distribution function that governs the possible values, can be determined based upon methods developed by Kolmogorov [27] and Smirnov [28]. These are termed Kolmogorov-Smirnov (distribution-free) confidence limits. The lower, $\underline{B}(y)$, and upper, $\overline{B}(y)$, limits for each estimate are given by

$$[\underline{B}(y), \overline{B}(y)] = \\ [\min(1, S_{LN}(y) + D_{\max}(\alpha, N)), \max(0, S_{RN}(y) - D_{\max}(\alpha, N))], \quad (7.88)$$

where S_{LN} is the fraction of the left end points of the estimate values that are at or below the magnitude y, S_{RN} is the fraction of the right end points of the estimate values that are at or above the magnitude y, and $D_{\max}$ is the Smirnov statistic that is a function of α and N. Statistically, $100(1 - \alpha)$ % of the time, the confidence limits will enclose the distribution function of the population from which the samples were drawn. When N is greater than $\sim$50 and $\alpha = 5$ % (95 % confidence limits), $D_{\max}(0.05, N) \approx 1.36/\sqrt{N}$. For smaller values of N, the value of $D_{\max}(\alpha, N)$ can be obtained from a statistical table [29], [30]. These confidence limits do not depend upon the type of the distribution function, only that the distribution is continuous and that the samples have been drawn randomly and independently from the distribution.

Applying these methods to the above example of the height estimates, assuming 95 % confidence and with $D_{\max}(0.05, 6) = 0.525$, yields the confidence limits shown by the dashed-line bounds in Figure 7.8. For example, 95 % of the time when height estimates are made in the manner described above, 50 % of the heights could be anywhere in the range from the minimum possible height (theoretically 0 cm) to 200 cm. Clearly, these are very extensive bounds. Their extent results from the small sample size, here $N = 6$, and the distribution-free interval assumption. Larger sample sizes and/or a specification of the interval distribution will reduce these limits. For example, if $N = 100$, $D_{\mathrm{max}}(0.05, 100) = 0.134$, 95 % of the time when height estimates are made, 50 % of the heights could be anywhere in the range from the minimum possible height to $\sim$187 cm.

It is important to emphasize that these confidence limits, which are based on a lack of knowledge about the specific values of the data, do not consider additional uncertainties, such as instrument resolution and measurement accuracy and precision. Ferson *et al.* [25] present a hybrid approach in which the methods of interval statistics are combined with the standard uncertainty method, thereby overcoming this limitation. Their results show that the confidence limits are the most extensive (widest) when distribution-free intervals are assumed. When normal-distribution intervals are assumed, the confidence limits become less extensive (narrower). The current ISO method gives the least extensive (narrowest) confidence limits because it assumes a normal distribution and neglects epistemic uncertainty.

7.14 Problem Topic Summary

Topic	Review Problems	Homework Problems
Basic Uncertainty	1, 3, 4, 8, 10, 12, 13, 14, 18, 20, 22, 23, 24	8, 21, 23
Result Uncertainty	5, 9, 19	1, 2, 7, 13, 14, 15, 16, 19, 21, 26, 27, 28
General Uncertainty	2, 6, 7, 9, 11, 15, 16, 17, 21	3, 5, 9, 10, 11, 18, 19, 23, 24, 25
Detailed Uncertainty		4, 6, 12
Finite-Difference Approach		17

TABLE 7.2
Chapter 7 Problem Summary

7.15 Review Problems

1. A researcher is measuring the length of a microscopic scratch in a microphone diaphragm using a stereoscope. A ruler incremented into ten-thousandths of an inch is placed next to the microphone as a distance reference. If the stereoscope magnification is increased 10 times, what property of the distance measurement system has been improved? (a) Sensitivity, (b) precision, (c) readability, (d) least count.

2. A multimeter, with a full-scale output of 5 V, retains two decimal digits of resolution. For instance, placing the multimeter probes across a slightly used AA battery results in a readout of 1.35 V. Through calibration, the instrument uncertainties established are sensitivity, 0.5 % of FSO, and offset, 0.1 % of FSO. What is the total design stage uncertainty in volts based on this information to 95 % confidence? (Note: The readout of the instrument dictates that the uncertainty should be expressed to three decimal places.)

3. Three students are playing darts. The results of the first round are shown in Figure 7.9, where the circle in the center is the bullseye. Circles = Player 1; squares = Player 2; triangles = Player 3. In terms of hitting the bullseye, which player best demonstrates precision, but not accuracy? (a) Player 1 (circles), (b) Player 2 (squares), (c) Player 3 (triangles).

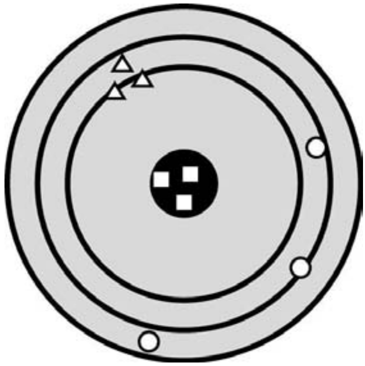

FIGURE 7.9
Dartboard.

4. Compare the precision of a metric ruler, incremented in millimeters, with a standard customary measure ruler, incremented into 16ths of an inch. How much more precise is the more precise instrument? Express your answer in millimeters and consider the increments on the rulers to be exact.

5. For a circular rod with density, ρ, diameter, D, and length, L, derive an expression for the uncertainty in computing its moment of inertia about the rod's end from the measurement of those three quantities. If $\rho = 2008 \pm 1$ kg/m^3, $D = 3.60$ mm $\pm$ 0.05 mm, and $L = 2.83$ m $\pm$ 0.01 m, then compute the uncertainty in the resulting moment of inertia to the correct number of significant figures in the SI units of kilograms per square meter. Significant figures are based on the measured quantities. The formula for the moment of inertia of a circular rod about its end is $I = \rho\pi/12D^2L^3$.

6. The velocity of the outer circumference of a spinning disk may be measured in two ways. Using an optical sensing device, the absolute uncertainty in the velocity is 0.1 %. Using a strobe and a ruler, the uncertainty in the angular velocity is 0.1 rad/s and the uncertainty in the diameter of the disk is 1 mm. Select the measurement method with the least uncertainty for the two methods if the disk is 0.25 m in diameter and is spinning at 10 rpm.

7. Using a pair of calipers with 0.001 in. resolution, a machinist measures the diameter of a pinball seven times with the following results in units of inches: 1.026, 1.053, 1.064, 1.012, 1.104, 1.098, and 1.079. He uses the average of the measurements as the correct value. Compute the uncertainty in the average diameter in inches to one-thousandth of an inch.

8. Match the following examples of experimental error (numbered 1 through 4) and uncertainty to the best categorization of that example (lettered a through d): (1) temperature fluctuations in the laboratory, (2) physically pushing a pendulum through its swing during one experimental trial, (3) the numerical bounds of the scatter in the distance traveled by the racquet ball about the mean value, (4) releasing the pendulum from an initial position so that the shaft has a small initial angle to the vertical for each trial; (a) uncertainty, (b) systematic error, (c) experimental mistake, (d) random error.

9. A geologist finds a rock of unknown composition and desires to measure its density. To measure the volume, she places the rock in a cylinder, which is graduated in 0.1 mL increments, half-filled with water, so that the rock is submerged in the water. She removes the rock from the cylinder and directly measures the rock's mass using a scale with a digital readout resolution of 0.1 g. No information is provided from the manufacturer about the scale uncertainty. She records the volume, V, and mass, m, as follows: V = 40.5 mL, m = 143.1 g. Determine the percent uncertainty in the density expressed with the correct number of significant figures.

10. In addition to the digital display, the manometer described in the previous problem has an analog voltage output that is proportional to the sensed differential pressure in units of in inches of water. A researcher calibrates the analog output by applying a series of known pressures across the manometer ports and observing both the analog and digital output for each input pressure. A linear fit to the data yields $P = 1.118E + 0.003$, where P is pressure (in. H_2O) and E is the analog voltage (V). If for zero input pressure, the manufacturer specifies that no output voltage should result, what magnitude of systematic error has been found?

11. The mean dynamic pressure in a wind tunnel is measured using the manometer described in the last two problems and a multimeter to measure the output voltage. The recorded voltages are converted to pressures using the calibration fit presented in Problem 10. The resolution of the multimeter is 1 mV and the manufacturer specifies the following instrument errors: sensitivity = 0.5 mV and linearity = 0.5 mV. If 150 multimeter readings with a standard deviation of 0.069 V are acquired with a mean voltage of 0.312 V, what is the uncertainty in the resulting computed mean pressure in inches of water? Express the answer to the precision of the digital readout of the manometer.

12. A technician uses a graduated container of water to measure the volume of odd-shaped objects. The changes in the density of the water caused by ambient temperature and pressure fluctuations directly contribute

to (a) systematic error, (b) redundant error, (c) random error, or (d) multiple measurement error.

13. A graduate student orders a set of very accurate weights to calibrate a digital scale. The desired accuracy of the scale is 0.1 g. The manufacturer of the weights states that the mass of each weight is accurate to 0.04 g. What is the maximum number of weights that may be used in combination to calibrate the scale?

14. A test engineer performs a first-run experiment to measure the time required for a 2010 prototype car to travel a fourth of a mile beginning from rest. When the car begins motion, a green light flashes in the engineer's field of vision, signaling him to start the time count with a hand-held stopwatch. Similarly, a red light flashes when the car reaches the finish line. The resulting times from four trials are 13.42 s, 13.05 s, 12.96 s, and 12.92 s. Outside of the test environment, another engineer measures the first test engineer's reaction time to the light signals. The results of the test show that the test engineer over-anticipates the green light and displays slowed reaction to the red light. Both reaction times were measured to be 0.13 s. Compute the average travel time in seconds, correcting for the systematic error in the experimental procedure.

15. A digital manometer measures the differential pressure across two inputs. The range of the manometer is 0 in. H_2O to 0.5 in. H_2O. The LED readout resolves pressure into 0.001 in. H_2O. Based on calibration, the manufacturer specifies the following instrument errors: hysteresis error = 0.1 % of FSO; linearity = 0.25 % of FSO; sensitivity = 0.1 % of FSO. Determine the design stage uncertainty of the digital manometer in inches of water to the least significant digit resolved by the manometer.

16. The smallest division marked on the dial of a pressure gage is 2 psi. The accuracy of the pressure gage as stated by the manufacturer is ±1 psi. Determine the design-stage uncertainty in psi and express it with the correct number of significant figures.

17. A student conducts an experiment in which the panel meter displaying the measurement system's output in volts fluctuates up and down in time. Being a conscientious experimenter, the student decides to estimate the temporal random error of the measurement. She takes 100 repeated measurements and finds that the standard deviation equals a whopping 1.0 V! Determine the temporal random error in volts at 95 % confidence and express the answer with the correct number of significant figures.

18. Standard measurement uncertainty is (a) the error in a measurement, (b) the probability of a measurement being correct, (c) the probability

of a measurement not being correct, (d) an estimate of the range of probable errors in a measurement, or (e) the sum of the systematic and random errors.

19. If the uncertainties in the length and diameter of a cylinder are 2 % and 3 % respectively, what is the percent uncertainty in its volume expressed with the correct number of significant figures?

20. Sixty-four pressure measurements have a sample mean equal to 200 N/m^2 and a sample variance equal to 16 N^2/m^4. What is the percent uncertainty in the pressure measurement if the only contributor to its uncertainty is the random error?

21. A voltmeter having three digits displays a reading of 8.66 V. What percent instrument uncertainty must the voltmeter have to yield a design-stage uncertainty of 0.01 V at 8.66 V?

22. Determine the temporal precision error, in V, in the estimate of the average value of a voltage based upon nine measurements at 95 % confidence and a sample variance of 9 V^2.

23. Match the following examples of experimental error and uncertainty (1 through 4) to the type of uncertainty they correspond to (a through d): (1) fluctuations in the humidity of the air during the summer while conducting a month-long series of experiments, (2) holding a ruler at an angle to the measurement plane for a series of measurements, (3) while taking data, bumping into a table that holds a pendulum experiment, (4) the numerical bounds of the scatter in the height a ball bounces during measurements of its coefficient of restitution; (a) systematic error, (b) experimental mistake, (c) random error, (d) uncertainty.

24. A student records a small sample of three voltage measurements: 1.000 V, 2.000 V, and 3.000 V. Determine the uncertainty in the population's true mean value of the voltage estimated with 50 % confidence. Express your answer with the correct number of significant figures.

7.16 Homework Problems

1. The supply reservoir to a water clock is constructed from a tube of circular section. The tube has a nominal length of 52 cm $\pm$ 0.5 cm, an outside diameter of 20 cm $\pm$ 0.04 cm, and an inside diameter of 15 cm $\pm$ 0.08 cm. Determine the percent uncertainty in the calculated volume.

2. A mechanical engineer is asked to design a cantilever beam to support a concentrated load at its end. The beam is of circular section and has a length, L, of 6 ft and a diameter, d, of 2.5 in. The concentrated load, F, of 350 lbf is applied at the beam end, perpendicular to the length of the beam. If the uncertainty in the length is ± 1.5 in., in the diameter is ± 0.08 in., and in the force is $\pm$ 5 lbf, what is the uncertainty in the calculated bending stress, σ? [Hint: $\sigma = 32FL/(\pi d^3)$.] Further, if the uncertainty in the bending stress may be no greater than 6 %, what maximum uncertainty may be tolerated in the diameter measurement if the other uncertainties remain unchanged?

3. An electrical engineer must decide on a power usage measurement method that yields the least uncertainty. There are two alternatives to measuring the power usage of a DC heater. Either (1) heater resistance and voltage drop can be measured simultaneously and then the power computed, or (2) heater voltage drop and current can be measured simultaneously and then the power computed. The manufacturers' specifications of the available instruments are as follows: ohmmeter (resolution 1 Ω and reading uncertainty = 0.5 %); ammeter (resolution 0.5 A and % reading uncertainty = 1 %); voltmeter (resolution 1 V and % reading uncertainty = 0.5 %). For loads of 10 W, 1 kW, and 10 kW each, determine the best method based on an appropriate uncertainty analysis. Assume nominal values as necessary for resistance and current based upon a fixed voltage of 100 V.

4. A new composite material is being developed for an advanced aerospace structure. The material's density is to be determined from the mass of a cylindrical specimen. The volume of the specimen is determined from diameter and length measurements. It is estimated that the mass, m, can be determined to be within 0.1 lbm using an available balance scale, the length, L, to within 0.05 in., and the diameter, D, to within 0.0005 in. Estimate the zero-order design stage uncertainty in the determination of the density. Which measurement would contribute most to the uncertainty in the density? Which measurement method should be improved first if the estimate in the uncertainty in the density is unacceptable? Use nominal values of $m = 4.5$ lbm, $L = 6$ in., and $D = 4$ in. Next, multiple measurements are performed yielding the data shown in Table 7.3. Using this information and what was given initially, provide an estimate of the true density at 95 % confidence. Compare the uncertainty in this result to that determined in the design stage.

5. High pressure air is to be supplied from a large storage tank to a plenum located immediately before a supersonic convergent-divergent nozzle. The engineer designing this system must estimate the uncertainty in the plenum's pressure measurement system. This system outputs a voltage that is proportional to pressure. It is calibrated against a transducer

$D = 3.9924$ in.	$\bar{m} = 4.4$ lbm	$L = 5.85$ in.
$S_D = 0.0028$ in.	$S_m = 0.1$ lbm	$S_L = 0.10$ in.
$N = 3$	$N = 21$	$N = 11$

TABLE 7.3
Composite material data.

standard (certified accuracy: within ±0.5 psi) over its 0 psi to 100 psi range with the results given below. The voltage is measured with a voltmeter (instrument error: within ±10 μV; resolution: 1 μV). The engineer estimates that installation effects can cause the indicated pressure to be off by another ±0.5 psi. Estimate the uncertainty at 95 % confidence in using this system based upon the following information given in Table 7.4.

E(mv):	0.004	0.399	0.771	1.624	2.147	4.121
p(psi):	0.1	10.2	19.5	40.5	51.2	99.6

TABLE 7.4
Storage tank calibration data.

6. One approach to determining the volume of a cylinder is to measure its diameter and length and then calculate the volume. If the length and diameter of the cylinder are measured at four different locations using a micrometer with an uncertainty of 0.05 in. with 95 % confidence, determine the percent uncertainty in the volume. The four diameters in inches are 3.9920, 3.9892, 3.9961, and 3.9995; those of the length in inches are 4.4940, 4.4991, 4.5110, and 4.5221.

7. Given $y = ax^2$ and that the uncertainty in a is 3 % and that in x is 2 %, determine the percent uncertainty in y for the nominal values of $a = 2$ and $x = 0.5$.

8. The lift force on a Wortmann airfoil is measured five times under the same experimental conditions. The acquired values are 10.5 N, 9.4 N, 9.1 N, 11.3 N, and 9.7 N. Assuming that the only uncertainty in the experiment is a temporal random error as manifested by the spread of the data, determine the uncertainty (in ±N) at the 95 % confidence level of the true mean value of the lift force.

9. A pressure transducer specification sheet lists the following instrument errors, all in units of percent span, where the span for the particular pressure transducer is 10 in. H_2O: combined null and sensitivity shift

= ±1.00, linearity = ±2.00, and repeatability and hysteresis = ±0.25. Estimate (a) the transducer's instrument uncertainty in the pressure in units of inches of water and (b) the % instrument uncertainty in a pressure reading of 1 in. H_2O. (c) Would this be a suitable transducer to use in an experiment in which the pressure ranged from 0 in. H_2O to 2 in. H_2O and the pressure reading must be accurate to within ± 10 %?

10. The mass of a golf ball is measured using an electronic balance that has a resolution of 1 mg and an instrument uncertainty of 0.5 %. Thirty-one measurements of the mass are made yielding an average mass of 45.3 g and a standard deviation of 0.1 g. Estimate the (a) zero-order, (b) design-stage, and (c) first-order uncertainties in the mass measurement. What uncertainty contributes the most to the first-order uncertainty?

11. A group of students wish to determine the density of a cylinder to be used in a design project. They plan to determine the density from measurements of the cylinder's mass, length, and diameter, which have instrument resolutions of 0.1 lbm, 0.05 in., and 0.0005 in., respectively. The balance used to measure the weight has an instrument uncertainty (accuracy) of 1 %. The rulers used to measure the length and diameter present negligible instrument uncertainties. Nominal values of the mass, length, and diameter are 4.5 lbm, 6.00 in., and 4.0000 in., respectively. (a) Estimate the zero-order uncertainty in the determination of the density. (b) Which measurement contributes the most to this uncertainty? (c) Estimate the design-stage uncertainty in the determination of the density.

12. The group of students in the previous problem now perform a series of measurements to determine the actual density of the cylinder. They perform 20 measurements of the mass, length, and diameter that yield average values for the mass, length, and diameter equal to 4.5 lbm, 5.85 in., and 3.9924 in., respectively, and standard deviations equal to 0.1 lbm, 0.10 in., and 0.0028 in., respectively. Using this information and that presented in the previous problem, estimate (a) the average density of the cylinder in lbm/in.^{3}, (b) the systematic errors of the mass, length, and diameter measurements, (c) the random errors of the mass, length, and diameter measurements, (d) the combined systematic errors of the density, (e) the combined random errors of the density, (f) the uncertainty in the density estimate at 95 % confidence (compare this to the design-stage uncertainty estimate, which should be smaller), and (g) an estimate of the true density at 95 % confidence.

13. Given King's law, $E^2 = A + B\sqrt{U}$, and the fractional uncertainties in A, B, and U of 5 %, 4 %, and 6 %, respectively, determine the percent fractional uncertainty in E with the correct number of significant figures.

14. The resistivity ρ of a piece of wire must be determined. To do this, the relationship $R = \rho L/A$ can be used and the appropriate measurements made. Nominal values of R, L, and the diameter of the wire, d, are 50 Ω, 10 ft, and 0.050 in., respectively. The error in L must be held to no more than 0.125 in. R will be measured with a voltmeter having an accuracy of ±0.2 % of the reading. How accurately will d need to be measured if the uncertainty in ρ is not to exceed 0.5 %?

15. The tip deflection of a cantilever beam with rectangular cross-section subjected to a point load at the tip is given by the formula

$$\delta = \frac{PL^3}{3EI}, \text{ where } I = \frac{bh^3}{12}.$$

Here, P is the load, L is the length of the beam, E is the Young's modulus of the material, b is the width of the cross-section, and h is the height of the cross-section. If the instrument uncertainties in P, L, E, b, and h are each 2 %, (a) estimate the fractional uncertainty in δ. This beam is used in an experiment to determine the value of an unknown load, P_x, by performing four repeated measurements of δ at that load under the same controlled conditions. The resulting sample standard deviation of these measurements is 8 μm and the average deflection is 20 μm. Determine (b) the overall uncertainty in the deflection measurements estimated at 90 % confidence assuming that the resolution of the instrument used to measure δ is so small that it produces negligible uncertainty.

16. The resistance of a wire is given by $R = R_o[1 + \alpha(T - T_o)]$ where T_o = 20 °C, R_o = 6 Ω ± 0.3 % is the resistance at 20 °C, α = 0.004/ °C ±1 % is the temperature coefficient of resistance, and the temperature of the wire is $T = 30 \pm 1$ °C. Determine (a) the normal resistance of the wire and (b) the uncertainty in the resistance of the wire, u_R.

17. Calculate the uncertainty in the wire resistance that was described in the previous problem using the first-order finite-difference technique.

18. An experiment is conducted to verify an acoustical theory. Sixty-one pressure measurements are made at a location using a pressure measurement system consisting of a pressure transducer and a display having units of kilopascals. A statistical analysis of the 61 measurand values yields a sample mean of 200 kPa and a sample standard deviation of 2 kPa. The resolution of the pressure display is 6 kPa. The pressure transducer states that the transducer has a combined hysteresis and linearity error of 2 kPa, a zero-drift error of 2 kPa, and sensitivity error of 1 kPa. (a) What classification is this experiment? Determine the system's (b) zero-order uncertainty, (c) instrument uncertainty, (d) uncertainty arising from pressure variations, and (e) combined standard uncertainty. Assume 95 % confidence in all of the estimates. Express all estimates with the correct number of significant figures.

19. A hand-held velocimeter uses a heated wire and, when air blows over the wire, correlates the change in temperature to the air speed. The reading on the velocimeter, v_{std}, is relative to standard conditions defined as $T_{std} = 70$ °F and $P_{std} = 14.7$ psia. To determine the actual velocity, v_{act}, in units of feet per minute, the equation

$$v_{act} = v_{std}[(460 + T)/(460 + T_{std})][P_{std}/P]$$

must be applied, where T is in ^{o}F and P is in psia. The accuracy of the reading on the velocimeter is ±5.0 % or ±5 ft/min, whichever is greater. The velocimeter also measures air temperature with an accuracy of ±1 °F. During an experiment, the measured air velocity is 400 ft/min and the temperature is 80 °F. The air pressure can be assumed to be at standard conditions. Determine (a) the actual air velocity, (b) the uncertainty in the actual air velocity, $u_{v_{act}}$, and (c) the percent uncertainty in the actual air velocity.

20. Compute the random uncertainty (precision limit) for each of the following and explain the reasoning for which equation was used.

(a) An engineer is trying to understand the traffic flow through a particularly busy intersection. In 2008, every official business day during the month of September (excluding holidays and weekends) he counts the number of cars that pass through from 10 AM until 1 PM. He found that the number of cars averaged 198 with a variance of 36. He wishes to know the uncertainty with 90 % confidence.

(b) A student wishes to determine the accuracy of a relative humidity gauge in the laboratory with 99 % confidence. He takes a reading every minute for one hour and determines that the mean is 48 % relative humidity with a standard deviation of 2 %.

(c) The student's partner enters the lab after the first student and wishes to determine the relative humidity in the room with 99 % confidence prior to running his experiments. He also takes a reading every minute for one hour and determines that the mean is 48 % relative humidity and the standard deviation is 2 %. (Assume he knows nothing about what his partner has done.)

(d) An engineer designing cranes is working with a manufacturing engineer to assess whether as-manufactured beams will be able to satisfy a ten-year guarantee for normal use at which time they will need refurbishment or replacement. On the drawing he specified an absolute minimum thickness of 3.750 in. The manufacturer measures 200 beams off the assembly floor and they have an average thickness of 4.125 in. with a standard deviation of 0.1500 in. Does the manufacturer have 99 % confidence that the as-manufactured beams will meet the ten-year guarantee?

21. The resistance of a wire is given by $R = R_o\,[1 + \alpha\,(T - T_o)]$ where T_o and R_o are fixed reference values of 20 °C and 100 Ω ± 2.5 %, respectively. The temperature coefficient is α = 0.004/°C ± 0.1 %. The development engineer is checking the resistance of the wire and measures the temperature to be T = 60 °C. When measuring the wire and reference temperatures, the engineer used the same thermocouple that had a manufacturer's accuracy of ± 1 °C. (a) Determine the nominal resistance of the wire and the nominal uncertainty. (b) Assess whether the certainty was positively or negatively affected by using the same thermocouple rather than two separate thermocouples with the same nominal accuracy. Note: When calculating percentages of temperatures, an absolute scale needs to be used.

22. An instrument has a stated accuracy of q %. An experiment is conducted in which the instrument is used to measure a variable, z, N times under controlled conditions. There are some temporal variations in the instrument's readings, characterized by S_z. Determine the overall uncertainty in z.

23. A design criterion for an experiment requires that the combined standard uncertainty in a measured pressure be 5 % or less based upon 95 % confidence. It is known from a previous experiment conducted in the same facility that the pressure varies slightly under 'fixed' conditions, as determined from a sample of 61 pressure measurements having a standard deviation of 2.5 kPa and mean of 90 kPa. The accuracy of the pressure measurement system, as stated by the manufacturer, is 3 %. Determine the value of the smallest division (in kPa) that the pressure indicator must have to meet the design criterion.

24. A calibration experiment is conducted in which the output of a secondary-standard pressure transducer having negligible uncertainty is read using a digital voltmeter. The pressure transducer has a range of 0 psi to 10 psi. The digital voltmeter has a resolution of 0.1 V, a stated accuracy of 1 % of full scale, and a range of 0 V to 10 V. The calibration based upon 61 measurements yields the least-squares linear regression relation $V(\text{volts}) = 0.50P(\text{psi})$, with a standard error of the fit, S_{yx}, equal to 0.10 V. Determine the combined standard uncertainty *in the voltage* at the 95 % confidence level.

25. An inclined manometer has a stated accuracy of 3 % of its full-scale reading. The range of the manometer is from 0 in. H_2O to 5 in. H_2O. The smallest marked division on the manometer's scale is 0.2 in. H_2O. An experiment is conducted under controlled conditions in which a pressure difference is measured 20 times. The mean and standard deviation of the pressure-difference measurements are 3 in. H_2O and 0.2 in. H_2O, respectively. Assuming 95 % confidence, determine (a) the zero-order uncertainty, u_0, (b) the temporal precision uncertainty that arises from

the variation in the pressure-difference during the controlled-conditions experiment, and (c) the combined standard uncertainty, u_c.

26. Determine the uncertainty (in ohms) in the total resistance, R_T, that is obtained by having two resistors, R_1 and R_2, in parallel. The resistances of R_1 and R_2 are 4 Ω and 6 Ω, respectively. The uncertainties in the resistances of R_1 and R_2 are 2 % and 5 %, respectively.

27. A student group postulates that the stride length, L, of a marathon runner is proportional to a runner's inseam, H, and inversely proportional to the square of a runner's weight, W. The inseam length is to be measured using a tape measure and the weight using a scale. The estimated uncertainties in H and W are 4 % and 3 %, respectively, based upon a typical inseam of 70 cm and a weight of 600 N. Determine (a) the percent uncertainty in L, (b) the resolution of the tape measure (in cm), and (c) the resolution of the scale (in N).

28. Given that the mass, M, of Saturn is 5.68×10^{26} kg, the radius, R, is 5.82×10^7 m, and g (m/s^2) $= GM/R^2$, where $G = 6.6742 \times 10^{-11}$ N·m^2/kg^2, determine the percent uncertainty in g on Saturn, assuming that the uncertainties in G, M, or R are expressed for each by the place of the least-significant digit (for example, $\mathrm{u}_R = 0.01 \times 10^7$ m).

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8

Regression and Correlation

CONTENTS

Of all the principles that can be proposed for this purpose, I think there is none more general, more exact, or easier to apply, than that which we have used in this work; it consists of making the sum of the squares of the errors a minimum. By this method, a kind of equilibrium is established among the errors which, since it prevents the extremes from dominating, is appropriate for revealing the state of the system which most nearly approaches the truth.

Adrien-Marie Legendre. 1805. *Nouvelles méthodes pour la détermination des orbites des comètes.* Paris.

Two variable organs are said to be co-related when the variation of the one is accompanied on the average by more or less variation of the other, and in the same direction.

Sir Francis Galton. 1888. *Proceedings of the Royal Society of London.* 45:135-145.

8.1 Chapter Overview

This chapter introduces two important areas of data analysis: **regression** and **correlation**. Regression analysis establishes a mathematical relation between two or more variables. Typically, it is used to obtain the best fit of data with an analytical expression. Correlation analysis quantifies the extent to which one variable is related to another, but it does not establish a mathematical relation between them. Statistical methods can be used to determine the confidence levels associated with regression and correlation estimates.

We begin this chapter by considering the least-squares approach to regression analysis. This approach enables us to obtain a best-fit relation between variables. We focus on linear regression analysis first. The statistical parameters that are used to characterize regression are introduced next. Then we consider regression analysis as applied to experiments along with their associated uncertainties and confidence limits. We further examine correlation analysis by considering how a random variable is correlated with itself and with another random variable. Finally, we examine extended methods, including higher-order regression analysis and multi-variable linear analysis.

8.2 Least-Squares Approach

Toward the end of the 18th century scientists faced an interesting problem. This was how to find the best agreement between measurements and an analytical model that contained the measured variables, given that repeated measurements were made, but with each containing error. Jean-Baptiste-Joseph Delambre (1749-1822) and Pierre-François-André Méchain (1744-1804) of France, for example [1] and [2], were in the process of measuring a 10° arc length of the meridian quadrant passing from the North Pole to the Equator through Paris. The measure of length for their newly proposed *Le Système International d'Unités*, the meter, would be defined as 1/10 000 000 the length of the meridian quadrant. So, the measured length of this quadrant had to be as accurate as possible.

Because it was not possible to measure the entire length of the 10° arc, measurements were made in arc lengths of approximately 65 000 modules (1 module ≅ 12.78 ft). From these measurements, an analytical expression involving the arc length and the astronomically determined latitudes of each of the arc's end points, the length of the meridian quadrant was determined. The solution essentially involved solving four equations containing four measured arc lengths with their associated errors for two unknowns, the ellipticity of the earth and a factor related to the diameter of the earth. Although many scientists proposed different solution methods, it was Adrien-Marie Legendre (1752-1833), a French mathematician, who arrived at the most accurate determination of the meter using the method of least squares, equal to 0.256 480 modules ($\sim$ 3.280 ft). Ironically, it was the more politically astute Pierre-Simon Laplace's (1749-1827) value of 0.256 537 modules ($\sim$ 3.281 ft) based upon a less accurate method that was adopted as the basis for the meter. Current geodetic measurements show that the quadrant from the North Pole to the Equator through Paris is 10 002 286 m long. This renders the meter as originally defined to be in error by 0.2 mm or 0.02 %.

Legendre's method of least squares, which originally appeared as a four-page appendix in a technical paper on comet orbits, was more far-reaching than simply determining the length of the meridian quadrant. It prescribed the methodology that would be used by countless scientists and engineers to this day. His method was elegant and straightforward, simply to express the errors as the squares of the differences between all measured and predicted values and then determine the values of the coefficients in the governing equation that minimize these errors. To quote Legendre [1] "...we are led to a system of equations of the form

$$E = a + bx + cy + fz + ..., \tag{8.1}$$

in which $a, b, c, f, ...$ are known coefficients, varying from one equation to the other, and $x, y, z, ...$ are unknown quantities, to be determined by the condition that each value of E is reduced either to zero, or to a very small quantity."

In the present notation, for a linear system

$$e_i = a + bx_i + cy_i = y_{c_i} - y_i, \tag{8.2}$$

where e_i is the i-th error for each of i equations based upon the measurement pair $[x_i, y_i]$ and the general analytical expression $y_{c_i} = a + bx_i$ with $c = -1$. Using Legendre's method, the minimum of the sum of the squares of the e_i's would be found by varying the values of coefficients a and b. Formally, these coefficients are known as **regression coefficients** and the process of obtaining their values is called **regression analysis**.

8.3 Least-Squares Regression Analysis

Least-squares regression analysis follows a very logical approach in which the coefficients of an analytical expression that best fits the data are found through the process of error minimization. The best fit occurs when the sum of the squares of the differences (the errors or residuals) between each y_{c_i}

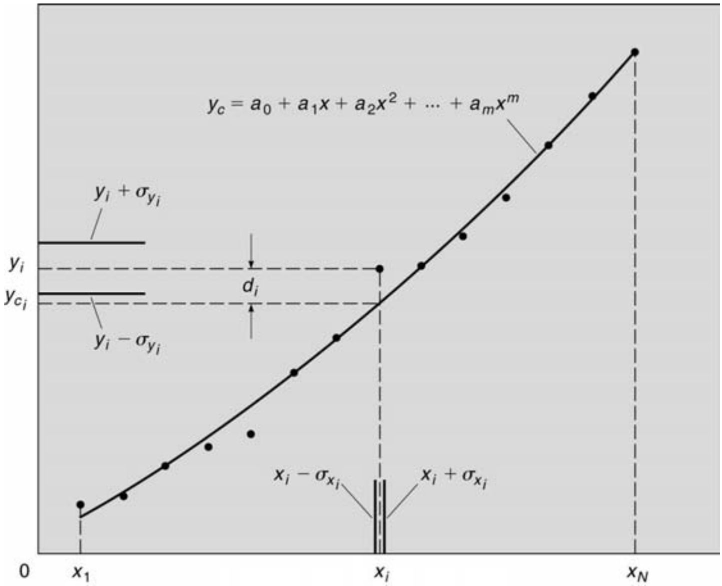

FIGURE 8.1
Least-squares regression analysis.

value calculated from the analytical expression and its corresponding measured y_i value is a minimum (the differences are squared to avoid adding compensating negative and positive differences). The best fit would be obtained by continually changing the coefficients (a_0 through a_m) in the analytical expression until the differences are minimized. This, however, can be quite tedious unless a formal approach is taken and some simplifying assumptions are made.

Consider the data presented in Figure 8.1. The goal is to find the values of the a coefficients in the analytical expression $y_c = a_0 + a_1x + a_2x^2 + ... + a_mx^m$ that best fits the data. To proceed formally, D is defined as the sum of the squares of all the vertical distances (the d_i's) between the measured and calculated values of y (between y_i and y_{c_i}), as

$$D = \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} (y_i - y_{c_i})^2 = \sum_{i=1}^{N} (y_i - \{a_0 + a_1x_i + ... + a_mx_i^m\})^2. \quad (8.3)$$

Implicitly, it is assumed in this process that y_i is normally distributed with a true mean value of y_i' and a true variance of $\sigma_{y_i}^2$. The independent variable x_i is assumed to have no or negligible variance. Thus, $x_i = x'$, where x' denotes the true mean value of x. Essentially, the value of x is fixed, known, and with no variance, and the value of y is sampled from a normally distributed population. Thus, all of the uncertainty results from the y value. If this were not the case, then the y_{c_i} value corresponding to a particular y_i value would not be vertically above or below it. This is because the x_i value would fall within a range of values. Consequently, the distances would not be vertical but rather at some angle with respect to the ordinate axis. Hence, the regression analysis approach being developed would be invalid.

Now D is to be minimized. That is, the value of the sum of the *squares* of the distances is to be the *least* of all possible values. This minimum is found by setting the total derivative of D equal to zero. This actually is a minimization of χ^2 (see [3]). Thus,

$$dD = 0 = \frac{\partial D}{\partial a_0} da_0 + \frac{\partial D}{\partial a_1} da_1 + ... + \frac{\partial D}{\partial a_m} da_m. \tag{8.4}$$

For this equation to be satisfied, a set of $m+1$ equations must be solved for $m+1$ unknowns. This set is

$$\begin{aligned} \frac{\partial D}{\partial a_0} &= 0 = \frac{\partial}{\partial a_0} \sum_{i=1}^{N} d_i^2, \\ \frac{\partial D}{\partial a_1} &= 0 = \frac{\partial}{\partial a_1} \sum_{i=1}^{N} d_i^2, \\ &... \text{, and} \\ \frac{\partial D}{\partial a_m} &= 0 = \frac{\partial}{\partial a_m} \sum_{i=1}^{N} d_i^2. \end{aligned} \tag{8.5}$$

This set of equations leads to what are called the **normal equations** (named by Carl Friedrich Gauss).

8.4 Linear Analysis

The simplest type of least-squares regression analysis that can be performed is for the linear case. Assume that y is linearly related to x by the expression $y_c = a_0 + a_1 x$. Proceeding along the same lines, for this case only two equations (here $m+1 = 1+1 = 2$) must be solved for two unknowns, a_0 and a_1, subject to the constraint that D is minimized.

When $dD = 0$,

$$\frac{\partial D}{\partial a_0} = 0 = \frac{\partial}{\partial a_0} \left(\sum_{i=1}^{N} [y_i - (a_0 + a_1 x_i)]^2 \right) = -2 \sum_{i=1}^{N} (y_i - a_0 - a_1 x_i). \tag{8.6}$$

Carrying through the summations on the right side of Equation 8.6 yields

$$\sum_{i=1}^{N} y_i = a_0 N + a_1 \sum_{i=1}^{N} x_i. \tag{8.7}$$

Also,

$$\frac{\partial D}{\partial a_1} = 0 = \frac{\partial}{\partial a_1}\left(\sum_{i=1}^{N}[y_i - (a_0 + a_1 x_i)]^2\right) = -2\sum_{i=1}^{N} x_i(y_i - a_0 - a_1 x_i). \tag{8.8}$$

This gives

$$\sum_{i=1}^{N} x_i y_i = a_0 \sum_{i=1}^{N} x_i + a_1 \sum_{i=1}^{N} x_i^2. \tag{8.9}$$

Thus, the two normal equations become Equations 8.7 and 8.9. These can be rewritten as

$$\bar{y} = a_0 + a_1 \bar{x} \tag{8.10}$$

and

$$\overline{xy} = a_0 \bar{x} + a_1 \overline{x^2}. \tag{8.11}$$

From the first normal equation it can be deduced that a linear least-squares regression analysis fit will *always* pass through the point $(\bar{x}, \bar{y})$. Equations 8.7 and 8.9 can be solved for a_0 and a_1 to yield

$$a_0 = \left(\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i\right) / \Delta, \tag{8.12}$$

$$a_1 = \left(N \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right) / \Delta, \text{ and} \tag{8.13}$$

$$\Delta = N \sum_{i=1}^{N} x_i^2 - \left[\sum_{i=1}^{N} x_i\right]^2. \tag{8.14}$$

Linear regression analysis also can be used for a higher-order expression if the variables in expression can be transformed to yield a linear expression. This sometimes is referred to as curvilinear regression analysis. Such variables are known as **intrinsically linear variables**. For this case, a least-squares linear regression analysis is performed on the transformed variables. Then the resulting regression coefficients are transformed back to yield the desired higher-order fit expression. For example, if $y = ax^b$, then $\log_{10} y = \log_{10} a + b \log_{10} x$. So, the least-squares linear regression fit of the data pairs $[\log_{10} x, \log_{10} y]$ will yield a line of intercept $\log_{10} a$ and slope b.

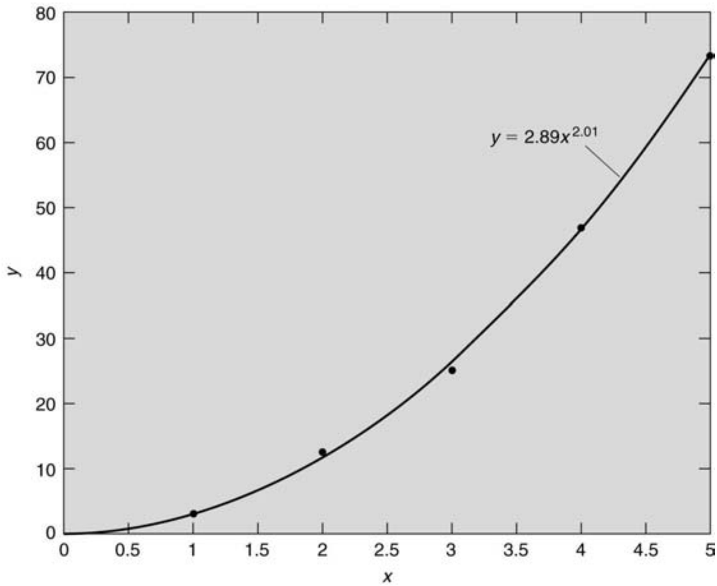

FIGURE 8.2
Regression fit of the model $y = ax^b$ with data.

The resulting best-fit values of a and b can be determined and then used in the original expression.

Example Problem 8.1

Statement: An experiment is conducted to validate a physical model of the form $y = ax^b$. Five $[x, y]$ pairs of data are acquired: [1.00, 2.80; 2.00, 12.5; 3.00, 25.2; 4.00, 47.0; 5.00 73.0]. Find the regression coefficients a and b using a *linear* least-squares regression analysis.

Solution: First express the data in the form of $[\log_{10} x, \log_{10} y]$ pairs. This yields the transformed data pairs [0.000, 0.447; 0.301, 1.10; 0.477, 1.40; 0.602, 1.67; 0.699, 1.86]. A linear regression analysis of the transformed data yields the best-fit expression: $\log_{10} y = 0.461 + 2.01 \log_{10} x$. This implies that $a = 2.89$ and $b = 2.01$. Thus, the best-fit expression for the data in its original form is $y = 2.89x^{2.01}$. This best-fit expression is compared with the original data in Figure 8.2.

A similar approach can be taken when using a linear least-squares regression analysis to fit the equation $E^2 = A + B\sqrt{U}$, which is King's law. This law relates the voltage, E, of a constant-temperature anemometer to a fluid's velocity, U. A regression analysis performed on the data pairs $[E^2, \sqrt{U}]$ will yield the best-fit values for A and B. This is considered in homework problem 7.

8.5 Regression Parameters

There are several statistical parameters that can be calculated from a set of data and its best-fit relation. Each of these parameters quantifies a different relationship between the quantities found from the data (the individual values x_i and y_i and the mean values $\overline{x}$ and $\overline{y}$) and from its best-fit relation (the calculated values).

Those quantities that are calculated directly from the data include the sum of the squares of x, S_{xx}, the sum of the squares of y, S_{yy}, and the sum of the product of x and y, S_{xy}. Their expressions are

$$S_{xx} \equiv \sum_{i=1}^{N}(x_i - \overline{x})^2 = \sum_{i=1}^{N} x_i^2 - N\overline{x}^2, \tag{8.15}$$

$$S_{yy} \equiv \sum_{i=1}^{N}(y_i - \overline{y})^2 = \sum_{i=1}^{N} y_i^2 - N\overline{y}^2, \tag{8.16}$$

and

$$S_{xy} \equiv \sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y}) = \sum_{i=1}^{N} x_i y_i - N\overline{xy}. \tag{8.17}$$

All three of these quantities can be viewed as measures of the square of the differences or product of the differences between the x_i and y_i values and their corresponding mean values. Equations 8.15 and 8.17 can be used with the normal equations of a linear least-squares regression analysis to simplify the expressions for the linear case's best-fit slope and intercept, where

$$b = S_{xy}/S_{xx} \tag{8.18}$$

and

$$a = \overline{y} - b\overline{x}. \tag{8.19}$$

Those quantities calculated from the data and the regression fit include the sum of the squares of the regression, SSR, the sum of the squares of the error, SSE, and the sum of the squares of the total error, SST. Their expressions are

$$SSR \equiv \sum_{i=1}^{N}(y_{c_i} - \overline{y})^2, \tag{8.20}$$

$$SSE \equiv \sum_{i=1}^{N}(y_i - y_{c_i})^2, \tag{8.21}$$

and

$$SST \equiv SSE + SSR = \sum_{i=1}^{N}(y_i - y_{c_i})^2 + \sum_{i=1}^{N}(y_{c_i} - \overline{y})^2. \tag{8.22}$$

All three of these can be viewed as quantitative measures of the square of the differences between the $\bar{y}$ and y_i values and their corresponding y_{c_i} values. *SSR* is also known as the *explained* variation and *SSE* as the *unexplained* variation. Their sum, *SST*, is called the *total* variation. *SSR* is a measure of the amount of variability in y_i accounted for by the regression line and *SSE* of the remaining amount of variation not explained by the regression line.

It can be shown further (see [5]) that

$$SST = \sum_{i=1}^{N}(y_i - \overline{y})^2 = S_{yy}. \tag{8.23}$$

The combination of Equations 8.22 and 8.23 yields what is known as the sum of squares partition [5] or the analysis of variance identity [6]

$$\sum_{i=1}^{N}(y_i - \overline{y})^2 = \sum_{i=1}^{N}(y_i - y_{c_i})^2 + \sum_{i=1}^{N}(y_{c_i} - \overline{y})^2. \tag{8.24}$$

This expresses the three quantities of interest (y_i, y_{c_i}, and $\overline{y}$) in one equation.

An additional and frequently used parameter that characterizes the quality of the best-fit is the standard error of the fit, S_{yx},

$$S_{yx} \equiv \sqrt{\frac{SSE}{\nu}} = \sqrt{\frac{SSE}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N}(y_i - y_{c_i})^2}{N-2}}. \tag{8.25}$$

This is equivalent to the standard deviation of the measured y_i values with respect to their calculated y_{c_i} values, where $\nu = N - (m+1) = N - 2$ for $m = 1$.

Example Problem 8.2

Statement: For the set of [*x*,*y*] data pairs [0.5, 0.6; 1.5, 1.6; 2.5, 2.3; 3.5, 3.7; 4.5, 4.2; 5.5, 5.4], determine $\bar{x}$, $\bar{y}$, S_{xx}, S_{yy}, and S_{xy}. Then determine the intercept and the slope of the regression line using Equations 8.18 and 8.19 and compare the values to those found by performing a linear least-squares regression analysis. Next, using the regression fit equation determine the values of y_{c_i}. Finally, calculate *SSE*, *SSR*, and *SST*. Show, using the results of these calculations, that $SST = SSR + SSE$.

Solution: Direct calculations yield $\bar{x} = 3.00$, $\bar{y} = 2.97$, $S_{xx} = 17.50$, $S_{yy} = 15.89$, and $S_{xy} = 16.60$. The intercept and the slope values are $a = 0.1210$ and $b = 0.9486$ from Equations 8.19 and 8.18, respectively. The same values are found from regression analysis. Thus, from the equation $y_{c_i} = 0.1210 + 0.9486x_i$ the y_{c_i} values are 0.5952, 1.5438, 2.4924, 3.4410, 4.3895, and 5.3381. Direct calculations then give $SSE = 0.1470$, $SSR = 15.7463$, and $SST = 15.8933$. This shows that $SSR + SSE = 15.7463 + 0.1470 = 15.8933 = SST$, which follows from Equation 8.24.

Historically, regression originally was called reversion. Reversion referred to the tendency of a variable to revert to the average of the population from which it came. It was Francis Galton who first elucidated the property of reversion ([14]) by demonstrating how certain characteristics of a progeny revert to the population average more than to the parents. So, in general terms, regression analysis relates variables to their mean quantities.

8.6 Confidence Intervals

Thus far it has been shown how measurement uncertainties and those introduced by assuming an incorrect order of the fit can contribute to differences between the measured and calculated y values. There are additional uncertainties that must be considered. These arise from the finite acquisition of data in an experiment. The presence of these additional uncertainties affects the confidence associated with various estimates related to the fit. For example, in some situations, the inverse of the best-fit relation established through calibration is used to determine unknown values of the independent variable and its associated uncertainty. A typical example would be to determine the value and uncertainty of an unknown force from a voltage measurement using an established voltage-versus-force calibration curve. To arrive at such estimates, the sources of these additional uncertainties must be examined first.

For simplicity, focus on the situation where the correct order of the fit is assumed and there is no measurement error in x. Here, $\sigma_{E_y} = \sigma_y$. That is, the uncertainty in determining a value of y from the regression fit is solely due to the measurement error in y.

Consider the following situation, as illustrated in Figure 8.3, in which best fits for two sets of data obtained under the same experimental conditions are plotted along with the data. Observe that different values of y_i are obtained for the same value of x_i each time the measurement is repeated (in this case there are two values of y_i for each x_i). This is because y is a random variable drawn from a normally distributed population. Because x is not a random variable, it is assumed to have no uncertainty. So, in all likelihood, the best-fit expression of the first set of data, $y = a_1 + b_1 x$, will be different from the second best-fit expression, $y = a_2 + b_2 x$, having different values for the intercepts ($a_1 \neq a_2$) and for the slopes ($b_1 \neq b_2$).

The true-mean regression line is given by Equation 8.45 in which $x = x'$. The true intercept and true slope values are those of the underlying population from which the finite samples are drawn. From another perspective, the true-mean regression line would be that found from the least-squares linear regression analysis of a very large set of data ($N >> 1$).

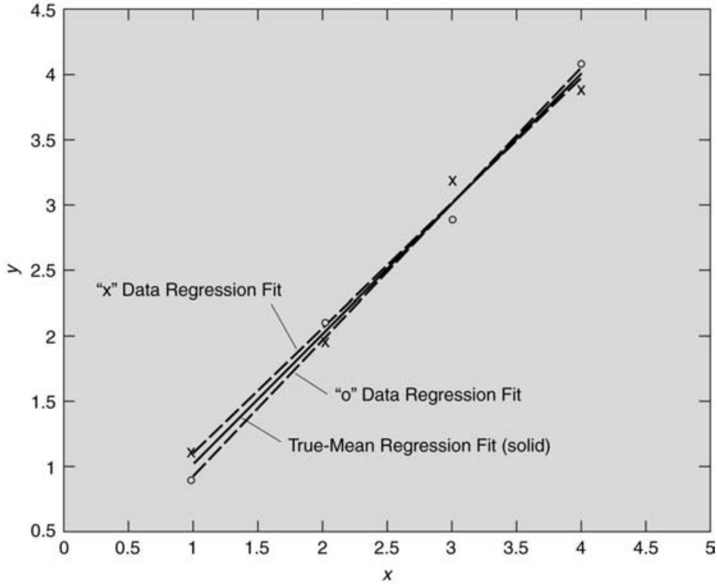

FIGURE 8.3
Linear regression fits for two finite samples and one very large sample.

Recognizing that such finite sampling uncertainties arise, how do they affect the estimates of the true intercept and true slope? The estimates for the true intercept and true slope values can be written in terms of the above expressions for S_{xx} and S_{yx} [5],[6]. The estimate of the true intercept of the true-mean regression line is

$$\alpha = a \pm t_{N-2,P} S_{yx} \sqrt{\frac{1}{N} + \frac{\bar{x}^2}{S_{xx}}}. \tag{8.26}$$

The estimate of the true slope of the true-mean regression line is

$$\beta = b \pm t_{N-2,P} S_{yx} \sqrt{\frac{1}{S_{xx}}}. \tag{8.27}$$

As N becomes larger, the sizes of the confidence intervals for the true intercept and true slope estimates become smaller. The value of a approaches that of α, and the value of b approaches that of β. This simply reflects the former statement, that any regression line based upon several N will approach the true-mean regression line as N becomes large.

Example Problem 8.3

Statement: For the set of [x,y] data pairs [1.0, 2.1; 2.0, 2.9; 3.0, 3.9; 4.0, 5.1; 5.0, 6.1] determine the linear best-fit relation using the method of least-squares regression

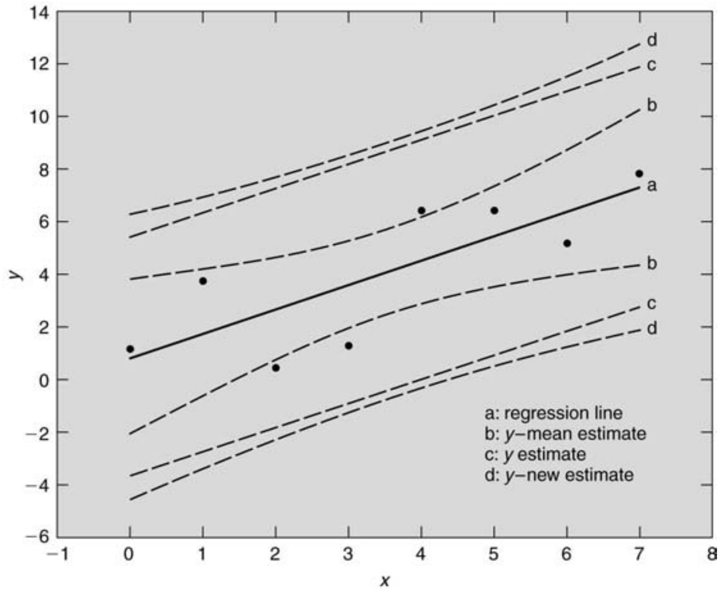

FIGURE 8.4
Various confidence intervals for linear regression estimates.

analysis. Then estimate at 95 % confidence the values of the true intercept and the true slope.

Solution: The best-fit relation is $y = 0.96 + 1.02x$ for $N = 5$ with $S_{yx} = 0.12$, $S_{xx} = 10$, $\bar{x} = 3$, and $t_{3,95} = 3.1824$. This yields $\alpha = 0.96 \pm 0.40$ (95 %) and $\beta = 1.02 \pm 0.12$ (95 %).

The values of some other useful quantities also can be estimated [5], [6], [7]. The estimate of the sample mean value of a large number of y_i values for a given value of x_i, denoted by $\bar{y}_i$, and also known as the mean response, is

$$\bar{y}_i = y_{c_i} \pm t_{N-2,P} S_{yx} \sqrt{\frac{1}{N} + \frac{(x_i - \bar{x})^2}{S_{xx}}}. \tag{8.28}$$

Note that the greater the difference between x_i and $\bar{x}$, the greater the uncertainty in estimating $\bar{y}_i$. This leads to confidence intervals that are hyperbolic, as shown in Figure 8.4 by the curves labeled b (based upon 95 % confidence), that are positioned above and below the regression line labeled by a. The confidence interval is the smallest at $x = \bar{x}$. Also, because of the factor $t_{\nu,P}$, the confidence interval width will decrease with decreasing percent confidence.

The range within which a new y value, y_n, added to the data set will be for a new value x_n is

$$y_n = y_{n_{c_i}} \pm t_{N-2,P} S_{yx} \sqrt{1 + \frac{1}{N} + \frac{(x_n - \bar{x})^2}{S_{xx}}}. \quad (8.29)$$

This interval is marked by the hyperbolic curves labeled d (based upon 95 % confidence). Note that the hyperbolic curves are farther from the regression line for this case than for the mean response case. This is because Equation 8.29 estimates a single new value of y, whereas Equation 8.28 estimates the mean of a large number of y values.

Finally, the range within which a y_i value probably will be, with respect to its corresponding y_{c_i} value, is

$$y_i = y_{c_i} \pm t_{\nu,P} S_{yx}, \quad (8.30)$$

where $t_{\nu,P} S_{yx}$ denotes the **precision interval**. This expression establishes the confidence intervals that always should be plotted whenever a regression line is present. Basically, Equation 8.30 defines the limits within which P percent of a large number of measured y_i values will be with respect to the y_{c_i} value for a given value of x_i. Its confidence intervals are denoted by the lines labeled c (based upon 95 % confidence) which are parallel to the regression line. Equation 8.30 also can be used for a higher mth-order regression fit to establish the confidence intervals. This is provided that ν is determined by $\nu = N - (m + 1)$ and that the general expression for S_{yx} given in Equation 8.25 is used.

Several useful inferences can be drawn from Equation 8.30. For a fixed number of measurements, N, the extent of the precision interval increases as the percent confidence is increased. The extent of the precision interval must be greater if more confidence is required in the estimate of y_i. For a given confidence, as N is increased the extent of the precision interval decreases. A smaller precision interval is required to estimate y_i if a greater number of measurements is acquired.

Example Problem 8.4

Statement: For the set of [x,y] data pairs [0.00, 1.15; 1.00, 3.76; 2.00, 0.41; 3.00, 1.30; 4.00, 6.42; 5.00, 6.42; 6.00, 5.20; 7.00, 7.87] determine the linear best-fit relation using least-squares regression analysis. Then estimate at 95 % confidence the intervals of $\bar{y}_i, y_n$, and y_i according to Equations 8.28 through 8.30 for $x = 2.00$.

Solution: From Equation 8.28 it follows directly that $\bar{y}_i = 2.68 \pm 1.93$. That is, there is a 95 % chance that the mean value of a large number of measured y_i values for $x = 2.00$ will be within ± 1.93 of the y_{c_i} value of 2.68. Further, from Equation 8.29, $y_n = 2.68 \pm 4.95$, which implies that there is a 95 % chance that a new measurement of y for $x = 2.00$ will be within ± 4.95 of 2.68. Finally, from Equation 8.30, $y_i = 2.68 \pm 4.56$. The confidence intervals for this data set for the range $0 \leq x \leq 7$ are shown in Figure 8.4.

Another confidence interval related to the regression fit can be established for the estimate of a value of x for a given value of y. This situation

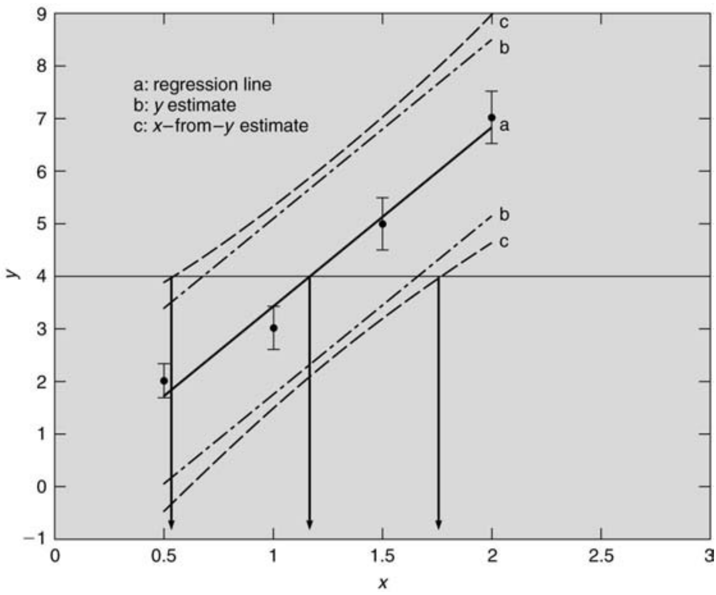

FIGURE 8.5
Regression fit with relatively large sensitivity.

is encountered when a calibration curve is used to determine unknown x values. Figures 8.5 and 8.6 each display a linear regression fit of the data (labeled by a) along with two different confidence intervals for $P = 95$ %. In addition to the usual estimate of the range within which a y_i value will be with respect to its calculated value (labeled by b), there is another estimate, the x-from-y estimate with its confidence interval (labeled by c). This new estimate's confidence interval should be greater in extent than that for the y estimate. This is because additional uncertainties arise when the best fit is used to project from a chosen y value back to an unknown x value.

The confidence interval for the estimate of x from y is represented by hyperbolic curves. The uncertainty forming the basis of this confidence interval results from three different uncertainties associated with y and the best-fit expression: from the measurement uncertainty in y, from the uncertainty in the true value of the intercept, and from the uncertainty in the true value of the slope. The latter two result from determining the regression fit based upon a finite amount of data. In essence, the hyperbolic curves can be viewed as bounds for the area within which all possible finite regression fits with their standard y-estimate confidence intervals are contained. When one projects from a chosen y value back to the x-axis, one does not know upon which regression fit the projected x value is based. The chosen y value could have resulted from an x value different than the one used to establish the fit. This new confidence interval accounts for this. The three

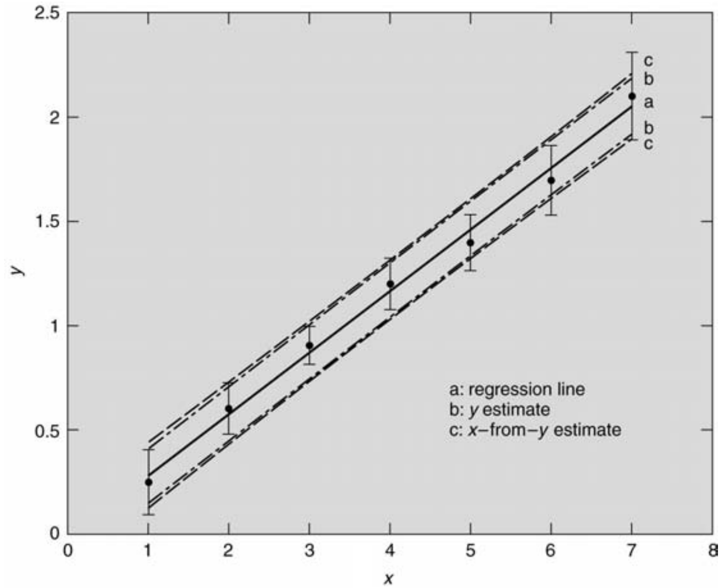

FIGURE 8.6
Regression fit with relatively small sensitivity.

contributory uncertainties *cannot* be combined in quadrature to yield the final uncertainty because the intercept and slope uncertainties are not statistically independent from one another. So, a more rigorous approach must be taken to determine this confidence interval. This was done by Finney [8], who established this confidence interval to be

$$y = y_c \pm t_{\nu,P} S_{yx} \sqrt{\frac{1}{n} + \frac{1}{N} + \frac{(x - \bar{x})^2}{S_{xx}}}, \tag{8.31}$$

where n denotes the number of replications of y measurements for a particular value of x ($n = 1$ for the examples shown in Figures 8.6 and 8.5).

A comparison of Figures 8.5 and 8.6 reveals several important facts. When the magnitude of the uncertainty in y is relatively small, the confidence limits are closer to the regression line. When there is more scatter in the data, both intervals are wider. Fewer $[x,y]$ data pairs result in a relatively larger difference between the confidence limits. The sensitivity of y with respect to x (the slope of the regression line) plays an important role in determining the level of uncertainty in x in relation to the x-from-y confidence interval. Lower sensitivities result in relatively large uncertainties in x. For example, the uncertainty range in x for a value of $y = 4.0$ in Figure 8.5 is from approximately 0.53 to 1.76, as noted by the arrows in the figure,

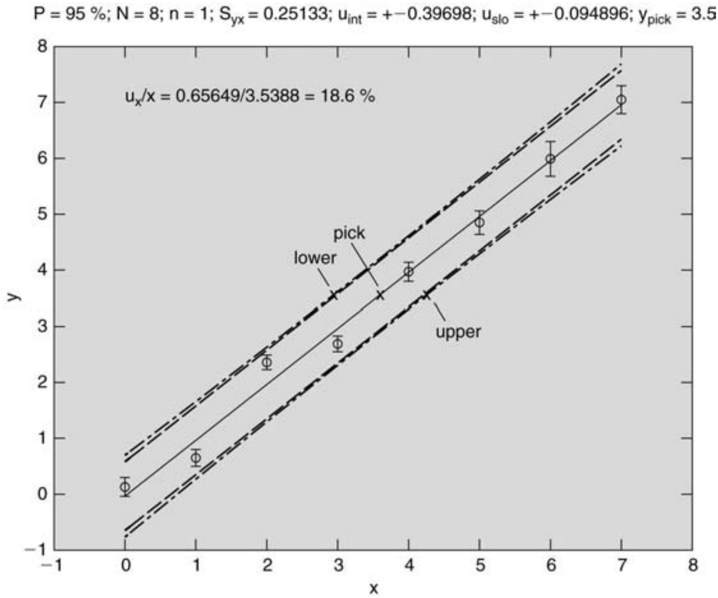

FIGURE 8.7
Regression fit with indicated x-from-y estimate uncertainty.

for a calculated value of $x = 1.18$. Note that the range of this uncertainty is *not* symmetric with respect to the calculated value of x.

The M-file `caley.m` performs a linear least-squares regression analysis on a set of $[x,\ y,\ ey]$ data pairs (where ey is the measurement error of y) and plots the regression fit and its associated confidence intervals, as given by Equations 8.30 and 8.31. This M-file was used to generate Figures 8.5 and 8.6. The M-file `caleyII.m` also determines the range in the x-from-y estimate for a user-specified value of y, as shown in Figure 8.7. The M-file `caleyIII.m` extends this type of analysis farther by determining the percent uncertainty in the x-from-y estimate for the entire range of y values. It plots the standard regression fit with the data and also the x-from-y estimate uncertainty versus x. The two resulting plots are shown in Figure 8.8.

Caution should be exercised when claims are made about trends in the data. Any claim must be made within the context of measurement uncertainty that is assessed at a particular confidence level. An example is illustrated in Figure 8.9. The same values of five trials are plotted in each of the two figures. The trend in the values appears to increase with increasing trial number. In the top figure, the error bars represent the measurement uncertainty assessed at a 95 % level of confidence. The solid line suggests an increasing trend, whereas the dotted line implies a decreasing trend. Both claims are valid to within the measurement uncertainty at 95 % confidence. In the bottom figure, the error bars represent the measurement uncertainty

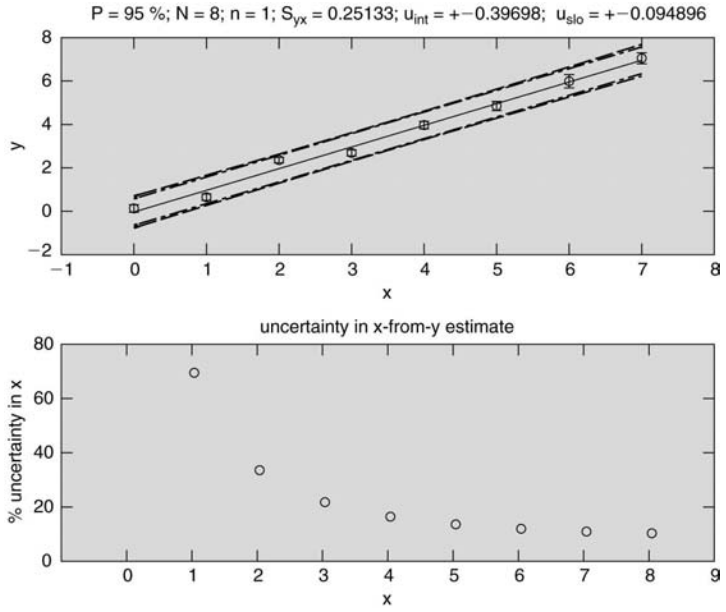

FIGURE 8.8
Regression fit and x-from-y estimate uncertainty.

assessed at a 68 % level of confidence. It is now possible to exclude the claim of a decreasing trend and support only that of an increasing trend. This, however, has been done at the cost of reducing the confidence level of the claim. In fact, if the level of confidence is reduced even further, the claim of an increasing trend cannot be supported. Thus, a specific trend in comparison with others can only be supported through accurate experimentation in which the error bars are small at a high level of confidence.

8.7 Linear Correlation Analysis

It was not until late in the 19th century that scientists considered how to quantify the extent of the relation between two random variables. Francis Galton in his landmark paper published in 1888 [16] quantitatively defined the word *co-relation*, now known as correlation. In that paper he presented for the first time the method for calculating the correlation coefficient and its confidence limits. He was able to correlate the height (stature) of 348 adult males to their forearm (cubit) lengths. This data is presented in Table 8.1. Galton designated the coefficient by the symbol r, which "measures the

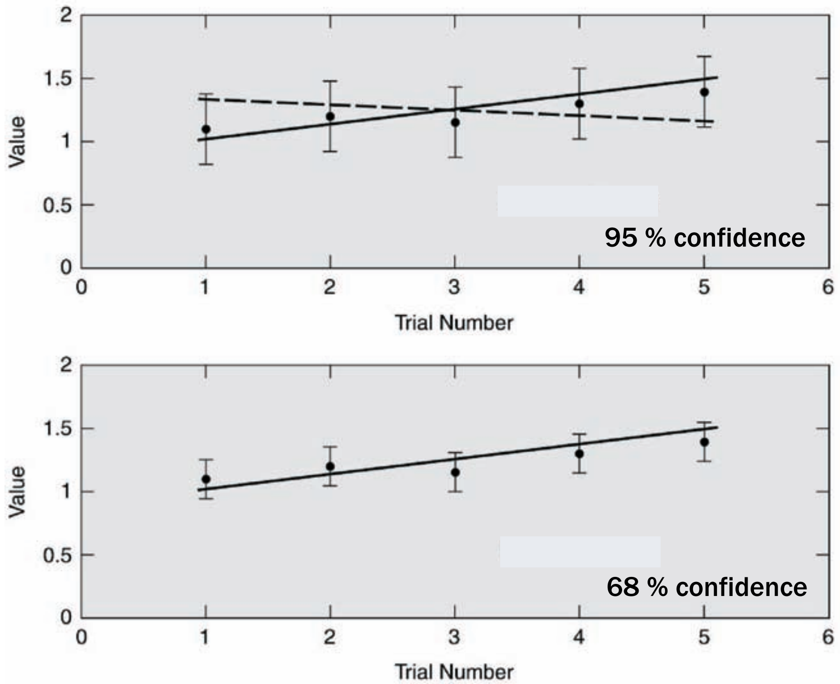

FIGURE 8.9
Data trends with respect to uncertainty.

closeness of co-relation." This symbol still is used today for the correlation coefficient.

Galton purposely presented his data in a particular tabular form, as shown in Table 8.1. In this manner, a possible co-relation between stature and cubit became immediately obvious to the reader, as indicated by the larger numbers along the table's diagonal. All that was left after realizing a co-relation was to quantify it. Galton approached this in an *ad hoc* manner by computing the mean value for each row (the mean cubit length for each stature) as well as the overall mean (the mean cubit length for all statures). He then expressed this data in terms of standard units (the number of probable measurement error units from the overall mean). He plotted the standardized unit values for each row (the standardized cubit lengths) versus the standardized unit value of the row (the standardized stature). This yielded the regression of cubit length upon stature. He followed a similar approach to determine the regression of stature upon cubit length by interchanging the rows and columns of data. He then established the composite best linear fit by eye and approximated the slope's value to be equal

	$C <$ 16.5	16.5 $< C <$ 17.0	17.0 $< C <$ 17.5	17.5 $< C <$ 18.0	18.0 $< C <$ 18.5	18.5 $< C <$ 19.0	19.0 $< C <$ 19.5	19.5 $< C$
$S > 71$	-	-	-	1	3	4	15	7
$71 > S > 70$	-	-	-	1	5	13	11	-
$70 > S > 69$	-	1	1	2	25	15	6	-
$69 > S > 68$	-	1	3	7	14	7	4	2
$68 > S > 67$	-	1	7	15	28	8	2	-
$67 > S > 66$	-	1	7	18	15	6	-	-
$66 > S > 65$	-	4	10	12	8	2	-	-
$65 > S > 64$	-	5	11	2	3	-	-	-
$64 > S$	9	12	10	3	1	-	-	-

TABLE 8.1
Galton's data of stature (S) versus cubit (C) length [16] (units of inches).

to 0.8, which was the regression coefficient. This approach was formalized later by the statisticians Francis Edgeworth and Karl Pearson.

But what exactly is the correlation coefficient and how can it be calculated? This relates to the general process of **correlation analysis**. In this section only *linear* correlation analysis is considered. In general, two random variables, x and y, are correlated if x's values can be related to the y's values to some extent. In the left graph of Figure 8.10, the variables show no correlation, whereas in the right graph, they are correlated moderately.

The extent of linear dependence between x and y is quantified through the correlation coefficient. This coefficient is related to the population variances of x and y, σ_x and σ_y, and the **population covariance**, σ_{xy}. The population correlation coefficient is defined as

$$\rho \equiv \frac{\sigma_{xy}}{\sqrt{\sigma_x \sigma_y}}, \tag{8.32}$$

where

$$\sigma_{xy} \equiv E[(x - x')(y - y')] = E[xy] - x'y', \tag{8.33}$$

$$\sigma_x \equiv \sqrt{E[(x - x')^2]}, \tag{8.34}$$

and

$$\sigma_y \equiv \sqrt{E[(y - y')^2]}. \tag{8.35}$$

E[ ] denotes the expectation or mean value of a quantity, which for any statistical parameter q raised to a power m involving N discrete values is

$$E[q^m] \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} q_i^m. \tag{8.36}$$

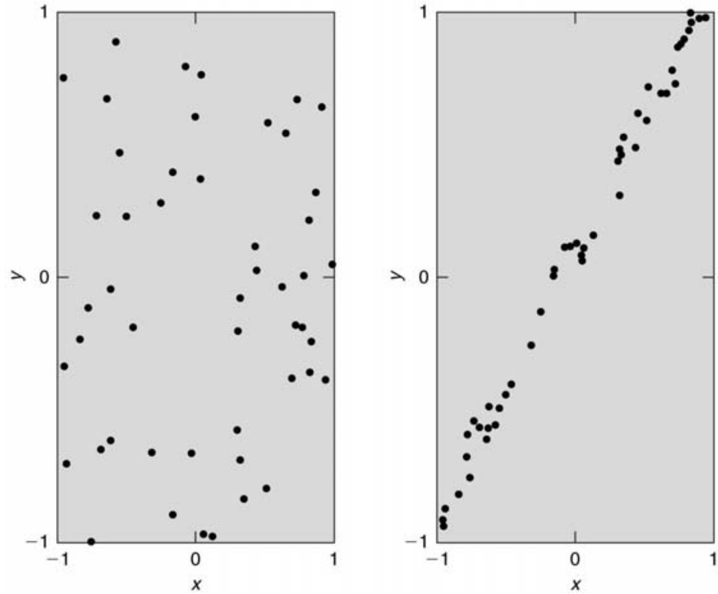

FIGURE 8.10
Uncorrelated and correlated data.

Two parameters, q and r, are statistically independent when

$$E[q^m \cdot r^n] = E[q^m] \cdot E[r^n], \tag{8.37}$$

where m and n are powers.

The covariance is the mean value of the product of the deviations of x and y from their true mean values. The population correlation coefficient is simply the ratio of the population covariance to the product of the x and y population variances. It measures the strength of the linear relationship between x and y. When $\rho = 0$, x and y are *uncorrelated*, which implies that y is *independent* of x. When $\rho = \pm 1$, there is *perfect* correlation, where $y = a \pm bx$ for *all* $[x,y]$ pairs.

The **sample correlation coefficient**, r, is an estimate of the **population correlation coefficient**, ρ. That is, the population correlation coefficient can be estimated but not determined exactly because a sample is finite and a population is infinite. The sample correlation coefficient is defined in a manner analogous to Equation 8.32 as

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}. \tag{8.38}$$

Squaring both sides of this equation, substituting Equation 8.18 for the slope of the regression line, and then taking the square root of both sides, Equation 8.38 becomes

$$b = r\sqrt{\frac{S_{yy}}{S_{xx}}}. \tag{8.39}$$

Thus, the slope of the regression fit equals the linear correlation coefficient times a scale factor. The scale factor is simply the square root of the ratio of the the spread of the y values to the spread of the x values. So, b and r are related closely, but they are *not* the same.

Using Equations 8.15, 8.16, and 8.17, Equation 8.38 can be rewritten as

$$r = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2 \sum_{i=1}^{N}(y_i - \bar{y})^2}}. \tag{8.40}$$

Equation 8.40 is known as the **product-moment formula**, which automatically keeps the proper sign of r. From this, r is calculated directly from the data without performing any regression analysis. It is evident from both of the above equations that r is a function, not only of the specific x_i and y_i values, but also of N. This point will be addressed shortly.

Example Problem 8.5

Statement: The Center on Addiction and Substance Abuse at Columbia University conducted a study on college-age drinking. They reported the following average drinks per week (DW) of alcohol consumption in relation to the average GPA (grade point average) for a large population of college students: 3.6, A; 5.5, B; 7.6, C; 10.6, D or F. Using an index of $A = 4$, $B = 3$, $C = 2$, and D or $F = 0.5$, determine the linear best-fit relation and the value of the linear correlation coefficient.

Solution: Using the M-file `plotfit.m`, a linear relation with $r = 0.99987$ and $GPA = 5.77 - 0.50DW$ can be determined for the range of $0 \leq GPA \leq 4$. These results are presented in Figure 8.11.

A more physical interpretation of r can be made by examining the quantity r^2, which is known as the **coefficient of determination.** Note that r given by Equation 8.38 says nothing about the relation that best-fits x and y. It can be shown [5] that

$$SSE = S_{yy}(1 - r^2) = SST(1 - r^2). \tag{8.41}$$

From Equations 8.22 and 8.41, it follows that

$$r^2 = 1 - \frac{SSE}{S_{yy}} = \frac{SSR}{S_{yy}}. \tag{8.42}$$

Equation 8.42 shows that the coefficient of determination is the ratio of the explained squared variation to the total squared variation. Because SSE and S_{yy} are always nonnegative, $1 - r^2 \geq 0$. So, the coefficient of determination is bounded as $0 \leq r^2 \leq 1$. It follows directly that $-1 \leq r \leq 1$.

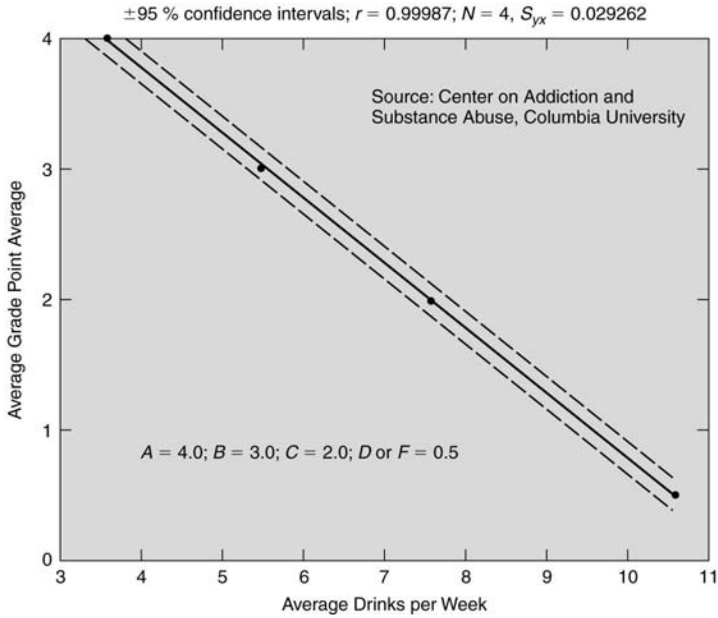

FIGURE 8.11
College student alcohol consumption.

When the correlation is *perfect*, there is no unexplained squared variation ($SSE = 0$) and $r = \pm 1$. Further, when there is no fit, all the y_i values are the same because they are completely independent of x. That is, all $y_{c_i} = \overline{y}$ and, by Equation 8.21, $SSE = 0$. Thus, $r = 0$. Values of $|r| > 0.99$ imply a *very significant* correlation; values of $|r| > 0.95$ imply a *significant* correlation. On the other extreme, values of $|r| < 0.05$ imply an *insignificant* correlation; values of $|r| < 0.01$ imply a *very insignificant* correlation.

Another expression for r can be obtained which relates it to the results of a regression analysis fit. Substituting Equations 8.16 and 8.20 into Equation 8.42 yields

$$r = \sqrt{\frac{\sum_{i=1}^{N}(y_{c_i} - \bar{y})^2}{\sum_{i=1}^{N}(y_i - \bar{y})^2}}. \tag{8.43}$$

This equation relates r to the y_{c_i} values obtained from regression analysis. This is in contrast to Equation 8.40, which yields r directly from data. These two equations help to underscore an important point. Correlation analysis and regression analysis are separate and distinct statistical approaches. Each is performed independently from the other. The results of a linear regression analysis, however, can be used for correlation analysis.

Caution should be exercised in interpreting various values of the linear correlation coefficient. For example, a value of $r \sim 0$ simply means that the two variables are not *linearly* correlated. They could be highly correlated *nonlinearly*. Further, a value of $r \sim \pm 1$ implies that there is a strong linear correlation. But the correlation could be casual, such as a correlation between the number of cars sold and pints of Guinness consumed in Ireland. Both are related to Ireland's population, but not directly to each other. Also, even if the linear correlation coefficient value is close to unity, that does not imply necessarily that the fit is the most appropriate. Although the spring's energy is related fundamentally to the square of its extension, a linear correlation coefficient value of 0.979 results for Case 2 in section 8.9 when correlating a spring's energy with its extension. This high value implies a strong linear correlation between energy and extension, but it does *not* imply that a linear relation is the most appropriate one.

Finally, when attempting to establish a correlation between two variables it is important to recognize the possibility that two uncorrelated variables can appear to be correlated simply by chance. This circumstance makes it imperative to go one step more than simply calculating the value of r. One must also determine the probability that N measurements of two *uncorrelated* variables will give a value of r equal to or larger than any particular r_o. This probability is determined by

$$P_N(\mid r \mid \geq \mid r_o \mid) = \frac{2\Gamma[(N-1)/2]}{\sqrt{\pi}\Gamma[(N-2)/2]} \int_{|r_o|}^{1} (1-r^2)^{(N-4)/2} dr = f(N, r), \quad (8.44)$$

where Γ denotes the gamma function. If $P_N(\mid r \mid \geq \mid r_o \mid)$ is small, then it is unlikely that the variables are uncorrelated. That is, it is likely that they are correlated. Thus, $1 - P_N(\mid r \mid \geq \mid r_o \mid)$ is the probability that two variables are correlated given $\mid r \mid \geq \mid r_o \mid$. If $1 - P_N(\mid r \mid \geq \mid r_o \mid) > 0.95$, then there is a *significant* correlation, and if $1 - P_N(\mid r \mid \geq \mid r_o \mid) > 0.99$, then there is a *very significant* correlation. Values of $1 - P_N(\mid r \mid \geq \mid r_o \mid)$ versus the number of measurements, N, are shown in Figure 8.12. For example, a value of $r_o = 0.6$ gives a 60 % chance of correlation for $N = 4$ and a 99.8 % chance of correlation for $N = 25$. Thus, whenever citing a value of r it is imperative to present the percent confidence of the correlation and the number of data points upon which it is based. Reporting a value of r alone is ambiguous.

8.8 Uncertainty from Measurement Error

One of the major contributors to the differences between the measured and calculated y values in a regression analysis is measurement error. This can be understood best by examining the linear case.

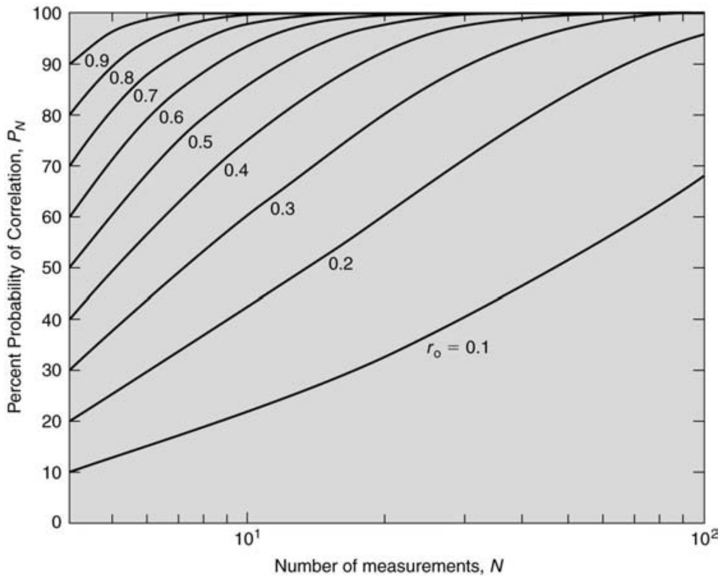

FIGURE 8.12
Probability of correlation.

For an error-free experiment in which the data pairs $[x_i, y_i]$ are linearly related, the best-fit relation would be

$$y_i' = \alpha + \beta x_i', \tag{8.45}$$

in which α and β are the true intercept and slope, respectively, and y_i' is the true mean value of y_i associated with the true mean value of x_i, x_i'. For an experiment in which measurement errors are present one can write

$$x_i = x_i' + \epsilon_x \tag{8.46}$$

and

$$y_i = y_i' + \epsilon_y, \tag{8.47}$$

where x_i and y_i denote the actual, measured values and ϵ_x and ϵ_y their *measurement* errors. Here, it is assumed that the value of all of the x_i errors is the same and equal to ϵ_x, and the value of all of the y_i errors is the same and equal to ϵ_y. That is, the x_i and y_i errors are independent of the particular data pair. This is true if each of the y_i measurements results from an independent measurement situation. Using Equations 8.46 and 8.47, Equation 8.45 becomes

$$y_i = \alpha + \beta x_i + (\epsilon_y - \beta\epsilon_x) = y_{c_i} + E_y. \tag{8.48}$$

The terms in parentheses represent the error term for y_i, which is denoted by E_y. Thus, the value of y_{c_i} will have an error of E_y with respect to its measured value, y_i. This error results from possible measurement errors in x and y or both.

This error is characterized best through its variance, σ_E^2. A subtle yet important point is that the variance of x_i is the same as that of ϵ_x and that the variance of y_i is the same as that of ϵ_y. This is because both x_i' and y_i' have no error. Thus, the variance in x_i is characterized by the variance in its error. This also is true for y_i. These variances are denoted by σ_x^2 and σ_y^2. If ϵ_y and $\beta\epsilon_x$ are statistically independent, then the variance of the combined errors, σ_{E_y}, is given by [4]

$$\sigma_{E_y}^2 = \sigma_y^2 + \beta^2\sigma_x^2. \tag{8.49}$$

This equation is valid only when either $\epsilon_x = 0$ or x is controlled such that its randomness is constrained. If either of these conditions are not met, then $\sigma_{E_y}^2$ cannot be subdivided into these two components. Then, the individual contributions of the ϵ_x and ϵ_y due to the difference between the measured and calculated value of y cannot be ascertained.

So, measurement errors lead to variances in x and y. These variances contribute to the combined variance, σ_{E_y}. It is σ_{E_y} that contributes to the differences between the y_i and y_{c_i} values.

8.9 Determining the Appropriate Fit

Even determining the linear best fit for a set of data and its associated precision can be more involved than it appears. How to determine a linear best fit of data already has been discussed. Here, implicitly it was assumed that the measurement uncertainties in x were negligible with respect to those in y and that the assumed mathematical expression was the most appropriate one to model the data. However, many common situations involving regression usually are more complicated. Examine the various cases that can occur when fitting data having uncertainty with a least-squares regression analysis.

There are six cases to consider, as listed in Table 8.2. Each assumes a level of measurement uncertainty in x, u_x, and in y, u_y, and whether or not the order of the regression is correct. The term *correct* implies that the underlying physical model that governs the relationship between x and y has the same order as the fit. The last two cases (5 and 6), in which both x and y have comparable uncertainties ($u_x \sim u_y$), are more difficult to analyze. Often, only special situations of these two cases are considered [10]. Each of the six cases is now discussed in more detail.

Case	u_x	u_y	Fit
1	0	0	correct
2	0	0	incorrect
3	0	$\neq 0$	correct
4	0	$\neq 0$	incorrect
5	$\neq$0	$\neq 0$	correct
6	$\neq$0	$\neq 0$	incorrect

TABLE 8.2
Cases involving uncertainties and the type of fit.

- Case 1: This corresponds to the ideal case in which there are no uncertainties in x and y ($u_x = u_y = 0$) and the order of the fit is the same as that of the underlying physical model (a correct fit). For example, consider a vertically-oriented, linear spring with a weight, W, attached to its end. The spring will extend downward from its unloaded equilibrium position a distance x proportional to W, as given by Hooke's law, $W = -kx$, where k is the spring constant and negative x corresponds to positive displacement (extension). Assuming that the experiment is performed without error, a first-order (linear) regression analysis would yield a perfect fit of the data with an intercept equal to zero and a slope equal to $-k$. Because there are no measurement errors in either x or y, the values of the intercept and slope will be true values, even if the data set is finite.

- Case 2: This case involves an error-free experiment in which the data is fit with an incorrect order. For example, continuing with the spring-weight example, the work done by the weight to extend the spring, Wx, could be plotted versus its displacement. This work equals the stored energy of the spring, E, which equals $0.5kx^2$. A linear regression fit of Wx versus x would result in a fit that does not correspond to the correct underlying physical model, as shown in Figure 8.13. A second-order fit would be appropriate because $E \sim x^2$. The resulting differences between the data and the linear fit come solely from the incorrect choice of the fit. These differences, however, easily could be misinterpreted as the result of errors in the experiment, as is the case for the data shown in Figure 8.13. Obviously, it is important to have a good understanding of the most appropriate order of the fit *before* the regression analysis is performed.

- Case 3: For this case there is uncertainty in y but not in x and the correct order of the fit is used. This is the type of situation encountered when regression analysis first was considered. The resulting differences between the measured and calculated y values result from the measure-

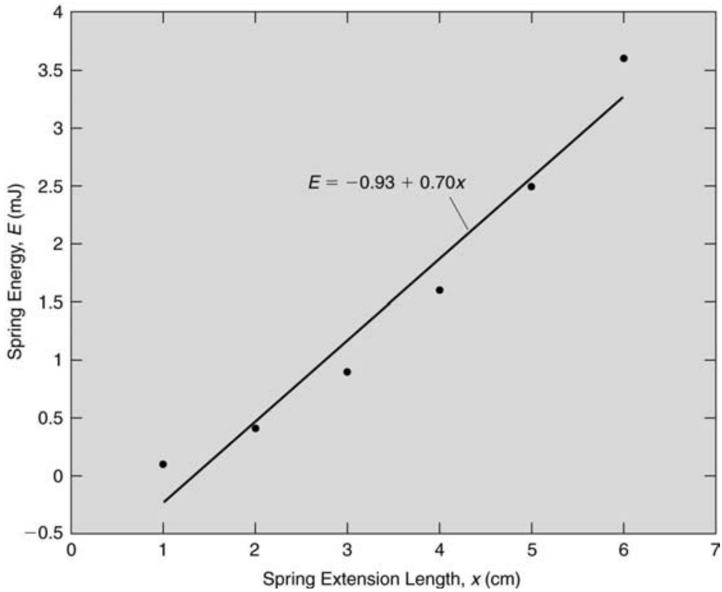

FIGURE 8.13
Example of Case 2.

ment uncertainties in y. Consequently, a correct regression fit will agree with the data to within its measurement uncertainty.

When the correct physical model is not known *a priori*, the standard approach is to increase the order of the fit within reason until an acceptable fit is obtained. What is acceptable is somewhat arbitrary. Ideally, all data points inclusive of their uncertainties should agree with the fit to within the confidence intervals specified by Equation 8.30. Although an n-th order polynomial will fit n-1 data points exactly, this usually does not correspond to a physically-realizable model. Very seldom does a physical law involve more than a fourth power of a variable. In fact, high-degree polynomial fits characteristically exhibit large excursions *between* data points and have coefficients that require many significant figures for repeatable accuracy [13]. So, caution should be exercised when using higher-order fits. Whenever possible, the order of the fit should correspond to the order of the physical model.

- Case 4: This case considers the situation in which there is uncertainty in y but not in x and an incorrect order of the fit is used. Two uncertainties in the calculated y values result in relation to the true fit. One is from the measurement uncertainty in y and the other is from the use of an incorrect model. Here it is difficult to determine directly the contribution of each uncertainty to the overall uncertainty. A systematic

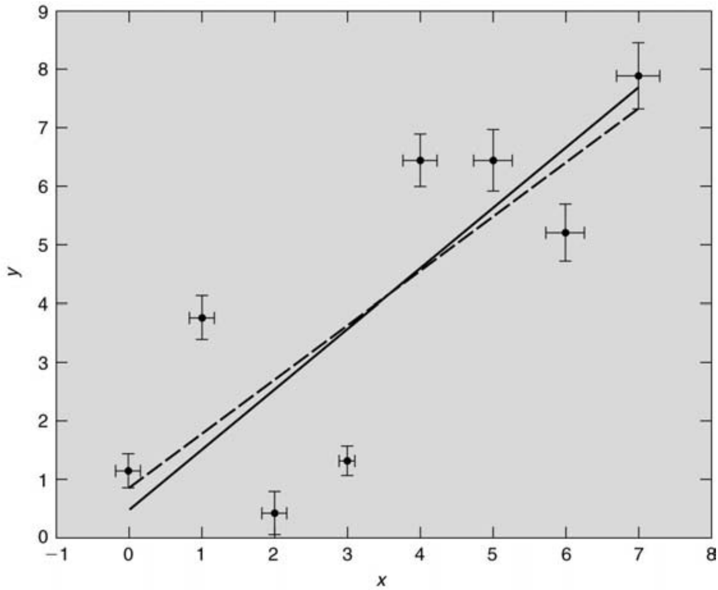

FIGURE 8.14
Two regression fits of the same data.

study involving either more accurate measurements of y or the use of a different model would be necessary to determine this.

Finally, there are two other cases that arise in which there is uncertainty in both x and y. The presence of both of these uncertainties leads to a best fit that is different from that when there is only uncertainty in y. This is illustrated in Figure 8.14 in which two regression fits are plotted for the *same* data. The dashed line represents the fit that considers only the uncertainty in y that was established using a linear least-squares regression analysis. The solid line is the fit that considers uncertainty in both x and y that was established using Deming's method (see [9]), which is considered in the following case. It is easy to see that when uncertainty is present in both x and y, a fit established using the linear least-squares regression analysis that does not consider the uncertainty in x will *not* yield the best fit.

Whenever $u_x \sim u_y$ and no further constraints are placed on them, more extensive regression techniques must be used to determine the best fit of the data (for example, see [3]). This topic is beyond the scope of this text. However, Mandel [10] has examined two special and practical situations in which uncertainty is present in x and linear regression analysis can be applied. These will now be examined.

- Case 5: The general situation for this case involves uncertainties in both x and y and a correct order of the fit.

For the first special situation in which the ratio of the variances of the x and y errors, $\lambda = \sigma_x^2/\sigma_y^2$, is known *a priori*, a linear best-fit equation can be determined using Deming's method of minimizing the *weighted* sum of squares of x and y. Further, estimates of the variances of the x and y can be obtained.

The slope of the regression line calculated by this method is

$$b = \frac{\lambda S_{yy} - S_{xx} + \sqrt{(S_{xx} - \lambda S_{yy})^2 + 4\lambda S_{xy}^2}}{2\lambda S_{xy}}, \tag{8.50}$$

and the intercept is given by the normal Equation 8.19.

The estimates for the variances of the x and y errors are, respectively,

$$\widetilde{S}_x^2 = (\frac{\lambda}{1+\lambda b^2})\frac{S_{yy} - 2bS_{xy} + b^2 S_{xx}}{N-2} \tag{8.51}$$

and

$$\widetilde{S}_y^2 = (\frac{1}{1+\lambda b^2})\frac{S_{yy} - 2bS_{xy} + b^2 S_{xx}}{N-2}. \tag{8.52}$$

Note that Equations 8.51 and 8.52 differ only by the factor λ. These equations can be used to estimate the final uncertainties in x and y for P percent confidence. These are the uncertainties in estimating x and y from the fit (as opposed to the measurement uncertainties in x and y). They are

$$u_{x_{final}} = t_{N-2,P}\widetilde{S}_x \tag{8.53}$$

and

$$u_{y_{final}} = t_{N-2,P}\widetilde{S}_y. \tag{8.54}$$

Using these equations, a regression fit can be plotted with data and its error bars, as shown in Figure 8.15, in addition to determining values of λ, $u_{x_{final}}$ and $u_{y_{final}}$. These values are 0.25, ± 4.0985, and ± 2.0492, respectively, for the data presented in the figure. The estimates for the variances of x and y are $\widetilde{S}_x^2 = 0.7014$ and $\widetilde{S}_y^2 = 2.8055$. The estimates of the final uncertainties in x and y appear relatively large at first sight. This is the result of the relatively large scatter in the data. So, for a specified value of x in this case, the value of y will be within ± 4.0985 of its best-fit value 95 % of the time. Likewise, for a specified value of y, the value of x will be within ± 2.0492 of its best-fit value 95 % of the time.

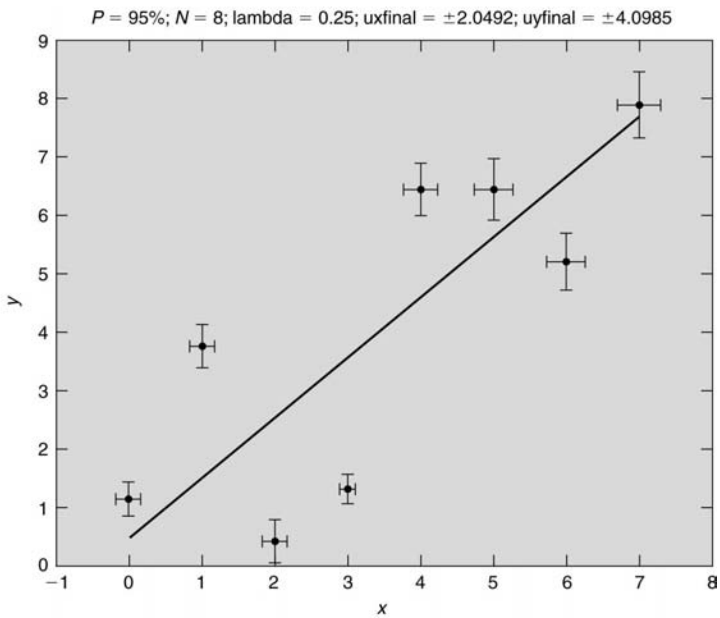

FIGURE 8.15
Example of Case 5 when λ is known.

The second special situation considers when x is a controlled variable. This is known as the Berkson case, in which the value of x is set as close as possible to its desired value, thereby constraining its randomness. This corresponds, for example, to a static calibration in which there is some uncertainty in x but the value of x is specified for each calibration point. For this situation a standard linear least-squares regression fit of the data is valid. Further, estimates can be made for all of the uncertainties presented beforehand for Case 3. The interpretation of the uncertainties, however, is somewhat different [10]. The uncertainty in y with respect to the regression fit must be interpreted according to Equation 8.49.

- Case 6: This is the most complicated case in which there are uncertainties in both x and y and an incorrect order of the fit is used. The same analytical approaches can be taken here as were done for the special situations in Case 5. However, the interpretation of the uncertainties is confounded further as a result of the additional uncertainty introduced by the incorrect order of the fit.

8.10 *Signal Correlations in Time

Thus far, the application of correlation analysis to discrete information has been considered. Correlation analysis also can be applied to information that is continuous in time.

Consider two signals, $x(t)$ and $y(t)$, of two experimental variables. Assume that these signals are statistically stationary and ergodic. These terms are defined in Chapter 9. For a stationary signal, the statistical properties determined by ensemble averaging values for an arbitrary time from the beginning of a number of the signal's time history records are independent of the time chosen. Further, if these average values are the same as those found from the time-average over a single time history record, then the signal is also ergodic. So, an ergodic signal is also a stationary signal. By examining how the amplitude of either signal's time history record at some time compares to its amplitude at another time, important information, such as on the repeatability of the signal, can be gathered. This can be quantified through the autocorrelation function of the signal, which literally correlates the signal with itself (thus the prefix *auto*). The amplitudes of the signals also can be compared to one another to examine the extent of their *co-relation*. This is quantified through the cross-correlation function, in which the cross product of the signals is examined.

8.10.1 *Autocorrelation

For an ergodic signal $x(t)$, the autocorrelation function is the average value of the product $x(t) \cdot x(t+\tau)$, where τ is some time delay. Formally, the **autocorrelation function**, $R_x(\tau)$, is defined as

$$R_x(\tau) \equiv E[x(t) \cdot x(t+\tau)] = \lim_{T \to \infty} \int_0^T x(t)x(t+\tau)dt. \tag{8.55}$$

Because the signal is stationary, $R_x(\tau)$, its mean and its variance are independent of time. So,

$$E[x(t)] = E[x(t+\tau)] = x' \tag{8.56}$$

and

$$\sigma^2_{x(t)} = \sigma^2_{x(t+\tau)} = \sigma^2_x = E[x^2(t)] - x'^2. \tag{8.57}$$

Analogous to Equation 8.33, the **autocorrelation coefficient** can be defined as

$$\rho_{xx}(\tau) \equiv \frac{E[(x(t) - x')(x(t+\tau) - x')]}{\sigma^2_x}. \tag{8.58}$$

The numerator in Equation 8.58 can be expanded to yield

$$\rho_{xx}(\tau) = \frac{E[x(t) \cdot x(t+\tau)] - x'E[x(t+\tau)] - x'E[x(t)] + x'^2}{\sigma_x^2}. \tag{8.59}$$

Substitution of Equations 8.55 and 8.56 into Equation 8.59 results in an expression that relates the autocorrelation function to its coefficient

$$\rho_{xx}(\tau) = \frac{R_x(\tau) - x'^2}{\sigma_x^2} \tag{8.60}$$

or

$$R_x(\tau) = \rho_{xx}(\tau)\sigma_x^2 + x'^2. \tag{8.61}$$

Some limits can be placed on the value of $R_x(\tau)$. Because $-1 \leq \rho_{xx}(\tau) \leq 1$, $R_x(\tau)$ is bounded as

$$-\sigma_x^2 + x'^2 \leq R_x(\tau) \leq \sigma_x^2 + x'^2. \tag{8.62}$$

Now it can be shown (see Chapter 5) that

$$E[x^2] = \sigma_x^2 + x'^2 \tag{8.63}$$

by expanding $E[(x - x')^2]$. So, the maximum value that $R_x(\tau)$ can have is $E[x^2]$. It follows from Equation 8.55 that

$$R_x(0) = E[x^2]. \tag{8.64}$$

That is, the maximum value of $R_x(\tau)$ occurs at $\tau = 0$. Using Equation 8.63 and the definition of the autocorrelation coefficient (Equation 8.60),

$$\rho_{xx}(0) = 1. \tag{8.65}$$

Further, as $\tau \to \infty$, there is no correlation between $x(t)$ and $x(t+\tau)$ because $x(t)$ is the signal of a random variable. That is,

$$\rho_{xx}(\tau \to \infty) = 0, \tag{8.66}$$

which implies that

$$R_x(\tau \to \infty) = x'^2. \tag{8.67}$$

Finally, $R_x(\tau)$ is an even function because

$$R_x(-\tau) = E[x(t)x(t-\tau)] = E[x(t-\tau)x(t)] = E[x(t^*)x(t^*+\tau)] = R_x(\tau), \tag{8.68}$$

where $t^* = t - \tau$, noting $x(t)$ is stationary. So, $R_x(\tau)$ is symmetric about the $\tau = 0$ axis.

A generic autocorrelation function and its corresponding autocorrelation coefficient having these properties is displayed in Figure 8.16. Values of $R_x(\tau)$ and $\rho_{xx}(\tau)$ that are greater than their respective limiting values

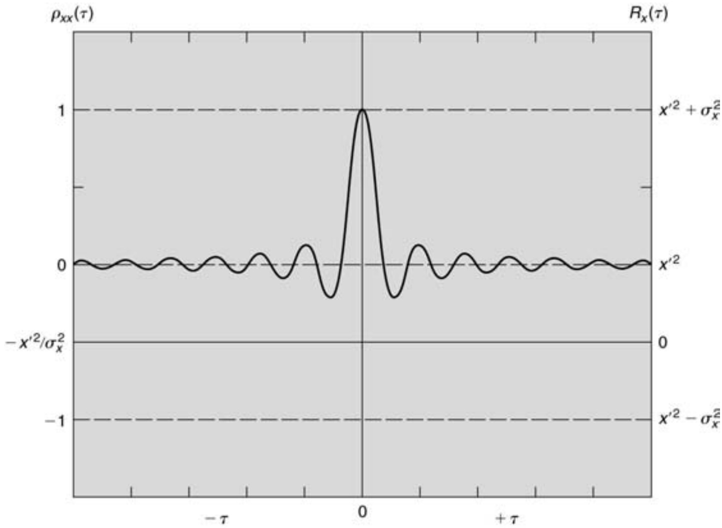

FIGURE 8.16
Typical autocorrelation function and coefficient.

as $\tau \to \infty$ indicate a positive correlation at that particular value of τ. Conversely, a negative correlation is indicated for values less than that limiting value. Values equal to the limiting value signify no correlation. Note that $\rho_{xx}(\tau)$ experiences decreasing oscillations about a value of 0 as $\tau \to \infty$. This always will be the case for a stationary signal provided there are no deterministic components in the signal other than a nonzero mean.

Example Problem 8.6

Statement: Determine the autocorrelation coefficient for the signal $x(t) = A\sin(\omega t)$. Then plot the coefficient for values of $\omega = 1$ rad/s and $A = 1$.

Solution: The autocorrelation function, only for the range $0 \geq \tau \geq T/2\pi$, needs to be examined, because $x(t)$ in this example is a periodic function of period $T = 2\pi/\omega$. Equation 8.55 for this periodic function becomes

$$R_x(\tau) = \frac{A^2}{T}\int_0^{2\pi/\omega} \sin(\omega t)\sin(\omega t + \phi)dt,$$

where $\phi = \omega\tau$. Performing and evaluating the integral,

$$R_x(\tau) = \frac{A^2}{T}[\sin^2(\omega t)\cos(\phi) + \sin(\omega t)\cos(\omega t)\sin(\phi)]_0^{2\pi/\omega} = \frac{1}{2}A^2\cos(\omega\tau).$$

Now $\sigma_x = \sqrt{A/2}$, so, according to Equation 8.58,

$$\rho_{xx} = \cos(\omega\tau).$$

The plot of $\rho_{xx}(\tau)$ for values of $\omega = 1$ rad/s and $A = 1$ is presented in Figure 8.17. It shows that the sine function has a positive, perfect autocorrelation at values of $\tau =$

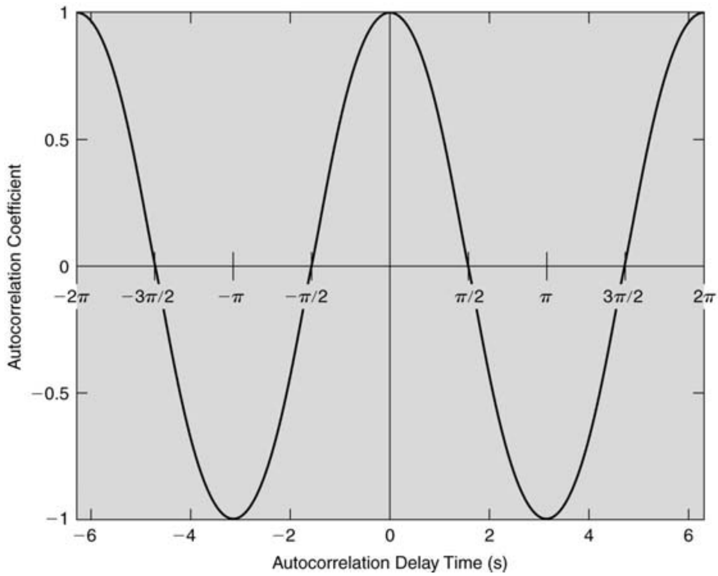

FIGURE 8.17
Autocorrelation of a sine function.

$T = 2\pi$ when $\omega = 1$ and a negative, perfect autocorrelation at values of $\tau = T/2 = \pi$. Further, when $\tau = T/4 = \pi/2$ or $\tau = 3T/4 = 3\pi/2$ there is no correlation of the sine function with itself.

8.10.2 *Cross-Correlation

Expressions for the cross-correlation function and coefficient can be developed in the same manner as that done for the case of autocorrelation.

For the stationary signals $x(t)$ and $y(t)$ there are two **cross-correlation functions** defined as

$$R_{xy}(\tau) \equiv E[x(t) \cdot y(t+\tau)] \tag{8.69}$$

and

$$R_{yx}(\tau) \equiv E[y(t) \cdot x(t+\tau)]. \tag{8.70}$$

$R_{xy}(\tau)$ denotes the cross-correlation of x with y and $R_{yx}(\tau)$ that of y with x. Further, because the signals are stationary

$$R_{xy}(\tau) = E[x(t-\tau)y(t)] = R_{yx}(-\tau) \tag{8.71}$$

and

$$R_{yx}(\tau) = E[y(t-\tau)x(t)] = R_{xy}(-\tau). \tag{8.72}$$

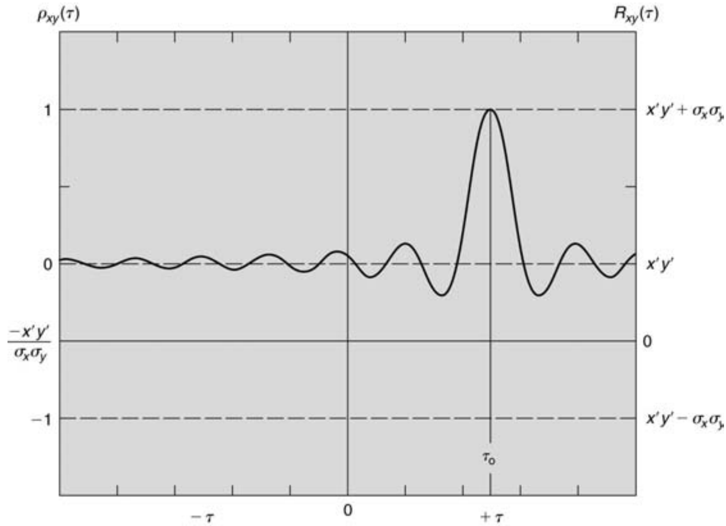

FIGURE 8.18
Typical cross-correlation of two signals.

So, in general, $R_{xy}(\tau) \neq R_{yx}(\tau)$ and both are not even with respect to τ.

The corresponding two **cross-correlation coefficients** are defined as

$$\rho_{xy}(\tau) \equiv \frac{R_{xy}(\tau) - x'y'}{\sigma_x \sigma_y} \tag{8.73}$$

and

$$\rho_{yx}(\tau) \equiv \frac{R_{yx}(\tau) - x'y'}{\sigma_x \sigma_y}, \tag{8.74}$$

where both coefficients are bounded between values of −1 and 1. Thus, both functions are bounded between values of $-\sigma_x\sigma_y + x'y'$ and $\sigma_x\sigma_y + x'y'$. Finally, as $\tau \to \infty$ both functions tend to the value of $x'y'$ because no correlation between the random signals $x(t)$ and $y(t)$ would be expected at that limit.

Typically, two signals will experience a maximum cross-correlation at some value of $\tau = \tau_o$, which corresponds to a phase lag between the two signals, where $\phi = \omega\tau_o$. This is shown for a typical cross-correlation in Figure 8.18.

Example Problem 8.7

Statement: Determine the cross-correlation coefficient $\rho_{xy}(\tau)$ for the the signals $x(t) = A\sin(\omega t)$ and $y(t) = B\cos(\omega t)$. Then plot the coefficient for the value of $\omega = 1$ rad/s.

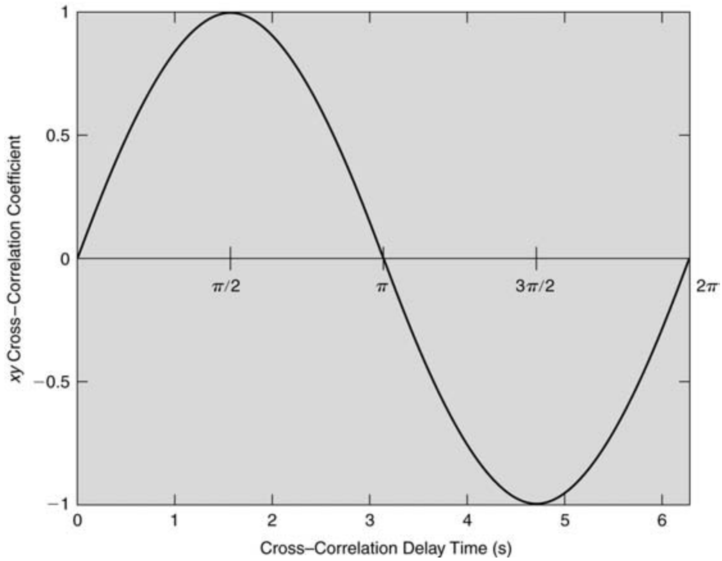

FIGURE 8.19
Cross-correlation of sine and cosine functions.

Solution: The cross-correlation function only for the range $0 \leq \tau \leq T/2\pi$ needs to be examined because $x(t)$ and $y(t)$ are periodic functions of period $T = 2\pi/\omega$. Equation 8.69 for these periodic functions becomes

$$R_{xy}(\tau) = \frac{AB}{T}\int_0^{2\pi/\omega} \sin(\omega t)\cos(\omega t + \phi)dt,$$

where $\phi = \omega\tau$. Performing and evaluating the integral,

$$R_{xy}(\tau) = \frac{AB}{\tau}[-\frac{1}{4}\omega\cos(2\omega t + \phi) - \frac{1}{2}t\sin(\phi)]_0^{2\pi/\omega} = \frac{-AB}{2}\sin(\phi).$$

Now $\sigma_x = A/\sqrt{2}$ and $\sigma_y = B/\sqrt{2}$, so, according to Equation 8.73,

$$\rho_{xy}(\tau) = -\sin(\omega\tau).$$

Note the minus sign in this expression. The plot of $\rho_{xy}(\tau)$ for the value of $\omega = 1$ rad/s is given in Figure 8.19. For a delay time value of $\tau = \pi/2$, $\rho_{xy}(\tau) = 1$. This is because the value of $\cos(t + \pi/2)$ exactly equals that of $\sin(t)$. Similar reasoning can be used to explain the value of $\rho_{xy}(\tau) = -1$ when $\tau = 3\pi/2$, where the cosine and sine values are equal but opposite in sign.

The M-file `sigcor.m` determines and plots the autocorrelations and cross-correlation of discrete data that is user-specified. This M-file normalizes the correlations such that the autocorrelations at zero time lag are identically 1.0. An example plot generated using `sigcor.m` for a file containing eight sequential measurements of x and y data is shown in Figure 8.20. Note that both autocorrelations have a value of 1.0 at zero time lag.

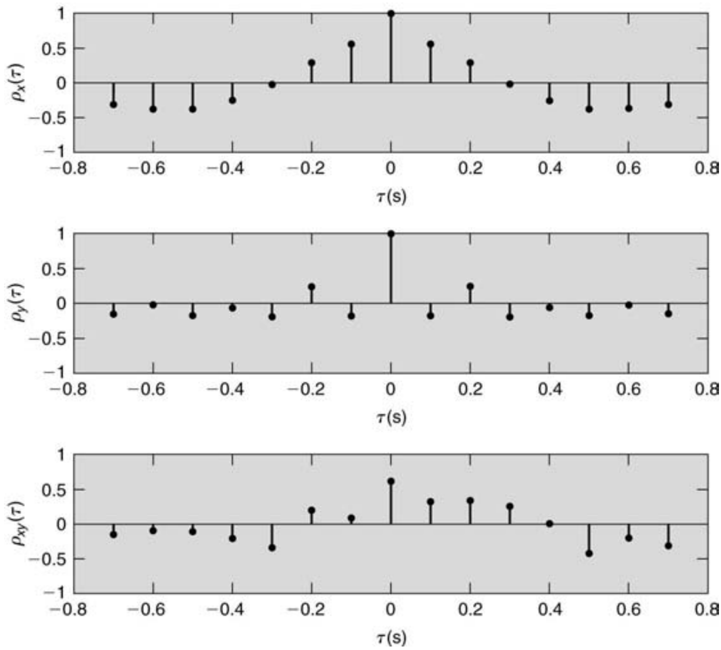

FIGURE 8.20
Autocorrelations and cross-correlation of discrete data.

8.11 *Higher-Order Analysis

Higher-order ($m > 2$) regression analysis can be performed in a manner similar to that developed for linear least-squares regression analysis. This will result in $m + 1$ algebraic normal equations. These can be solved most easily using methods of linear algebra to obtain the expressions for the $m+1$ regression coefficients.

For higher-order regression analysis, the coefficients a_0 through a_m in the expression

$$a_0 + a_1 x_i + a_2 x_i^2 + ... + a_m x_i^m = y_{c_i} \tag{8.75}$$

are found using the method of minimizing D as described in the previous section. The resulting equations are

$$
\begin{aligned}
a_0 N + a_1 \sum_{i=1}^{N} x_i + a_2 \sum_{i=1}^{N} x_i^2 + \quad \cdots \quad + a_m \sum_{i=1}^{N} x_i^m &= \sum_{i=1}^{N} y_i, \\
a_0 \sum_{i=1}^{N} x_i + a_1 \sum_{i=1}^{N} x_i^2 + a_2 \sum_{i=1}^{N} x_i^3 + \quad \cdots \quad + a_m \sum_{i=1}^{N} x_i^{m+1} &= \sum_{i=1}^{N} x_i y_i, \\
a_0 \sum_{i=1}^{N} x_i^2 + a_1 \sum_{i=1}^{N} x_i^3 + a_2 \sum_{i=1}^{N} x_i^4 + \quad \cdots \quad + a_m \sum_{i=1}^{N} x_i^{m+2} &= \sum_{i=1}^{N} x_i^2 y_i, \\
\vdots \quad , \text{ and} \\
a_0 \sum_{i=1}^{N} x_i^m + a_1 \sum_{i=1}^{N} x_i^{m+1} + a_2 \sum_{i=1}^{N} x_i^{m+2} + \quad \cdots \quad + a_m \sum_{i=1}^{N} x_i^{2m} &= \sum_{i=1}^{N} x_i^m y_i.
\end{aligned}
\tag{8.76}
$$

In expanded matrix notation, the set of Equations 8.76 becomes

$$
\begin{bmatrix}
N & \sum x_i & \sum x_i^2 & \cdots & \sum x_i^m \\
\sum x_i & \sum x_i^2 & \sum x_i^3 & \cdots & \sum x_i^{m+1} \\
\sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \cdots & \sum x_i^{m+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sum x_i^m & \sum x_i^{m+1} & \sum x_i^{m+2} & \cdots & \sum x_i^{2m}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}
=
\begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \\ \vdots \\ \sum x_i^m y_i \end{bmatrix},
$$

where the summations are from $i = 1$ to N. Or, in matrix notation, this becomes

$$[E][a] = [F]. \tag{8.77}$$

$[E]$, $[a]$, and $[F]$ represent the matrices shown in the expanded form. The solution to Equation 8.77 for the regression coefficients is

$$[a] = [E]^{-1}[F]. \tag{8.78}$$

$[E]^{-1}$ is the inverse of the coefficient matrix.

Example Problem 8.8

Statement: An experiment is conducted in which a ball is dropped with an initial downward velocity, v_0, from a tall tower. The distance fallen, y(m), is recorded at time, t, intervals of one second for a period of six seconds. The resulting time, distance data are 0, 0; 1, 7; 2, 21; 3, 48; 4, 81; 5, 131; and 6, 185. The distance equation is $y = y_0 + v_0 t + 0.5 g t^2$, where g is the local gravitational acceleration. This equation is of the form $y = a_0 + a_1 t + a_2 t^2$. Using higher-order regression analysis, determine the values of the regression coefficients a_0, a_1, and a_2. From their values determine the values of v_0 and g. Finally, plot the data with the regression fit.

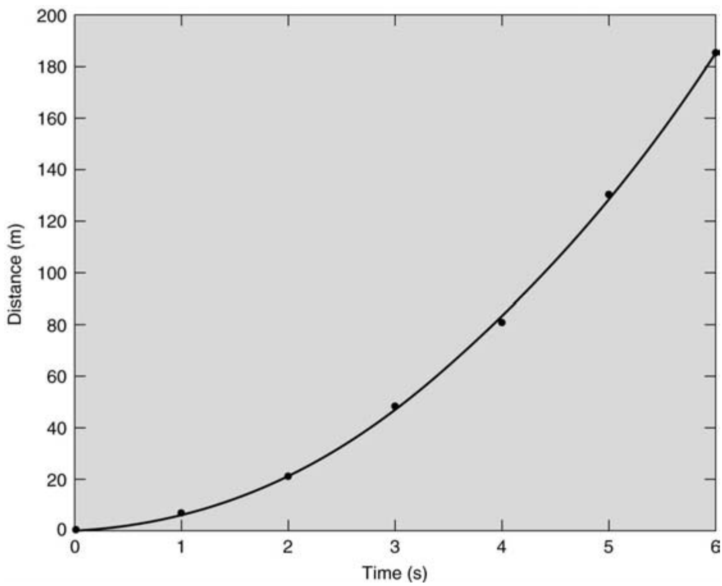

FIGURE 8.21
Higher-order regression fit example.

Solution: The solution is obtained using MATLAB's left-division method by typing $t\backslash y$, where $[t]$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$$

and $[y]$ is

$$\begin{bmatrix} 0 \\ 7 \\ 21 \\ 48 \\ 81 \\ 131 \\ 185 \end{bmatrix}.$$

The resulting regression coefficient matrix is

$$\begin{bmatrix} 0.5238 \\ 0.3214 \\ 5.0833 \end{bmatrix}.$$

Thus, the best-fit expression is

$$0.5238 + 0.3214t + 5.0833t^2 = y.$$

The data and Equation 8.11 are shown in Figure 8.21. Also, $v_0 = 0.3214$ and $g = 10.1666$.

8.12 *Multi-Variable Linear Analysis

Linear least-squares regression analysis can be extended to situations involving more than one independent variable. This is known as multi-variable linear regression analysis and results in $m+1$ algebraic equations with $m+1$ regression coefficient unknowns. This system of equations can be solved using methods of linear algebra.

Multi-variable linear regression analysis for a system of three independent variables [15] with the regression coefficients a_0 through a_3 in the expression

$$a_0 + a_1 x_i + a_2 y_i + a_3 z_i = R_{c_i} \tag{8.79}$$

yields a system of four equations, which is

$$\begin{aligned}
a_0 N + a_1 \sum_{i=1}^{N} x_i + a_2 \sum_{i=1}^{N} y_i + a_3 \sum_{i=1}^{N} z_i &= \sum_{i=1}^{N} R_i, \\
a_0 \sum_{i=1}^{N} x_i + a_1 \sum_{i=1}^{N} x_i^2 + a_2 \sum_{i=1}^{N} x_i y_i + a_3 \sum_{i=1}^{N} x_i z_i &= \sum_{i=1}^{N} R_i x_i, \\
a_0 \sum_{i=1}^{N} y_i + a_1 \sum_{i=1}^{N} x_i y_i + a_2 \sum_{i=1}^{N} y_i^2 + a_3 \sum_{i=1}^{N} y_i z_i &= \sum_{i=1}^{N} R_i y_i, \text{ and} \\
a_0 \sum_{i=1}^{N} z_i + a_1 \sum_{i=1}^{N} x_i z_i + a_2 \sum_{i=1}^{N} y_i z_i + a_3 \sum_{i=1}^{N} z_i^2 &= \sum_{i=1}^{N} R_i z_i.
\end{aligned} \tag{8.80}$$

In expanded matrix notation, the set of Equations 8.80 becomes

$$\begin{bmatrix} N & \sum x_i & \sum y_i & \sum z_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i & \sum x_i z_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 & \sum y_i z_i \\ \sum z_i & \sum x_i z_i & \sum y_i z_i & \sum z_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum R_i \\ \sum R_i x_i \\ \sum R_i y_i \\ \sum R_i z_i \end{bmatrix},$$

where the summations are from $i = 1$ to N. Or, in matrix notation, this becomes

$$[G][a] = [R]. \tag{8.81}$$

$[G]$, $[a]$, and $[R]$ represent the matrices shown in the expanded form. The solution to Equation 8.81 for the regression coefficients is

$$[a] = [G]^{-1}[R]. \tag{8.82}$$

$[G]^{-1}$ is the inverse of the coefficient matrix.

Example Problem 8.9

Statement: An experiment is conducted in which the values of three independent variables, x, y, and z, are selected and then the resulting value of the dependent variable R is measured. This procedure is repeated six times for different combinations of x, y, and z values. The $[x,y,z,R]$ data values are [1,3,1,17; 2,4,2,24; 3,5,1,25; 4,4,2,30; 5,3,1,24; 6,3,2,31]. Determine the regression coefficients for the multi-variable regression fit of the data. Then, using the resulting best-fit expression, determine the calculated values of R in comparison to their respective measured values.

Solution: The solution is obtained using MATLAB's left-division method by typing $G\backslash R$, where $[G]$ for this example is

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 5 & 1 \\ 1 & 4 & 4 & 2 \\ 1 & 5 & 3 & 1 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

and $[R]$ is

$$\begin{bmatrix} 17 \\ 24 \\ 25 \\ 30 \\ 24 \\ 31 \end{bmatrix}.$$

The resulting regression coefficient matrix is

$$\begin{bmatrix} 3.4865 \\ 2.0270 \\ 2.2162 \\ 4.3063 \end{bmatrix}.$$

Thus, the best-fit expression is

$$3.4865 + 2.0270x + 2.2162y + 4.3063z = R. \tag{8.83}$$

The calculated values of R are obtained by typing $G*ans$ after the regression coefficient solution is obtained. The values are 16.4685, 25.0180, 24.9550, 29.0721, 24.5766, and 30.9099. All calculated values agree with their respective values to within a difference of less than 1.0.

8.13 Problem Topic Summary

Topic	Review Problems	Homework Problems
Regression Analysis	1, 2, 3, 4, 7, 8	1, 2, 3, 6, 8, 11, 12
Linearly Intrinsic		4, 5, 7, 9, 10
Regression Parameters	4, 5, 6	6

TABLE 8.3
Chapter 8 Problem Summary

8.14 Review Problems

1. Consider the following set of three (x, y) data pairs: (0, 0), (3, 2), and (6, 7). Determine the intercept of the best-fit line for the data to two decimal places.

2. Consider the following set of three (x, y) data pairs: (0, 0),(3, 0) and (9, 5). Determine the slope of the best-fit line for the data to two decimal places.

3. Who is the famous mathematician who developed the method of least squares?

4. Consider the following set of three (x, y) data pairs: (1.0, 1.7), (2.0, 4.3), and (3.0, 5.7). A linear least-squares regression analysis yields the best-fit equation $y = -0.10 + 2.00x$. Determine the standard error of the fit rounded off to two decimal places.

5. Consider the following set of three (x, y) data pairs: (1.0, 1.7), (2.0, 4.3), and (3.0, 5.7). A linear least-squares regression analysis yields the best-fit equation $y = -0.10 + 2.00x$. Determine the precision interval based upon 95 % confidence rounded off to two decimal places. Assume a value of 2 for the Student-t factor.

6. An experimenter determines the precision interval, PI-1, for a set of data by performing a linear least-squares regression analysis. This interval is based upon three measurements and 50 % confidence. Then the same experiment is repeated under identical conditions and a new precision interval, PI-2, is determined based upon 15 measurements and 95 %

confidence. The ratio of PI-2 to PI-1 is (a) less than one, (b) greater than one, (c) equal to one, or (d) could be any of the above.

7. A strain gage-instrumented beam was calibrated by hanging weights of 1.0 N, 2.0 N, and 3.0 N at the end of the beam and measuring the corresponding output voltages. A linear least-squares regression analysis of the data yielded a best-fit intercept equal to 1.00 V and a best-fit slope equal to 2.75 V/N. At 1.0 N, the recorded voltage was 3.6 V and at 2.0 N it was 6.8 V. What was the recorded voltage at 3.0 N?

8. A linear least-squares regression analysis fit of the (x, y) pairs (0, 1), (1, 3.5), (2, 5.5), (3, 7), and (4, 9.5) *must* pass through (a) (0, 1), (b) (1, 3.5), (c) (2, 5.3), (d) (3, 8), or (e) (4, 9.5)? Why (give one reason)?

8.15 Homework Problems

1. Prove that a least-squares linear regression analysis fit always goes through the point ($\bar{x}$,$\bar{y}$).

2. Starting with the equation $y_i - \bar{y} = (y_i - y_{c_i}) + (y_{c_i} - \bar{y})$ and using the normal equations, prove that $\sum_{i=1}^{N}(y_i - \overline{y})^2 = \sum_{i=1}^{N}(y_i - y_{c_i})^2 + \sum_{i=1}^{N}(y_{c_i} - \overline{y})^2$.

3. Find the linear equation that best fits the data shown in Table 8.4.

x:	10	20	30	40
y:	5.1	10.5	14.7	20.3

TABLE 8.4
Calibration data.

4. Determine the best-fit values of the coefficients a and b in the expression $y = 1 \,/\, (a + bx)$ for the (x, y) data pairs (1.00, −1.11), (2.00, −0.91), (3.00, −0.34), (4.00, −0.20), and (5.00, −0.14).

5. For an ideal gas, $pV^{\gamma} = C$. Using regression analysis, determine the best-fit value for γ given the data shown in Table 8.5.

6. The data presented in Table 8.6 was obtained during the calibration of a cantilever-beam force-measurement system. The beam is instrumented with four strain gages that serve as the legs of a Wheatstone bridge. In the table F(N) denotes the applied force, E(V) the measured output

p (psi)	V (in.^{3})
16.6	50
39.7	30
78.5	20
115.5	15
195.3	10
546.1	5

TABLE 8.5
Gas pressure-volume data.

voltage, and u_E(V) the measurement uncertainty in E. Based upon a knowledge of how such a system operates, what order of the fit would model the physics of the system most appropriately? Perform a regression analysis of the data for various orders of the fit. What is the order of the fit that has the lowest value of S_{yx}? What is the order of the fit that has the smallest precision interval, $\pm t_{\nu,P} S_{yx}$, that is required to have the actual fit curve agree with *all* of the data to within the uncertainty of E?

F(N)	E(V)	u_E(V)
0.4	2.7	0.1
1.1	3.6	0.2
1.9	4.4	0.2
3.0	5.2	0.3
5.0	9.2	0.5

TABLE 8.6
Strain-gage force-balance calibration data with uncertainty.

7. A hot-wire anemometry system probe inserted into a wind tunnel is used to measure the tunnel's centerline velocity, U. The output of the system is a voltage, E. During a calibration of this probe, the data listed in Table 8.7 was acquired. Assume that the uncertainty in the voltage measurement is 2 % of the indicated value. Using a linear least-squares regression analysis determine the best fit values of A and B in the relation $E^2 = A + B\sqrt{U}$. Finally, plot the fit with 95 % confidence intervals and the data with error bars as voltage versus velocity. Is the assumed relation appropriate?

Velocity (m/s)	Voltage (V)
0.00	3.19
3.05	3.99
6.10	4.30
9.14	4.48
12.20	4.65

TABLE 8.7
Hot-wire probe calibration data.

8. The April 3, 2000 issue of *Time Magazine* published the body mass index (BMI) of each Miss America from 1922 to 1999. The BMI is defined as "the weight divided by the square of the height". (Note: The units of the BMI are specified as kg/m^2, which strictly is mass divided by the square of the height.) The author argues, based on the data, that Miss America may dwindle away to nothing if the BMI-versus-year progression continues. Perform a linear least-squares regression analysis on the data and determine the linear regression coefficient. How statistically justified is the author's claim? Also determine how many Miss Americas have BMIs that are *below* the World Health Organization's cutoff for undernutrition, which is a BMI equal to 18.6. Use the data file `missamer.dat` that contains two columns, the year and the BMI.

9. For the ideal gas data presented in Table 8.5, determine the standard error of the fit, S_{xy}, for the best fit found using linear regression analysis. Plot the best-fit relation of p as a function of V along with the data and the precision intervals of 90 % and 99 % confidence, all on the same graph.

10. Given the (x, y) data pairs (0, 0.2), (1, 1.3), (2, 4.8), and (3, 10.7), (a) develop the expressions (but do not solve them) for variables x and y such that a least-squares linear-regression analysis could be used to fit the data to the non-linear expression $y = axe^x + b$, where a and b are best-fit constants. Then, (b) determine the value of b.

11. The four (x, y) data pairs (2, 2), (4, 3), (6, 5), and (8, 6) are fitted using the method of linear least-squares regression. Determine the calculated y value through which the fit passes when $x = 5$ *without* doing the regression analysis.

12. The data presented in the Table 8.8 was obtained during a strain-gage force balance calibration, where F(N) denotes the applied force in N and E(V) the measured output voltage in V. Determine a suitable least-squares fit of the data using the appropriate functions in MATLAB. Quantify the best fit through the standard error of the fit, S_{yx}. Which

F(N)	E(V)
0.4	2.7
1.1	3.6
1.9	4.4
3.0	5.2
5.0	9.2

TABLE 8.8
Strain-gage force-balance calibration data.

polynomial fit has the lowest value of S_{yx}? Which polynomial fit is the most suitable (realistic) based on your knowledge of strain-gage system calibrations? Plot each polynomial fit on a separate graph and include error bars for the y-variable, with the magnitude of error bar estimated at 95 % confidence and based on S_{yx}. Use the M-file `plotfit.m`. This M-file requires three columns of data, where the third column is the measurement uncertainty in y. Table 8.8 does not give those, so they must be added. Assume that each y measurement uncertainty is 5 % of the measured E(V) value. S_{yx} is actually the curve-fit uncertainty. `plotfit.m` shows this uncertainty as dotted lines (not as error bars as in the problem statement). The M-file `plotfit.m` also prints out the S_{yx} value and calculates the precision interval ($\pm t_{\nu,P} S_{yx}$) to help plot the dotted lines. Calculate the y measurement uncertainties and use them as a third column input to the M-file, then run `plotfit.m` for each order of the curve fit desired. Also be sure to label the axes appropriately.

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9

Signal Characteristics

CONTENTS

... there is a tendency in all observations, scientific and otherwise, to see what one is looking for ...

D.J. Bennett. 1998. *Randomness.* Cambridge: Harvard University Press.

But you perceive, my boy, that it is not so, and that facts, as usual, are very stubborn things, overruling all theories.

Professor VonHardwigg in *Voyage au centre de la terra* by Jules Gabriel Verne, 1864.

9.1 Chapter Overview

One of the key requirements in performing a successful experiment is a knowledge of signal characteristics. Signals contain vital information about the process under investigation. Much information can be extracted from them, provided the experimenter is aware of the methods that can be used and their limitations. In this chapter, the types of signals and their characteristics are identified. Formulations of the statistical parameters of signals are presented. Fourier analysis and synthesis are introduced and used to find the amplitude, frequency, and power content of signals. These tools are applied to continuous signals, first to some classic periodic signals and then to aperiodic signals. In the following chapter, these methods are extended to digital signal analysis.

9.2 Signal Characterization

In the context of measurements, a **signal** is a measurement system's representation of a physical variable that is sensed by the system. More broadly, it is defined as a detectable, physical quantity or impulse (as a voltage, current, or magnetic field strength) by which messages and information can be transmitted [1]. The information contained in a signal is related to its size and extent. The size is characterized by the **amplitude** (magnitude) and the extent (timewise or samplewise variation) by the **frequency**. The actual shape of a signal is called its **waveform**. A plot of a signal's amplitude versus time is called a **time history record**. A collection of N time history records is called an **ensemble**, as illustrated in Figure 9.1. An ensemble also can refer to a set of many measurements made of a single entity, such as the weight of an object determined by each student in a science class, and of many entities of the same kind made at the same time, such as everyone's weight on New Year's morning.

Signals can be classified as either **deterministic** or **nondeterministic** (**random**). A deterministic signal can be described by an explicit mathematical relation. Its future behavior, therefore, is predictable. Each time history record of a random signal is unique. Its future behavior cannot be determined exactly but to within some limits with a certain confidence.

Deterministic signals can be classified into static and dynamic signals, which are subdivided further, as shown in Figure 9.2. **Static** signals are steady *in time*. Their amplitude remains constant. Dynamic signals are either periodic or aperiodic. A **periodic** signal, $y(t)$, repeats itself at regular intervals, nT, where $n = 1, 2, 3, ...$ Analytically, this is expressed as

$$y(t + T) = y(t) \tag{9.1}$$

for all t. The smallest value of T for which Equation 9.1 holds true is called the **fundamental period**. If signals $y(t)$ and $z(t)$ are periodic, then their product $y(t)z(t)$ and the sum of any linear combination of them, $c_1 y(t) + c_2 z(t)$, are periodic.

A **simple** periodic signal has one period. A **complex** periodic signal has more than one period. An **almost-periodic** signal is comprised of two or more sinusoids of arbitrary frequencies. However, if the ratios of all possible pairs of frequencies are rational numbers, then an almost-periodic signal is periodic.

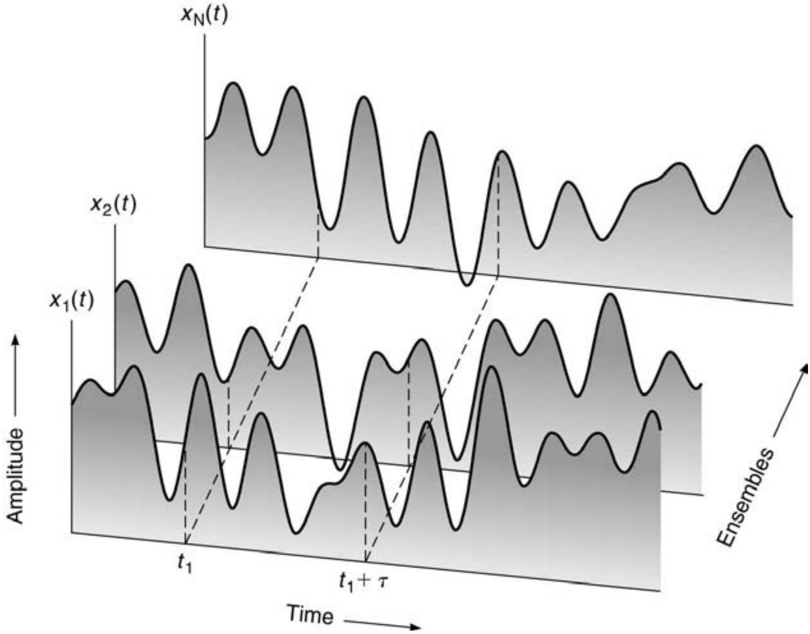

FIGURE 9.1
An ensemble of N time history records.

Nondeterministic signals are classified as shown in Figure 9.3. Properties of the ensemble of the nondeterministic signals shown in Figure 9.1 can be computed by taking the average of the instantaneous property values acquired from each of the time histories at an arbitrary time, t_1. The ensemble mean value, $\mu_x(t_1)$, and the ensemble autocorrelation function (see Chapter 8 for more on the autocorrelation), $R_x(t_1, t_1 + \tau)$, are

$$\mu_x(t_1) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t_1) \tag{9.2}$$

and

$$R_x(t_1, t_1 + \tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t_1) x_i(t_1 + \tau), \tag{9.3}$$

in which τ denotes an arbitrary time measured from time t_1. Both equations represent **ensemble averages**. This is because $\mu_x(t_1)$ and $R_x(t_1, t_1 + \tau)$ are determined by performing averages over the ensemble at time t_1.

If the values of $\mu_x(t_1)$ and $R_x(t_1, t_1 + \tau)$ change with t_1, then the signal is **nonstationary**. Otherwise, it is **stationary** (stationary in the *wide* sense). A nondeterministic signal is considered to be **weakly stationary**

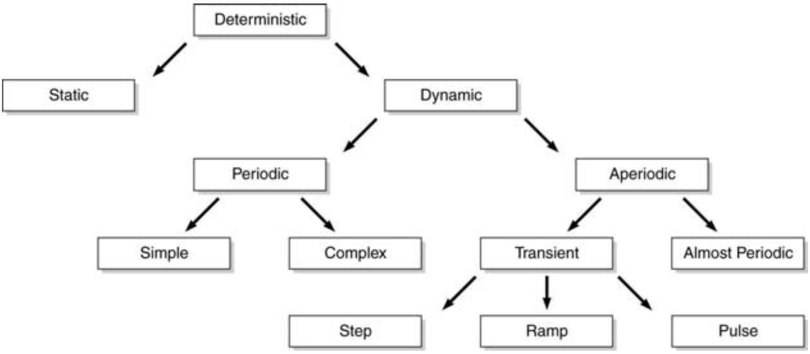

FIGURE 9.2
Deterministic signal subdivisions (adapted from [2]).

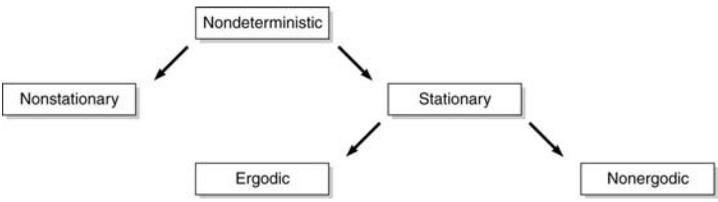

FIGURE 9.3
Nondeterministic signal subdivisions (adapted from [2]).

when only $\mu_x(t_1) = \mu_x$ and $R_x(t_1, t_1 + \tau) = R_x(\tau)$, that is, when only the signal's ensemble mean and autocorrelation function are time invariant. In a more restrictive sense, if all other ensemble higher-order moments and joint moments (see Chapter 5 for more about moments) also are time invariant, the signal is **strongly stationary** (stationary in the *strict* sense). So, the term *stationary* means that each of a signal's ensemble-averaged statistical properties are constant with respect to t_1. It does *not* mean that the amplitude of the signal is constant over time. In fact, a random signal is never completely stationary in time!

For a single time history, the temporal mean value, μ_x, and the temporal autocorrelation coefficient, $R_x(\tau)$, are

$$\mu_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)dt \tag{9.4}$$

and

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)x(t + \tau)dt. \tag{9.5}$$

For most stationary data, the ensemble averages at an arbitrary time, t_1,

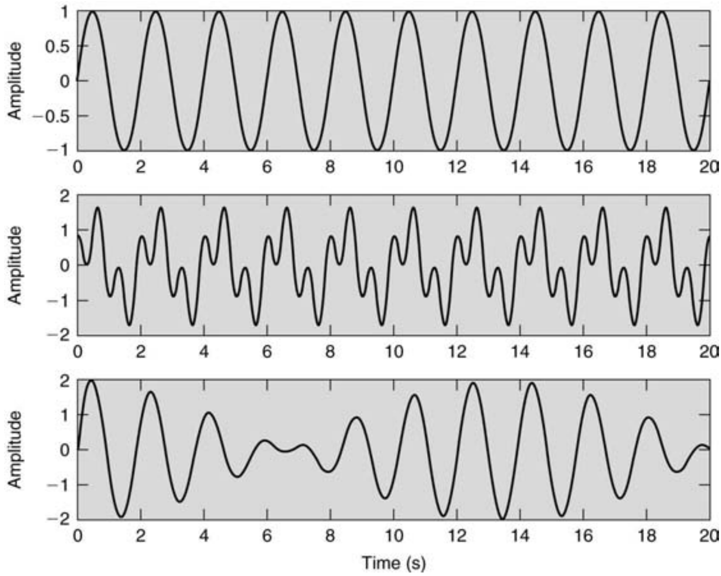

FIGURE 9.4
Various signals comprised of sines and cosines.

will equal their corresponding temporal averages computed for an arbitrary single time history in the ensemble. When this is true, the signal is **ergodic**. If the signal is periodic, then the limits in Equations 9.4 and 9.5 do not exist because averaging over one time period is sufficient. Ergodic signals are important because all of their properties can be determined by performing time averages over a *single* time history record. This greatly simplifies data acquisition and reduction. Most random signals representing stationary physical phenomena are ergodic.

A finite record of data of an ergodic random process can be used in conjunction with probabilistic methods to quantify the statistical properties of an underlying process. For example, it can be used to determine a random variable's true mean value within a certain confidence limit. These methods also can be applied to deterministic signals, which are considered next.

9.3 Signal Variables

Most waveforms can be written in terms of sums of sines and cosines, as will be shown later in Section 9.5. Before examining more complex waveform expressions, the variables involved in simple waveform expressions must be defined. This can be done by examining the following expression for a simple, periodic sine function,

$$y(t) = C\sin(n\omega t + \phi) = C\sin(2\pi n f t + \phi), \tag{9.6}$$

in which the argument of the sine is in units of radians. The variables and their units (given in brackets) are as follows:

- C: amplitude [units of $y(t)$]
- n: number of cycles [dimensionless]
- ω: *circular* frequency [rad/s]
- f: *cyclic* frequency [cycles/s = Hz]
- t: time [s]
- T: period ($= 2\pi/\omega = 1/f$) [s/cycle]
- ϕ: phase [rad] where $\phi = 2\pi(t/T) = 2\pi(\theta^\circ/360^\circ)$

Also note that 2π rad = 1 cycle = 360° and $\sin(\omega t + \pi/2) = \cos(\omega t)$. The top plot in Figure 9.4 displays the signal $y(t) = \sin(\pi t)$. Its period equals $2\pi/\pi$ = 2 s, as seen in the plot. The above definitions can be applied readily to determine the frequencies of a periodic signal, as in the following example.

Example Problem 9.1

Statement: Determine the circular and cyclic frequencies for the signal $y(t) = 10\sin(628t)$.

Solution: Using the above definitions,

circular frequency, $\omega = 628$ rad/s (assuming $n = 1$ cycle), and

cyclic frequency, $f = \omega/2\pi = 628/2\pi = 100$ cycles/s= 100 Hz.

When various sine and cosine waveforms are combined by addition, more complex waveforms result. Such waveforms occur in many practical situations. For example, the differential equations describing the behavior of many systems have sine and cosine solutions of the form

$$y(t) = A\cos(\omega t) + B\sin(\omega t). \tag{9.7}$$

By introducing the phase angle, ϕ, $y(t)$ can be expressed as either a cosine function,

$$y(t) = C\cos(\omega t - \phi), \tag{9.8}$$

or a sine function,

$$y(t) = C\sin(\omega t - \phi + \pi/2) = C\sin(\omega t + \phi^*), \tag{9.9}$$

where C, ϕ, and ϕ^* are given by

$$C = \sqrt{A^2 + B^2}, \tag{9.10}$$

$$\phi = \tan^{-1}(B/A), \tag{9.11}$$

and

$$\phi^* = \tan^{-1}(A/B), \tag{9.12}$$

noting that $\phi^* = (\pi/2) - \phi$. Reducing the waveform in Equation 9.7 to either Equation 9.8 or Equation 9.9 often is useful in interpreting results. The middle plot in Figure 9.4 shows the signal $y(t) = \sin(\pi t) + 0.8\cos(3\pi t)$. This signal is complex and has two frequencies, $\omega_1 = \pi$ and $\omega_2 = 3\pi$ rad/s. This leads to two periods, $T_1 = 2$ s and $T_2 = 2/3$ s. Because $T_1 = 3T_2$, the period T_1 will contain one cycle of $\sin(\omega_1 t)$ and three periods of $0.8\cos(\omega_2 t)$. So, $T_2 = 2$ s is the fundamental period of this complex signal. In general, the fundamental period of a complex signal will be the least common multiple of the contributory periods.

An interesting situation arises when two waves of equal amplitude and nearly equal frequencies are added. The resulting wave exhibits a relatively slow beat with a frequency called the **beat frequency**. In general, the sum of two sine waves of frequencies, f and $f + \Delta f$, combines trigonometrically to yield a signal whose amplitude is modulated as the $\cos(\Delta f/2)$. The frequency $\Delta f/2$ is defined conventionally as the beat frequency. An example of the resultant beating for the signal $y(t) = \sin(\pi t) + \sin(1.15\pi t)$ is displayed in the bottom plot of Figure 9.4. As can be seen, the signal repeats itself every 13.33 s. This corresponds to a cyclic frequency of 0.075 Hz, which equals $\Delta f/2$ (0.15/2). The phenomenon of producing a signal (wave) having a new frequency from the mixing of two signals (waves) is called **heterodyning** and is used in tuning musical instruments and in laser-Doppler velocimeters.

9.4 Signal Statistical Parameters

Signals can be either continuous in time or discrete. Discrete signals usually arise from the digitization of a continuous signal, to be discussed in Chapter 10, and from sample-to-sample experiments, which were considered in Chapter 7. A large number of statistical parameters can be determined from either continuous or discrete signal information. The parameters most frequently of interest are the signal's mean, variance, standard deviation, and rms. For continuous signals, these parameters are computed from integrals

Quantity	Continuous	Discrete
Mean	$\bar{x} = \frac{1}{T}\int_0^T x(t)dt$	$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$
Variance	$S_x^2 = \frac{1}{T}\int_0^T [x(t) - \bar{x}]^2 dt$	$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}[x_i - \bar{x}]^2$
Standard Deviation	$S_x = \sqrt{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^2 dt}$	$S_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}[x_i - \bar{x}]^2}$
rms	$x_{rms} = \sqrt{\frac{1}{T}\int_0^T x(t)^2 dt}$	$x_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} x_i^2}$

TABLE 9.1
Statistical parameters for continuous and discrete signals.

of the signal over time. For discrete signals, these parameters are determined from summations over the number of samples. The expressions for these properties are presented in Table 9.1. Note that as $T \to \infty$ or $N \to \infty$, the statistical parameter values approach the true values of the underlying process.

The choice of a time period that is used to determine the statistical parameters of a signal, called the **signal sample period**, depends upon the type of waveform. When the waveform is periodic, either simple or complex, the signal sample period should be the fundamental period. When the waveform is almost periodic or nondeterministic, no single signal sample period will produce exact results. For this situation, it is best to keep increasing the signal's sample period until the statistical parameter values of interest become constant to within acceptable limits.

Determining an appropriate sample period is not always straightforward. The values of the running mean, variance, skewness, and kurtosis of two data samples are shown in Figure 9.5. The adjective *running* implies that the value of a statistical moment is averaged over the time period from an initial time to each time of interest. The first sample, indicated by solid curves, was drawn randomly from a normal population having a mean value of 3.0 and a standard deviation of 0.5. The second sample, indicated by dotted curves, was the same as the first but with an additional amplitude decrease in time equal to 0.001/s. The mean of the first sample reaches its final value at approximately 100 s. The mean of the second sample exhibits a decrease in time, which is linear after approximately 100 s. The variance, skewness,

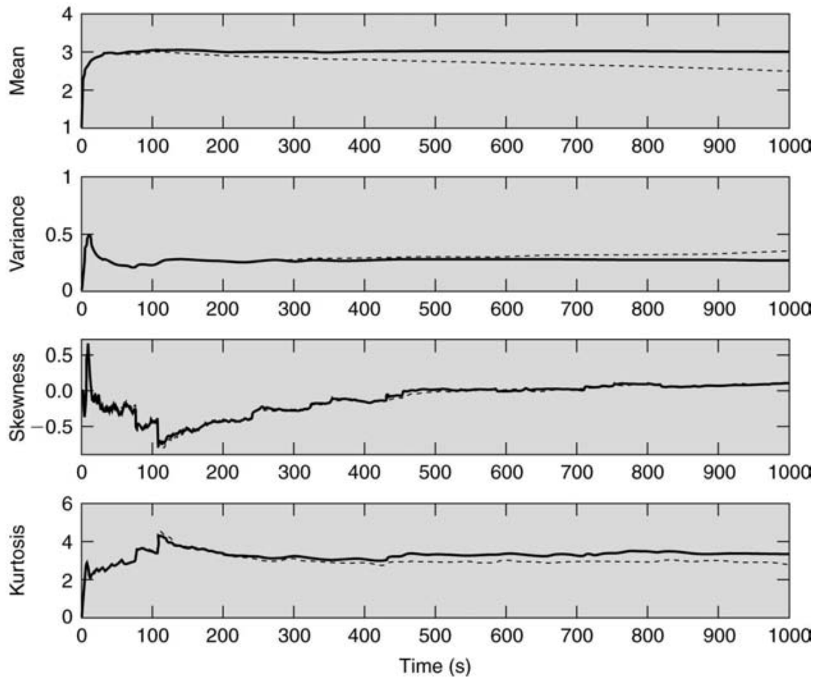

FIGURE 9.5
Statistical properties versus sample time.

and kurtosis values of both samples vary with respect to sample time and between samples during most of the entire sample time. The variances of the two samples agree up to approximately 300 s. Then, they deviate from one another because of the second sample's mean value decrease in time. This example illustrates the complexity in determining an appropriate sample time, especially if the signal being sampled has a gradual change in time over the sample period in addition to short-time fluctuations.

Sometimes it is important to examine the fluctuating component of a signal. The average value of a signal (its DC component) can be subtracted from the original signal to reveal more clearly the signal's fluctuating behavior (its AC component). This is shown in Figure 9.6, in which the left plot is the complete signal (DC plus AC components), the middle plot is the DC component and the AC component, each shown separately, and the right plot is the AC component amplified 10 times.

The concepts of the mean, variance, and standard deviation were presented in Chapter 5. The **root mean square (rms)** is another important statistical parameter. It is defined as the positive square root of the mean of the squares. Its continuous and discrete representations are presented in Table 9.1. The rms characterizes the dynamic portion (AC component) of the

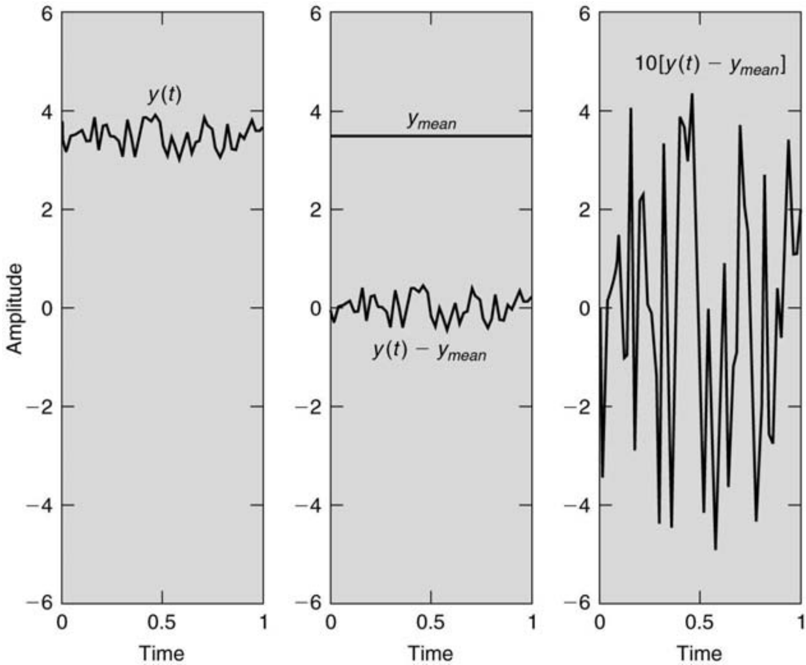

FIGURE 9.6
Subtraction of the mean value from a signal.

signal and the mean characterizes its static portion (DC component). The magnitudes of these components for a typical signal are shown in Figure 9.7. When no fluctuation is present in the signal, $x(t)$ is constant and equal to its mean value, $\bar{x}$. So, $x_{rms} \geq \bar{x}$ always. x_{rms}^2 is the temporal average of the square of the amplitude of x.

The following two applications of the rms concept show its utility:

1. The total energy dissipated over a period of time by a resistor in a circuit is

$$E_T = \int_{t_1}^{t_2} P(t)dt = R\int_{t_1}^{t_2} [I(t)]^2 dt = R(t_2 - t_1)I_{rms}^2, \tag{9.13}$$

   where

$$I_{rms}^2 = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [I(t)]^2 dt. \tag{9.14}$$

2. The temporal-averaged kinetic energy per unit volume in a fluid at a point in a flow is

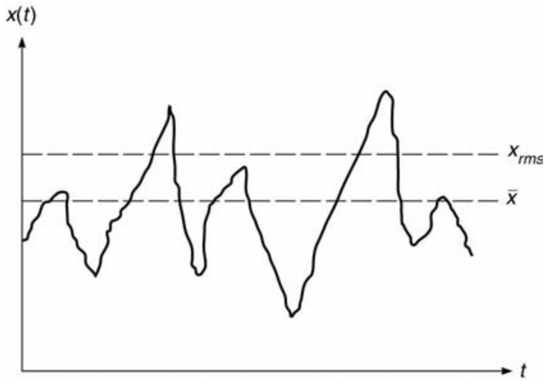

FIGURE 9.7
A signal showing its mean and rms values.

$$\bar{E} = \frac{\rho}{2(t_2 - t_1)} \int_{t_1}^{t_2} [U(t)]^2 dt = \frac{1}{2}\rho U_{rms}^2, \tag{9.15}$$

where

$$U_{rms}^2 = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [U(t)]^2 dt. \tag{9.16}$$

Sometimes, the term *rms* refers to the rms of the *fluctuating* component of the signal and *not* to the rms of the signal itself. For example, the fluctuating component of a fluid velocity, $u(t)$, can be written as the difference between a total velocity, $U(t)$, and a mean velocity, $\bar{U}(t)$, as

$$u(t) = U(t) - \bar{U}(t). \tag{9.17}$$

So, the rms of the fluctuating component is

$$u_{rms} = \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \{U(t) - \bar{U}(t)\}^2 dt\right]^{1/2}, \tag{9.18}$$

where

$$\bar{U}(t) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} U(t) dt. \tag{9.19}$$

By comparing Equations 9.16 and 9.18, it is evident that $U_{rms} \neq u_{rms}$.

Example Problem 9.2

Statement: Determine the rms of the ramp function $y(t) = A(t/T)$ in which A is the amplitude and T is the period.

Solution: Because $y(t)$ is a deterministic periodic function, the rms needs to be computed for only one period, from $t = 0$ to $t = T$. Application of the rms equation from Table 9.1 for $y(t)$, which is a continuous signal, yields

$$y_{rms} = [\frac{A^2}{T^2(t_2 - t_1)} \int_{t_1}^{t_2} t^2 dt]^{1/2} = \frac{A^2(t_2^3 - t_1^3)}{3T^2(t_2 - t_1)}.$$

For $t_1 = 0$ and $t_2 = T$, the rms becomes

$$y_{rms} = \frac{A}{\sqrt{3}}.$$

What is $\bar{y}(t)$? (Answer: $A/2$) What is the rms of a sine wave of amplitude A? (Answer: $A/\sqrt{2}$)

9.5 Fourier Series of a Periodic Signal

Before considering the Fourier series, the definition of orthogonality must be examined. The **inner product** (dot product), (x, y), of two real-valued functions $x(t)$ and $y(t)$ over the interval $a \leq t \leq b$ is defined as

$$(x, y) = \int_a^b x(t)y(t)dt. \tag{9.20}$$

If $(x, y) = 0$ over that interval, then the functions x and y are **orthogonal** in the interval. If each *distinct* pair of functions in a set of functions is orthogonal, then the set of functions is **mutually orthogonal**.

For example, the set of functions $\sin(2\pi mt/T)$ and $\cos(2\pi mt/T)$, $m = 1, 2, ...$, form one distinct pair and are mutually orthogonal because

$$\int_{-T/2}^{T/2} \sin(2\pi mt/T)\cos(2\pi nt/T)dt = 0 \text{ for all } m, n. \tag{9.21}$$

Also, these functions satisfy the other orthogonality relations

$$y(t) = \int_{-T/2}^{T/2} \cos(2\pi mt/T)\cos(2\pi nt/T)dt = \begin{cases} 0 & m \neq n \\ T & m = n \end{cases} \tag{9.22}$$

and

$$y(t) = \int_{-T/2}^{T/2} \sin(2\pi mt/T)\sin(2\pi nt/T)dt = \begin{cases} 0 & m \neq n \\ T & m = n. \end{cases} \tag{9.23}$$

Knowing these facts is useful when performing certain integrals, such as those that occur when determining the Fourier coefficients.

Fourier analysis and synthesis, named after Jean-Baptiste-Joseph Fourier (1768-1830), a French mathematician, now can be examined. Fourier showed that the temperature distribution through a body could be represented by a series of harmonically related sinusoids. The mathematical theory for this, however, actually was developed by others [3]. Fourier methods allow complex signals to be approximated in terms of a series of sines and cosines. This is called the trigonometric **Fourier series**. The Fourier trigonometric series that represents a signal of period T can be expressed as

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left[\frac{2\pi nt}{T}\right] + B_n \sin\left[\frac{2\pi nt}{T}\right] \right), \tag{9.24}$$

where

$$A_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)dt, \tag{9.25}$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos\left(\frac{2\pi nt}{T}\right) dt \quad n = 1, 2, ..., \tag{9.26}$$

and

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin\left(\frac{2\pi nt}{T}\right) dt \quad n = 1, 2, ... \tag{9.27}$$

The frequencies associated with the sines and cosines are integer multiples (**n-th harmonics**) of the fundamental frequency. The **fundamental** or **primary** frequency, the first harmonic, is denoted by $n = 1$, the second harmonic by $n = 2$, the third harmonic by $n = 3$, and so on. A_0 is twice the average of $y(t)$ over one period. A_n and B_n are called the **Fourier coefficients** of the Fourier amplitudes. The expression for A_n can be determined by multiplying both sides of the original series expression for $y(t)$ by $\cos(2\pi nt/T)$, then integrating over one period from $t = -T/2$ to $t = T/2$. The expression for B_n is found similarly, but instead, by multiplying by $\sin(2\pi nt/T)$. This is called Fourier's trick.

The procedure by which the Fourier amplitudes for any specified $y(t)$ are found is called **Fourier analysis**. Fourier analysis is the analog of a prism that separates white light (a complex signal) into colors (simple periodic sine functions). **Fourier synthesis** is the reverse procedure by which $y(t)$ is constructed from a series of appropriately weighted sines and cosines. The Fourier synthesis of a signal is useful because the amplitude and frequency components of the signal can be identified.

A Fourier series representation of $y(t)$ exists if $y(t)$ satisfies the following Dirichlet conditions:

1. $y(t)$ has a finite number of discontinuities within the period T (it is piece-wise differentiable).

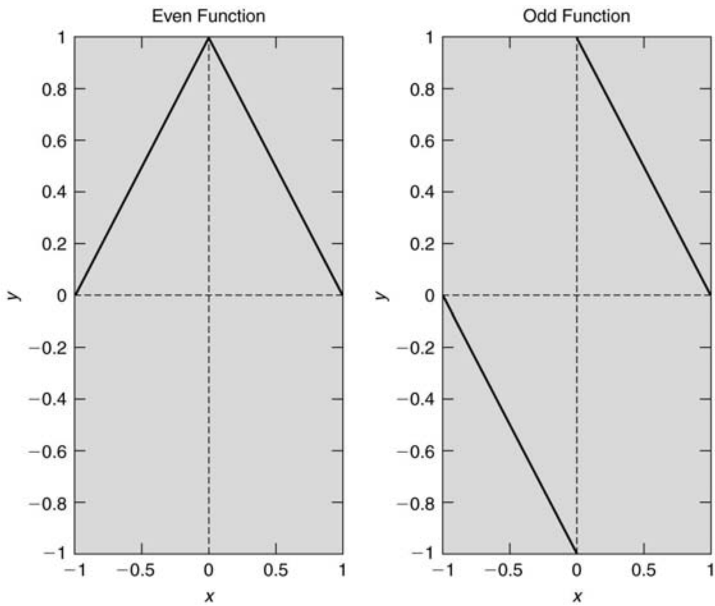

FIGURE 9.8
Even and odd functions.

2. $y(t)$ has a finite average value.
3. $y(t)$ has a finite number of relative maxima and minima within the period T.

If these conditions are met, then the series converges to $y(t)$ at the values of t where $y(t)$ is continuous and converges to the mean of $y(t^+)$ and $y(t^-)$ at a finite discontinuity. Fortunately, these conditions hold for most situations.

Recall that a periodic function with period T satisfies $y(t+T) = y(t)$ for all t. It follows that if $y(t)$ is an integrable periodic function with a period T, then the integral of $y(t)$ over *any* interval of length T has the same value. Hence, the limits from $-T/2$ to $T/2$ of the Fourier coefficient integrals can be replaced by, for example, from 0 to T or from $-T/4$ to $3T/4$. Changing these limits sometimes simplifies the integration procedure.

The process of arriving at the Fourier coefficients also can be simplified by examining whether the integrands are either even or odd functions. Example even and odd functions are shown in Figure 9.8. If $y(t)$ is an **even function**, where it is symmetric about the y-axis, then $g(x) = g(-x)$. Thus,

$$\int_{-T}^{T} g(x)dx = 2\int_{0}^{T} g(x)dx. \qquad (9.28)$$

The cosine is an even function. Likewise, if $y(t)$ is an **odd function**, where it is symmetric about the origin, then $g(x) = -g(-x)$. So,

$$\int_{-T}^{T} g(x)dx = 0. \tag{9.29}$$

The sine is an odd function. Other properties of even and odd functions include the following:

1. The sum, difference, product, or quotient of two even functions is even.

2. The sum or difference of two odd functions is odd.

3. The product or quotient of two odd functions is even.

4. The product or quotient of an even function and an odd function is odd.

5. The sum or difference of an even function and an odd function is neither even nor odd, unless one of the functions is identically zero.

6. A general function can be decomposed into a sum of even plus odd functions.

From these properties, Equation 9.24 and the Fourier coefficient equations, it follows that, when $y(t)$ is an even periodic function, $B_n = 0$ and $y(t)$ has the Fourier series

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\left[\frac{2\pi nt}{T}\right]\right). \tag{9.30}$$

This is called the **Fourier cosine series**. Further, when $y(t)$ is an odd periodic function, $A_0 = A_n = 0$ and $y(t)$ has the Fourier series

$$y(t) = \sum_{n=1}^{\infty}\left(B_n \sin\left[\frac{2\pi nt}{T}\right]\right). \tag{9.31}$$

This is called the **Fourier sine series**.

Example Problem 9.3

Statement: Find the frequency spectrum of the step function

$$y(t) = \begin{cases} -A & -\pi \leq t < 0 \\ +A & 0 \leq t < \pi \end{cases}$$

Solution: This is an odd function, therefore $A_0 = A_n = 0$.

$$
\begin{aligned}
B_n &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin\left(\frac{2\pi n t}{T}\right) dt \\
&\quad \text{note} : \omega = \frac{2\pi}{T} \text{ where } T = 2\pi \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-A)\sin(nt)dt + \int_{0}^{\pi} (A)\sin(nt)dt\right] \\
&= \frac{1}{\pi}\left[\left[\frac{A}{n}\cos(nt)\right]_{-\pi}^{0} - \left[\frac{A}{n}\cos(nt)\right]_{0}^{\pi}\right] \\
&= \frac{A}{n\pi}\{1 - \cos(-n\pi) - \cos(n\pi) + 1\} \\
&= \frac{2A}{n\pi}[1 - \cos(n\pi)] \\
&= 4A/n\pi \text{ for } n \text{ odd} \\
&= 0 \text{ for } n \text{ even.}
\end{aligned}
$$

Note that B_n involves only n and constants.

$$
\begin{aligned}
\Rightarrow y(t) &= \sum_{n=1}^{\infty} B_n \sin(nt) \\
&= \sum_{n=1}^{\infty} \frac{2A}{n\pi}[1 - \cos(n\pi)]\sin(nt) \\
&= \frac{4A}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}[\frac{1}{n}]\sin(2\pi n f t) \\
&= \frac{4A}{\pi}[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots].
\end{aligned}
$$

$y(t)$ involves both n and t. The frequencies of each of the sine terms are 1, 3, 5, ..., in units of ω (rad/s), or $\frac{1}{2\pi}$, $\frac{3}{2\pi}$, $\frac{5}{2\pi}$, ..., in units of f (cycles/s = Hz). Generally, this can be written as $f_n = (2n-1)f_1$. Likewise the corresponding amplitudes can be expressed as $A_n = \frac{A_1}{(2n-1)}$, where $A_1 = 4A/\pi$. $y(t)$ is shown in Figure 9.10 for three different partial sums for $A = 5$.

The contributions of each of the harmonics to the amplitude of the square wave are illustrated in Figure 9.9. The square wave shown along the back plane is the sum of the first 500 harmonics. Only the first five harmonics are given in the figure. The decreasing amplitude and increasing frequency contributions of the next higher harmonic tend to fill in the contributions of the previously summed harmonics such that the resulting wave approaches a square wave.

The partial Fourier series sums of the step function are shown in Figure 9.10 for $N = 1$, 10, and 500. Clearly, the more terms that are included in the sum, the closer the sum approximates the actual step function. Relatively small fluctuations at the end of the step can be seen, especially in the $N = 500$ sum. This is known as the Gibbs phenomenon. The inclusion of more terms in the sum will attenuate these fluctuations but never completely eliminate them.

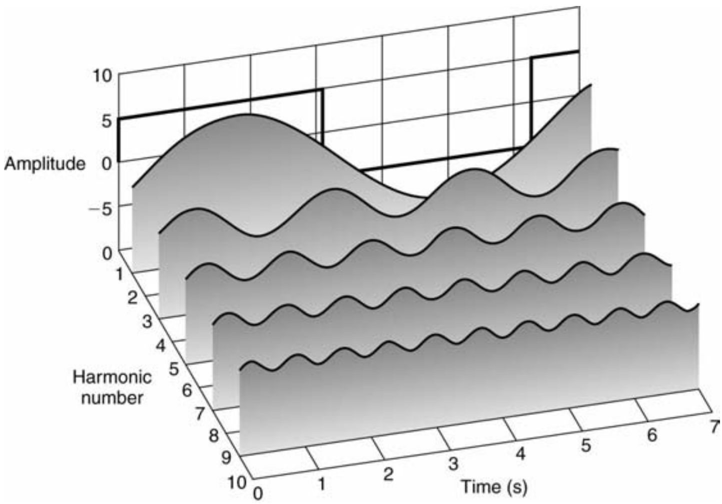

FIGURE 9.9
Contributions of the first five harmonics to the Fourier series of a step function.

A plot of amplitude versus frequency can be constructed for a square wave, as presented in Figure 9.11 for the first eight harmonics. This figure illustrates in another way that the Fourier series representation of a square wave consists of multiple frequencies of decreasing amplitudes.

Example Problem 9.4

Statement: Find the frequency spectrum of the ramp function:

$$y(t) = \begin{cases} 2At & 0 \le t < 1/2 \\ 0 & 1/2 \le t < 1 \end{cases}$$

with $T = 1$ s.

Solution: This function is neither even nor odd.

$$A_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)dt = \frac{2}{1}\left[\int_{-0.5}^{0} 0dt + \int_{0}^{0.5} 2Atdt\right] = 4A\left.\frac{t^2}{2}\right|_0^{0.5} = \frac{A}{2}$$

$$\begin{aligned} A_n &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos\frac{2\pi nt}{T}dt \\ &= 2\int_0^{0.5} 2At\cos\frac{2\pi nt}{T}dt \\ &= 4A\left.\left\{\frac{\cos(2\pi nt/T)}{(2\pi n/T)^2} + \frac{t\sin(2\pi nt/T)}{(2\pi n/T)}\right\}\right|_0^{0.5} \end{aligned}$$

because $\int t\cos mtdt = \frac{1}{m^2}\cos mt + \frac{t}{m}\sin mt$ (from integration by parts). Continuing,

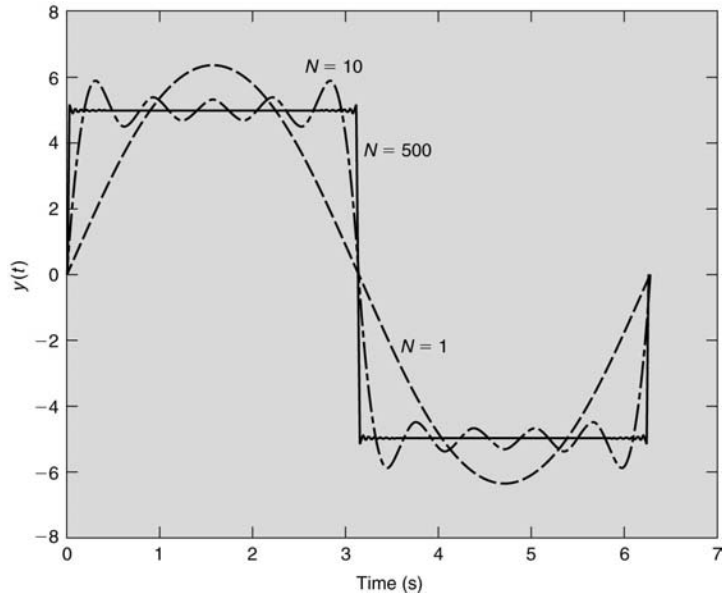

FIGURE 9.10
Partial Fourier series sums for the step function.

$$\begin{aligned} A_n &= 4A\left\{\frac{\cos n\pi - \cos 0}{(2\pi n/T)^2} + \frac{0.5\sin n\pi - 0}{(2\pi n/T)}\right\} \\ &= (4A)\left(\frac{T^2}{4\pi^2 n^2}\right)(\cos n\pi - 1) = \frac{A}{\pi^2 n^2}(\cos n\pi - 1). \end{aligned}$$

Further,

$$\begin{aligned} B_n &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin\frac{2\pi nt}{T}dt \\ &= 2\int_0^{0.5} 2At\sin\frac{2\pi nt}{T}dt \\ &= 4A\left\{\frac{\sin(2\pi nt/T)}{(2\pi n/T)^2} - \frac{t\cos(2\pi nt/T)}{(2\pi n/T)}\right\}\Bigg|_0^{0.5} \end{aligned}$$

because $\int t\sin mtdt = \frac{1}{m^2}\sin mt - \frac{t}{m}\cos mt$ (from integration by parts). So,

$$\begin{aligned} B_n &= 4A\left\{\frac{\sin n\pi - \sin 0}{(2\pi n/T)^2} + \frac{0.5\cos n\pi - 0}{(2\pi n/T)}\right\} \\ &= (4A)\left(\frac{-0.5\cos n\pi}{(2\pi n/T)}\right) = \frac{-A}{\pi n}\cos n\pi. \end{aligned}$$

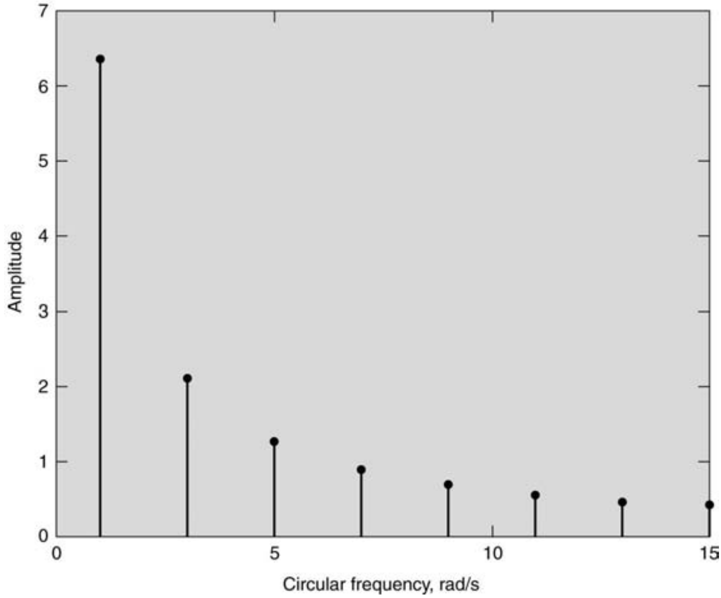

FIGURE 9.11
Amplitude spectrum for the first eight terms of the step function.

Thus,

$$\begin{aligned} y(t) &= A\left(\frac{1}{4} + \sum_{n=1}^{\infty}\left\{\frac{(\cos n\pi - 1)}{\pi^2 n^2}\cos\frac{2\pi nt}{T} - \frac{\cos n\pi}{\pi n}\sin\frac{2\pi nt}{T}\right\}\right) \\ &= A\left(\frac{1}{4} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left\{\frac{(-1+(-1)^n)}{\pi n}\cos\frac{2\pi nt}{T} - (-1)^n\sin\frac{2\pi nt}{T}\right\}\right). \end{aligned}$$

This $y(t)$ is shown (with $A = 1$) for three different partial sums in Figure 9.12.

9.6 Complex Numbers and Waves

Complex numbers can be used to simplify waveform notation. Waves, such as electromagnetic waves that are all around us, also can be expressed using complex notation.

The complex exponential function is defined as

$$\exp(z) = e^z = e^{(x+iy)} = e^x e^{iy} \equiv e^x(\cos y + i\sin y), \tag{9.32}$$

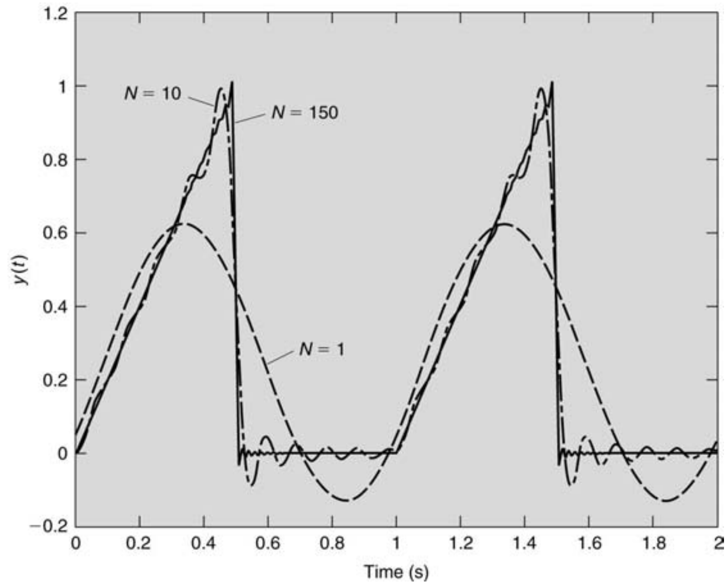

FIGURE 9.12
Three partial Fourier series sums for a ramp function.

where $z = x + iy$, with the complex number $i \equiv \sqrt{-1}$ and x and y as real numbers. The **complex conjugate** of z, denoted by z^*, is $z^* = x - iy$. The **modulus** or absolute value of z is given by $|z| = \sqrt{zz^*} = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2+y^2}$, which is a real number. Using Equation 9.32, the **Euler formula** results,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{9.33}$$

which also leads to

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{9.34}$$

The complex expressions for the sine and cosine functions can be found from Equations 9.33 and 9.34,

$$\cos\theta = \frac{1}{2}\left[\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right] \tag{9.35}$$

and

$$\sin\theta = \frac{1}{2i}\left[\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right]. \tag{9.36}$$

A wave can be represented by sine and cosine functions. Such representations are advantageous because [1] these functions are periodic, like many waves in nature, [2] linear math operations on them, such as integration and differentiation, yield waveforms of the same frequency but different amplitude and phase, and [3] they form complex waveforms that can be expressed in terms of Fourier series.

A wave can be represented by the general expression

$$y(t) = A_r \cos \frac{2\pi}{\lambda}(x - ct) + iA_i \sin \frac{2\pi}{\lambda}(x - ct), \tag{9.37}$$

in which A_r is the real amplitude, A_i the imaginary amplitude, x the distance, λ the wavelength, and c the wave speed. This expression can be written in another form, as

$$y(t) = A_r \cos(\kappa x - \omega t) + iA_i \sin(\kappa x - \omega t), \tag{9.38}$$

in which κ is the (angular) wave number and ω the circular frequency. The **wave number** denotes the number of waves in 2π units of length, where $\kappa = 2\pi/\lambda$. The wave speed is related to the wave number by $c = \omega/\kappa$. The cosine term represents the real part of the wave and the sine term the imaginary part. Further, the phase lag is defined as

$$\alpha = \tan^{-1}(A_i/A_r). \tag{9.39}$$

Equations 9.38 and 9.39 imply that

$$y(t) = \sqrt{A_r^2 + A_i^2} \cos(\kappa x - \omega t - \alpha), \tag{9.40}$$

in which the complex part of the wave manifests itself as a phase lag.

Example Problem 9.5

Statement: Determine the phase lag of the wave given by $z(t) = 20e^{i(4x-3t)}$.

Solution: The given wave equation, when expanded using Euler's formula, reveals that both the real and imaginary amplitudes equal 20. Thus, according to Equation 9.39, $\alpha = \tan^{-1}(20/20) = \pi/4$ radians.

9.7 Exponential Fourier Series

The trigonometric Fourier series can be simplified using complex number notation. Starting with the trigonometric Fourier series

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left[\frac{2\pi nt}{T}\right] + B_n \sin\left[\frac{2\pi nt}{T}\right] \right) \tag{9.41}$$

and substituting Equations 9.35 and 9.36 into Equation 9.41 yields

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{A_n}{2} \left[e^{in\omega_0 t} + e^{-in\omega_0 t} \right] + \frac{B_n}{2i} \left[e^{in\omega_0 t} - e^{-in\omega_0 t} \right] \right), \quad (9.42)$$

where $\theta = n\omega_0 t = 2\pi nt/T$. Rearranging the terms in this equation gives

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(e^{in\omega_0 t} \left[\frac{A_n}{2} - \frac{iB_n}{2} \right] + e^{-in\omega_0 t} \left[\frac{A_n}{2} + \frac{iB_n}{2} \right] \right), \quad (9.43)$$

noting $1/i = -i$.

Using the definitions

$$C_n \equiv \frac{A_n}{2} - \frac{iB_n}{2} \quad (9.44)$$

and

$$C_{-n} \equiv \frac{A_n}{2} + \frac{iB_n}{2}, \quad (9.45)$$

this equation can be simplified as follows:

$$\begin{aligned} y(t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(C_n e^{in\omega_0 t} + C_{-n} e^{-in\omega_0 t} \right) \\ &= \frac{A_0}{2} + \sum_{n=-\infty}^{-1} C_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{in\omega_0 t}. \quad (9.46) \end{aligned}$$

(9.47)

Combining the two summations yields

$$y(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}, \quad (9.48)$$

where $C_0 = A_0/2$.

The coefficients C_n can be found by multiplying the above equation by $e^{-in\omega_0 t}$ and then integrating from 0 to T. The integral of the right side equals zero except where $m = n$, which then yields the integral equal to T. Thus,

$$C_m = \frac{1}{T} \int_0^T y(t) e^{-im\omega_0 t} dt. \quad (9.49)$$

What has been done here is noteworthy. An expression (Equation 9.41) involving two coefficients, A_n and B_n with sums from $n = 1$ to $n = \infty$, was reduced to a simpler form having one coefficient, C_n, with a sum from $n = -\infty$ to $n = \infty$ (Equation 9.48). This illustrates the power of complex notation.

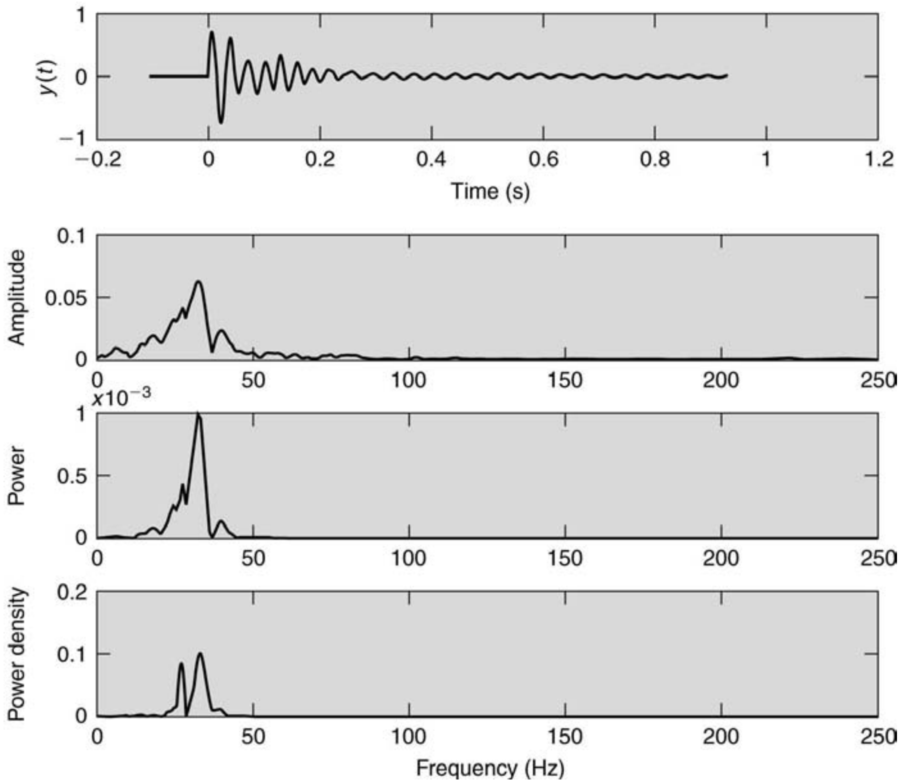

FIGURE 9.13
A signal and its spectra.

9.8 Spectral Representations

Additional information about the process represented by a signal can be gathered by displaying the signal's amplitude components versus their corresponding frequency components. This results in representations of the signal's amplitude, power, and power density versus frequency, which are termed the **amplitude spectrum**, **power spectrum**, and **power density spectrum**, respectively. These spectra can be determined from the Fourier series of the signal.

Consider the time average of the square of $y(t)$,

$$\left\langle [y(t)]^2 \right\rangle \equiv \frac{1}{T} \int_0^T [y(t)]^2 dt. \tag{9.50}$$

The square of $y(t)$ in terms of the Fourier complex exponential sums is

$$[y(t)]^2 = \left(\sum_{m=-\infty}^{\infty} C_m e^{im\omega_0 t} \right) \left(\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m C_n e^{i(m+n)\omega_0 t}. \tag{9.51}$$

If Equation 9.51 is substituted into Equation 9.50 and the integral on the right side is performed, the integral will equal zero unless $m = -n$, where, in that case, it will equal T. This immediately leads to

$$\begin{aligned} \langle [y(t)]^2 \rangle &= \sum_{n=-\infty}^{\infty} C_n C_{-n} = 2\sum_{n=0}^{\infty} \left(\frac{A_n}{2} - \frac{iB_n}{2} \right) \left(\frac{A_n}{2} + \frac{iB_n}{2} \right) \\ &= 2\sum_{n=0}^{\infty} \left(\frac{A_n^2}{4} + \frac{B_n^2}{4} \right) = 2\sum_{n=0}^{\infty} |C_n|^2 = \sum_{n=-\infty}^{\infty} |C_n|^2 , \end{aligned} \tag{9.52}$$

where $y(t)$ is assumed to be real and $\langle [y(t)]^2 \rangle$ is termed the **mean squared amplitude** or average power. Note that this equation is an approximation to the actual mean squared amplitude whenever the number of summed terms is finite. By comparing Equation 9.51 with the definition of the rms for a discrete signal (see Table 9.1), it can be seen that the power is simply the square of the rms. Recall also that the average power over a time interval equals the product of the total energy expended over that interval and the reciprocal of the time interval.

Equations 9.50 and 9.51 can be combined to yield Parseval's relation for continuous-time periodic signals

$$\frac{1}{T} \int_0^T [y(t)]^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2 . \tag{9.53}$$

This states that the total average power in a periodic signal equals the sum of the average powers of all its harmonic components.

The n-th amplitude equals $2\sqrt{|C_n|^2} = \sqrt{A_n^2 + B_n^2}$. The plot of $2\sqrt{|C_n|^2}$ versus the frequency is called the amplitude spectrum of the signal $y(t)$. The ordinate units are those of the amplitude. The plot of $|C_n|^2$ versus the frequency is called the power spectrum of the signal $y(t)$. The ordinate units are those of the amplitude squared. The absolute value squared of the C_n Fourier coefficient, which equals one-quarter of the amplitude squared, gives the amount of power associated with the n-th harmonic. Both spectra are two-sided because they involve summations on both sides of $n = 0$. They can be made one-sided by multiplying their values by two and then summing from $n = 0$ to ∞. The power density spectrum is the derivative of the power spectrum. Its ordinate units are those of amplitude squared divided by frequency. So, the integral of the power density spectrum over a particular frequency range yields the power contained in the signal in that range.

The spectra of a signal obtained by impulsively tapping a cantilever beam supported on its end, shown in Figure 9.13, displays a dominant frequency at approximately 30 Hz.

9.9 Continuous Fourier Transform

Fourier analysis of a periodic signal can be extended to an aperiodic signal by treating the aperiodic signal as a periodic signal with an infinite period. From Equation 9.24, the Fourier trigonometric series representation of a signal with zero mean ($A_0 = 0$) is

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty}\left(A_n \cos\left[\frac{2\pi nt}{T}\right] + B_n \sin\left[\frac{2\pi nt}{T}\right]\right) \\ &= \sum_{n=1}^{\infty}\left\{\frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos\left(\frac{2\pi nt}{T}\right)dt\right\}\cos\left(\frac{2\pi nt}{T}\right) \\ &\quad + \sum_{n=1}^{\infty}\left\{\frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin\left(\frac{2\pi nt}{T}\right)dt\right\}\sin\left(\frac{2\pi nt}{T}\right). \end{aligned} \tag{9.54}$$

Noting $\omega_n = 2\pi n/T$ and $\Delta\omega = 2\pi/T$, where $\Delta\omega$ is the spacing between adjacent harmonics, gives

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty}\left\{\frac{\Delta\omega}{\pi}\int_{-T/2}^{T/2} y(t)\cos(\omega_n t)dt\right\}\cos(\omega_n t) \\ &\quad + \sum_{n=1}^{\infty}\left\{\frac{\Delta\omega}{\pi}\int_{-T/2}^{T/2} y(t)\sin(\omega_n t)dt\right\}\sin(\omega_n t). \end{aligned} \tag{9.55}$$

As $T \to \infty$ and $\Delta\omega \to d\omega$, $\omega_n \to \omega$ and the summations become integrals with the limits $\omega = 0$ and $\omega = \infty$,

$$\begin{aligned} y(t) &= \int_0^{\infty}\left\{\frac{d\omega}{\pi}\int_{-\infty}^{+\infty} y(t)\cos(\omega t)dt\right\}\cos(\omega t) \\ &\quad + \int_0^{\infty}\left\{\frac{d\omega}{\pi}\int_{-\infty}^{+\infty} y(t)\sin(\omega t)dt\right\}\sin(\omega t). \end{aligned} \tag{9.56}$$

Equation 9.56 can be simplified by defining

$$A(\omega) = \int_{-\infty}^{+\infty} y(t)\cos(\omega t)dt \tag{9.57}$$

and

$$B(\omega) = \int_{-\infty}^{+\infty} y(t)\sin(\omega t)dt. \tag{9.58}$$

$A(\omega)$ and $B(\omega)$ are the components of the Fourier transform of $y(t)$. Note that $A(\omega)$ and $B(\omega)$ have units of y/ω, whereas A_n and B_n in Equations 9.26 and 9.27, respectively, have units of y. $A(\omega)$ is an even function of ω and $B(\omega)$ is an odd function of ω. Substituting these definitions into Equation 9.56 yields

$$\begin{aligned} y(t) &= \frac{1}{\pi}\int_0^{\infty} A(\omega)\cos(\omega t)d\omega + \frac{1}{\pi}\int_0^{\infty} B(\omega)\sin(\omega t)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)\cos(\omega t)d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\sin(\omega t)d\omega. \end{aligned} \tag{9.59}$$

Note that these integrals involve negative frequencies. The negative frequencies are simply a consequence of the mathematics and have no mystical significance.

The **complex Fourier coefficient** is defined as

$$Y(\omega) \equiv A(\omega) - iB(\omega). \tag{9.60}$$

Substituting the definitions of the Fourier coefficients gives

$$Y(\omega) = \int_{-\infty}^{+\infty} y(t)[\cos(\omega t) - i\sin(\omega t)]dt = \int_{-\infty}^{+\infty} y(t)e^{-i\omega t}dt. \tag{9.61}$$

This equation expresses that $Y(\omega)$ is the Fourier transform of $y(t)$. $|Y(\omega)|^2$ corresponds to the amount of power contained in the frequency range from ω to $\omega + d\omega$.

The inverse Fourier transform of $Y(\omega)$ can be developed. Note that

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty} A(\omega)\sin(\omega t)d\omega = 0, \tag{9.62}$$

because $A(\omega)$ and $\sin(\omega t)$ are orthogonal. Likewise,

$$\frac{-i}{2\pi}\int_{-\infty}^{+\infty} B(\omega)\cos(\omega t)d\omega = 0, \tag{9.63}$$

because $B(\omega)$ and $\cos(\omega t)$ are orthogonal. These integral terms can be added to those in Equation 9.59, which results in

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)\cos(\omega t)d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\sin(\omega t)d\omega \\ &+ \frac{i}{2\pi}\int_{-\infty}^{\infty} A(\omega)\sin(\omega t)d\omega - \frac{i}{2\pi}\int_{-\infty}^{\infty} B(\omega)\cos(\omega t)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [A(\omega) - iB(\omega)][\cos(\omega t) + i\sin(\omega t)]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}d\omega. \end{aligned} \tag{9.64}$$

Equations 9.61 and 9.64 form the **Fourier transform pair**, which consists of the **Fourier transform** of $y(t)$,

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt, \tag{9.65}$$

and the **inverse Fourier transform** of $Y(\omega)$,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}d\omega. \tag{9.66}$$

Equation 9.65 determines the amplitude-frequency characteristics of the signal $y(t)$ from its amplitude-time characteristics. Equation 9.66 constructs the amplitude-time characteristics of the signal from its amplitude-frequency characteristics. Taking the inverse Fourier transform of the Fourier transform always should recover the original $y(t)$.

9.10 *Continuous Fourier Transform Properties

Many useful properties can be derived from the Fourier transform pair [3]. These properties are relevant to understanding how signals in the time domain are represented in the frequency domain. Examine the Fourier transform

$$Y(\omega) = 2\pi\delta(\omega - \omega_o), \tag{9.67}$$

where $\delta(x)$ denotes the delta function, which has the properties $\int_{-\infty}^{\infty}\delta(0) = 1$ and $\int_{-\infty}^{\infty}\delta(\neq 0) = 0$. Substitution of this expression into Equation 9.66 yields

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_o)e^{i\omega t}d\omega = e^{i\omega_o t}. \tag{9.68}$$

Thus, the Fourier transform of $y(t) = e^{i\omega_o t}$ is $2\pi\delta(\omega - \omega_o)$. This transform occurs in those involving sine and cosine functions, which are the foundations of the Fourier series.

$y(t)$	$Y(\omega)$	Property
$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$	linearity
$x(t - t_o)$	$X(\omega)e^{-i\omega t_o}$	time shifting
$x(t)e^{i\omega_o t}$	$X(\omega - \omega_o)$	frequency shifting
$x(at)$	$\frac{1}{\lvert a\rvert}X(\omega/a)$	time scaling
$\frac{dx(t)}{dt}$	$i\omega X(\omega)$	time differentiation
$\int_{-\infty}^{t} x(\tau)d\tau$	$\pi X(0)\delta(\omega) + \frac{1}{i\omega}X(\omega)$	integration
$x_e(t)$	$Re[X(\omega)] = a(\omega)$	even signal
$x_o(t)$	$iIm[X(\omega)] = iB(\omega)$	odd signal
$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$	multiplication
$x_1(t) * x_2(t)$	$X_1(\omega) + X_2(\omega)$	convolution

TABLE 9.2
Properties of the continuous Fourier transform.

As an example, consider the Fourier transform of $y(t) = \cos(\omega_o t)$ and, consequently, how that signal is represented in the frequency domain. Substituting the complex expression for the cosine function (Equation 9.35) into Equation 9.65 gives

$$\begin{aligned} Y(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty}[e^{i\omega_o t} + e^{-i\omega_o t}]e^{-i\omega t}dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty}[e^{i\omega_o t}e^{-i\omega t} + e^{-i\omega_o t}e^{-i\omega t}]dt \\ &= \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)], \end{aligned} \tag{9.69}$$

using Equations 9.67 and 9.68. This implies that the signal $y(t) = \cos(\omega_o t)$ in the time domain appears as impulses of amplitude π in the frequency domain at $\omega = -\omega_o$ and $\omega = \omega_o$.

In a similar manner, the Fourier transform of $y(t) = x(t)e^{i\omega_o t}$ becomes

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(t)e^{i\omega_o t}e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-i(\omega-\omega_o)t}dt \\ &= X(\omega - \omega_o). \end{aligned} \tag{9.70}$$

The multiplication of the signal $x(t)$ by $e^{i\omega_o t}$ is called **complex modulation**. Equation 9.70 implies that the Fourier transform of $x(t)e^{i\omega_o t}$ results in a frequency shift, from ω to $\omega - \omega_o$, in the frequency domain.

The multiplication of two functions in one domain is related to the convolution the two functions' transforms in the transformed domain. The **con-**

volution of the functions $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) = \int_0^t x_1(\tau)x_2(t-\tau)d\tau, \tag{9.71}$$

in which the $*$ denotes the convolution operator. This leads immediately to the multiplication and convolution properties of the Fourier transform that are presented in Table 9.2. These two properties are quite useful because one function often can be expressed as the product of two functions whose Fourier transforms or inverse transforms are known.

Example Problem 9.6

Statement: Determine $3 * \sin 2\omega$.

Solution: Let $X_1(\omega) = 3$ and $X_2(\omega) = \sin 2\omega$. Thus, $X_1(\tau) = 3$ and $X_2(\omega - \tau) = \sin 2(\omega - \tau)$. Applying Equation 9.71 gives

$$3 * \sin 2\omega = \int_0^\omega 3 \sin 2(\omega - \tau) d\tau = \frac{3}{2} \cos 2(\omega - \tau)\Big|_0^\omega = \frac{3}{2}\left[1 - \cos 2\omega\right].$$

9.11 Problem Topic Summary

Topic	Review Problems	Homework Problems
Signal Characteristics	5, 6, 7, 8, 9, 11, 12	3, 7, 13, 15
Signal Parameters	1, 2, 10	1, 2, 3, 4, 7, 9, 10, 14
Fourier Series	3, 4	5, 6, 7, 8, 11, 12, 14

TABLE 9.3
Chapter 9 Problem Summary

9.12 Review Problems

1. Consider the deterministic signal $y(t) = 3.8\sin(\omega t)$, where ω is the circular frequency. Determine the rms value of the signal to three decimal places.

2. Compute the rms of the dimensionless data set in the file `data10.dat`.

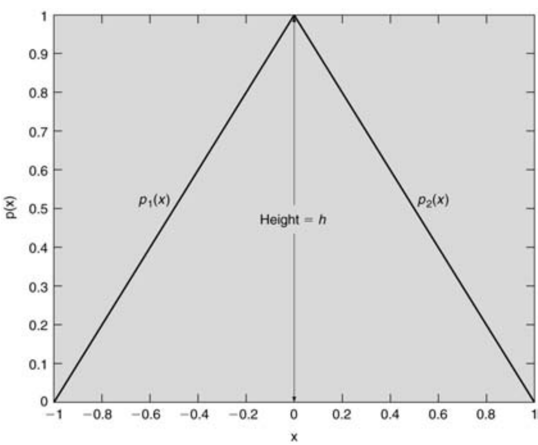

FIGURE 9.14
Triangular function.

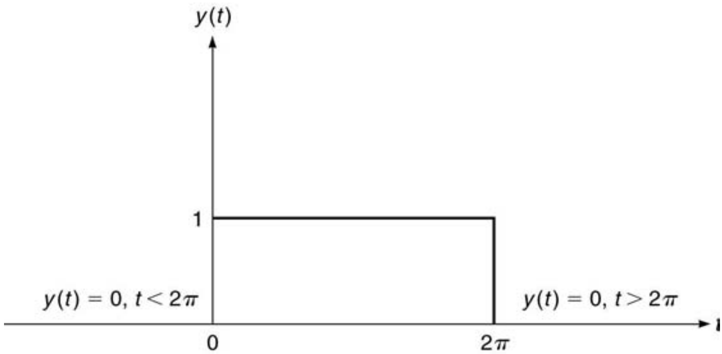

FIGURE 9.15
Rectangular function.

3. Find the third Fourier coefficient of the function pictured in Figure 9.14, where $h = 1$. (Note: Use $n = 3$ to find the desired coefficient.) Consider the function to be periodic. Respond to three significant figures.

4. What is the value of the power spectrum at the cyclic frequency 1 Hz for the function given by the Figure 9.15? Respond to three significant figures. (The desired result is achieved by representing the function on an infinite interval.)

5. Which one of the following functions is periodic? (a) $x(t) = 5\sin(2\pi t)$ or (b) $x(t) = \cos(2\pi t)\exp(-5t)$.

6. Which one of the following is true? A stationary random process must (a) be continuous, (b) be discrete, (c) be ergodic, (d) have ensemble averaged properties that are independent of time, or (e) have time averaged properties that are equal to the ensemble averaged properties.

7. Which of the following are true? An ergodic random process must (a) be discrete, (b) be continuous, (c) be stationary, (d) have ensemble averaged properties that are independent of time, or (e) have time averaged properties that are equal to the ensemble averaged properties.

8. Which of the following are true? A single time history record can be used to find all the statistical properties of a process if the process is (a) deterministic, (b) ergodic, (c) stationary, or (d) all of the above.

9. Which of the following are true? The autocorrelation function of a stationary random process (a) must decrease as $|\tau|$ increases, (b) is a function of $|\tau|$ only, (c) must approach a constant as $|\tau|$ increases, or (d) must always be non-negative.

10. Determine for the time period from 0 to $2T$ the rms value of a square wave of period T given by $y(t) = 0$ from 0 to $T/2$ and $y(t) = A$ from $T/2$ to T.

11. Which of the following functions are periodic? (a) $y(t) = 5\sin(5t) + 3\cos(5t)$, (b) $y(t) = 5\sin(5t)\, e^{\frac{t+2}{12}}$, (c) $y(t) = 5\sin(5t) + e^{\frac{t+2}{12}}$, (d) $y(t) = 15\sin(5t)\cos(5t)$.

12. A speed of a turbine shaft is 13 000 revolutions per minute. What are its cyclic frequency (in Hz), period (in s), and circular frequency (in rad/s)?

9.13 Homework Problems

1. Determine the autocorrelation of $x(t)$ for (a) $x(t) = c$, where c is a constant, (b) $x(t) = \sin(2\pi t)$, and (c) $x(t) = \cos(2\pi t)$.

2. Determine the average and rms values for the function $y(t) = 30 + 2\cos(6\pi t)$ over the following time periods: (a) 0 s to 0.1 s, (b) 0.4 s to 0.5 s, (c) 0 s to $\frac{1}{3}$ s, and (d) 0 s to 20 s.

3. Consider the deterministic signal $y(t) = 7\sin(4t)$ with t in units of seconds and 7 (the signal's amplitude) in units of volts. Determine the signal's (a) cyclic frequency, (b) circular frequency, (c) period for one cycle, (d) mean value, and (e) rms value. Put the correct units with each answer. Below are two integrals that may or may not be needed:

$$\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x)$$

and

$$\int \cos^2(x)dx = \frac{1}{2}x + \frac{1}{4}\sin(2x).$$

4. For the continuous periodic function $y(t) = y_1(t) - y_2(t)$, where $y_1(t) = A(t/T)^{1/2}$ and $y_2(t) = B(t/T)$, determine for one period (a) the mean value of $y(t)$ and (b) the rms of $y_1(t)$.

5. Determine the Fourier series for the period T of the function described by

$$y(t) = \frac{4At}{T} + A \text{ for } -\frac{T}{2} \le t \le 0$$

and

$$y(t) = \frac{-4At}{T} + A \text{ for } 0 \le t \le \frac{T}{2}.$$

Do this without using any computer programs or spreadsheets. Show all work. Then, on one graph, plot the three resulting series for 2, 10, and 50 terms along with the original function $y(t)$.

6. Determine the Fourier series of the function

$$y(t) = t \text{ for } -5 < t < 5.$$

(This function repeats itself every 10 units, such as from 5 to 15, 15 to 25, ...). Do this without using any computer programs or spreadsheets. Show all work. Then, on one graph, plot the three resulting series for 1, 2, and 3 terms along with the original function $y(t)$.

7. Consider the signal $y(t) = 2 + 4\sin(3\pi t) + 3\cos(3\pi t)$ with t in units of seconds. Determine (a) the fundamental frequency (in Hz) contained in the signal and (b) the mean value of $y(t)$ over the time period from 0 s to 2/3 s. Also (c) sketch the amplitude-frequency spectrum of $y(t)$.

8. For the Fourier series

$$y(t) = (20/\pi)[\sin(4\pi t/7) + 4\sin(8\pi t/7) + 3\sin(12\pi t/7) + 5\sin(16\pi t/7)],$$

determine the amplitude of the third harmonic.

9. Calculate the mean value of a rectified sine wave given by

$$y = |A \sin \tfrac{2\pi t}{T}|$$

during the time period $0 < t < 1000T$.

10. Determine the rms (in V) of the signal $y(t) = 7\sin(4t)$ where y is in units of V and t is in units of s. An integral that may be helpful is $\int \sin^2 ax dx = x/2 - (1/4a)\sin(2ax)$.

11. Determine the Fourier coefficients A_0, A_n, and B_n, and the trigonometric Fourier series for the function $y(t) = At$, where the function has a period of 2 s with $y(-1) = -A$ and $y(1) = A$.

12. Consider the following combination of sinusoidal inputs:

$$y(t) = \sin(t) + 2\cos(2t) + 3\sin(2t) + \cos(t).$$

(a) Rewrite this equation in terms of only cosine functions. (b) Rewrite this equation in terms of only sine functions. (c) What is the fundamental period of this combination of inputs?

13. Consider the signal

$$y(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t),$$

where $\omega_1 = 56/500$ rad/hr and ω_2 is 8 % greater in magnitude than ω_1. (a) What is the period of the corresponding slow beat in minutes (the formal definition of slow beat)? (b) What is the period at which the slow beat manifests itself in the output signal in minutes?

14. The following problems use the data file `signal.dat` that contains two columns of data, each with 5000 rows (the first column is the measured velocity in m/s, and the second column is the sample time in s). The velocities were measured behind an obstruction that contained several cables of different diameters. The data was taken over a period of 5 s at a sample rate of 1 kHz (1000 samples/s). Assume that the sample rate was fast enough such that the sampled signal represents the actual signal in terms of its amplitude and frequency. The following M-files may be useful: `propintime.m` and `sstol.m`. Write a program or spreadsheet for this problem. (a) Plot the velocities versus time *for the first 250 ms* using points (dots) for each data point. (b) Plot the running mean and running rms versus time. (c) Determine the times at which the running mean and also the running rms for them to remain within 1 %, 2 %, 3 %, and 4 % of their final values. Note that there will be different times for each running value for each percent tolerance.

15. Determine the rms of one period of a square wave in which $y(t) = 0$ from $t = 0$ to $t = 0.5$ and $y(t) = 2$ from $t > 0.5$ to $t = 1.0$.

Bibliography

[1] *Merriam-Webster OnLine Collegiate Dictionary* at http://www.m-w.com/.

[2] Bendat, J.S. and A.G. Piersol. 1986. *Random Data: Analysis and Measurement Procedures.* 2nd ed. New York: John Wiley and Sons.

[3] Oppenheim, A.V. and A.S. Willsky. 1997. *Signals & Systems.* 2nd ed. New Jersey: Prentice Hall.

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10

Signal Analysis

CONTENTS

For even the most stupid of men, by some instinct of nature, by himself and without any instruction (which is a remarkable thing), is convinced that the more observations have been made, the less danger there is of wandering from one's goal.

Jacob Bernoulli. 1713. *Ars Conjectandi.*

10.1 Chapter Overview

Today, most data is acquired and stored digitally. This format is advantageous because of relatively rapid acquisition rates and minimal storage requirements. Digital data acquisition, however, introduces errors. Fortunately, these can be minimized with some foresight. So, how are signal acquisition and analysis done digitally? What errors are introduced? How can these errors be minimized such that the acquired information truly represents that of the process under investigation? Such questions are addressed and answered in this chapter.

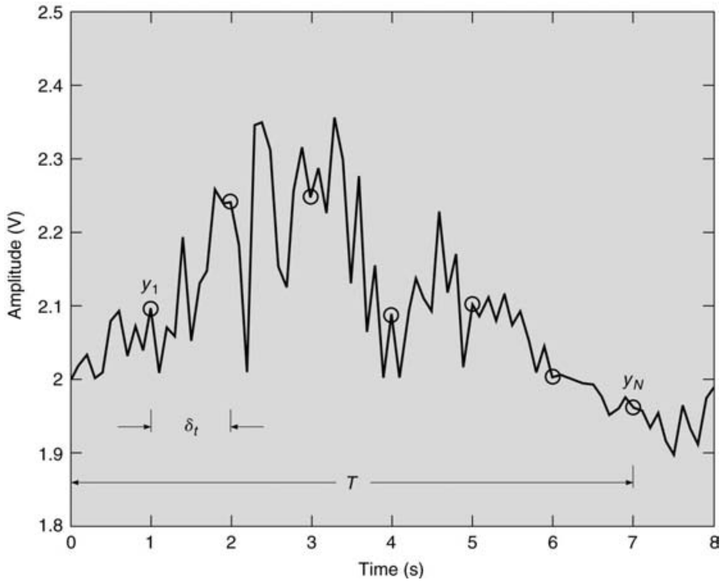

FIGURE 10.1
Discrete sampling of an analog signal.

10.2 Digital Sampling

Consider the analog signal, $y(t)$, shown in Figure 10.1 as a solid curve. This signal is sampled digitally over a period of T seconds at a rate of one sample every δt seconds. The resulting discrete signal, $y(r\delta t)$, is comprised of the analog signal's amplitude values y_1 through y_N at the times $r\delta t$, where $r = 1, 2, ..., N$ for N samples. The discrete signal is represented by circles in Figure 10.1. The accurate representation of the analog signal by the discrete signal depends upon a number of factors. These include, at a minimum, the frequency content of $y(t)$, the time-record length of the signal, $T = N\delta t$, and the frequency at which the signal is sampled, $f_s = 1/\delta t = N/T$.

Further assume that the signal contains frequencies ranging from 0 to W Hz, which implies that the signal's **bandwidth** is from 0 to W Hz. The minimum resolvable frequency, f_{min}, will be $1/T = 1/(N\delta t)$. If the sampling rate is chosen such that $f_s = 2W$, then, as will be seen shortly, the maximum resolvable frequency, f_{max}, will be $W = 1/(2\delta t)$. Thus, the number of discrete frequencies, N_f, that can be resolved from f_{min} to f_{max} will be

$$N_f = \frac{f_{max} - f_{min}}{\delta f} = \frac{1/(2\delta t) - 1/(N\delta t)}{1/(N\delta t)} = \frac{N}{2} - 1. \tag{10.1}$$

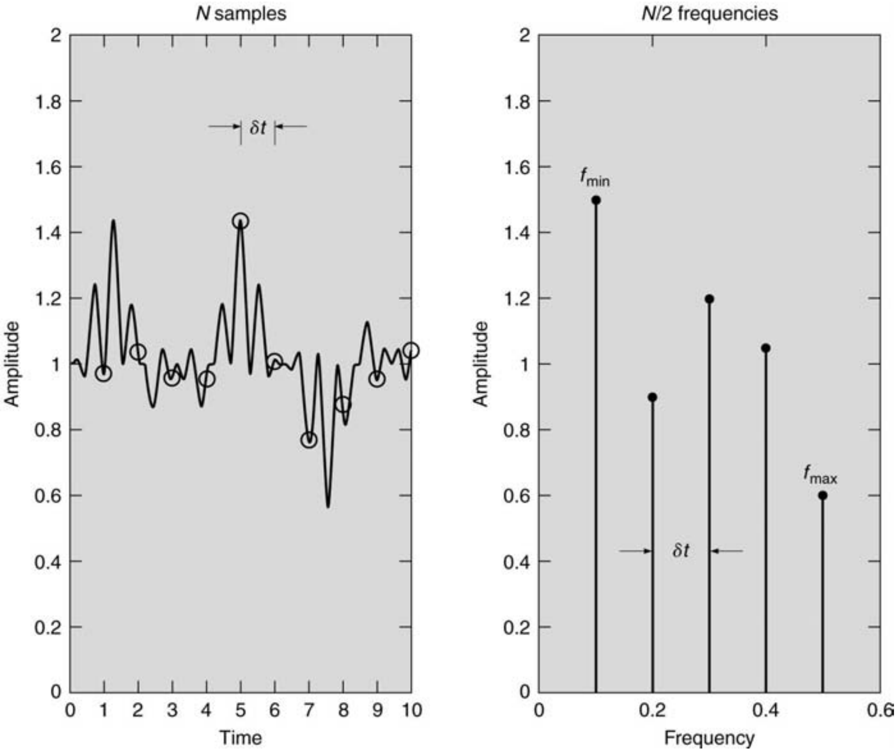

FIGURE 10.2
Amplitude-time-frequency mapping.

This implies that there will be $N/2$ discrete frequencies from f_{min} to *and including* f_{max}. This process is illustrated in Figure 10.2.

Situations arise which introduce errors into the acquired information. For example, if the sampling frequency is too low, the discrete signal will contain false or alias amplitudes at lower frequencies, which is termed **aliasing**. Further, if the total sample period is not an integer multiple of *all* of the signal's contributory periods, **amplitude ambiguity** will result. That is, false or ambiguous amplitudes will occur at frequencies that are immediately adjacent to the actual frequency. Thus, by not using the correct sampling frequency and sampling period, incorrect amplitudes and frequencies result. This, obviously, is undesirable.

How can these problems be avoided? Signal aliasing can be eliminated simply by choosing a sampling frequency, f_s, equal to at least twice the highest frequency, f_{max}, contained in the signal. However, it is difficult to avoid amplitude ambiguity. Its effect only can be minimized. This is accom-

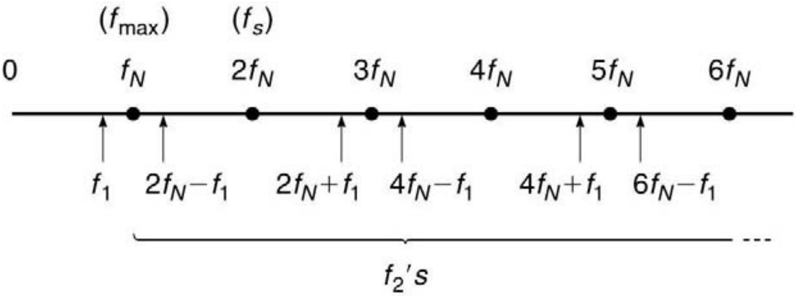

FIGURE 10.3
Frequency map to illustrate aliasing.

plished by reducing the magnitude of the signal at the beginning and the end of the sample period through a process called **windowing**.

To fully understand each of these effects, the discrete version of the Fourier transform must be considered. This transform yields the amplitude-frequency spectrum of the discrete data. The spectrum can be determined for the discrete representation of a known periodic signal, which best illustrates the effects of aliasing and amplitude ambiguity.

10.3 Aliasing

Ambiguities arise in the digitized signal's frequency content whenever the analog signal is not sampled at a high enough rate. Shannon's sampling theorem basically states that for aliasing not to occur, the signal should be sampled at a frequency which is greater than twice the maximum frequency contained in the signal, which often is termed the maximum frequency of interest. That is, $f_s > 2f_{max}$. At the sampling frequency $f_s = 2f_{max}$, f_{max} also is known as the Nyquist frequency, f_N. So, $f_s = 2f_N$.

To illustrate analytically how aliasing occurs, consider the two signals $y_1(t) = \cos(2\pi f_1 t)$ and $y_2(t) = \cos(2\pi f_2 t)$, in which f_2 is chosen subject to two conditions: [1] $f_2 = 2mf_N \pm f_1$ with $m = 1, 2, ...$, and [2] $f_2 > f_N$. These conditions yield specific f_2 frequencies above the Nyquist frequency, all of which alias down to the frequency f_1. The resulting frequencies are displayed on the frequency map shown in Figure 10.3.

Assume that these two periodic signals are sampled at δt time increments, r times. Then

$$y_1(t) = \cos(2\pi f_1 t) \text{ becomes } y_1(r\delta t) = \cos(2\pi r f_1/f_s), \tag{10.2}$$

and

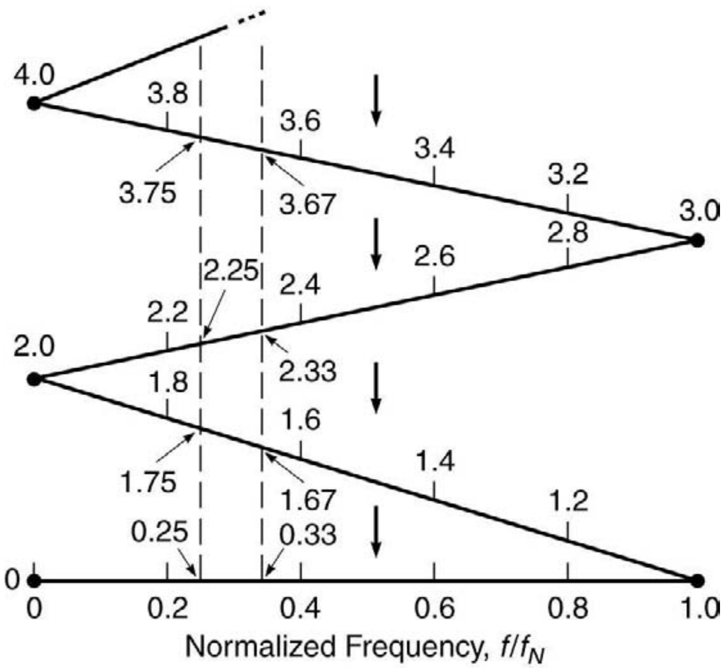

FIGURE 10.4
The folding diagram.

$$y_2(t) = \cos(2\pi f_2 t) \text{ becomes } y_2(r\delta t) = \cos(2\pi r f_2/f_s). \tag{10.3}$$

Further reduction of Equation 10.3 reveals that

$$\begin{aligned} y_2(r\delta t) &= \cos(2\pi r[2mf_N \pm f_1]/f_s) \\ &= \cos(2\pi rm \pm 2\pi r f_1/f_s) \\ &= \cos(2\pi r(m \pm f_1/f_s)) \\ &= \cos(2\pi r f_1/f_s) \\ &= y_1(r\delta t). \end{aligned} \tag{10.4}$$

Thus, the sampled signal $y_2(r\delta t)$ will be identical to the sampled signal $y_1(r\delta t)$, and the frequencies f_1 and f_2 will be indistinguishable. In other words, *all* of the signal content at the f_2 frequencies will appear at the f_1 frequency. Their amplitudes will combine in quadrature with the signal's original amplitude at frequency f_1, thereby producing a false amplitude at frequency f_1.

When aliasing occurs, the higher f_2 frequencies can be said to *fold into* the lower frequency f_1. This mapping of the f_2 frequencies into f_1 is illustrated by the folding diagram, as shown in Figure 10.4. The frequency, f_a, into which a frequency f is folded, assuming $f > f_n$, is identified as follows:

1. Determine k, where $k = f/f_N$. Note that $f_N = f_{max} = f_s/2$.
2. Find the value k_a that k folds into, where k_a occurs on the bottom line $(0 \leq k_a \leq 1)$.
3. Calculate f_a, where $f_a = k_a f_N$.

The following example illustrates aliasing.

Example Problem 10.1

Statement: Assume that there is an analog signal whose highest frequency of interest is 200 Hz ($=f_N$), although there may be frequencies higher than that contained in the signal. According to the sampling theorem, the sampling frequency must be set at $f_s > 400$ Hz for the digitized signal to accurately represent any signal content at and below 200 Hz. However, the signal content above 200 Hz will be aliased. At what frequency will an arbitrary aliased frequency appear?

Solution: According to the folding diagram, for example, the f_2 frequencies of 350 ($1.75f_N$), 450 ($2.25f_N$), 750 ($3.75f_N$), and 850 ($4.25f_N$), all will map into $f_1 = 50\ Hz$ ($0.25f_N$). Likewise, other frequencies greater than f_N will map down to frequencies less than f_N. A frequency of 334 Hz will map down to 67 Hz, and so forth.

Thus, for aliasing of a signal not to occur and for the digitized signal not to be contaminated by unwanted higher-frequency content, f_N first must be identified and then set such that $f_s > 2f_N$. Second, a filter must be used to eliminate all frequency content in the signal above f_N. In an experiment, this can be accomplished readily by filtering the signal with an **anti-alias** (low-pass) filter prior to sampling, with the filter cut off set at f_N.

Example Problem 10.2

Statement: The signal $y(t) = \sin(2\pi 10t)$ is sampled at 12 Hz. Will the signal be aliased and, if so, to what frequency?

Solution: Here $f = 10$ Hz. For signal aliasing not to occur, the signal should be sampled at a frequency that is at least twice the maximum frequency of interest. For this case, the required sampling frequency would be higher than 20 Hz. Because the signal actually is sampled at only 12 Hz, aliasing will occur. If $f_s = 12$ Hz, then $f_N = 6$ Hz, which is one-half of the sampling frequency. Thus, $k = f/f_N = 10/6 = 1.67$. This gives $k_a = 0.33$, which implies that $f_a = 0.33f_N = (0.33)(6) = 2$ Hz. So, the aliased signal will appear as a sine wave with a frequency of 2 Hz. This is illustrated in Figure 10.5.

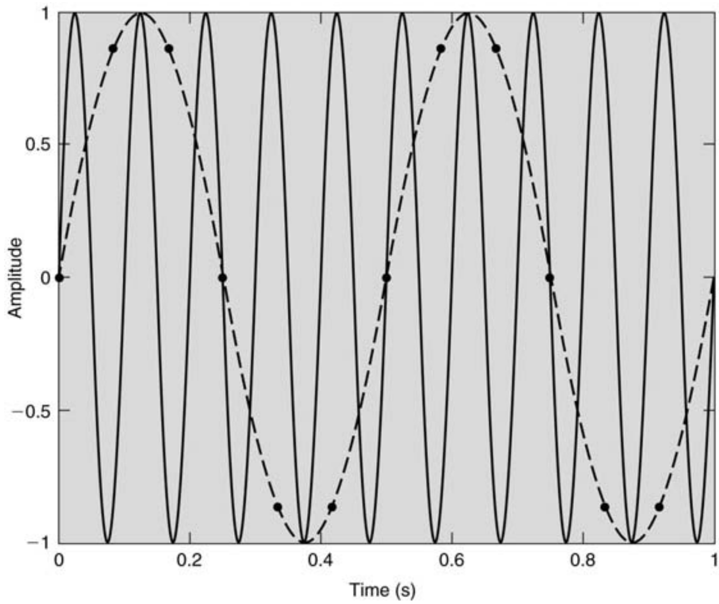

FIGURE 10.5
Aliasing of $y(t) = \sin(2\pi 10t)$ (solid curve is $y(t)$; dashed curve is aliased signal from sampling at 12 Hz).

10.4 Discrete Fourier Transform

The discrete Fourier transform is a method used to obtain the frequency content of a signal by implementing the discrete version of the Fourier transform. A more detailed discussion of the discrete and fast Fourier transforms is presented in [2].

Consider a sample of the signal $y(t)$ with a finite record length, T, which is its fundamental period. This signal is sampled N times at δt increments of time. N values of the signal are obtained, $y_n = y(r\delta t)$, where $r = 1, 2, ..., N$. The discrete signal becomes

$$y(r\delta t) = y(t) \cdot \tilde{\delta}(t - r\delta t). \tag{10.5}$$

The impulse function, $\tilde{\delta}$, is defined such that $\tilde{\delta}(0) = 1$ and $\tilde{\delta}(\neq 0) = 0$.

Now recall the Fourier series representation of $y(t)$,

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left[\frac{2\pi nt}{T}\right] + B_n \sin\left[\frac{2\pi nt}{T}\right] \right), \tag{10.6}$$

with the Fourier coefficients given by

$$\begin{aligned} A_0 &= \frac{2}{T}\int_0^T y(t)dt, \\ A_n &= \frac{2}{T}\int_0^T y(t)\cos\left(\frac{2\pi nt}{T}\right)dt \qquad n = 1, 2, \ldots, \infty, \end{aligned} \tag{10.7}$$

and

$$B_n = \frac{2}{T}\int_0^T y(t)\sin\left(\frac{2\pi nt}{T}\right)dt \qquad n = 1, 2, \ldots, \infty. \tag{10.8}$$

Fourier analysis of a discrete signal is accomplished by replacing the following in Equations 10.6 and 10.8:

1. The integrals over t in Equation 10.8 by summations over δt.
2. Continuous time t by discrete time $r\delta t$, where $r = 1, 2, \ldots, N$.
3. $T = N\delta t$, where N is an even number.
4. n from 1 to ∞ with k from 0 to $N/2$.

In doing so, A_n becomes a_k, where

$$\begin{aligned} a_k &= \frac{2}{N\delta t}\sum_{r=1}^{N} y(r\delta t)\cos\left[\frac{2\pi kr\delta t}{N\delta t}\right]\delta t \\ &= \frac{2}{N}\sum_{r=1}^{N} y(r\delta t)\cos\left[\frac{2\pi kr}{N}\right] \qquad k = 0, 1, \ldots, \frac{N}{2}. \end{aligned} \tag{10.9}$$

Likewise,

$$b_k = \frac{2}{N}\sum_{r=1}^{N} y(r\delta t)\sin\left[\frac{2\pi kr}{N}\right] \qquad k = 1, 2, \ldots, \frac{N}{2} - 1 \tag{10.10}$$

and

$$c_k = \frac{2}{N}\sum_{r=1}^{N}\sqrt{a_r^2 + b_r^2}. \tag{10.11}$$

Note that k represents the discrete frequency and r the discrete sample point. Each discrete sample point can contribute to a discrete frequency. Every sample point's contribution to a particular discrete frequency is included by summing over all sample points at that frequency. This yields the corresponding discrete expression for $y(t)$,

$$y(r\delta t) = \frac{a_0}{2} + \sum_{k=1}^{(N/2)-1} \left(a_k \cos\left[\frac{2\pi rk}{N}\right] + b_k \sin\left[\frac{2\pi rk}{N}\right] \right) + \frac{a_{N/2}}{2} \cos(\pi r). \quad (10.12)$$

The last term corresponds to $f_{max} = f_N$. The equations for a_k and b_k comprise the **discrete Fourier transform** or DFT of $y(r\delta t)$. The equation $y(r\delta t)$ is the **discrete Fourier series**.

A computer program or M-file can be written to perform the DFT, which would include the following steps:

1. Fix k.
2. Evaluate $2\pi rk/N$ for all r.
3. Compute $\cos[2\pi rk/N]$ and $\sin[2\pi rk/N]$.
4. Compute $y(r\delta t)\cos[2\pi rk/N]$ and $y(r\delta t)\sin[2\pi rk/N]$.
5. Sum these values from $r = 1$ to N to give a_k and b_k as given in Equations 10.9 and 10.10.
6. Repeat for next k
7. After completing for all k, determine c_k using Equation 10.11.

This method involves N^2 real multiply-add operations.

Alternatively, the DFT can be written using complex notation. Using the Fourier coefficient definitions in Equation 10.8, and introducing Y_n, which was called C_n in Chapter 9, gives

$$Y_n = \frac{A_n}{2} - i\frac{B_n}{2}. \quad (10.13)$$

This leads to

$$\begin{aligned} Y_n(t) &= \frac{1}{T}\int_0^T y(t)\left[\cos\left(\frac{2\pi nt}{T}\right) - i\sin\left(\frac{2\pi nt}{T}\right)\right] dt \\ &= \frac{1}{T}\int_0^T y(t)\exp[-i(2\pi nt/T)]dt. \end{aligned} \quad (10.14)$$

By making the appropriate substitutions for T, δt, and n in Equation 10.14, the discrete Fourier transform in complex form becomes

$$
\begin{aligned}
Y_k &= \frac{1}{N\delta t}\sum_{r=1}^{N} y(r\delta t)\exp\left[-i\left(\frac{2\pi k r\delta t}{N\delta t}\right)\right] dt \\
&= \frac{1}{N}\sum_{r=1}^{N} y(r\delta t)\exp[-i(2\pi kr/N)] \\
&= \frac{1}{N}\sum_{r=1}^{N} y_r \exp[-i(2\pi kr/N)]. \qquad (10.15)
\end{aligned}
$$

Again, k represents the discrete frequency and r represents the discrete sample point. This method requires N^2 complex multiplications. Note also that $2\pi rk/N$ can be replaced by $2\pi r f_k \delta t$ because $f_k = k/T = k/(N\delta t)$.

10.5 Fast Fourier Transform

The **fast Fourier transform**, or FFT, is a specific type of DFT that is computationally faster than the original DFT. Danielson and Lanczos produced one such FFT algorithm in 1942. Cooley and Turkey developed the most frequently used one in the mid-1960's. Danielson and Lanczos showed that a DFT of length N can be rewritten as the sum of two DFTs, each of length $N/2$, one coming from the even-numbered points of the original N, the other from the odd-numbered points [3]. Equation 10.15 can be rearranged to conform to this format as

$$
\begin{aligned}
Y_k &= \frac{1}{N}\sum_{r=0}^{N-1} y_r\, e^{-i\left(\frac{2\pi rk}{N}\right)} \qquad (10.16) \\
&= \frac{1}{N}\left\{\sum_{r=0}^{(N/2)-1} y_{2r}\, e^{-i\left[\frac{2\pi(2r)k}{N}\right]} + \sum_{r=0}^{(N/2)-1} y_{2r+1}\, e^{-i\left[\frac{2\pi(2r+1)k}{N}\right]}\right\} \\
&= \frac{1}{N}\left\{\sum_{r=0}^{(N/2)-1} y_{2r}\, e^{-i\left[\frac{2\pi rk}{(N/2)}\right]} + W^k \sum_{r=0}^{(N/2)-1} y_{2r+1}\, e^{-i\left[\frac{2\pi rk}{(N/2)}\right]}\right\},
\end{aligned}
$$

where $W^k \equiv e^{-i[2\pi k/N]}$. Equation 10.17 can be written in a more condensed form, $Y_k = Y_k^{\text{even}} + W^k Y_k^{\text{odd}}$, where Y_k^{even} is the kth component of the DFT of length $N/2$ formed from the even-numbered y_k values and Y_k^{odd} is the kth component of the DFT of length $N/2$ formed from the odd-numbered y_k values. This approach can be applied successively until the last transforms

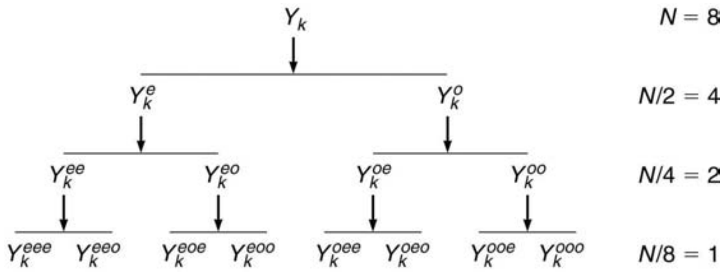

FIGURE 10.6
DFT sequence for N = 8.

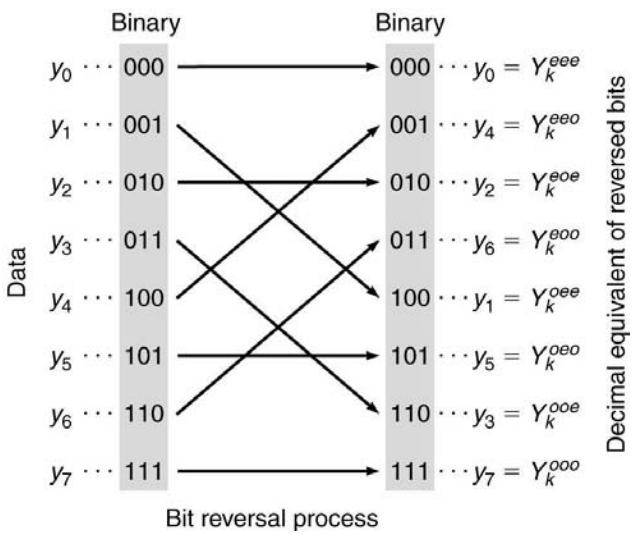

FIGURE 10.7
DFT sequence for N = 8 showing bit reversal.

have only one term each. At that point, the DFT of the term equals the term itself, where Y_k (for $k = 0,\ r = 0,\ N = 1) = (1/1)\, y_0\, e^{-i\cdot 0} = y_0 = Y_0$.

The sequence of the computational breakdown for $N = 8$ is displayed in Figure 10.6. Symmetry is maintained when $N = 2^M$. Here Y_k^{xxx} are the DFTs of length one. They equal the values of the discrete sample points, $y(r\delta t)$. For a given N, the particular y_k values can be related to a pattern of e's and o's in the sequence. By reversing the pattern of e's and o's (with $e = 0$ and $o = 1$), the value of k in binary is obtained. This is called *bit reversal*. This process is illustrated in Figure 10.7. The speed of this FFT is $\sim O(N \log_2 N)$ vs $O(N^2)$ for the DFT, which is approximately 40 000 times faster than the original DFT!

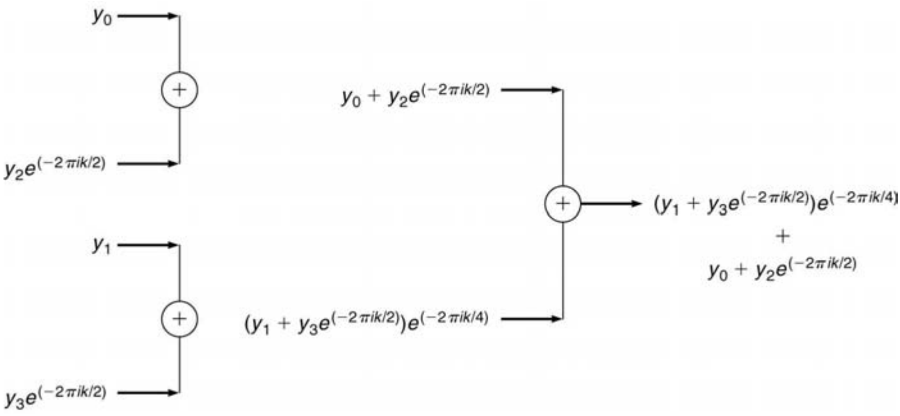

FIGURE 10.8
More efficient DFT sequence for N = 4.

Example Problem 10.3

Statement: For the case of $N = 4$, determine the four DFT terms y_0, y_1, y_2 and y_3 for $r = 0, \ldots, N-1$.

Solution: Direct implementation of Equation 10.17 yields

$$\begin{aligned} Y_k &= \frac{1}{4}\sum_{r=0}^{3} y_r \, e^{-i(2\pi rk/4)} \\ &= \frac{1}{4}\left\{y_0 + y_1 \, e^{-i(2\pi k/4)} + y_2 \, e^{-i(2\pi 2k/4)} + y_3 \, e^{-i(2\pi 3k/4)}\right\} \\ &= \frac{1}{4}\left\{y_0 + y_2 \, e^{-i(2\pi k/2)} + e^{-i(2\pi k/4)} \cdot \left[y_1 + y_3 \, e^{-i(2\pi k/2)}\right] . \right\} \end{aligned}$$

Thus, the DFT could be performed computationally faster in the sequence, as illustrated in Figure 10.8, by starting with the even (y_0 and y_2) and odd (y_1 and y_3) pairs.

10.6 Amplitude Ambiguity

Amplitude ambiguity also arises when the sample time period, T_r, is not an integer multiple of the fundamental period of the signal. If the signal has more than one period or is aperiodic, this will complicate matters. For complex periodic signals, T_r must be equal to the least common integer

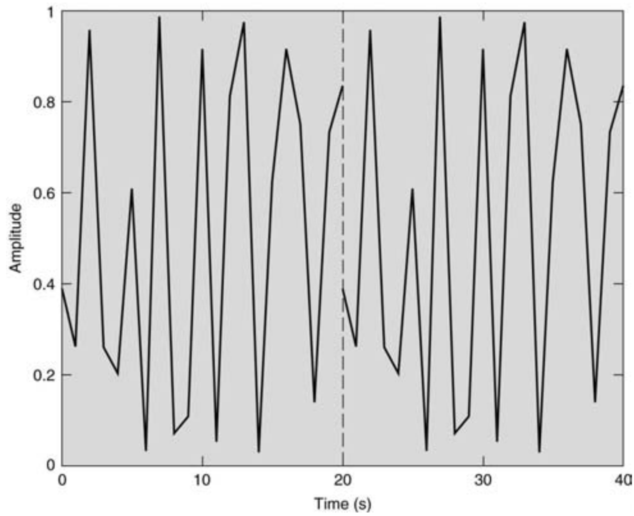

FIGURE 10.9
Two repeated segments of the same random signal.

multiple of all frequencies contained in the signal. For aperiodic signals, T_r theoretically must be infinite. Practically, finite records of length T_r are considered and windowing must be used to minimize the effect of amplitude ambiguity. Application of the DFT or FFT to an aperiodic signal implicitly assumes that the signal is infinite in length and formed by repeating the signal of length T_r an infinite number of times. This leads to discontinuities in the amplitude that occur at each integer multiple of T_r, as shown in Figure 10.9 at the time equal to 20 s. These discontinuities are step-like, which introduce false amplitudes that decrease around the main frequencies similar to those observed in the Fourier transform of a step function (see Chapter 9).

Thus, the amplitudes of simple or complex periodic waveforms will be accurately represented in the DFT when $f_s > 2f_{max}$ and $T_r = mT_1$, where $m = 1, 2, ...,$. T_1 is the fundamental period ($= 1/f_1$) and T_r the total sample period ($= N\delta t = N/f_s$), which implies that $N = m(f_s/f_1)$. If the latter condition is not met, leakage will occur in the DFT, appearing as amplitudes at f_1 spilling over into other adjacent frequencies. Further, for DFT computations to be fast, N must be set equal to 2^M, which yields $2^M = m(f_s/f_1)$, where m and N are positive integers. These conditions are summarized as follows:

1. Set $f_{max} = f_N \Rightarrow f_s = 2f_{max}$, assuming that f_1 and f_{max} are known.
2. Find a suitable N by the steps:
   (a) Choose a value for m, keeping $m \geq 10$).

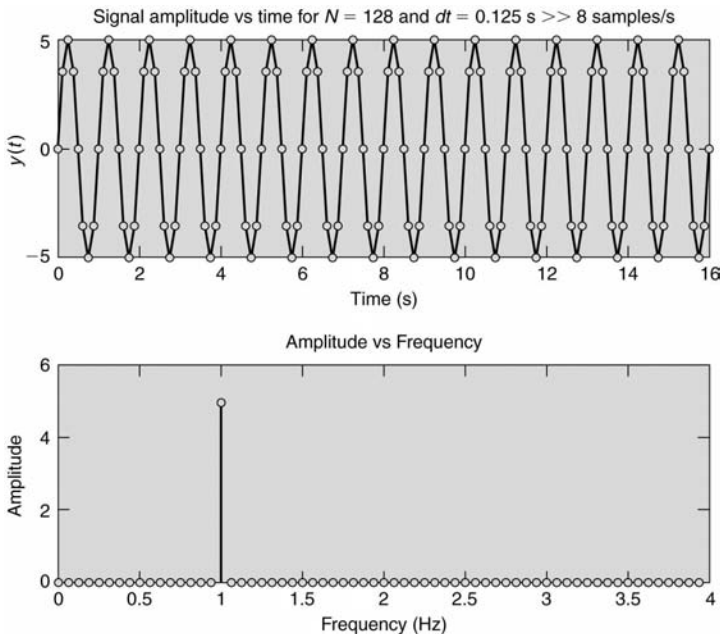

FIGURE 10.10
Signal and frequency spectrum with $dt = 0.125$ s.

(b) Is there an integer solution for M, where $2^M = m(f_s/f_1)$?

(c) If so, stop.

(d) If not, iterate until an integer M is found. Thus, $N = 2^M$ and $T_r = N\delta t$.

For aperiodic and nondeterministic waveforms, the frequency resolution δf ($= 1/N\delta t$) is varied until leakage is minimized. Sometimes, all frequencies are not known. In that case, to avoid leakage, windowing must be used.

The following example illustrates the effect of sampling rate on the resulting amplitude-frequency spectrum in terms of either aliasing or amplitude ambiguity.

Example Problem 10.4

Statement: Convert the analog voltage, $E(t) = 5\sin(2\pi t)$ mV, into a discrete time signal. Specifically, using sample time increments of (a) 0.125 s, (b) 0.30 s, and (c) 0.75 s, plot each series as a function of time over at least one period. Discuss apparent differences between the discrete representation of the analog signal. Also, compute the DFT for each of the three discrete signals. Discuss apparent differences. Use a data set of 128 points.

Solution:

$$y(t) = 5\sin(2\pi t) \Rightarrow f = 1 \text{ Hz}.$$

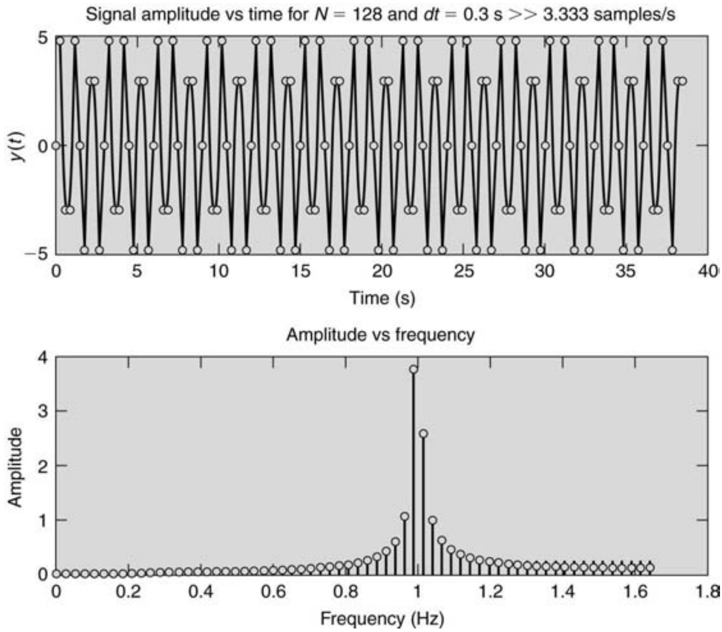

FIGURE 10.11
Signal and frequency spectrum with $dt = 0.3$ s.

Aliasing will *not* occur when $f_s(= 1/dt) > 2f$ ($f = 1$ Hz). Amplitude ambiguity will *not* occur when $T = mT_1 \Rightarrow m = fNdt$ (m : integer).

For part (a) $f_s > 2f$ and $m = (1)(128)(.125) = 16 \Rightarrow$ no aliasing or amplitude ambiguity. The result is shown in Figure 10.10, which was presented previously to illustrate the FFT.

For part (b) $f_s > 2f \Rightarrow$ no aliasing, and $m = (1)(128)(0.3) = 38.4 \Rightarrow$ amplitude ambiguity will occur. This is displayed in Figure 10.11. The amplitude, however, is less than the actual amplitude (here it is less than 4). Around that frequency the amplitude appears to leak into adjacent frequencies.

For part (c) $f_s < 2f \Rightarrow$ aliasing will occur, and $m = (1)(128)(0.75) = 96 \Rightarrow$ no amplitude ambiguity will be present. This is shown in Figure 10.12. The aliased frequency can be determined using the aforementioned folding-diagram procedure. Here, $f_s = 4/3$, $f_N = 2/3$, and $f = 1$. This leads to $k = 3/2$, which implies $k_a = 1/2$ using the folding diagram. Thus, $f_a = (1/2)(2/3) = 1/3$ Hz.

Now consider an example where both aliasing and amplitude ambiguity can occur simultaneously.

Example Problem 10.5

Statement: Compute the DFT for the discrete time signal that results from sampling the analog signal, $T(t) = 2\sin(4\pi t)$ °C, at sample rates of 3 Hz and 8 Hz. Use a data set of 128 points. Discuss and compare your results.

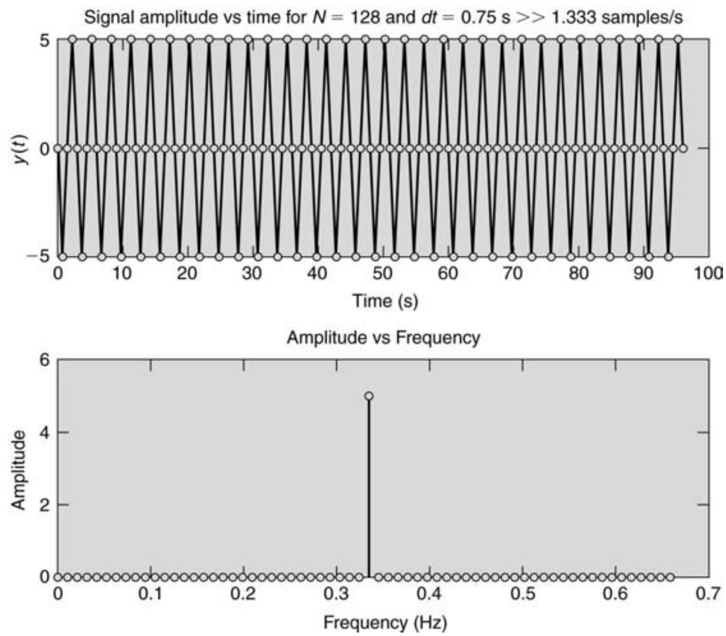

FIGURE 10.12
Signal and frequency spectrum with $dt = 0.75$ s.

Solution:

$$T(t) = 2\sin(4\pi t) \Rightarrow 2 \text{ Hz}.$$

For the sample rate of 3 Hz, $f_s = 2f \Rightarrow$ aliasing will occur and $m = (1)(128)(1/3) = 42.67 \Rightarrow$ amplitude ambiguity will be present. The results are presented in Figure 10.14. The aliased frequency occurs where the amplitude is maximum, at 1 Hz. This can be determined using the aforementioned folding-diagram procedure. Here, $f_s = 3$, $f_N = 3/2$, and $f = 2$. This leads to $k = 4/3$, which implies $k_a = 2/3$ using the folding diagram. Thus, $f_a = (2/3)(3/2) = 1$. Also, note the distortion of the signal's time record that occurs because of the low sampling rate.

When the sampling rate is increased to 8 Hz, $f_s > 2f \Rightarrow$ no aliasing occurs. Also $m = (1)(128)(0.125) = 16 \Rightarrow$ no amplitude ambiguity occurs. This is shown in Figure 10.13, which is the correct spectrum.

Analysis becomes more complicated when more than one frequency is present in the signal. Next, consider an example that involves a signal containing two frequencies.

Example Problem 10.6

Statement: Consider the signal $y(t) = 3.61\sin(4\pi t + 0.59) + 5\sin(8\pi t)$. Plot $y(t)$ versus time and the resulting frequency spectrum for the following cases and discuss what is observed with respect to aliasing and amplitude ambiguity:

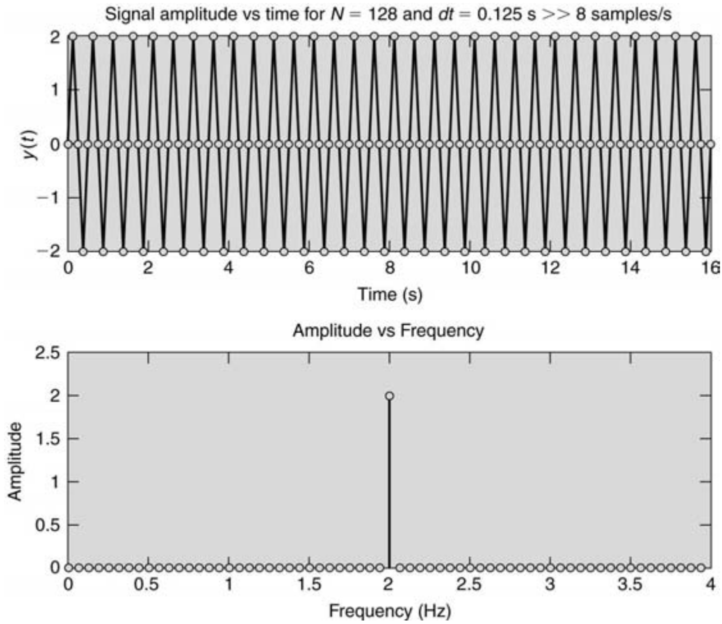

FIGURE 10.13
Signal and frequency spectrum with $dt = 0.125$ s.

(i) $N = 100$, $f_s = 50$
(ii) $N = 20$, $f_s = 10$
(iii) $N = 10$, $f_s = 5$
(iv) $N = 96$, $f_s = 5$
(v) $N = 96$, $f_s = 10$

Solution: $y(t) = 3.61\sin(4\pi t + 0.59) + 5\sin(8\pi t)$. So, $f_1 = 2$ Hz and $f_2 = 4$ Hz, which implies that $f_{max} = 4$ Hz. If $f_s = 5$ samples/s,

$$\frac{f_s}{f_1} = \frac{5}{2} = 2.5 > 2 \Rightarrow \text{no aliasing, and}$$
$$\frac{f_s}{f_2} = \frac{5}{4} = 1.25 < 2 \Rightarrow \text{aliasing will occur.}$$

To where will the 4 Hz component be aliased?

$$f_N = \frac{f_s}{2} = \frac{5}{2} = 2.5 \text{ Hz} \Rightarrow \frac{f_2}{f_N} = \frac{4}{2.5} = 1.6.$$

Using the folding diagram $1.6f_N$ is folded down to $0.4f_N = (0.4)(2.5) = 1$ Hz. That is, the 4 Hz component appears as a 1 Hz component.

But what about amplitude ambiguity? $T_1 = 1/f_1 = 1/2$ s, and $T_2 = 1/f_2 = 1/4$ s. The total sample period, T, must contain integer multiples of both T_1 and T_2 so as not to have amplitude ambiguity in both components. This can be easily met by having $T = mT_1 = m/2$ s (since $T_2 = T_1/2$). In essence, the least common integer multiple

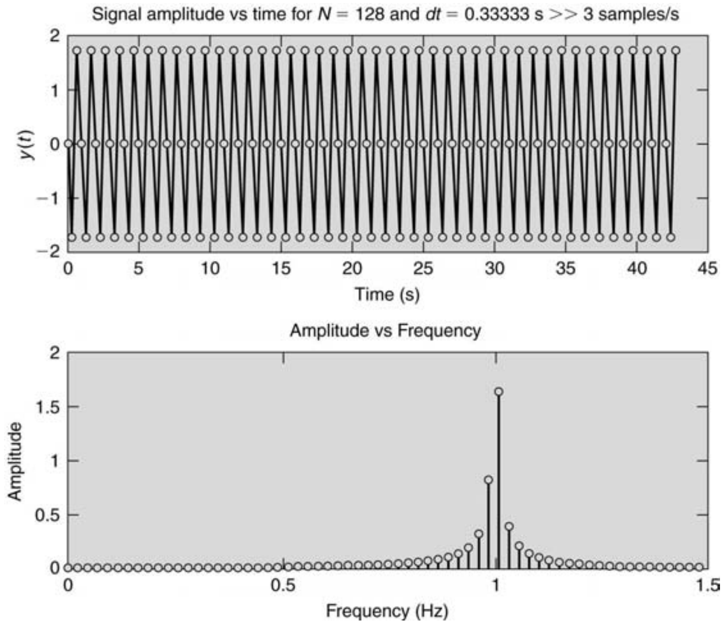

FIGURE 10.14
Signal and frequency spectrum with $dt = 1/3$ s.

of T_1 and T_2 is sought. Recalling that $T = N\delta t = N/f_s$, if $m/2 = N/f_s$ no amplitude ambiguity will be present. That is, when $N = f_s(m/2) = (5/2)m$, with m and N integers, there will be no amplitude ambiguity. This occurs, for example, when $m = 2$ (with $N = 5$) and $m = 4$ (with $N = 10$). However, all the frequencies of interest should be seen in the spectrum. The highest frequency of interest is $f_{max} = f_N = f_s/2$. Because there are $N/2$ discrete frequencies and assuming that $f_{min} = 0$ needs to be considered, this yields

$$f_{max} = \frac{1}{T}(\frac{N}{2} - 1) = \frac{f_s}{2} = \frac{f_s}{N}(\frac{N}{2} - 1) \;\; (T = N/f_s).$$

Solving for N,

$$N = 2f_s/(f_s - 4).$$

So, when $f_s = 5$, $N = 10/(5 - 4)$. Thus, $N = 10$ is the minimum N needed to see both components.

For case (i), the discrete signal and the amplitude-frequency spectrum are correct. This is shown in Figure 10.15.

For case (ii), the spectrum remains correct and the discrete signal, although still correct, does not represent the signal well because of the lower sampling rate. This is illustrated in Figure 10.16.

For case (iii), the 4 Hz component is aliased down to 1 Hz and the 2 Hz component is correct. No amplitude ambiguity has occurred. This is displayed in Figure 10.17.

For case (iv), amplitude ambiguity has occurred for both components and only the 4 Hz component is aliased down to 1 Hz. This is shown in Figure 10.18.

For case (v), amplitude ambiguity has occurred but not aliasing. This is presented in Figure 10.19.

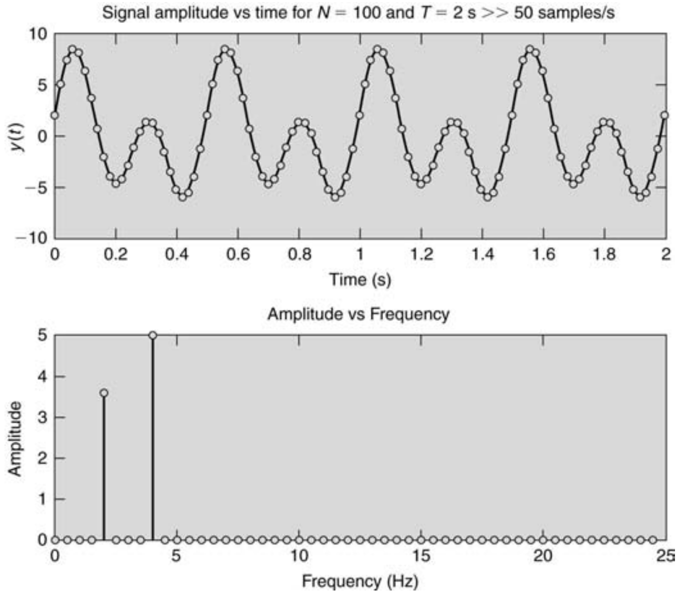

FIGURE 10.15
Signal and frequency spectrum with $N = 100$, $f_s = 50$.

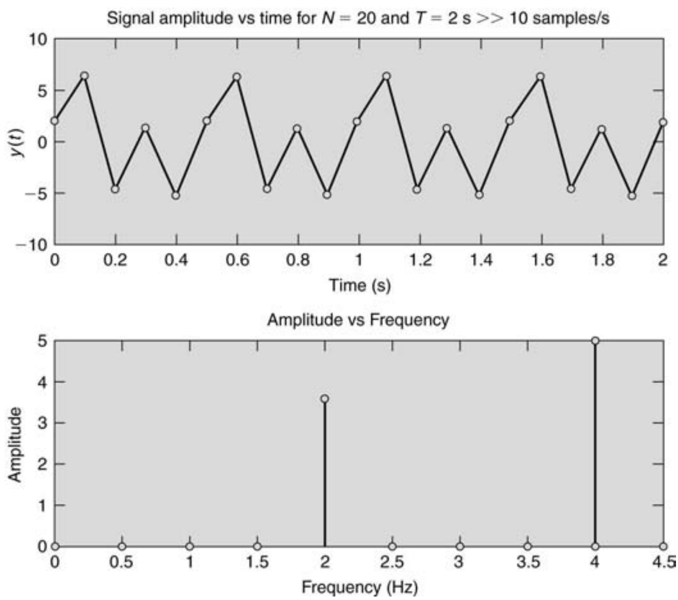

FIGURE 10.16
Signal and frequency spectrum with $N = 20$, $f_s = 10$.

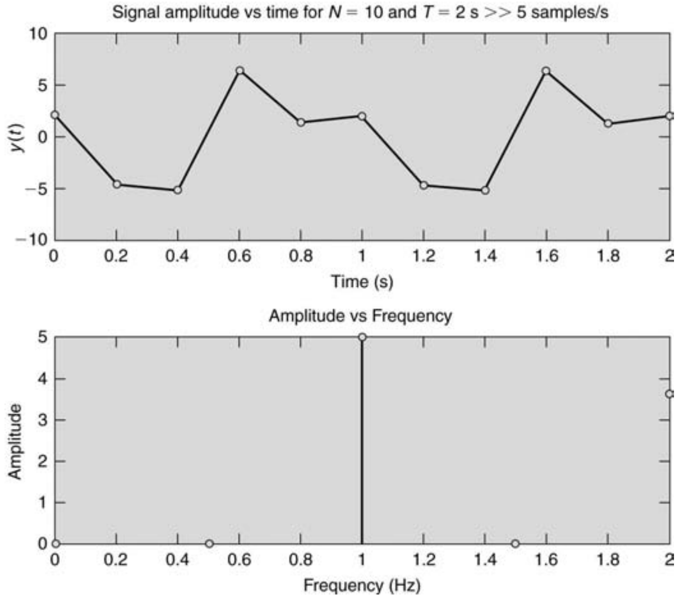

FIGURE 10.17
Signal and frequency spectrum with $N = 10$, $f_s = 5$.

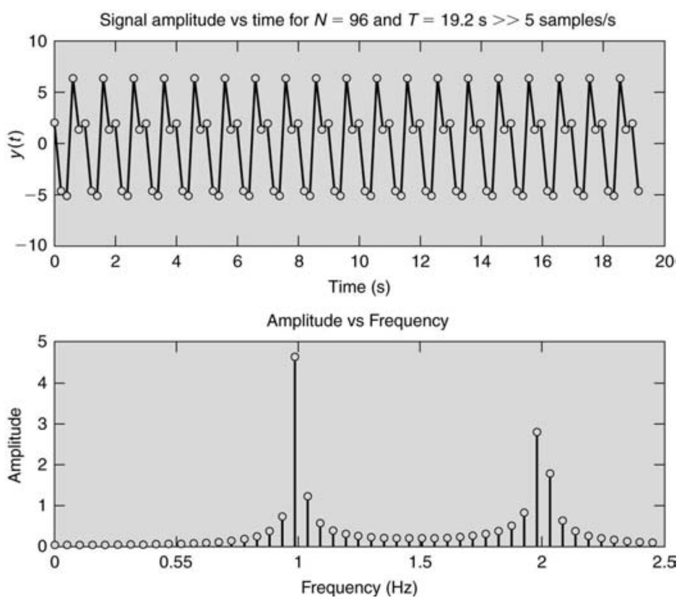

FIGURE 10.18
Signal and frequency spectrum with $N = 96$, $f_s = 5$.

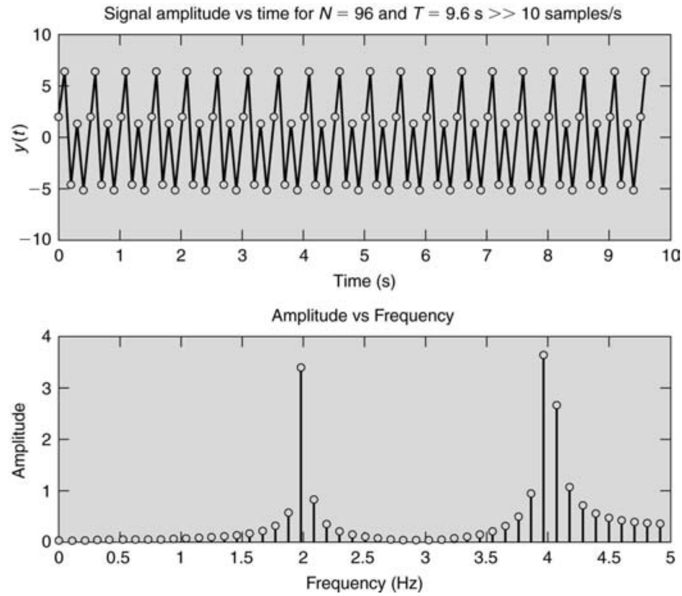

FIGURE 10.19
Signal and frequency spectrum with $N = 96$, $f_s = 10$.

10.7 *Windowing

Because either an aperiodic or random signal does not have a period, the Fourier transform applied to such a signal's finite record length produces leakage in its spectrum. This effect can be minimized by applying a **windowing function**. This process effectively attenuates the signal's amplitude near the discontinuities that were discussed previously in Section 10.6, thereby leading to less leakage. The windowing function actually is a function that weights the signal's amplitude in time. The effect of a windowing function on the spectrum can be seen by examining the convolution of two Fourier transforms, one of the signal and the other of the windowing function. This is considered next.

The discrete Fourier transform of $y(t)$ can be viewed as the Fourier transform of an unlimited time history record $v(t)$ multiplied by a rectangular time window $u(t)$ where

$$u(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

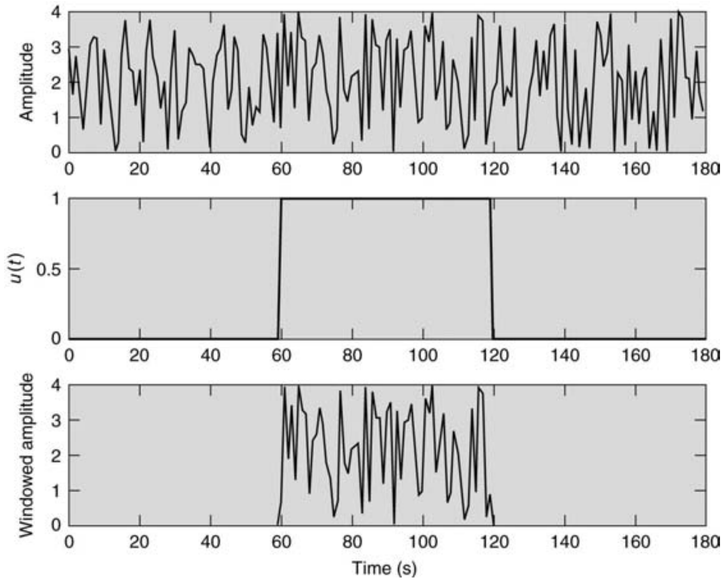

FIGURE 10.20
Rectangular windowing.

This is illustrated in Figure 10.20. Now,

$$Y_n(f,T) = \frac{1}{T}\int_0^T y_n(t)\exp[-i2\pi ft]dt, \tag{10.17}$$

which leads to

$$Y(f) = \int_{-\infty}^{\infty} U(\alpha)V(f-\alpha)d\alpha. \tag{10.18}$$

This is the convolution integral [3]. So, the Fourier transform of $y(t)$, $Y(f)$, is the convolution of the Fourier transforms of $u(t)$ and $v(t)$, which are denoted by $U(\alpha)$ and $V(f-\alpha)$, respectively. For the present application, $u(t)$ represents the windowing function and $y(t)$ is $v(t)$. The record length is denoted by T. Various windowing functions can be used. They yield different amounts of leakage suppression.

The rectangular windowing function $u_{rect}(t)$ has the Fourier transform $U_{rect}(f)$, given by

$$U_{rect}(f) = T\left(\frac{\sin \pi ft}{\pi ft}\right). \tag{10.19}$$

The relatively large side lobes of $|U(f)/U(0)|$ produce a leakage at frequencies separated from the main lobe. This produces a distortion throughout the spectra, especially when the signal consists of a narrow band of frequencies. This is illustrated in Figure 10.21.

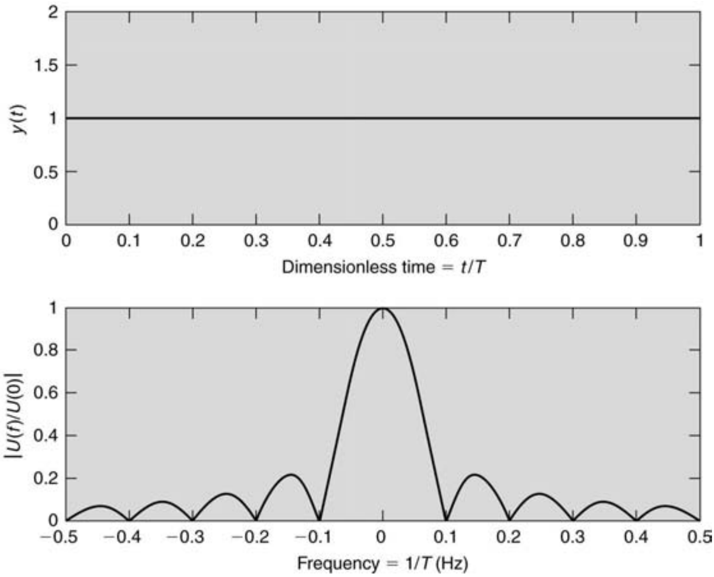

FIGURE 10.21
The rectangular (boxcar) window.

It is better to taper the signal to eliminate the discontinuity at the beginning and end of the data record. There are many types of tapering windows available. The cosine-squared window, also known as the Hanning window, is used most commonly. This is given by

$$u_{hanning}(t) = \frac{1}{2}\left(1 - \cos\frac{2\pi t}{T}\right) = 1 - \cos^2\left(\frac{\pi t}{T},\right) \tag{10.20}$$

when $0 \leq t \leq T$. Otherwise, $u_{hanning} = 0$. Further,

$$U_{hanning}(f) = \frac{1}{2}U(f) + \frac{1}{4}U(f - f_1) + \frac{1}{4}U(f + f_1), \tag{10.21}$$

where $f_1 = 1/T$ and $U(f)$ is defined as before with

$$U(f - f_1) = T\left[\frac{\sin\pi(f - f_1)T}{\pi(f - f_1)T}\right] \tag{10.22}$$

and

$$U(f + f_1) = T\left[\frac{\sin\pi(f + f_1)T}{\pi(f + f_1)T}\right]. \tag{10.23}$$

The Hanning window is presented in Figure 10.22.

Finally, it should be noted that windows reduce the amplitudes of the spectrum. For a given window, this loss factor can be calculated [2]. For

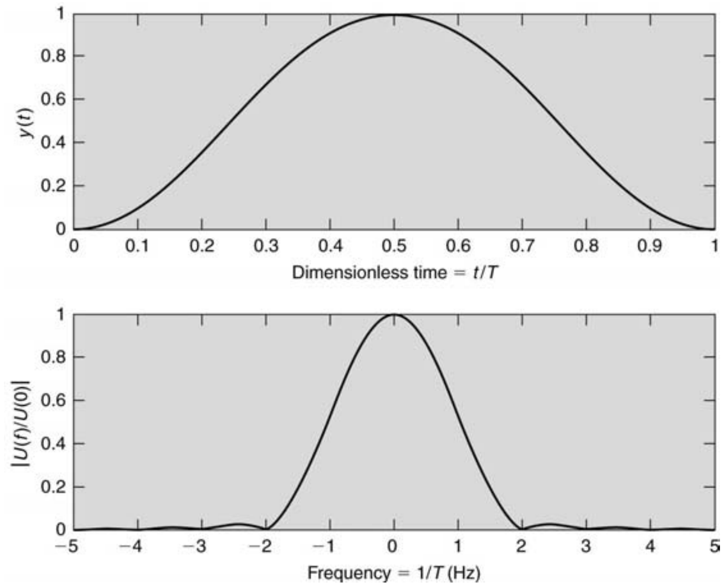

FIGURE 10.22
The Hanning window.

the Hanning window, the amplitude spectrum must be scaled by the factor $\sqrt{8/3}$ to compensate for this attenuation. Thus,

$$Y_n(f_k) = \delta t \sqrt{\frac{8}{3}} \sum_{n=0}^{N-1} y_{nk} \left(1 - \cos^2 \frac{\pi n}{N}\right) \exp\left[-i \frac{2\pi k n}{N}\right], \qquad (10.24)$$

with $f_k = k/(N\delta t)$, where $k = 0, 1, 2, \ldots, N/2$ and

$$G_y(f_k) = \frac{2}{n_d N \delta t} \sum_{i=1}^{n_d} |Y_n(f_k)|^2. \qquad (10.25)$$

The recommended procedure [2] for computing a smoothed amplitude spectrum is the following:

1. Divide data into n_d blocks, each of size $N = 2^M$.
2. Taper the data values in each block $\{y_n\}$ ($n = 0, 1, 2, \ldots, N-1$) with a Hanning or other window.
3. Compute the N-point FFT for each data block yielding $Y_n(f_k)$, adjusting the scale to account for the tapering loss (for example, multiply by $\sqrt{8/3}$ for the Hanning window).
4. Compute $G_y(f_k)$ for n_d blocks.

10.8 Problem Topic Summary

Topic	Review Problems	Homework Problems
Sampling	1, 2, 5	1, 2, 3, 4, 5, 6, 7, 8, 9
Aliasing	4	2, 3, 4, 5, 6, 7, 8, 9,
Amplitude Ambiguity	3	2, 4, 7, 9

TABLE 10.1
Chapter 10 Problem Summary

10.9 Review Problems

1. Determine the number of discrete frequencies from the minimum to and including the maximum frequency that will appear in an amplitude-frequency plot of a signal sampled every 0.2 s. The signal's minimum frequency is 0.5 Hz.

2. Determine the frequency resolution of a signal sampled 256 times for a period of 4 s.

3. Does windowing of a signal produce a signal with no amplitude distortion?

4. Determine the aliased frequency, in Hz, of a 100-Hz sine wave sampled at 50 Hz.

5. The frequency resolution, in Hz, of a signal sampled 256 times for a period of 4 s is (a) 256 (b) 1/4, (c) 4/256, or (d) 4.

10.10 Homework Problems

1. A discrete Fourier transform of the signal $B(t) = \cos(30t)$ is made to obtain its power-frequency spectrum. $N = 4000$ is chosen. Determine (a) the period of $B(t)$ (in s), (b) the cyclic frequency of $B(t)$ (in Hz), (c) the appropriate sampling rate (in samples/s), and (d) the highest

resolvable frequency, f_{max} (in Hz). Finally, (e) if $N = 4096$ was chosen instead, would the computations of the Fourier transform be faster or slower and why?

2. Using a computer program written by yourself or constructed from available subroutines, calculate and plot the following: one plot containing the continuous signal $y(t)$ and its discrete version versus time, and the other plot containing the amplitude spectrum of the discrete sample. Provide a complete listing of the program. Do this for each of the cases below. Support any observed aliasing or leakage of the sample by appropriate calculations. State, for each case, whether or not aliasing and/or leakage occur. The continuous signal is given by

$$y(t) = 5\sin(2\pi t + 0.8) + 2\sin(4\pi t) + 3\cos(4\pi t) + 7\sin(7\pi t).$$

The cases to examine are (a) $N = 100$, $T = 10$ s, (b) $N = 100$, $T = 18$ s, (c) $N = 100$, $T = 20$ s, (d) $N = 100$, $T = 15$ s, and (e) $N = 50$, $T = 15$ s, where N represents the number of sample points and T the sample period.

3. Consider the signal $y(t) = 5 + 10\cos(30t) + 15\cos(90t)$. Determine (a) the frequencies (in Hz) contained in the signal, (b) the *minimum* sample rate (in samples/s) to avoid aliasing, and (c) the frequency resolution of the frequency spectrum if the signal is sampled at that rate for 2 seconds. Finally, sketch (d) the amplitude-frequency spectrum of $y(t)$ and (e) the amplitude-frequency spectrum if the signal is sampled at 20 samples/s.

4. A velocity sensor is placed in the wake behind an airfoil subjected to a periodic pitching motion. The output signal of the velocity transducer is $y(t) = 2\cos(10\pi t) + 3\cos(30\pi t) + 5\cos(60\pi t)$. Determine (a) the fundamental frequency of the signal (in Hz), (b) the maximum frequency of the signal (in Hz), (c) the range of acceptable frequencies (in Hz) that will avoid signal aliasing, and (d) the minimum sampling frequency (in Hz) that will avoid *both* signal aliasing and amplitude ambiguity if 20 samples of the signal are taken during the sample period. Finally, if the signal is sampled at 20 Hz, determine (e) the frequency content of the resulting discrete series, $y(\delta nt)$, and (f) the resulting discrete series $y(\delta nt)$.

5. The signal $y(t) = 3\cos(\omega t)$ has a period of 4 seconds. Determine the following for the signal: (a) its amplitude, (b) its cyclic frequency, (c) the *minimum* sampling rate to avoid aliasing, (d) its mean value over *three* periods, and (e) its rms value over *two* periods. The formula $\int [\cos(ax)]^2 dx = \frac{1}{a}[-\frac{1}{2}\cos(ax)\sin(ax) + \frac{1}{2}ax]$ may or may not be useful.

6. At what cyclic frequency will the signal $y(t) = 3\sin(4\pi t)$ appear if (a) $f_s = 6$ Hz, (b) $f_s = 4$ Hz, (c) $f_s = 2$ Hz, and (d) $f_s = 1.5$ Hz?

7. For the deterministic signal $y(t) = 2 + 3\sin(6\pi t) + 4\sin(18\pi t)$, sketch the amplitude-frequency spectrum of $y(t)$ (a) when the signal is sampled at a rate of 24 Hz (indicate by solid lines) and (b) when it is sampled at a rate of 12 Hz (indicate by dashed lines). Finally, (c) determine the *minimum* sample period (in s) to avoid amplitude ambiguity in the amplitude-frequency spectrum.

8. At what cyclic frequency will the signal $y(t) = 12\cos(3\pi t)$ appear if sampled at (a) $f_s = 6$ Hz, (b) $f_s = 2.75$ Hz, (c) $f_s = 3$ Hz, and (d) $f_s = 1$ Hz?

9. Consider the signal $z(t) = 3\cos(8\pi t) + 4\sin(5\pi t + 0.25)$. (a) Classify the signal by its main division plus all subdivisions (for example, nondeterministic/stationary/ergodic). Next, determine (b) the *cyclic* frequency of each component, (c) the shortest sample period to avoid amplitude ambiguity, and (d) the minimum sampling rate to avoid aliasing. Finally, determine, if any, (e) the aliased frequencies if the signal is sampled at 7 Hz.

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11

**Units and Significant Figures*

CONTENTS

Units thus resemble sports officials: the only time you pay real attention to them is when something stupid happens.
Steve Minsky, "Measure for Measure," *Scientific American*, August 2000, 96.

Scientists lost a $125 million spacecraft as it approached Mars last week essentially because they confused feet and pounds with meters and kilograms, according to the National Aeronautics and Space Administration.
"A Little Metric Misstep Cost NASA $125 Million," *International Herald Tribune*, October 2, 1999.

Maximum Height: 11'3"
3.4290 metres
A sign on a bus in Edinburgh, Scotland, September, 1998.

11.1 Chapter Overview

This chapter introduces two important topics: systems of units and significant figures. These topics often are considered too mundane to occupy valuable lecture time and therefore are left to students to learn on their own time. In fact, most students and their teachers spend very little time on these topics. Consequently, they often cannot identify the proper units of a particular dimension, convert its value from one system of units to another, and express it with the proper number of significant figures. It cannot be over-emphasized that it is essential for a good scientist or engineer to have an excellent grasp of systems of units and significant figures.

11.2 English and Metric Systems

We live in a world in which we are constantly barraged by numbers and units. Examples include 100 megabyte per second ethernet connections, 64 gigabyte USB flash drives, a pint of Guinness quaffed in an English pub by an American tourist weighing 15 stones, 103 mile per hour fastballs, and over two-meter-high aerial dunk shots. A visitor from the early 1900's would have no idea about what we are talking! We speak a foreign language that appears confusing to most. But units and measures are not meant to confuse. They were developed for us to communicate effectively, both commercially and technically. They are the structure behind our technical accomplishments. Without them, the Tower of Babel still would be under construction!

In the United States, two *languages* of systems currently are *spoken.* These loosely are referred to as the English and the metric systems. This *bilingual* situation can lead to some serious mistakes. A contemporary example of this is the loss of a $125 million Mars Climate Orbiter on September 23, 1999, which was referred to in the headlines quoted at the beginning of this chapter. Basically, one group of scientists calculated the thrust of the orbiter's engines in units of pounds, but NASA assumed the values were in units of newtons. This led to approximately a 100 mile difference from its intended orbit, causing the spacecraft to burn up during orbital insertion! So, effective technical communication requires the abilities to *speak* the language of both systems of units and to be able to *translate* between them. Before studying each system, however, it would be good to delve into a little of their history.

The English system of units evolved over centuries starting from the Babylonians, followed by the Egyptians, Greeks, Romans, Anglo-Saxons, and Norman-French. It was the Romans who introduced the base of 12 in the English system, where one Roman *pes* (foot) equaled 12 Roman *unciae* (our inch). It was not until around the early 1500s that man began to consider quantifying and standardizing dimensions such as time and length. The yard, for example, has its origin with Saxon kings, whose *gird* was the circumference of the waist. It was King Edgar who, in an apparent attempt to provide a standard of measurement, declared that the yard should be the distance from the tip of his outstretched fingers to his nose. Other royal declarations, such as one made by Queen Elizabeth I defining the *statute* mile to be 5280 feet (8 furlongs at 220 yards per furlong) instead of the Roman mile (the distance of 1000 Roman soldier paces or 5000 feet), served to standardize what has become known as the English system of units.

The metric system, on the other hand, was not burdened with units of anthropometric origin, as was the English system. The metric system did not arise until near the end of the Period of Enlightenment, around the end of the 17th century. Thus, its development followed a more rational and scientific approach. Prior to its introduction, practically no single unit of measure was consistent. Footplates abounded, which marked the lengths of the most common *footmeasures* in Europe. In Rhineland, a foot was 31 centimeters, whereas in Gelderland it was 27 centimeters. The pound in Amsterdam was 494 grams. Slightly farther south in the Hague, it was 469 grams. This presented considerable confusion and impeded intercity commerce.

In 1670, a decimal system based on the length of one arc minute of the great circle of the earth was proposed by Gabriel Mounton. Jean Picard, in 1671, proposed that the length standard be defined as the length of a clock's pendulum whose period was a specified time. It was not until 1790 when a commission appointed by the French Academy of Sciences developed and formalized a decimal-based system defining length, mass, and volume. The unit of length, the meter, equaled one ten-millionth of the distance from the north pole to the equator along the meridian of the earth running from Dunkerque, France, through Paris to Barcelona, Spain. The unit of mass, the gram, was defined in terms of a liquid volume, where one gram equaled the mass of one cubic centimeter of water at its temperature of maximum density. The unit of volume, the liter, equaled one cubic decimeter. This approach established mass and volume as supplementary units in terms of a base unit (the meter), which was to a physical standard (the earth's circumference).

In 1866, the United States Congress made it lawful to use the metric system in the United States in contracts, dealings, and court proceedings. Various metric units were defined in terms of their English counterparts. For example, the meter was defined as *exactly* 39.37 inches. In 1875, the United States signed in Sèvres, France, along with 16 other countries, an international treaty called the Metric Conversion. This treaty established a permanent international bureau of standards and the standards for length and mass. In 1893, the US customary units (those based on the English system) were redefined in terms of their metric standards (which was opposite the approach taken in 1866). The yard became *exactly* 0.9144 meters (hence, the foot became *exactly* 0.3048 meters and the inch became *exactly* 0.0254 meters) and the pound *exactly* 0.453 592 4 kilograms. Since the treaty, almost all world countries have officially accepted this system. Over the years it was revised and simplified, eventually resulting in the *Le Système International d'Unités* (International System of Units). This system, abbreviated as SI, was adopted by the General Conference in 1960 and is what people today call the metric system.

The United States made a valiant attempt to adopt the metric system in 1975 with the Metric Conversion Act, which required Federal agencies to use the metric system by 1992. Some signs along interstate highways showed distances to cities in both English and metric units. Soft drinks appeared in the market in liter bottles. A national chain of stores began selling metric tools. Beyond that, however, little happened with the general public and industry. Since then, most of these highways signs have disappeared. The liter-size plastic bottles and the metric tools remain as epitaphs. As of 2010, the United States, Liberia, and Myanmar are the only three countries out of the 193 countries in the world that formally have not adopted the metric system.

Today, the responsibility of maintaining the standards of measure in the US rests with the National Institute of Standards and Technology (NIST). NIST provides a wealth of information on systems of units and their origin [1].

11.3 Systems of Units

The measurement of a physical quantity involves the process of assigning a specific value with units to the physical quantity. The quantity has a **dimension**. Its **unit** determines its **measure** or magnitude. For example, a sheet of European A4 paper is 210 millimeters wide by 297 millimeters long. The dimension is length, the unit is millimeters, and the measures are 210 and 297. If this information is expressed in another system of units, the dimension still is length but the unit and measures will be different. There are seven **fundamental dimensions**: length, mass, time, temperature, electrical current, amount of substance, and luminous intensity.

A **system of units** is necessary to provide a framework in which physical quantities can be expressed and also related to one another through physical laws. Five different systems of units are presented in Table 11.1. SI is the universally accepted system. Unfortunately, the English Engineering system (US Standard Engineering or old English) and the Technical English system (US Customary or British Gravitational) still are championed by US industry (*vox clamantis in deserto*). Use of the other two systems, Absolute Metric and Absolute English, continues to appear in some publications.

Physical quantities of different dimensions are related to one another through equations in the form of definitions and natural laws. Consider, for example, Newton's second law: $F = ma$, where F denotes force, m mass, and a acceleration. In SI, a gravitational force of 9.81 newtons (N) (9.806 65 *exactly*) is required for a 1 kilogram (kg) mass to accelerate by 9.81 meters (m) per second (s) squared. The equation $F = ma$ gives two equations, a **numerical equation** that contains only the measures of the physical quantities, $9.81 = 1 \cdot 9.81$, and a **unit equation**, $\mathrm{N} = 1 \cdot \mathrm{kg} \cdot \mathrm{m/s^2}$. The units within a system are **consistent** or **coherent** if no numerical factors other than 1 occur in all unit equations, as in this example.

A system of units is comprised of **base**, **supplementary**, and **derived** units. Base units are *dimensionally independent.* There is a base unit for every fundamental dimension contained in a particular system of units. Supplementary units, such as the radian, are considered dimensionless and do not represent a fundamental dimension. Derived units literally are derived from the base and supplemental units and, therefore, are comprised of products, quotients, and powers of base and supplemental units. In the SI system, for example, the kilogram, meter, and second are base units. The newton is a derived unit because it represents a force, which is derived from the base units of mass times acceleration (as expressed by Newton's second law). Hence, a newton equals a kilogram times a meter divided by a second squared ($\mathrm{N} = \mathrm{kg\ m/s^2}$). The base units of the quantities listed in Table 11.1 are printed in normal font; those of the derived quantities are in italics.

Example Problem 11.1

Statement: Give the fundamental dimensions, unit, and measure of your weight in the Technical English system and in the International system.

Solution: Assume an example weight of 172 pounds in the Technical English system. Weight is a force, which is mass times the local gravitational acceleration. So, for both the Technical English and International systems of units, the fundamental dimensions of weight are mass, length, and time. The unit of force is lbf in the Technical English system and N in the International system. The measure is 172 in the Technical English system and 765, which equals 172/0.2248, in the International system.

Example Problem 11.2

Statement: In the English Engineering system 1 lbf is required to accelerate 1 lbm 32.174 ft/s^2. Is this system of units consistent?

Solution: The unit equation for this circumstance is $1\ \mathrm{lbf} = 1\ \mathrm{lbm} \times 32.174\ \mathrm{ft/s^2}$. That is, 1 lbf equals $32.174\ \mathrm{lbm} \times \mathrm{ft/s^2}$. A numerical factor other than 1 (here 32.174) appears in the unit equation, so the English Engineering system is *not* consistent.

Example Problem 11.3

Statement: What are the base units of mass and of force in the Technical English and English Engineering systems of units?

Solution: In both the Technical English and English Engineering systems of units, force is a base unit and its unit is lbf. In the Technical English system, mass is a derived unit, the slug. Its base units are lbf, ft, and s (1 slug = 1 lbf/1 ft/s^2). In the English Engineering system, mass is a base unit and its unit is lbm.

For SI there are seven base units corresponding to the seven fundamental dimensions. These include the meter (m) for length, the kilogram (kg) for mass, the second (s) for time, the kelvin (K) for temperature, the ampere (A) for electric current, the mole (mol) for the amount of substance or quantity of matter, and the candela (cd) for luminous intensity. The corresponding fundamental symbols used for dimensional analysis are L for length, M for mass, T for time, Θ for temperature, $\mathcal{A}$ for electric current, $\mathcal{M}$ for the amount of substance or quantity of matter, and $\mathcal{K}$ for luminous intensity. There are two supplemental units, the radian (rad), which defines a plane angle, and the steradian (sr), which defines a solid angle. The seven base and two supplementary units of the SI system are discussed further in the following section. All other SI units are derived from these nine units, some having symbols and some not. For example, the volt, denoted by the symbol, V, is a derived SI unit that represents the electric potential. Expressed in base units it equals kg m^2/(s^3 A).

There is a convention for the capitalization of unit abbreviations. A unit abbreviation named in honor of a person begins with a capital letter. For example, the pascal (Pa) is named after Blaise Pascal (1623-1662), and the hertz (Hz) is named after Heinrich Hertz (1857-1894). All other units, when spelled out, begin with lower-case letters, with a few exceptions. One is the SI abbreviation for volume, the liter (L). This abbreviation is capitalized to avoid confusion with the lower-case letter l and with the numeral 1.

A system of units can be created from a given number of base units. The MKSA system of units (actually a subsystem of the SI system) is a consistent system of units used for mechanics, electricity, and magnetism, that has the four base units m, kg, s, and A. A coherent system for mechanics is the MKS system having the three base units m, kg, and s. Many other systems abound such as the electrostatic CGS and the electromagnetic CGS systems. Their ubiquitous presence implicitly cautions us when converting from one system of units to another and further supports the use of one consistent system by everyone. Consider the beginning sentence in an article describing the construction of Washington DC's Metro transit system that appeared in the July, 1976, *Newsletter of the National Safety Council*, "Mining recently has been completed for a 2200-foot-long (670 millimeters) twin tunnel section..." Apparently this system was designed for either humans or insects, depending upon which system of units was used!

Quantity	**International System (SI)**	**Absolute Metric System (CGS)**	**English Engineering System (EE)**	
Length	meter (m)	centimeter (cm)	foot (ft)	
Time	second (s)	second (s)	second (s)	
Mass	kilogram (kg)	gram (g)	pound-mass (lbm)	pound-mass
Force	*newton* (N)	*dyne*	pound-force (lbf)	
g_c	$1\ \frac{\text{kg·m}}{\text{N·s}^2}$	$1\ \frac{\text{g·cm}}{\text{dyne·s}^2}$	$32.174\ \frac{\text{lbm·ft}}{\text{lbf·s}^2}$	

TABLE 11.1
Five systems of units (adapted from [2] and [3])

Now return to the five systems of units presented in Table 11.1. There are basically four dimensions involved in each of these systems when used in mechanics: length, time, mass, and force. The English Engineering system is unique in that each of these four dimensions is defined to be independent. That is, for this particular system, the foot, second, pound-mass, and pound-force are base units. The unit for force is defined as the force with which the standard pound-mass is attracted to Earth at a location where the gravitational acceleration equals 32.1740 ft/s^2. The four dimensions are related through the equation $F = m \cdot a/g_c$, where F denotes the force in units of lbf, m the mass in units of lbm, a the gravitational acceleration (32.1740 ft/s^2), and g_c a constant that relates the units of force, mass, length, and time. From this equation, for *only* the English Engineering system, $g_c = 32.1740$ lbm·ft/lbf·s^2. Thus, this system is *not* consistent, as was shown in a previous example problem.

Example Problem 11.4

Statement: A sounding rocket travelling at a constant velocity of 200 miles per hour in steady, level flight ejects 0.700 lbm/s of exhaust gas from its exit nozzle. Determine the rocket's thrust, T, for both the English Engineering and International system of units.

Solution: Thrust is the equal and opposite reaction to the force that the exhaust gas exerts on the rocket nozzle. Because the rocket is traveling at a constant velocity, Newton's second law tells us that the thrust equals the velocity times the exhaust-gas mass flow rate.

In the English Engineering system,

$$T = \underbrace{200\frac{\text{mile}}{\text{hour}} \times 5280\frac{\text{ft}}{\text{mile}} \times \frac{1}{3600}\frac{\text{hour}}{\text{s}}}_{293\frac{\text{ft}}{\text{s}}} \times 0.700\frac{\text{lbm}}{\text{s}} \times \underbrace{\frac{1}{32.174}\frac{\text{lbf} \times \text{s}^2}{\text{lbm} \times \text{ft}}}_{g_c} = 6.37 \text{ lbf}.$$

In the International system of units,

$$T = \underbrace{293\frac{\text{ft}}{\text{s}} \times 12\frac{\text{in.}}{\text{ft}} \times 0.0254\frac{\text{m}}{\text{in.}}}_{89.3\frac{\text{m}}{\text{s}}} \times \underbrace{0.700\frac{\text{lbm}}{\text{s}} \times \frac{1}{2.2046}\frac{\text{kg}}{\text{lbm}}}_{0.318\frac{\text{kg}}{\text{s}}} = 28.4 \text{ N}.$$

Each of the other four systems derives one of its four dimensions from the other three. The derived unit for each of these systems (force for three of these systems and mass for one) is given in italics in Table 11.1. For example, in the Absolute Metric system, the derived unit is the *dyne*, which, when expressed in terms of the base units, becomes g·cm/s^2. Note that each of these four systems is consistent, as a consequence of this approach. This is indicated by the numerical factor of 1 for g_c.

11.4 SI Standards

Next, examine the current definitions of the seven base and two supplementary units of the SI system.

The SI base unit of the dimension of time is the **second (s)**. It is defined as the duration of 9 192 631 770 cycles of the radiation associated with the transition between two hyperfine levels of the ground state of cesium-133. The conversion of this duration of cycles into time is accomplished by passing many cesium-133 atoms through a system of magnets and a resonant cavity driven by an oscillator into a detector (this device is called an atomic beam spectrometer). Only those atoms that have undergone transition reach the detector. When 9 192 631 770 cycles of a detected atom in transition have occurred, the atomic clock advances 1 s.

The SI base unit of length is the **meter (m)**. The meter was defined in 1983 to be the length that light travels in a vacuum during the interval of time equal to 1/299 792 458 s. Although it is related to the dimension of time, it is a base unit because it is not derived from other units. This definition uncoupled the meter from its 200-year-old terrestrial origin. People certainly have come a long way since defining the meter in terms of a geophysical dimension that is changing constantly.

The SI base unit of mass is the **kilogram (kg)**. This is the only base unit still defined in terms of an artifact. The international standard is a cylinder of platinum-iridium alloy kept by the International Bureau of Weights and Measures in Sèvres, France. A copy of this cylinder, a secondary standard, is at the NIST in Gaithersburg, Maryland, where it serves as the primary standard in the United States. The kilogram is the only SI base unit linked to a unique physical object. This will end soon when the kilogram is redefined in terms of a more accurate, atom-based standard [10].

The **kelvin (K)** is the SI base unit of temperature. The kelvin is based upon the triple point of pure water where pure water coexists in solid, liquid, and vapor states. This occurs at 273.16 K and 0.0060 atmospheres of pressure. Thus, a kelvin is 1/273.16 of the thermodynamic temperature of the triple point of pure water. Absolute zero, at which all molecular motion ceases, is 0 K.

The SI base unit of electric current is the **ampere (A)**. The ampere is defined in terms of the force produced between two parallel, current carrying wires. Specifically, an ampere is the amount of current that must be maintained between two wires separated by one meter in free space in order to produce a force between the two wires equal to 2×10^{-7} N/m of wire length.

The **mole (mol)** is the SI base unit for the amount of substance. It is the amount of substance of a system that contains as many elementary entities as the number of atoms in 0.012 kg of carbon 12 ($6.022\ 142 \times 10^{23} = \mathrm{N}_a$).

That is, 1 mol contains N_a entities, where N_a is Avogadro's number. The entities can be either atoms, molecules, ions, electrons, other particles, or groups of such particles. The entities could even be golf balls! So, 1 mole of carbon 12 has a mass of 0.012 kg, 1 mole of monatomic oxygen has 0.016 kg, and 1 mole of diatomic oxygen has 0.032 kg. Each contains $6.022\,142 \times 10^{23}$ entities, which would be atoms for carbon 12 and for monatomic oxygen and molecules for diatomic oxygen. The mass of 1 mole of a substance is determined from its molecular (atomic) weight. Its SI units are kg/kg-mole. The atomic mass unit, typically designated by the symbol *amu*, *exactly* equals 1/12 the mass of one atom of the most abundant isotope of carbon, carbon-12, which is 1.6603×10^{-27} kg. This unit of mass is called a dalton.

The SI base unit of luminous intensity is the **candela (cd)**. One candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watts per steradian. A 100 watt light bulb has the luminous intensity of approximately 135 cd and a candle has approximately 1 cd.

There are two SI supplementary (dimensionless) units, the **radian (rad)** and the **steradian (sr)**. The radian is based upon a circle and the steradian upon a sphere. One radian is the plane angle with its vertex at the center of the circle that is subtended by an arc whose length is equal to the radius of the circle. Hence, there are 2π radians over the circumference of a circle. The steradian is the solid angle at the center of a sphere that subtends an area on the surface of the sphere equal to the *square* of the radius. Thus, there are 4π steradians over the surface of a sphere.

The base units of time, electric current, and amount of substance are the same in both the SI and English systems. The systems differ only in the units for the dimensions of length, mass, temperature, and luminous intensity. Presently, the level of accuracy for most base units is 1 part in 10 million [10].

11.5 Technical English and SI Conversion Factors

People working in technical fields today must learn both the Technical English and SI systems and be proficient in converting between them. This is particularly true for the dimensions of mechanical, thermal, rotational, acoustical, photometric, electric, magnetic, and chemical systems. The units used in the SI and Technical English systems for these dimensions are presented in tables on the text web site. Often, the knowledge of one conversion factor for each dimension is sufficient to construct other conversion factors for that dimension. Table 11.2 lists some conversion factors between units in SI, English Engineering, and Technical English. The SI units for electric

Dimension	Units with Factors
Length	1 m = 3.2808 ft 1 km = 0.621 mi
Volume	1 L = 0.001 m^3 = 61.02 $in.^3$
Mass	1 kg = 2.2046 lbm = 0.068 522 slug
Force	1 N = 0.2248 lbf
Work, Energy	1 kJ = 737.562 ft·lbf = 0.947 817 Btu
Power	1 kW = 1.341 02 hp = 3414.42 Btu/hr
Pressure, Stress	1 atm = 14.696 psi = 101 325 Pa = 407.189 in. H_2O = 760.00 mm Hg = 1 bar
Density	1 slug/ft^3 = 512.38 kg/m^3
Temperature	K = °C +273.15 K = (5/9) × °F + 255.38 K = (5/9) × °R °F = (9/5) × °C + 32.0 °F = °R - 459.69

TABLE 11.2
Some useful conversion factors

and magnetic systems are presented in Chapter 2. There are many electronic work sheets available on the Internet that automatically perform unit conversions [11]. Also refer to the standards used for SI unit conversion [12].

11.5.1 Length

For the dimension of length, 1 in. equals 2.54 cm *exactly*. Using this conversion, 1 ft = 0.3048 m *exactly*, 1 yd = 0.9144 m *exactly* and 1 mi = 1.609 344 km *exactly*. A 10 km race is approximately 6.2 mi. Note that a period is used after the abbreviation for inch. This is the *only* unit abbreviation that is followed by a period, so as not to confuse it with 'in', the English preposition. No other unit abbreviations are followed by periods.

11.5.2 Area and Volume

For area and volume, the square, and the cube of the length dimension, respectively, are considered. The SI units of area and volume are m^2 and m^3. However, the liter (L), which equals 1 cubic decimeter or 1/1000 m^3, often is used. One L of liquid is approximately 1.06 quarts or 0.26 gallons. A 350 cubic inch engine has a total cylinder displacement volume of approximately 5.7 L. Curiously, when the American tourist drinks an English pint of Guinness, he consumes 20 liquid UK ounces. A pint in his home country is 16 liquid ounces. In the United States, 1 liquid gallon (gal) = 4 liquid quarts (qt) = 8 liquid pints (pt) = 16 liquid cups (c). Further, 1 liquid cup (c) =

8 liquid ounces (oz) = 16 liquid tablespoons (Tbl) = 48 liquid teaspoons (tsp). The liquid (fluid) ounce is a unit of volume. The ounce when specified *without* the liquid prefix is a unit of mass, where 16 oz = 1 lbm.

11.5.3 Density

The SI unit for density is kg/m^3. Most gases have densities on the order of 1 kg/m^3 and most liquids and solids on the order of 1000 kg/m^3 to 10 000 kg/m^3. For example, at 1 atm and 300 K air has a density of 1.161 kg/m^3, water 1000 kg/m^3, and steel 7854 kg/m^3. The density of air can be determined over the temperature range from approximately 160 K to 2200 K using the equation of state for a perfect gas, which is

$$\rho = \frac{p}{R \cdot T} = \frac{p \cdot \mathcal{MW}}{\mathcal{R} \cdot T}, \tag{11.1}$$

where ρ is the density, p the pressure, T the temperature, $\mathcal{R}$ the universal gas constant equal to 8313.3 J/(kg-mole·K), $\mathcal{MW}$ the molecular weight, and R the gas constant, which equals $\mathcal{R}/\mathcal{MW}$. For air, $R = 287.04$ J/(kg·K) based upon its molecular weight of 28.966 kg/kg-mole. The density of air at sea level is 1.2250 kg/m^3. The density of water (±0.2 %) at 1 atm over the temperature range from 0 °C to 100 °C is given by the curve fit [4]

$$\rho = 1000 - 0.0178|T - 4|^{1.7}, \tag{11.2}$$

where the density is expressed in units of kg/m^3 and the temperature in units of degrees Celsius.

11.5.4 Mass and Weight

The conversion of mass is straightforward. One lbm equals 0.453 592 4 kg *exactly*. So, 1 slug is approximately 14.59 kg. In terms of base units, 1 slug equals 1 lbf·s^2/ft. Thus, the mass of the 15 stone American tourist in the English pub is 6.52 slugs in the Technical English system and 210 lbm in the English Engineering system (1 stone = 14 lbm).

Weight, which is a force, is the product of mass and acceleration. The tourist's weight is 210 lbf in both the Technical English and English Engineering systems. This seems confusing. The units and measures of the tourist's mass are different in these two systems. Yet, the weight units and measures are the same! Such system conversion confusion usually arises when those speaking the English system do not specify what *dialect* they are using (Technical or Engineering). This invariably leads to the common question, "Should the mass be divided by 32.2 to compute the force or not?" The answer is yes if you are *speaking* English Engineering and no if you're *speaking* Technical English. Let us see why.

To avoid confusion in problems involving mass, acceleration, and force for the different systems, Newton's second law can be written as $F = ma/g_c$. This effectively keeps the measures of the dimensions correct for all systems. For consistent systems, the measure of g_c is unity. So $F = ma$ can be used directly. For example, in the Technical English system 1 lbf will accelerate 1 slug at 1 ft/s^2. For the inconsistent English Engineering system, g_c equals 32.174 lbm ft/lbf s^2. So, $F = ma/g_c$ must be used. For example, 1 lbf will accelerate 32.174 lbm at 1 ft/s^2 or 1 lbm at 32.174 ft/s^2. By comparing the units of mass between the two English systems, 1 slug = 32.174 lbm. Such confusion usually compels unit-challenged individuals to learn the SI system for the sake of simplification.

Example Problem 11.5

Statement: Compute for both the Technical English and International systems of units the mass and weight of air at 300 K in a room with internal dimensions of 12 ft × 12 ft × 10 ft.

Solution: The volume of the air in the room is 1440 ft^3. The density of air at 1 atm and 300 K is 1.16 kg/m^3 = 0.002 26 slug/ft^3. So, in Technical English, the mass of the air is 3.26 slugs and its weight is 3.26 slugs × 32.174 ft/s^2 = 105 lbf. In SI the mass is 47.6 kg and its weight is 47.6 kg × 9.81 m/s^2 = 467 N. Also note that in the English Engineering system the density of air would be equal to 0.0727 lbm/ft^3. Thus, in English Engineering, the mass of the air is 105 lbm and its weight is 105 lbm × g/g$_c$ = 105 lbm × 32.174 ft/s^2 / 32.174 lbm × ft/lbf × s^2 = 105 lbf. Note that the force in both the Technical English and English Engineering systems has the same measure but the mass does not.

Keep in mind that for an object of a given mass, its acceleration and weight change with distance from the center of the gravitational field of the body to which it is attracted. The weight, w, of a body is related to its mass, m, through Newton's law of gravitational attraction as

$$w(z) = mg_o \left(\frac{R_b}{R_b + z} \right)^2 = mg(z), \tag{11.3}$$

where R_b is the radius of the body ($R_b = 6\ 378\ 150$ m for Earth), g_o the *local* gravitational acceleration (g_o equals 9.806 65 m/s^2 at sea level on Earth), and z is the distance away from the body ($z = 0$ at sea level).

Example Problem 11.6

Statement: Compute the gravitational acceleration in SI units at an altitude of 35 000 ft, where commercial jet airplanes fly.

Solution: First the altitude is converted in the SI unit of meters. Here 35 000 ft/3.2808 ft/m = 10 668 m. Then, using the expression for $g(z)$ from Equation 11.3 yields $g(10\ 668\ \text{m}) = 0.996\ 66 \times g_o$. So the change in the gravitational acceleration from that at sea level is very small, less than a half of one percent.

11.5.5 Force

The unit of force in SI is the newton (N), named after Sir Isaac Newton (1642-1727). A force of 1 N accelerates a 1 kg mass at 1 m/s^2. One N is approximately 0.225 lbf. Curiously, this is the approximate weight of an apple or, alternatively, the force felt by your hand when holding an apple. So, if a popular hamburger chain converted to metric, then its quarter pound hamburger would become a newton burger!

In the English Engineering system there are pounds of force and pounds of mass, which are designated by lbf and lbm, respectively. In the Technical English system there is only one pound, the pound-force, which is designated by lbf. The unit lbf (as opposed to lb) is used in Technical English to designate the pound-force in order to avoid any ambiguity.

Force per unit area is pressure or stress. The SI unit for this is the pascal (Pa), which equals one N/m^2. One atmosphere, approximately 14.696 psia (pounds per square inch absolute), equals 101.325 kPa. The pressure at the center of the Earth is 5.8×10^7 kPa and that of the best laboratory vacuum is 1.45×10^{-16} kPa [6].

Example Problem 11.7

Statement: At an altitude of 35 000 ft above sea level the atmospheric pressure is 205 mm Hg. Assuming that the typical area of an airplane's passenger window is 80 in.^{2}, determine the net force on the window during flight at that altitude.

Solution: Assume that the pressure inside the airplane is 1 atm = 760 mm Hg. So, the force on the window will be outward because of the higher pressure inside the cabin. The pressure difference will be equal to 760 mm Hg − 205 mm Hg = 555 mm Hg = 10.7 lbf/in.^{2}. Thus, the net outward force is (10.7 lbf/in.^{2}) × (80 in.^{2}) = 859 lbf = 3.82 kN.

11.5.6 Work and Energy

The SI unit for work or energy is the joule (J), named after the British scientist James Joule (1818-1889). Joule is best known for the classic experiment in which he demonstrated the equivalence of energy and work. In fact, energy is *defined* as the ability to do work. One J is 0.2288 calories (cal), or approximately 0.738 ft·lbf, or approximately 9.48×10^{-4} Btus (British thermal units). One Calorie (with a capital C, abbreviated Cal) is 1000 calories = 1 kcal. Thus, 1 kJ = 0.2288 Cal.

Most people count Calories when on a diet. It takes approximately 0.016 J of work to lift a teaspoonful of ice cream from the table to your mouth (a distance of approximately 1/3 m) to gain approximately 35 000 J of energy from the ice cream. That is not much caloric expenditure for a lot of caloric gain!

Example Problem 11.8

Statement: A person eats a cup of high quality ice cream. How many miles would the person have to jog to expend the energy he just consumed? How much weight would he gain if he did not jog off the calories added by eating the ice cream?

Solution: For energy to be conserved and the person not to change weight, the energy contained in the ice cream must equal the energy expended in jogging. Assume that 100 Cal are expended for each mile jogged. A cup of ice cream contains approximately 400 Cal. Thus, he would have to jog 4 miles. If he did not jog, the 400 Cal would be converted into a mass of body fat whose weight is approximately 1/8 lbf on earth. This is because 1 g of fat produces 9 Cal of energy. So, 400 Cal is converted into 44.4 g of body fat. The weight of this mass on earth is 0.436 N, which is approximately 1/8 lbf.

11.5.7 Power

Power is work or energy per unit time. The SI unit of power is the watt (W), which is a joule per second (J/s). This is named after the British engineer James Watt (1736-1819). Intensity usually refers to power per unit area, or in SI units, W/m^2. Flux often denotes intensity per unit time, or in SI units, $W/(m^2{\cdot}s)$ in many transport processes. However, be sure to check the units when the term *flux* is used. For example, the "solar flux" at Earth's surface is approximately 1 370 W/m^2, which is an intensity.

11.5.8 Temperature

The temperature scales are related as shown in Table 11.2. Water boils at approximately 212 °F, 100 °C, 373.15 K, and 671.67 °R, depending upon the local pressure. The unit °C denotes degrees Celsius (not degrees Centigrade, which is no longer preferable) and K stands for kelvin (not degrees kelvin). Note the lack of the degree symbol with K). The Kelvin and Rankine scales are *absolute* (thermodynamic) temperature scales. An absolute temperature is independent of the properties of a particular system and is based upon the second law of thermodynamics. The temperatures 0 K and 0 °R represent absolute zero. In the Kelvin scale, a value of 273.16 is assigned to the triple point of water.

An International Practical Temperature scale (IPTS-68) was adopted in 1968 by the International Committee of Weights and Measurements. This scale covers the temperature range from 13.81 K (the triple point of hydrogen) to 1377.58 K (the freezing point of gold at 1 atmosphere). It specifies a series of 11 temperatures based upon the triple, freezing, and boiling points of various substances, the temperature measurement instruments to be used for calibration purposes over a specified temperature range, and the equations for interpolating temperatures among the 11 fixed points.

Example Problem 11.9

Statement: Sir Isaac Newton developed his own temperature scale in 1701 where water was "just freezing" at 0 units and "boyles vehemently" at 34.4 units. Six units of his temperature scale corresponded to "air at midsummer." What was the temperature of the midsummer air in London in units of his contemporary Gabriel Fahrenheit's temperature scale?

Solution: From the temperature difference between the boiling and freezing of water, it is known that 212 °F − 32 °F = 180 °F, which corresponds to 34.4 units of Newton's scale. Thus, the conversion factor from Newton's to Fahrenheit's units is 5.23, *assuming* that both scales are linear in between the two temperatures. This implies that the midsummer's air is 6 × 5.23 °F/Newton unit + 32 °F = 63.4 °F.

11.5.9 Other Properties

The properties of gases, liquids, and solids can be expressed in terms of base and supplementary units. Absolute or dynamic viscosity, μ, is a fluid property that is related to the fluid's shear stress (force per unit area), τ, and rate of shear strain (strain per unit time), $d\theta/dt$, by the expression $\tau = \mu d\theta/dt$. Thus, the fundamental dimensions of absolute viscosity are $\mathrm{ML^{-1}T^{-1}}$, and the SI base units are kg/(m·s). Kinematic viscosity, ν, is the ratio of absolute viscosity to density, μ/ρ, and has the SI units of $\mathrm{m^2/s}$. The absolute viscosity of air and water are affected weakly by pressure and strongly by temperature. The absolute viscosity for air can be determined using Sutherland's law [4]

$$\mu = \mu_o \left(\frac{T}{T_o}\right)^{3/2} \left(\frac{T_o + S}{T + S}\right), \tag{11.4}$$

where μ_o equals 1.71 × 10^{-5} kg/(m·s), T_o = 273 K, and S = 110.4 K for air. The absolute viscosity for water (±1 %) can be determined by the curve fit [4]

$$\mu = \mu_o \exp\left[-1.94 - 4.80\left(\frac{T_o}{T}\right) + 6.74\left(\frac{T_o}{T}\right)^2\right], \tag{11.5}$$

where μ_o equals 1.792×10^{-3} kg/(m·s) and T_o = 273.16 K.

The units of most properties can be found using an expression, typically a physical law or definition, that relates the property to other terms in which the units are known. For example, the gas constant, R, is related to the speed of sound (distance per unit time), a, by the expression $a = \sqrt{\gamma RT}$, where T is the temperature, and γ equals the ratio of specific heats, $\mathrm{C}_p/\mathrm{C}_v$. Thus, the base units of R are $\mathrm{m^2/s^2{\cdot}K}$, or equivalently J/kg·K.

Note that C_p and C_v are functions of temperature. For air at 300 K, C_p = 1.0035 kJ/(kg·K) and C_v = 0.7165 kJ/(kg·K), which yields γ_{air} = 1.4. The temperature and pressure of the standard atmosphere of air at sea level are 288.15 K and 101 325 Pa, respectively. The speed of sound for air at these conditions is 340.43 m/s. Variations of atmospheric pressure, temperature, density, and speed of sound with altitude for the 1976 Standard Atmosphere are available in graphical and computational forms [13].

Finally, a word of caution is necessary. A unit balance always needs to be done when performing calculations involving unfamiliar quantities. Do this even when working within one system of units and using units that can be expressed directly in terms of base units. Conversion factors may be needed, especially when dealing with electric and magnetic units. The following example serves to illustrate this point.

Example Problem 11.10

Statement: Determine the charge in units of coulombs of a 1 μm diameter oil droplet that is charged to the Rayleigh limit. Also express this in terms of the number of elementary charges. The Rayleigh limit charge, q_{Ray}, is given by the expression

$$q_{Ray} = \sqrt{2\pi\sigma_l d_p^3},$$

where $\sigma_l = 0.04$ N/m and d_p denotes the droplet diameter.

Solution: Noting that $d_p = 1 \times 10^{-6}$ m and making substitutions into the expression for charge in terms of SI units yields

$$q_{Ray} = \sqrt{(2\pi)(0.04)(1 \times 10^{-18})} = 5.02 \times 10^{-10}\sqrt{J \cdot m}.$$

But is this the value of q_{Ray} in units of coulombs? In other words, is the unit C equal to the units $\sqrt{J \cdot m}$? The answer is no. A conversion factor of $\sqrt{4\pi\epsilon_o}$ that has units of $C/\sqrt{J \cdot m})$ is required, in which ϵ_o is the permittivity of free space that equals 8.85 $\times$ 10^{-12} F/m. This is because the units F/m equal the units $C^2/(J{\cdot}m)$. The measure of the conversion factor is 1.06 $\times$ 10^{-5}. Another useful unit conversion is $(4\pi\epsilon_o)(V^2)$ = N. Thus,

$$q_{Ray} = 5.02 \times 10^{-10}\sqrt{4\pi\epsilon_o} = (5.02 \times 10^{-10})(1.06 \times 10^{-5}) = 5.32 \times 10^{-15}\mathrm{C}.$$

The number of elementary charges, n_e, equals $q_{Ray}/e = 5.32 \times 10^{-15}/1.60 \times 10^{-19}$ = 33 000.

11.6 Prefixes

Often it is convenient to use **scientific notation** to avoid writing very large and very small numbers, as in the previous example. Positive and negative powers of 10 are used to shorten numbers by moving the decimal point. Examples are 1000 = 1 $\times$ 10^3, 0.001 = 1 $\times$ 10^{-3}, and as seen earlier, 6.022 137 $\times$ 10^{23}, which was used to represent 602 213 700 000 000 000 000 000. Sometimes the notation E $\pm$ n is used to replace $\times$ 10^n, particularly with computer output where exponents cannot be generated. For example, 3.254

Factor	Prefix	Symbol
10^{24}	yotta	Y
10^{21}	zeta	Z
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10^{1}	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a
10^{-21}	zepto	z
10^{-24}	yocto	y

TABLE 11.3
Prefixes for Units

$\times\ 10^8 = 3.254$ E+8. Prefixes also can be used with units to shorten the writing of numbers.

Table 11.3 lists the decimal prefixes to be used when expressing large or small numbers as an alternative to using scientific notation. Using this approach, for example, the mass of the earth, which is 5.98×10^{24} kg becomes 5.98 Ykg. Along these lines, there are approximately 10π Ms in one year.

In some situations, the magnitude of the number goes beyond where these prefixes can be used. Consider the estimate for the energy released in the Big Bang, 10^{68} J. When using the American system of numeration (see the Table of Numbers in a dictionary), this becomes 0.1 million vigintillion J! This system uses Latin prefixes for the "illion" in the unit and follows the simple formulation that $number = 10^{3(n+1)}$, where n specifies the name of the prefix. For example, the prefix "tri" means three, which corresponds to the number $10^{3(3+1)} = 10^{12}$ or one *tri*llion. Likewise, one quattuordecillion ("qua" is 4 plus "dec", which is 10) equals $10^{3(4+10+1)} = 10^{45}$. For really large numbers, use the *googol*, which was "coined in the late 1930s by the nine-year-old nephew of the American mathematician Edward Kasner when he was asked to come up with the name for a very large number." [14]. One googol equals 10^{100}. Its cousin, one googolplex, equals $10^{10^{100}}$. Obviously, the use of scientific notation usually is preferred in such cases unless you are out to impress your colleagues with your extensive vocabulary!

Example Problem 11.11

Statement: The American system of numeration differs from the British system of numeration, which also is used by most European countries. In the British system 10^6 is a 'thousand thousands', 10^9 is a 'milliard' and 10^{12} is a 'billion'. Beyond a British billion, the formulation is $number = 10^{6n}$, where n specifies the Latin name of the prefix of 'illion'. Determine whether a British billionaire is richer than an American trillionaire.

Solution: For numbers equal to and beyond 10^{12}, the formulations given for each of the American and British systems can be used. Thus, the ratio of a British number to an American number equal to and beyond 10^{12} will be given by the formulation $ratio = 10^{6n}/10^{3(n+1)} = 10^{3(n-1)}$. So, a British billionaire ($n = 2$) is actually a thousand times richer than an American billionaire, provided that the exchange rate between British pounds and American dollars is 1:1, which it is not. In fact, a British billionaire is approximately 1600 times richer than an American billionaire and 1.6 times richer than an American trillionaire!

11.7 Significant Figures

The ubiquitous use of calculators and computers has led to assignments and lab reports with far too many digits in every number! This situation begs for us to revisit the concept of **significant figures**. This is especially important because the number of digits present in a result implies the *precision* of the result. It goes without saying that the proper use of significant figures is an essential element in the presentation of both experimental and calculated results and their uncertainties.

How is the number of significant figures determined? The number of significant figures is the number of digits between and including the least and the most significant digits. The leftmost *nonzero* digit is called the **most significant digit**; the rightmost *nonzero* digit, the **least significant digit**. If there is a decimal point in the number, then the rightmost digit is the least significant digit even if it is a zero. These rules imply that the following numbers have five significant figures: 1.0000, 2734.2, 53 267., 428 970, 10 101 and 0.008 976 0.

But what happens when no decimal point is present, a zero is the rightmost digit and it *is* significant? This situation is ambiguous and can be avoided by expressing the number in scientific notation, where 428 970 becomes $4.289\,70 \times 10^5$. The convention here is that *all* of the digits present in scientific notation are significant. In this case, there are six significant figures.

How is a number **rounded off** to drop its insignificant figures? Again, there are rules. To round off a number, the number first is truncated to its desired length. Then the *excess* digits are expressed as a decimal fraction. Depending upon whether this fraction is less than, equal to, or greater than 1/2 determines the fate of the least significant digit in our truncated number. If it is greater than 1/2, we round up the least significant digit by one; if it is less than 1/2, it is left alone. If the fraction equals 1/2, the least significant digit is rounded up by one only if that digit is odd. This method reduces any systematic errors that can arise if that number actually resulted from a rounding at a previous step in the calculations. In this light, it is better *not* to round off numbers during the sequential analysis of data and only round off the final results. If numbers are rounded off every time during many sequential calculations, the results are skewed and a systematic error is introduced.

Example Problem 11.12

Statement: Round off the following numbers to three significant figures: 23 421, 16.024, 273.61, 5.6850×10^3, and 5.6750×10^3.

Solution: The answers are 23 400, 16.0, 274, 5.68×10^3, and 5.68×10^3. Note that 16.024 when rounded off to three significant figures is 16.0, where the 0 is significant because it is to the right of the decimal point. Also note that the last two numbers when rounded off to three significant figures become the same. This is because of our rule for round-off when the truncated fraction equals 1/2.

The misuse of significant figures occurs everywhere, from the laboratory reports of college students in Indiana to the buses in Edinburgh, Scotland. Let's examine the sign in the front of the Edinburgh bus that was presented at the beginning of this chapter. The maximum height was 11'3" or 135 inches. There are three significant figures. So, the maximum height in meters should be written as 3.43 m, not 3.4290 as shown having five significant figures. If the maximum height in meters was treated to have the correct number of significant figures, then the English system equivalent should have been written as 11'3.00".

Consider another example. The weight of a large steel cylinder is computed from measurements of its diameter and length. Let its length, L, be equal to 3.32 m (three significant figures) and its diameter, d, equal to 0.3605 m (four significant figures). Its volume, V, would be computed using the formula $V = \pi d^2 L/4$ and be equal to 0.339 m^3. This results from rounding off the computed value of 0.338 874 1... to the required number of three significant figures. This is because the number of significant figures in a computed result equals the *minimum* number of significant figures in any number used in the computation. Now, to convert from this volume to mass, suppose that the density of the steel ingot equals 7835 kg/m^3. This yields a mass of 2660 kg (rounded off from 2655.0793). Note that although the density has four significant figures, only three significant figures are retained in the result. Converting to weight gives 26 000 N (rounded off from 26 037.4), assuming a gravitational acceleration of 9.806 65 m/s^2. There are three significant figures in the final result, although it appears that there are only two. For this situation, the result should be expressed in scientific notation as 2.60×10^4 N, which implies three significant figures.

Example Problem 11.13

Statement: Determine in the appropriate SI units the value with the correct number of significant figures of the work done by a 1.460×10^6 lbf force over a 2.3476 m distance.

Solution: There are four significant figures for the force and five for the distance. Because work is the product of force and distance (assuming that the force is applied along the direction of motion), work will have four significant figures. The SI unit of work is the joule, where J = N·m. Now 1.460×10^6 lbf equals 6.495×10^6 N. So, the work is 1.525×10^6 J or 1.525 MJ.

Finally, what happens to the number of significant figures when you are converting from one unit of a dimension to another, say from inches to feet? The number of significant figures does not change (assuming, which usually is the case, that the conversion factors are known *exactly*).

For example, consider a distance measurement with an uncertainty of 0.125 in. In units of feet with the correct number of significant figures would be 0.0104 ft. Note there are three significant figures in both numbers, even though when converting to units of feet, inches are divided by 12 (which appears to have only two significant figures). The number of significant figures remains three because the conversion from inches to feet is an *exact* conversion (it could be divided by 12.000 when converting from inches to feet).

Example Problem 11.14

Statement: Convert 100.0185 °C to temperature in K.

Solution: The conversion factor from degrees Celsius to kelvin is K = 273.15 + °C. At first hand it appears that there are only five significant figures in the conversion equation. But this is not so because the conversion equation is *exact*. So, 100.0185 °C = 373.1685 K, where both temperatures have seven significant figures.

Thus far, the rules for applying significant figures to numerical calculations appear straightforward. However, applying them directly to experimental results and their uncertainties sometimes leads to ambiguous situations which require common sense and good judgment to resolve.

When expressing a measured value with its associated uncertainty, the precision should be the same between the measured value and its uncertainty. This is an accepted convention in uncertainty analysis. In the previous example of determining the weight of the ingot, the two dimensions of length had different numbers of significant figures. This could result from using one type of instrument to measure the ingot's length and another to measure its diameter. The number of significant figures should correspond with the uncertainty in the measurement. For example, if the uncertainty in a measurement is ± 0.05, then the measurement should be expressed with the same precision, say 1.23 ± 0.05.

Next consider the following apparent dilemma. A measured temperature of 54.0 °C is specified with its uncertainty of ±0.5 °C. Convert it into units of kelvin. Following the rules of significant figures the temperature becomes 327 K, where three significant figures are maintained in the conversion. Now our uncertainty of ±0.5 °C translates directly into an uncertainty of ±0.5 K because the conversion relation between °C and K is linear. Thus, a change in +0.5 °C or −0.5 °C from a temperature in °C is a change in +0.5 K or −0.5 K from the corresponding temperature in K. But look at what has happened. The measured temperature in K has lost the precision specified by the uncertainty. If the level of precision must be the same between a measured value and its uncertainty, then the converted, measured temperature needs to be 327.2 K. Following this convention, it appears that a significant figure was gained in the process!

In the end there is no single, correct answer for this problem. It all depends upon the purpose in performing the conversion. If temperature is only computed, then the converted, measured temperature is 327 K according to the rules of significant figures. If an experimental result is expressed with its uncertainty, then the converted, measured temperature is 327.2 K ± 0.5 K according to convention in uncertainty analysis.

The relation between significant figures and measurement uncertainty has been covered only briefly. This topic is important because many measured quantities are often reported with more significant figures than the uncertainty of the instruments used to measure them. This consequently leads to reporting the corresponding experimental results with more significant figures than the measurement uncertainty. This approach is wrong and can be very misleading when interpreting experimental results.

11.8 Problem Topic Summary

Topic	Review Problems	Homework Problems
Units	1, 3, 4, 5, 6, 7, 8, 18, 20, 23, 25	1, 7, 8, 9, 13, 16
Conversions	1, 2, 6, 7, 9, 10, 11, 12, 13, 18, 19, 20, 24	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17
Significant Figures	6, 14, 15, 16, 17, 21, 22	17

TABLE 11.4
Chapter 11 Problem Summary

11.9 Review Problems

1. What has the most mass? (a) one slug, (b) one kg, (c) one lbm, (d) one g, (e) one N.

2. Four ounces weigh approximately how many newtons?

3. A scientist developing instruments for use in nanotechnology would be most interested in measuring which of the following lengths? (a) the mileage between San Francisco and New York, (b) the length of the leg of an ant, (c) the diagonal of a unit cell of iron, (d) the chord of an airfoil.

4. Which of the following is *not* equivalent to the SI unit of energy? (a) $\text{kg}\cdot\text{m}^2/\text{s}^2$, (b) $\text{Pa}\cdot\text{m}^2$, (c) N·m, (d) W·s.

5. The SI system has how many base units? (a) 2, (b) 3, (c) 4, (d) 7, (e) 250.

6. What is the weight in newtons of a mass of 51 slugs with the correct number of significant figures?

7. An astronaut weighs 162 pounds on Earth (assume that Technical English is spoken). What is the astronaut's mass (expressed in the appropriate SI unit and with the correct number of significant figures) on the surface of Mars where the gravitational acceleration is 12.2 feet per second squared?

8. Which four base units are the same for both the SI and English Engineering systems?

9. A robotic manipulator weighs 393 lbf (Technical English) on Earth. What is the weight of the probe on the moon's surface in newtons (to the nearest tenth of a newton) if the lunar gravitational acceleration is 1/6 of that on Earth?

10. How much work is required to raise a 50 g ball 23 in. vertically upward? Express your answer in units of ft-lbf to the nearest one-thousandth of a ft-lbf.

11. How many molecules of water are there in 36 g of water?

12. If the Mars Rover weighs 742 N on Mars where g = 3.71 m/s^2, what would be its mass (in slugs) on Jupiter where g = 23.12 m/s^2 expressed with the correct number of significant figures?

13. The pressure acting on a 1.25 in^2 test specimen equals 15 MPa. What is the force (in N) acting on the specimen expressed with the correct number of significant figures?

14. If $w = (5.50/0.4) + 0.06$, what is the value of w with the correct number of significant figures?

15. The number 4 578.500 rounded off to four significant figures is (a) 4580, (b) 4579.0, (c) 4579, (d) 4578, or (e) 4570.

16. The number 001 001.0110 has how many significant figures? (a) 10, (b) 9, (c) 8, (d) 7, (e) 4.

17. The number $11.285\ 00 \times 10^{12}$ has how many significant figures? (a) 5, (b) 6, (c) 7, (d) 12, (e) 13.

18. A light-year is a unit that is used by astronomers. (a) What is the dimension of this unit? (b) Is it a base, supplementary, or derived unit? (c) What is the basic definition of this unit? (d) Convert 1.0 light-years into SI base units, and round off the answer to four significant figures.

19. In the manual of a water pump, the pump performance is characterized as 0.12 cipr, where **cipr** stands for 'cubic inch per revolution'. Determine the rpm (revolutions per minute) for this pump when the mass flow rate is 36.87 g/s.

20. Which is the greatest pressure? (a) 1 atm, (b) 100 kPa, (c) 14 psia, (d) 2000 psfa.

21. What is the number of significant figures in the number product of the numbers 037.0160 and $\sqrt{1234567}$? (a) 7, (b) 6, (c) 5, (d) 4.

22. Determine 0.250350 rounded off to four significant figures.

23. What is the approximate weight of a regular-sized apple? (a) 4 lbf, (b) 1 lbm, (c) 1 N, (d) 4 slugs, (e) 1/9.81 kg.

24. Which is greater, 700 Btu/h or 55 cal/s?

25. A hot-wire anemometry probe is used to acquire an output voltage, E (mV), as a function of velocity, U (m/s). The regression analysis determines that the linear relationship between voltage and velocity takes the form

$$E^2 = A + B\sqrt{U}.$$

What are the units of A and B?

11.10 Homework Problems

1. The following presents the original definitions of some of the more customary English units. Try to guess the unit's name for each: (a) the distance from the outstretched fingers to the tip of the nose of King Edgar, (b) the distance covered by 36 barleycorns laid end to end, (c) the width of the thumb of a king or 3 barleycorns laid end to end, (d) the distance a Roman soldier travelled in a thousand paces, (e) the length of a Viking's outstretched arms, (f) the amount of land that could be plowed with a yoke of oxen in a day.

2. Determine your (a) mass in kilograms, (b) weight in newtons, (c) height in meters, (d) volume in liters, and (e) density in kilograms per cubic meter. Finally, (f) compare your density to that of water at standard conditions.

3. Show that a quarter pound hamburger sold in a metric country would be (approximately) a 'Newton Burger'.

4. Show that 1 microcentury approximately equals 1 hr.

5. Compute how many seconds there are in one year and express this result in scientific notation and a familiar numerical constant.

6. On Earth an astronaut weighs 145 pounds (assume Technical English is spoken). Compute (a) this astronaut's weight (in the appropriate SI unit) on the surface of Mars where the gravitational acceleration is 12.2 ft/s^2, (b) her mass on Mars in the appropriate Technical English unit, (c) her mass on Mars in the appropriate SI unit, and (d) her mass on Earth in the appropriate SI unit.

7. The Reynolds number, Re, is a dimensionless number used in fluid mechanics and is defined as $Re = \rho VD/\mu$, where ρ is the fluid density, μ the fluid absolute viscosity, V the fluid velocity, and D the characteristic length dimension of the body immersed in the moving fluid. Because this number has no units, it should be independent of the system of units chosen for ρ, μ, V, and D. In the International system of units, $\rho = 1.16$ kg/m^3, $\mu = 1.85 \times 10^{-5}$ N·s/m^2, $V = 5.0$ m/s, and $D = 0.254$ m. Using this information, compute (a) values for ρ, μ, V, and D in the English Engineering system, (b) Re based on the International System, and (c) Re based on the English Engineering system.

8. The power coefficient, C_P, for a propeller is a nondimensional number that is defined as $C_P = P/(\rho n^3 D^5)$, where P is the power input to the propeller, ρ the density of the fluid (usually air), n the propeller's revolutions per second, and D the propeller diameter. For $\rho = 0.002\ 11$ slug/ft^3, $n = 2400$ rpm, $D = 6.17$ ft, and $P = 139$ hp, (a) express these four values in SI units and (b) compute C_P based on the SI units.

9. The advance ratio J for a propeller is defined as $J = V/(nD)$, where V is the velocity, n the propeller's revolutions per second, and D the propeller diameter. For $V = 198$ ft/s, $n = 2400$ rpm, and $D = 6.17$ ft, (a) show that J is a nondimensional number by "balancing" the units and (b) compute the value of J.

10. An engineering student measures an ambient lab pressure and temperature of 405.35 in. H_2O and 70.5 °F, respectively, and a wind tunnel dynamic pressure (using a pitot-static tube) of 1.056 kN/m^2. Assume that $R_{air} = 287.04$ J/(kg · K). Determine with the correct number of significant figures (a) the room density using the perfect gas law in SI units (state the units with the answer) and (b) the wind tunnel velocity using Bernoulli's equation in units of ft/s. Bernoulli's equation states that for irrotational, incompressible flow the dynamic pressure equals one-half the product of the density times the square of the velocity.

11. An engineer using a barometer measures the laboratory temperature and pressure to be 70.0 °F and 29.92 in. Hg, respectively. He then conducts a wind tunnel experiment using a pitot-static tube and an inclined manometer to determine the wind tunnel velocity through Bernoulli's equation. He measures a pressure difference of 3.22 in. H_2O. Determine the tunnel velocity and express it with the correct number of significant figures in units of m/s.

12. A capacitor consists of two round plates, each of radius $r = 5$ cm. The gap between the plates is $d = 5$ mm. Determine the maximum charge q_{max} of the capacitor in coulombs if the breakdown potential of the air is $U_{max} = 10$ kV. Find the capacitor energy in both the International (SI) and the English Engineering (EE) systems.

13. A wheel of radius $R = 50$ cm and of mass $m = 1$ kg rolls on the surface without slipping. The velocity of the center of the wheel is $v = 5$ m/s. Determine the kinetic energy E of the wheel and its angular velocity ω. What units are used for E and ω? Give the fundamental dimensions.

14. The average mass of the male human brain is 1361 g (Jerison, 1973). Gravitational acceleration on the surface of Mars is 0.35 times that of Earth. The mass of Mars is 10 % that of Earth. Determine (a) the average weight (in lbf) of the male human brain on Mars, (b) the average mass (in kg) of the male human brain on Mars, and (c) the kinetic energy (in MJ) of the average male human brain the instant that it is launched from Earth at a velocity of 1 km/s.

15. Express the gas flow rate of 100 sccm (standard cubic centimeters per minute), which is a standard unit of measure in vacuum-based microfabrication equipment, in (a) liters per minute and (b) cubic miles per millisecond.

16. Guess the names of the following common units: (a) amount of seed to sow an acre of ground, (b) distance from head to wrist, (c) distance an arrow would fly, (d) distance walked on foot in one hour, (e) distance a shout would carry, (f) actual (approximate) distance seen when squatting beneath a horse.

17. Given that the mass, M, of Saturn is 5.68×10^{26} kg, the radius, R, is 5.82×10^{7} m, and g (m/s^2) $= GM/R^2$, where $G = 6.6742 \times 10^{-11}$ N·m^2/kg^2, determine (a) the correct number of significant figures in g and (b) the mass on Earth (in kg) of a rock that weighs 200 lbf on Saturn.

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12

*Technical Communication

CONTENTS

The horror of the moment, the King went on, I shall never forget. You will, though, the Queen said, if you don't make a memorandum of it.

Lewis Carroll. 1945. *Alice's Adventures in Wonderland.* Racine: Whitman Publishing Company.

... the computation leads to rather reasonable results at least for first estimates.

From the draft of an anonymous student's 2009 Ph.D. dissertation.

12.1 Chapter Overview

The queen was right. All is lost if you do not make a memorandum of the results. This chapter describes the tools needed to help you prepare for technical communication. Suggested formats for technical memos and technical reports are presented. Guidelines for proper writing in general as well as specific to technical memoranda are given. Graphical presentation is discussed. Also, guidelines for oral technical presentations are summarized.

12.2 Guidelines for Writing

A short list of critical writing rules is presented in this section. These rules relate to neither the style nor the content of the writing. They only account for the most fundamental aspects of clearly written communication. All technical memoranda and reports must adhere to these rules. There are many good texts that present the styles for writing, including Strunk and White [1] and Baker [2].

12.2.1 Writing in General

- Words must be spelled properly.
  - incorrect: Mispellings should be avoided.
  - correct: Misspellings should be avoided.
- Sentence fragments must be avoided.
  - incorrect: First, a look behind the scenes.
  - correct: First, we will look behind the scenes.
- The subject and verb within the sentence must agree.
  - incorrect: A motion picture can improve upon a book, but they usually do not.
  - correct: A motion picture can improve upon a book, but it usually does not.
- Avoid abrupt changes in tense; past tense is best.
  - incorrect: We weigh the sample ....
  - correct: The investigators weighed the sample ....
- Avoid abrupt changes in person; third person is best.
  - incorrect: We weigh the sample ....
  - correct: The investigators weighed the sample ....
- Avoid abrupt changes in voice; active voice is best.
  - incorrect: It was decided ....
  - correct: The investigators decided ....

- Contractions should be avoided.
  - incorrect: Don't use contractions.
  - correct: Do not use contractions.
- Avoid splitting infinitives.
  - incorrect: ...to enable us to effectively plan our advertising ....
  - correct: ...to enable us to plan effective advertising ....
- Avoid dangling participles.
  - incorrect: Going home, the walk was slippery.
  - correct: When I was going home, the walk was slippery.
- Compound modifiers must be hyphenated properly.
  - incorrect: ...the red, hot flame ....
  - correct: ...the red-hot flame ....
- A sentence should not end with a preposition.
  - incorrect: What did she write with?
  - correct: With what did she write?
- Proper end punctuation must be used.
  - incorrect: Be careful
  - correct: Be careful.

12.2.2 Writing Technical Memoranda

Writing a good technical memorandum (also termed a *memo*) requires practice. Very few people have the innate ability to write memoranda well, especially technical memoranda. They learn to do so through experience. Some guidelines can be followed that relate to style. The following suggestions help to make a better document:

- Write technical memoranda in the third person.
- Use the past tense throughout technical memoranda.
- Limit the length of sentences. Break long sentences into shorter ones. Scientists and engineers tend to write long sentences.

- Segment ideas into paragraphs such that the reader is led through the presentation in a smooth and effortless fashion.

- Type all memoranda. Choose a word-processing software package and learn how to use it effectively. This will help to produce a professionally presented document which usually includes text, figures, tables, and equations.

- Proofread and check for spelling errors. It is best to have someone else do the proofreading.

- Provide a plausible explanation based upon scientific principles whenever predictions and measurements differ. Mistakes or 'human error', for example, are not acceptable reasons. Support any explanation by some calculations.

- Report the average value with its uncertainty whenever reporting results based upon multiple measurements. Avoid simply listing all the measured values.

- Use correct English. Do not confuse commonly used words such as 'its' and 'it's'. The former is possessive; the latter is a contraction. Other examples include 'affect' and 'effect', 'farther' and 'further', 'ensure' and 'insure', 'because' and 'since', 'approximately' and 'about', and 'decrease' and 'drop'.

One of the most frustrating experiences for a reader is to read a document having many mistakes and missing essential information. Some absolute rules can be established for writing technical memoranda. A memorandum that violates any one of these rules is incomplete.

1. Every variable and symbol, either measured or analytical, must be identified.

2. Every variable's units must be presented.

3. The proper number of significant figures must be used with all numbers.

4. Uncertainties must be given for every measured and predicted variable. Nominal values must be included. The assumed confidence level must be stated. Often it is easiest to present uncertainties in a table and include supporting calculations in an appendix.

5. The physical concepts behind a model must be explained. Do not simply present the model's equations.

6. Do not use relative words, such as 'good', 'reasonable', 'acceptable', 'significant', and so forth, when describing an agreement between values. See the second quotation at the beginning of this chapter as an incorrect example. Quantitative statements must be made when making a comparison, such as "x differed from y by z %."

7. Each figure or table included must be referred to and discussed in the text. Do not say "calibration data is shown in Figure 1" or "results are presented in Figures 1 through 6" and then fail to discuss what is shown in each figure.

8. Equations must be punctuated with commas or periods, as if they were part of a sentence. Do not let them dangle in space.

9. A '0' must be included in the front of the decimal point if no other number is present. The decimal point can be missed by the reader when the '0' is absent.

10. All pages must be numbered consecutively except the cover sheet.

12.2.3 Number and Unit Formats

The presentation of numbers and units should follow specific formats [3]. A few of these guidelines that are very appropriate to presenting technical information include the following:

- Use SI units. Give the equivalent values in other units in parentheses following the SI unit values only if necessary.

- Avoid using unacceptable abbreviations such as sec for second, cc for cubic centimeter, l for liter, ppm for parts per million. For example, express 7 ppm as 7 μL/L.

- Include units for each number when using composite expressions, such as those involving areas, volumes, and ranges. Volume, for example, would be written as 2 m $\times$ 3 m $\times$ 5 m. The correct expression for a range of values would be 23 L/s to 45 L/s. Use the word 'to' instead of a dash when expressing a range. For example, write 5 to 10 rather than 5-10.

- Use Arabic numerals and symbols for units, such as "the mass was 15 kg." Keep a space between the numeral and the symbol. This is also true for percentages, which should be expressed as x % (note the space) and not as x%. However, a degree sign indicating an angle does *not* have a space between it and its symbol.

- Italicize quantity symbols, such as l, V, and t for length, volume, and time, respectively.

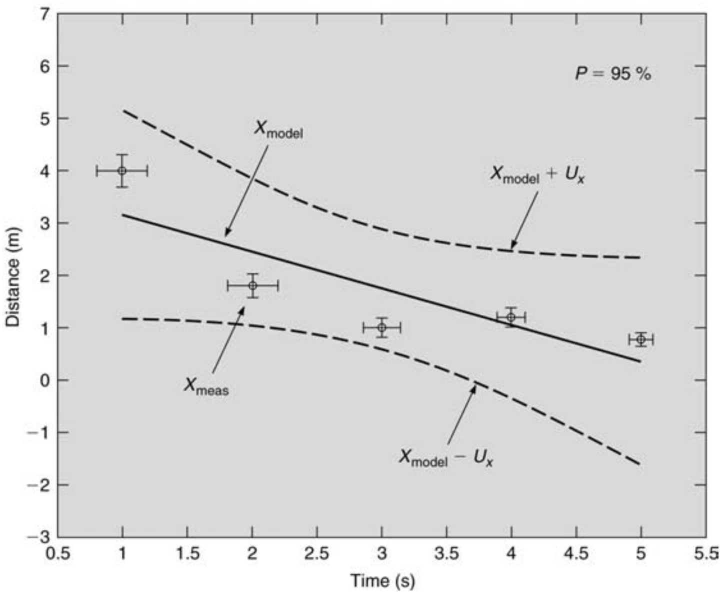

FIGURE 12.1
Graphical presentation of a comparison between predictive and experimental results.

- Put unit symbols in Roman type. Subscripts and superscripts may be in either Roman or italic type.
- When there are more than four digits in a number on either side of the decimal marker, use spaces instead of commas to separate numbers into groups of three, counting in both directions from the decimal marker, such as 3.141 592 654.
- Express all logarithms using log, with their bases as subscripts, such as $\log_e(x)$ and $\log_{10}(x)$. Do not use $\ln(x)$ for the natural logarithm.
- Use decimal prefixes with a number's units, keeping its numerical value between 0.1 and 1000, such as 1.05 MJ instead of 1.05×10^6 J.
- When forming adjectival phrases involving symbols for SI units, do not use a hyphen. For example, write 2 kg laptop. When units are spelled out, a hyphen is used. For example, write 2-kilogram laptop.

12.2.4 Graphical Presentation

The proper presentation of quantitative information is essential to good technical communication. Most quantitative information is presented graph-

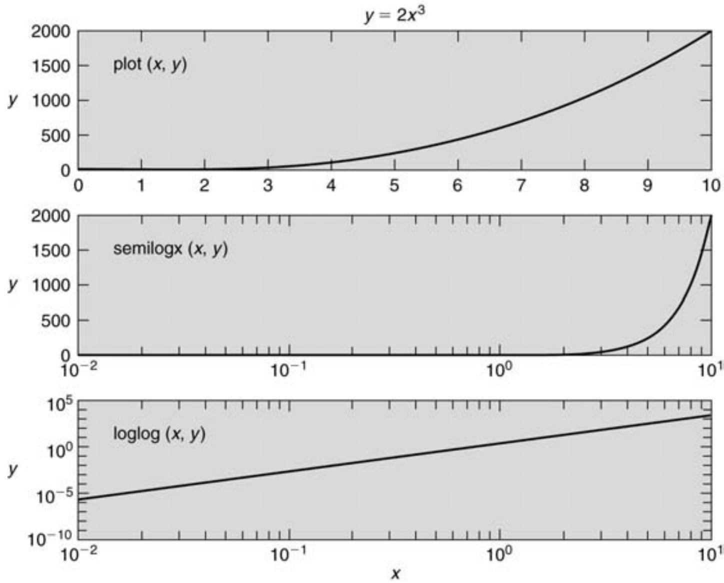

FIGURE 12.2
The same function obtained using the MATLAB `plot` (top), `semilogx` (middle), and `loglog` (bottom) commands.

ically in a variety of ways [4]. The types of plots commonly used for the graphical presentation of experimental data include Cartesian, semilogarithmic, logarithmic, stem, stair, and bar formats.

The Cartesian plot is the most common type of plot, in which the values of the dependent variable are plotted versus the values of the independent variable. Typically one or two dependent variables are included in one plot, having either two or three dimensions. An example of a Cartesian plot is shown in Figure 12.1.

Some physical systems respond in an exponential manner to external forcing. Many physical variables are related to one another through a power law, either linear, quadratic, or higher order. Possible relations can be ascertained by plotting the dependent variable versus the independent variable, such as time. Figure 12.2 shows three graphical representations of the same power-law relation. For $y = a\exp(bt)$, a semi-log plot with the y-axis as the logarithmic axis gives a line of slope value $b/\log_e 10$ and intercept value $\log_{10} a$. For $y = ax^b$, a log-log plot of y versus x will yield a line of slope value b and intercept value $\log_{10} a$.

Sometimes a series of values that were acquired sequentially needs to be examined. This usually is done to observe the trend of the values in time. Figure 12.3 displays three types of plots of the same data. A stem plot

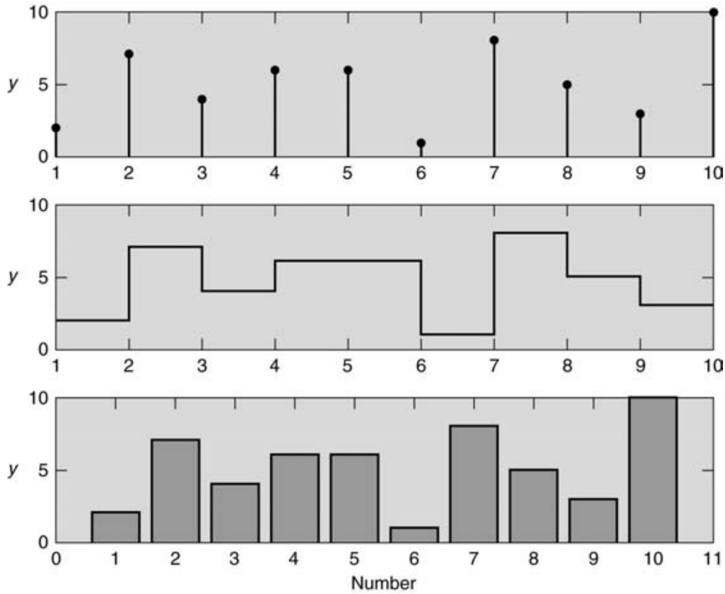

FIGURE 12.3
Ten sequential values plotted using the MATLAB `stem` (top), `stairs` (middle), and `bar` (bottom) commands.

extends a line from the abscissa up to the ordinate value that is designated by a marker. A stairs plot connects lines from each ordinate value to the next, mimicking a stairway. The bar plot gives a rectangular bar of a fixed width for each ordinate value. The type of presentation determines which plot is best.

Often when the amounts that contribute to a whole need to be displayed, the pie chart is used. This graphically is in the shape of a circular pie, as shown in Figure 12.4, in which each contributing amount is displayed as a sector of the pie. Each sector's area is its proportional contribution to the total area. Usually, each sector has a different color.

Some data may follow an angular dependence, such as those representing acoustic radiation or the surface pressure distribution around an object. A polar plot, such as that shown in Figure 12.5, can be constructed for this purpose. The magnitude of the variable for a given θ is plotted as a distance from the origin. Data that represent cyclic processes will pass more than once through the origin.

The following guidelines should be followed when constructing plots (refer to Figure 12.1 as an example):

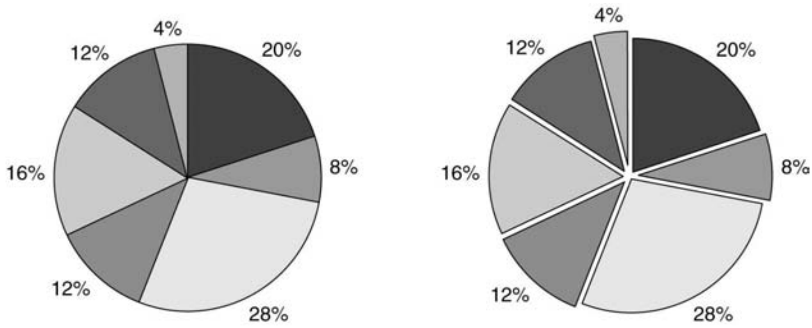

FIGURE 12.4
A standard pie chart (left) and a pie chart with "exploded" areas (right).

- A title or caption must be present.
- Both the abscissa and ordinate must be labeled with the name of the quantity plotted and its units in brackets or parentheses.
- Tick marks should be used on each side of both axes. Internal tick marks are preferable.
- All curves must be labeled either on the plot using arrow indicators or in the plot's legend when more than one curve is plotted.
- Analytical results must be plotted as a solid curve. Do not use symbols.
- Numerical results must be plotted as a dashed or dotted curve. Do not use symbols.
- Experimental results must be denoted with symbols and error bars, using the same symbol for a given data set.
- Any curve representing an estimate must be presented with ± confidence limit curves evaluated at P percent confidence.

12.3 Technical Memo

The format of a technical memo is similar to that commonly used in industry and at national laboratories. It is a concise, formal presentation of findings on a particular technical issue. It is *not* a comprehensive explanation of the theoretical or experimental methods used, but rather it is a summary highlighting the results obtained. Typically, the body of a technical memo should

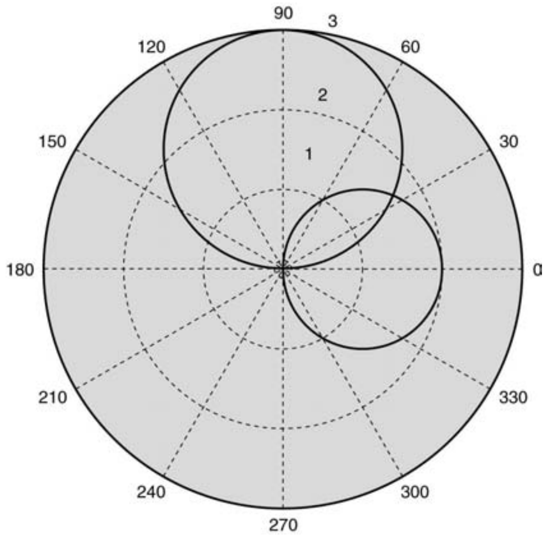

FIGURE 12.5
A polar plot of 3sin(θ) centered along the $\theta = 90$ axis and 2cos(θ) centered along the $\theta = 0$ axis.

not exceed two to three pages in length, excluding any supporting material that usually is placed in appendices. The following format is suggested.

Date: 1/18/10
To: Professor P.F. Dunn
From: I.M.A. Student
Subject: Rocket Thrust Measurement

Summary: This should be one paragraph that summarizes the important results and states the significant conclusions. When writing this section, assume that this may be the only part of the technical memo that actually is read. Thus, it needs to be self-contained and not refer to any other written section, graphs, and tables that are contained in the body of the memo. Key results must be presented. Values of important experimental parameter ranges must be included. If theory also is presented, a quantitative statement about agreement or disagreement with the experimental results needs to be made.

Findings: This part covers in more detail what was summarized above. Enough information must be provided such that an engineer at your level could critically evaluate the approach and methods and understand how you arrived at the results and conclusions, without your providing any information beyond what is written. Only the most important figures and tables need to be included here. Supporting material such as additional plots, program listings, or flow charts can be included as attachments (appendices) to the memo. All figures and tables must be numbered and referenced properly in the text. Include only the material that you consider necessary to support the conclusions. Never present results, especially in figures and tables, that you do not specifically discuss in this section. Do not attach volumes of computer-generated output without any explanation. The reader is impressed not by the volume of data collected but by the value of it.

References: References must be numbered in consecutive order. Do *not* include any references that are not cited in the memo. The following reference format conforms to that specified by *The Chicago Manual of Style* [5]:

Journal article references: 1. Student, N.O.D., and S.A.M. College. 2010. Measurements through a Hard Semester. *J. Heat Mass Transfer* 11:548-556.

This includes the last names and first and middle initials of all of the authors, the year of publication, the title of the article, the abbreviated title of the journal (italicized), the volume number, and inclusive page numbers of the article.

Book references: 2. M. A. Saad. 1992. *Compressible Fluid Flow.* 2nd ed. New Jersey: Prentice Hall Inc.

The book title is italicized.

Appendices: These addenda contain supplementary material, such as detailed derivations or calculations. They are not meant to contain a record of everything that was done on the memo's subject. Include only what is needed to support material presented in the body of the memo. Do not place the results here and then refer to them from the memo.

12.4 Technical Report

This section describes the format that typically is required for laboratory reports. This format essentially parallels that of many journal publications. Each of the following sections need to be included in the report. As stated for the technical memo, write the report as if you were writing to another engineer at your level. The suggested format is described below.

1. **Title Sheet**: List the class and its number, report title and number, your name, all group members' names, and the date. This is the cover sheet of the report. It does not have a page number.

2. **Abstract**: The primary purpose of the abstract is to provide the reader with a brief and sufficient summary of the project and its results. It is to be short (no more than approximately 100 words) and informative. It must indicate clearly the nature and range of the results contained in the report. The abstract must stand alone. No citing of numbered references, symbols, and so forth, must be made unless they are obvious without any reference to the report. The easiest procedure is to write the abstract summarizing the entire body of the report *after* the report has been written.

3. **Table of Contents**: Each of the subsequent sections should be listed with its corresponding first page number in the report.

4. **List of Symbols (Nomenclature)**: English symbols are first listed in alphabetical order, then Greek symbols in alphabetical order. Be sure to describe adequately your nomenclature, for example, not just 'viscosity' but either 'absolute (dynamic) viscosity' or 'kinematic viscosity'. Also note that in some cases the mere descriptive name of the symbol is not sufficient. For example, when listing coordinates be sure to specify the coordinates' directions with respect to some reference point. Also, when describing nondimensional numbers, specify their definitions in terms of the other symbols listed. The best procedure in gathering the nomenclature is to construct the list of symbols *after* the body of the report has been written.

5. **Introduction**: This section introduces the reader to the nature of the problem under investigation. It explains the history and relevance of such an experiment and its application. Previously published papers relevant to the experiment should be cited here. The general objectives of the experiment should be stated. Do not simply summarize the experimental objectives. Provide a guide for the reader as to what will follow in the report.

6. **Approach**: This section sometimes is referred to as *methods* or *procedure*. It needs to describe briefly the experimental, analytical, and numerical methods used to arrive at the results. There must be sufficient detail to permit a critical evaluation of the methods used and replication of the results by another party. It is not necessary, however, to give full descriptions of all of the methods that are described in detail elsewhere, for example, how a particular numerical integration scheme works step-by-step. Uncertainty estimates for all parameters and procedures used to arrive at the results must be provided. Usually, it is preferable to present these estimates in a table. A block diagram or flow chart of the steps in the approach can be very helpful to the reader. Alternatively, a step-by-step approach can be put in narrative form. A flow chart should be included for each computer program used. A listing of each program should be presented in an appendix and documented with sufficient comments such that it can be followed easily by the reader.

7. **Results**: The results of your experiment are presented here, usually facilitated by graphs, figures, and tables. The findings of the experiment

are presented but neither discussed nor evaluated in this section. Keep in mind that you want to be concise when presenting the results. Put results in graphical form whenever possible. Sample calculations can be put in an appendix. When the results cover several aspects of the project, subdivide this section such that each part deals with one major aspect. The results of an uncertainty analysis must be provided. Detailed, supporting calculations should be presented in an appendix. Mention specifically what is contained in *each* figure and table. Do *not* say, for example, that "the results are shown in figures 1 through 6," and then fail to explain what is presented in each figure.

8. **Discussion**: This section should include a discussion and evaluation of the results obtained and their relation to other pertinent studies, if any. The findings of your experiment are interpreted in this section. Express your scientifically justified opinion in this section about the facts that were presented in the previous section. Remember the distinction between fact and opinion. Point out the limitations of how you approached the experiment and how you would improve on your approach. Be constructively critical. Describe what you have learned from the experiment.

9. **Conclusions**: Briefly conclude the major findings of your experiment. Do not introduce anything new or continue to discuss the results.

10. **References**: These follow the same guidelines as for the technical memo.

11. **Appendices**: These follow the same guidelines as for the technical memo.

12.5 Oral Technical Presentation

Many technical societies now have web pages that provide instructions for speakers [6]. The mode of oral presentation has changed considerably over the past several years. Most professional presentations now are made using standard software packages. The resulting, user-designed slides are projected digitally. Thus, the visual format of the presentation becomes very important.

The success of an excellent presentation lies in its organization, preparation, and delivery. The amount of time spent in preparing an oral presentation should be equivalent to that spent in writing a technical paper. Sometimes preparing an oral presentation is even more challenging. The listener is not as attentive as a reader. Presentation time is limited. The presenter must speak with confidence and enthusiasm for the talk to be effective. Adequate time for preparation and practice must be spent to produce a

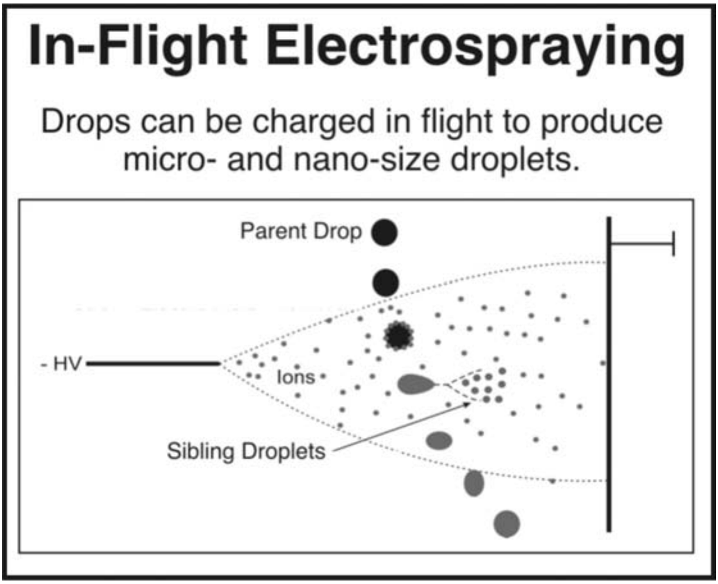

FIGURE 12.6
Example slide with suggested font size.

professional and well-received presentation. Practice also allows timing to be rehearsed.

Most presentations have two implicit goals: to deliver information and to have the audience understand it. This requires that information be presented clearly and concisely such that it is easily understood. The presentation must be substantive, including a statement of the problem, followed by a description of its solution, and the results.

Some guidelines for a good oral presentation are as follows:

- Start with a title slide, followed immediately by a slide that outlines the talk.
- Break the body of the talk into sections, each making a specific point.
- Guide the listeners through the presentation by referring back to the outline at appropriate times.
- Conclude with a slide that summarizes the main points.
- Minimize the number of words and information presented on a slide. Keep it simple. Going into unnecessary detail will only lose the audience.
- Use an appropriate font size and type that is supported by a contrasting, simple, and pleasing background. Place an 8 in. by 10 in. copy of a slide

on the floor. You should be able to read it clearly while standing directly over it. Refer to Figure 12.6 as an example.

- Follow the same rules for figures and equations that would be done for a written document. Label the axes. Provide units. Define all variables.
- Use your notes as a guide. Do not read directly from them.
- Stay focused on the topic. Avoid rambling beyond what was planned.
- Watch the time. Pace yourself. Do not exceed the time allotment.
- Speak enthusiastically. Do not speak in a monotone. Vary the rate and pitch.
- Try to stand near one place. Avoid walking aimlessly around.
- Avoid unconscious gestures, especially with your hands.
- Use a pointer to focus attention on an area of a slide. Do not unconsciously wave the pointer around, especially a laser pointer, as it is very distracting.
- Look directly at the audience. Do not look over peoples' heads or stare at the floor.
- Avoid using vocal pauses, such as "you know" and "ah". Have someone listen to a practice presentation and note the number of vocal pauses made. You may be surprised at how many times you pause. Learn to break the habit.
- If you make a mistake, correct it and go on. Avoid joking about it or making excuses for it. Be confident!
- Finally, relax and enjoy giving the presentation. It represents your hard work and interest.

12.6 Problem Topic Summary

Topic	Review Problems	Homework Problems
Writing	1, 2	
Formatting		1, 2, 3, 4, 5

TABLE 12.1
Chapter 12 Problem Summary

12.7 Review Problems

1. A technical memo should contain only the following sections: (a) abstract, introduction, results, (b) abstract, results, conclusions, (c) introduction, results, conclusions, (d) summary, list of symbols, results, references, (e) summary, findings, references.

2. Technical memoranda are presented customarily in what tense? (a) future, (b) passive, (c) past, (d) present, (e) subjunctive.

12.8 Homework Problems

1. Correct each of the following if it contains any format errors. (a) $2 - 3$ m/s, (b) 35 %, (c) 23 045.62, (d) 1.2E-4 J, (e) 23.4kg.

2. Identify all of the instances of incorrect format in the sentence: A circular arc's angle of 47.2 °, which is 4.2 in. long, is equivalent to 4×10^{-3} sr on a sphere.

3. The text below was taken from an abstract written to describe the results of an experiment. For each numbered and underlined group, identify, *if any*, the improperly formatted terms by providing the corresponding correctly formatted versions.

4. An experiment was conducted to determine the heat transfer from a hot heat exchanger tube to air. The velocity of the air was set as (1) 20 ft./sec (240 in./sec). Its flow rate was (2) 525 liters/min. An optical pyrometer recorded the surface temperature of the tube, (3) T, to be (4) 2,000°F. The temperature of the air was (5) 65% less.

5. The text below was taken from an abstract written to describe the results of an experiment. For each numbered and underlined group, identify, if any, the improperly formatted terms, and then provide the corresponding correctly formatted versions.

   An experiment was conducted to determine the cross-sectional (1) (x and y) distribution of the velocity of air at various (2) z locations along the length of a wind tunnel having a (3) 1 × 2 m cross-section. The velocity measurements made were in the range of (4) 14-30 m/sec. The air temperature, (5) T, was (6) 72°F and the volumetric flow rate, (7) Q, was set at (8) 30 liters/min ±2% for the entire duration of the experiment.

6. Correct each of the following if it contains any format errors.

   (a) 23 045.62, (b) 1.2E-4J, (c) 23.4kg.

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A

Glossary

absolute quantization error one-half the instrument resolution

absolute sensitivity coefficient change in a result due to an incremental change in a particular variable

absolute uncertainty uncertainty in a particular variable

accuracy closeness of agreement between a measured and true value

active requiring no external power supply to produce a voltage or current

active filter filter composed of operational amplifiers, resistors, and capacitors

aleatory caused by inherent randomness and/or finite sampling

aliasing false lower frequencies created by a sampling rate less than twice the highest frequency of interest

almost-periodic comprised of two or more sinusoids of arbitrary frequencies

alternating current (AC) current varying cyclically in time

amplitude magnitude or size

amplitude ambiguity false amplitudes occurring in the amplitude spectrum at the fundamental and adjacent frequencies

amplitude spectrum plot of a signal's amplitude versus frequency

analog continuous in time and magnitude

anode positively charged terminal that loses electrons

anti-alias filter filter that prevents signal aliasing

autocorrelation correlation of a signal with itself at various time delays

autocorrelation coefficient number between −1 and 1 that characterizes the extent of autocorrelation

autocorrelation function function used to determine the autocorrelation

balanced bridge Wheatstone bridge when the products of the cross-bridge resistances are equal

bandpass filter filter passing a signal's amplitude over a range of frequencies, but neither above nor below that range

base center element of a transistor

base units dimensionally independent units

beat frequency relatively low frequency equal to one-half the difference in two nearly equal frequencies

bias error *see* systematic error

binomial distribution discrete distribution describing the probability of one of two possible outcomes

calibration process of comparing the response of an instrument to a standard input over some range

calibration curve plot of a calibration's input versus output data

capacitance ratio of the charge on one of a pair of conductors to the electrical potential difference between them

cathode negatively charged terminal that gains electrons

central moments statistical moments calculated with respect to the centroid or mean

central tendency tendency to scatter about an average value

charge electrical quantity representative of the excess or deficiency of electrons

coefficient of determination square of the sample correlation coefficient

coherent no numerical factors other than 1 occur in all unit equations

collector one of the end elements of a transistor

combination the number of possible ways that members of a set can be arranged irrespective of their order

combined estimated variance *see* combined uncertainty

combined standard uncertainty combination of all individual uncertainties

complement *see* null set

complex having more than one period

complex conjugate complex number with a complex part that is opposite in sign to its complex number counterpart

conditional probability the probability of an event given that specified events have occurred previously

consistent no numerical factors other than 1 occur for all unit equations in a system of units

continuous without a break or cessation

control experiment an experiment in which *all* variables are identified and can be controlled

conventional current flow current flow from anode to cathode

conventionalism process in which experiments are performed to illustrate an aspect of nature

correlation analysis process of calculating the correlation coefficient

correlation coefficient number between −1 and 1 that characterizes the amount of correlation

covariance mathematical function that characterizes the relationship between two random variables

coverage factor number representing an assured probability of occurrence that multiplies the combined standard uncertainty to determine the overall uncertainty

critically damped having a damping ratio of unity

cross-correlation correlation between two variables

cross-correlation coefficient number between −1 and 1 that characterizes the extent of cross-correlation

cumulative probability distribution function *see* probability distribution function

current charge per unit time

cutoff frequency the frequency at which a signal's amplitude is attenuated

damping ratio nondimensional parameter characterizing a second-order system

decade frequency ratio of 10:1

deflection method method using a balanced Wheatstone bridge to achieve an output voltage proportional to a change in resistance

dependent affected by changes in an independent variable

degrees of freedom number of data points minus the number of constraints used for the required calculations

derived type of unit composed of base and supplementary units

detailed uncertainty analysis uncertainty analysis that identifies the systematic and random errors contributing to each measurand's overall uncertainty and then propagates them into the final result

deterministic signal signal that is predictable in time or space, such as a sine wave or a ramp function

digital having discrete values at discrete times or locations

dimension measure of spatial extent

direct current (DC) current constant in time

Discrete Fourier Transform (DFT) discrete representation of the Fourier transform

discrete having values at distinct times or locations

discrete Fourier series discrete representation of a Fourier series

dynamic varying in time

dynamic calibration calibration using a time-dependent input

dynamic error error related to the amplitude difference between a system's input and output

electric field electric force acting on a positive charge divided by the magnitude of the charge

electric potential potential energy per unit charge

electric power electric energy transferred per unit time

electric resistivity material property related to its resistance

emitter electrode in a transistor where electrons originate

ensemble collection of time history records

epistemic caused by a lack of knowledge

ergodic ensemble-averaged values equal the corresponding average values computed over time from an arbitrary, single time history in the ensemble

even function function symmetric about the ordinate

event outcome

exhaustive space spanned by a set and its complement

expectation *see* expected value

expected value probabilistic average value

experiment act in which one physically intervenes with the process under investigation and records the results

explorational conducted to explore an idea or possible theory

extraneous variable that cannot be controlled

fallibilism process in which experiments are performed to test the validity of a conjecture

Fast Fourier Transform (FFT) method that recursively divides the sample points in one-half down to two-point samples before it performs the Fourier transform

finite bounded or limited in magnitude or in spatial or temporal extent

first central moment mean

first-order replication level level that considers the additional random error resulting from small uncontrolled factors

Fourier analysis procedure that identifies the Fourier amplitudes of a signal

Fourier coefficients coefficients in a Fourier series

Fourier series series represented by sines and cosines of different periods and amplitudes that are added together to form an infinite series

Fourier synthesis procedure that constructs a signal representation from a series of appropriately weighted sines and cosines

Fourier transform mathematical transformation of a signal that gives the signal's amplitude versus frequency

Fourier transform pairs pair of equations consisting of the Fourier transform and the inverse Fourier transform

fourth central moment *see* kurtosis

fractional uncertainty uncertainty in a result divided by the value of the result

frequency distribution plot of the number of occurrences of a certain value divided by the total number of occurrences versus the value of the occurrence

frequency measure of a signal's temporal variation

fundamental dimensions length, mass, time, temperature, electrical current, amount of substance, and luminous intensity

Gaussian distribution *see* normal distribution

general uncertainty analysis simplified approach to uncertainty analysis that considers each measurand's overall uncertainty and its propagation into the final result

high-pass filter filter that passes a signal's amplitude above but not below a specific frequency

histogram literally means *picture of cells*; plot of the number of occurrences of a certain value versus the value of the occurrence

hysteresis difference in the indicated value obtained when approaching a particular input value in increasing versus decreasing directions

impedance electrical resistance of a circuit containing linear passive components (resistors, capacitors, and inductors)

impulse rapid change of a variable in time

independent not dependent upon another variable

inductivism process arriving at the laws and theories of nature based upon facts gained from experiments

infer estimate statistically

instrument error sum of an instrument's elemental errors identified through calibration

intersection set of all members common to both sets

intrinsically linear variables variables in a higher-order equation that can be transformed to yield a linear expression

inverse Fourier transform inverse of the Fourier transform that gives the signal's amplitude versus time

kurtosis fourth central moment normalized by the square of the variance

least significant digit rightmost nonzero digit

level of significance one minus the χ^2 probability

linear device device in which the output amplitude is linearly proportional to its input amplitude

lognormal distribution continuous distribution of the logarithm of a normally distributed variable

loop closed path in a circuit going from one node back to itself without passing through any intermediate node more than once

low-pass filter filter that passes a signal's amplitude below but not above a specific frequency

magnitude extent of dimension; size

magnitude ratio ratio of a dynamic system's output amplitude to its input amplitude

measure *see* magnitude

measurand measured variable

measurement error true, unknown difference between measured value and true value

measurement uncertainty estimate of the error in a measurement

mode most frequently occurring value

modulus absolute value of a complex number

most significant digit leftmost nonzero digit

mutually exclusive two sets not sharing any common members

mutually orthogonal set in which each pair of functions is orthogonal

node point in a circuit where two or more elements meet

nondeterministic random

nonstationary not stationary (*see* stationary)

normal continuous distribution caused by a very large number of small, uncontrollable factors that influence the outcome

normal equations equations resulting from the method of least squares

normalized z-variable a nondimensional variable indicating the number of standard deviations that a specific value deviates from the mean value

notch filter filter that passes a signal's amplitude over a range of frequencies above and below a specified range

Nth-order replication level level at which more than one random error beyond that in the first-order replication level is considered

null method use of a Wheatstone bridge to determine an unknown resistance by having two of its other four resistances fixed and varying the fourth until the bridge is balanced

null set set of all occurrences in which a desired event is not the outcome

numerical equation equation containing only the measures of physical quantities

octave frequency ratio of 2:1

odd function function symmetric about the origin

order degree in a continuum of size or quantity

orthogonal property of two functions whose inner product is equal to zero over an interval

outcome result of a test

over-damped having a damping ratio greater than unity

overall uncertainty measure of the uncertainty in a variable; the product of the coverage factor and the combined standard uncertainty

parameter variable or function of variables that is fixed during an experiment

passive requiring an external power supply to produce a voltage or current

pedagogical class of experiment designed to teach the novice or to demonstrate something that is already known

periodic repeating itself in time

permutations number of ways that a set can be arranged respective of its members' order

phase lag lag of an output signal with respect to an input signal

Poisson distribution a continuous distribution describing rarely occurring events

pooled formed into one set from a set of replicated experiments each involving multiple measurements

population collection of all possible values of a random variable

potential difference difference between two electric potentials

power spectrum plot of a signal's power versus frequency

precision variation of a variable's values obtained by repeated measurements

precision error *see* random error

precision interval interval characterized by the product of a coverage factor and a random uncertainty

probability number of specific occurrences over the total number of occurrences

probability density function (pdf) function when integrated yields the probability

probability distribution function (PDF) integral of the probability density function; also known as the cumulative probability distribution function

ramp method method to perform electronically analog-to-digital conversion by increasing a voltage and comparing it to the analog input signal's voltage

random having no particular order

random error error related to the scatter in the data obtained under fixed conditions; also known as the precision error

random uncertainty estimate of the random error

random variable variable whose value has no deterministic relation to any of its other values

range lower to upper limits of an instrument or test

reactance influence of a coil of wire upon an alternating current passing through it that impedes the current

readability closeness with which the scale of the instrument is read

reduced-chi square variable χ^2 variable normalized by the number of degrees of freedom

regression analysis process identifying the regression coefficients in the method of least squares

regression coefficients coefficients found in the method of least squares

relative accuracy accuracy divided by the true value

relative systematic uncertainty ratio of the reliability of the systematic uncertainty to the systematic uncertainty

reliability estimate of the accuracy of a systematic uncertainty

repeatability ability to achieve the same value upon repeated measurement

repetition repeated measurements made during the same test under the same operating conditions

replacement return of members to their set after selection, thereby allowing for their re-occurrence

replicates experiments identical to the original

replication duplication of an experiment under similar operating conditions

resistance defined by Ohm's law as the ratio of voltage to current

resolution smallest physically indicated division that an instrument displays or is marked

result variable that is a function of one or more measurands

ringing frequency frequency at which a second-order system rings or continually oscillates

rise time time required for a first-order system to respond to 90 % of a step change

root mean square (rms) positive square root of the mean of the squares

round off truncate a number to its desired length

sample subset of the population

sample mean mean of a sample

sample variance variance of a sample

sample-to-sample measurand values are recorded for multiple samples

scattergram discrete representation of an analog signal

scientific method method of investigation involving observation and theory to test scientific hypotheses

second central moment the variance

sensor device that senses a physical stimulus and converts it into an impulse

sequential systematically increased

set group of all occurrences in which a desired event is the outcome

settling time time beyond which a second-order system's response remains within ±10 % of its steady-state value

signal measurement system's representation of the temporal variation of a measurand

signal conditioning preparing the signal in its final form to be processed optimally and then recorded

signal processing operating on a signal to obtain desired results

signal sample period time period used to determine the statistical properties of a signal

significant figures number of digits required to express a result

simple RC **filter** filter comprised of a resistor and a capacitor

skewness third central moment normalized by the cube of the standard deviation

source groups groups that help to categorize sources of error, which are typically grouped as calibration, data acquisition, and data reduction

standard known value usually used as a basis of calibration

standard deviation square root of the variance, which characterizes the width of the probability distribution

standard deviation of the means (SDOM) standard deviation of the mean values obtained from groups of repeated measurements

standard error of the fit error characterizing the differences between data and its curve fit

standardized normal variate nondimensional variable indicating the number of standard deviations that a variable deviates from its mean value

static steady in time

static calibration calibration performed when the system is static

static sensitivity slope of a static calibration curve at a particular input value

stationary each of a signal's ensemble-averaged statistical properties are time invariant

statistics branch of applied mathematics concerned with the collection and interpretation of quantitative data and the use of probability theory to estimate population parameters

steady-state response the periodic part of a second-order system's response

strongly stationary having all ensemble moments invariant with respect to the record's time

successive approximation method method to perform analog-to-digital conversions electronically by subtracting the analog input signal from a digital-to-analog converter's output signal

supplementary nondimensional units that do not represent a fundamental dimension

system of units system in which physical quantities can be expressed and related to one another through physical laws

systematic error error related to the difference between a measured and true value; sometimes called the bias error

systematic uncertainty estimate of the systematic error

third central moment *see* skewness

time history record plot of a signal's amplitude versus time for a given period of time

time lag delay in time between a signal's input and output through a device

timewise experiment in which measurand values are recorded sequentially in time

transducer device that changes an impulse into a desired quantity

transient response part of a second-order system's response that decays in time

transient solutions homogeneous solutions to a differential equation that decay to zero in time

true mean value mean value of the population

true value error-free value of a variable

true variance variance of the population

uncertainty estimate of error in a variable

under-damped having a damping ratio less than unity

union set of all members of two sets that are in only one, only in the other, or in both

unit precisely specified quantity in terms of which the magnitudes of other quantities of the same kind can be stated

unit equation equation in which only units are used or defined

validational experiment conducted to validate a specific hypothesis

variables physical quantities involved in the process that can undergo change and thereby affect the process

variance statistical measure of the spread of values with respect to their mean

variational experiment quantifying the mathematical relationships between experimental variables

waveform actual shape of a signal

weakly stationary having the ensemble mean and autocorrelation invariant with respect to the record's time

Weibull distribution continuous distribution describing the time to failure of a physical system

Wheatstone bridge electrical circuit consisting of four resistors in a specific configuration and a voltage source

windowing mathematical method that reduces the magnitude of a signal's record at its beginning and end

zeroth central moment integral of the probability density function; equals unity if the probability density function is normalized correctly

zeroth-order replication level level at which only measurement system errors are present

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B

Symbols

Chapter 2:

AC alternating current

C capacitance

DC direct current

E electric field

E_{Th} Thévenin open-circuit voltage

F_e electric force

I current

L inductance

P electric power

q charge

R resistance

T_0 reference temperature

Z impedance

Z_C capacitive reactance

Z_L inductive reactance

Z_{Th} Thévenin circuit impedance

$\Delta\Phi$ potential difference

ρ electric resistivity

ρ_0 resistivity at reference temperature T_0

Φ electric potential

Chapter 3:

C capacitance

E_i input signal

E_o output signal

f_c cutoff frequency

G gain

L inductance

R resistance

R_0 resistance at reference temperature

T temperature

T_0 reference temperature

α, β, γ coefficients of thermal expansion

Δt time lag

ϵ relative permittivity

ϵ_0 permittivity of free space

η material constant

ν Poisson's ratio

ρ resistivity

ϕ phase lag

Chapter 4:

A relative accuracy

C_v specific heat at constant volume

E total energy

$F(t)$ input forcing function

h convective heat transfer coefficient

I current

K static sensitivity

k spring constant

n order

M magnitude ratio

q charge

Q rate of heat transfer

R resistance

y_h homogeneous solution

y_p particular solution

β lag time

δ fractional dynamic error

ϵ absolute error

τ time constant

ϕ phase shift

ω_d ringing frequency

Chapter 5:

$\bigcup$ union

$\bigcap$ intersection

$|$ subject to the condition

C^n_m combinations of n objects taken m at a time

$E[\ ]$ or $\langle\ \rangle$ expected value

f_j frequency density

f^*_j normalized frequency density

Ku kurtosis

P^n_m permutations of n objects taken m at a time

$p(x)$ probability density function

$P(x)$ probability distribution function

$Pr[\ ]$ probability of an event

S_x^2 sample variance

Sk skewness

x' true mean of x or population mean

μ_m m-th central moment

σ^2 true variance

Chapter 6:

$p(x)$ probability density function

$P(x)$ probability distribution function

$p(z_1)$ normal error function

S_x sample standard deviation

$\pm t_{\nu,P} S_x/\sqrt{N}$ precision interval of the true mean

$\bar{x}$ sample mean

x' mean value of population

$\{\bar{x}\}$ pooled mean

$\{S_x\}$ pooled standard deviation

$\{S_{\bar{x}}\}$ pooled standard deviation of the means

$\{S_x^2\}$ pooled variance

$\{\bar{x}\}$ pooled weighted mean

$\{S_x\}_w$ pooled weighted standard deviation

$\{S_{\bar{x}}\}_w$ pooled weighted standard deviation of the means

α level of significance

Γ gamma function

ν degrees of freedom

σ true standard deviation

σ^2 true variance

χ^2 chi-square

$\tilde{\chi}^2$ reduced chi-square

Chapter 7:

B_i source bias limit

B_x systematic uncertainty

C confidence level

$\Delta B_x/B_x$ relative systematic uncertainty

e_i elemental instrument error (see text for specific types)

e_I overall instrument error

FSO maximum or full-scale output

N number of measurements in a sample

P_i source precision index

P_x random uncertainty

$P_{\bar{x}}$ random uncertainty in the average value of the measurand x

r result

$t_{\nu,P}$ Student's t factor

u_0 zeroth-order instrument uncertainty

u_I instrument uncertainty

u_c combined standard uncertainty

u_c^2 combined estimated variance

u_d design-stage uncertainty

u_N N-th order uncertainty

u_r^2 combined estimated variance

u_r uncertainty in a result

u_x uncertainty in x

U_x overall uncertainty

x' true mean value or population mean of x

X_{true} true value of x

$\bar{x}$ sample mean of x

ν degrees of freedom

ν_{B_x} degrees of freedom for the systematic uncertainty

ν_{P_x} degrees of freedom for the random uncertainty

θ_i absolute sensitivity coefficient

Chapter 8:

D the sum of the squares of all the vertical distances between the measured and calculated values used in regression analysis

d_i the i-th vertical distance between the i-th measured and calculated values

e_i the error between the i-th calculated and measured values

G a coefficient matrix used in multi-variable linear analysis

R a coefficient matrix used in multi-variable linear analysis

r sample correlation coefficient

R_{xx} autocorrelation function

R_{xy}, R_{yx} cross-correlation functions

$\widetilde{S}_x^2$ variance of the x errors

$\widetilde{S}_y^2$ variance of the y errors

S_{yx} standard error of the fit

SSE sum of the squares of the error

SSR sum of the squares of the regression

SST sum of the squares of the total error

$t_{\nu,P}S_{yx}$ precision interval

$\bar{x}$ sample mean

$\bar{x^2}$ sample mean of the sum of the squares

$\bar{y}_i$ estimate of the sample mean

y_i' true mean of y_i

α true intercept

β true slope

Γ gamma function

ρ population correlation coefficient

ρ_{xx} population autocorrelation coefficient

ρ_{xy}, ρ_{yx} population cross-correlation coefficients

σ_{xy} population covariance

σ_x population variance of x

$\sigma_{y_i}^2$ population variance of y_i

Chapter 9:

A_0, A_n, B_n, C_n Fourier coefficients

A_i imaginary or complex amplitude

A_r real amplitude

$A(\omega)$ real component of the Fourier transform

$B(\omega)$ imaginary component of the Fourier transform

c wave speed

C amplitude

f cyclic frequency

i imaginary number $= \sqrt{-1}$

n integer

$R_x(\tau)$ temporal-averaged autocorrelation coefficient

$R_x(t_1, t_1 + \tau)$ ensemble-averaged autocorrelation coefficient

S_x sample standard deviation of x

S_x^2 sample variance of x

t time

T fundamental period

$\bar{x}$ sample mean value of x

x_{rms} root-mean-square of x

$Y(\omega)$ complex Fourier coefficient

z complex number

z^* complex conjugate

α phase angle resulting from complex amplitude

κ wave number

λ wavelength

μ_x temporal-averaged mean value

$\mu_x(t_1)$ ensemble-averaged mean value

ω circular frequency

ϕ phase angle

Chapter 10:

A_0, A_n, B_n, C_n Fourier coefficients

a_0, a_k, b_k, c_k discrete Fourier coefficients

f cyclic frequency

f_a aliased frequency

f_1 fundamental frequency

f_N Nyquist frequency

f_s sampled frequency

k folding diagram factor

k_a folding diagram aliased factor

m positive integer

N number of sample points

T_1 fundamental period

T_r time-record length

$u(t)$ windowing function

$U(f)$ Fourier transform of the windowing function

Y_k discrete Fourier transform

$Y(f)$ Fourier transform

$y(r\delta t)$ discretized signal

δf frequency increment

δt time increment

Chapter 11:

g gravitational acceleration

g_o local gravitational acceleration

m mass

$\mathcal{MW}$ molecular weight

p pressure

R gas constant

R_b radius of a body

$\mathcal{R}$ universal gas constant

S Sutherland constant

T temperature

T_o reference temperature

w weight

z distance

μ absolute viscosity

μ_o reference absolute viscosity

ρ density

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C

Review Problem Answers

The following lists the answers to the odd-numbered Review Problems.

Chapter 1: [1] independent; [3] a beaker of ice water; [5] to validate a particular hypothesis and, in the process, to determine the range of validity of that hypothesis; [7] (a) parameter, (b) independent, (c) independent or extraneous, (d), (e), and (f) dependent

Chapter 2: [1] 4.0 μF; [3] 1 V; [5] 3.47 V; [7] current = Ampere; charge = Coulomb; electric field work = Joule; electric potential = Volta; resistance = Ohm; power = Watt; inductance = Henry; capacitance = Faraday; [9] 25.2 Ω; [11] 200; [13] deflection method; [15] compression on the lower-side gage causes an increase in resistance; [17] 11 Ω; [19] (a) -50 W, (b) 33 W; [21] 0.1 $\Omega/^\circ$R; [23] $< 100\ \Omega$; [25] (d)

Chapter 3: [1] radiator fluid temperature = physical variable; thermistor = sensor; Wheatstone bridge = transducer; car computer = signal processor; [3] 0.47 Ω; [5] 34.2 μF; [7] (c)

Chapter 4: [1] the larger-diameter thermocouple; [3] 25 %; [5] zero; [7] -84.3°; [9] (b); [11] (d); [13] 250; [15] (a) zero, (b) second, (c) second, (d) first, (e) first, (f) zero; [17] overdamped ($R > 2\sqrt{L/C}$), critically damped ($R = 2\sqrt{L/C}$), underdamped ($R < 2\sqrt{L/C}$); [19] (a) -6.47 x 10^{-5}, (b) -5.74×10^{-5}, (c) -4.20×10^{-5}, thermistor; [21] 2; [23] 49.4

Chapter 5: [1] 31st-of-the-month birthday = 0.0192, August birthday = 0.0849, February 29, 1979 birthday = 0, month-with-30-days birthday = 0.329; [3] 0.0625; [5] 1; [7] 2.4; [9] 0.4082; [11] (c); [13] the fourth central moment; [15] 0.49 % with replacement, 0.47 % without replacement; [17] 31 %

Chapter 6: [1] 0.9544; [3] 0.1621; [5] 0.8742; [7] 0.149; [9] 99.94 %; [11] 10.82; [13] 28.6 %; [15] (d); [17] from 27 to 93; [19] 1 %; [21] 0.1359; [23] 0.1331; [25] both are nondimensional

Chapter 7: [1] readability; [3] player 3; [5] 0.004 59 kg m^2; [7] 0.024 in.; [9] 0.1 %; [11] 0.013; [13] 6; [15] 0.002; [17] 0.20 V; [19] 6 %; [21] 0.1 %; [23] (1) with (c), (2) with (a), (3) with (b), (4) with (d)

Chapter 8: [1] -0.50; [3] Legendre; [5] 0.98; [7] 9.1 V

Chapter 9: [1] 2.687; [3] 0.0450; [5] (a) periodic with 1 s period, (b) aperiodic because of the exponential term; [7] (c), (d) and (e); [9] (b) and (c); [11] (a) and (d) periodic, (b) and (c) aperiodic

Chapter 10: [1] 5; [3] 0 Hz; [5] (b)

Chapter 11: [1] (a); [3] (c); [5] (d); [7] 73.5 kg; [9] 291.3 N; [11] 12.04 $\times 10^{23}$; [13] 12 000 N; [15] (d); [17] (c); [19] 1125 rpm; [21] (b); [23] (c); [25] A: mV^2 and B: $mV^2/\sqrt{m/s}$

Chapter 12: [1] (e)

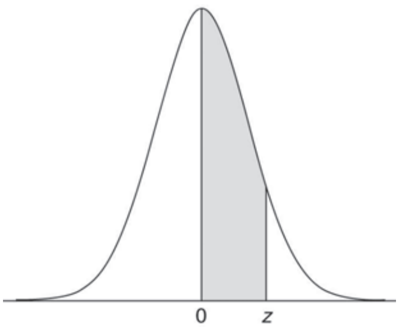
Index

Length	1 m =	100 cm	1×10^{-3} km	39.37 in.	3.281 ft	6.214×10^{-4} mi	3.937×10^{4} mil
Area	$1\ m^2$ =	$1 \times 10^{4}\ cm^2$	$10.76\ ft^2$	$1550\ in.^2$	2.471×10^{-4} acre	1×10^{-4} ha	3.861×10^{-7} mi
Volume	$1\ m^3$ =	$1 \times 10^{6}\ cm^3$	1000 L	$35.31\ ft^3$	$6.102 \times 10^{4}\ in.^3$	264.17 US gallon	1056.7 liquid qt
Time	1 s =	1000 ms	1.667×10^{-2} min	2.778×10^{-4} h	1.157×10^{-5} d	3.169×10^{-8} y	3×10^{8} light m
Speed	1 m/s =	100 cm/s	3.281 ft/s	3.6 km/h	2.237 mi/h	1.944 nautical mi/h	
Mass	1 kg =	1000 g	6.852×10^{-2} slug	2.2046 lbm	1×10^{-3} metric ton	6.023×10^{26} amu	500 carat
Mass Density	$1\ kg/m^3$ =	$0.001\ g/cm^3$	$1.940 \times 10^{-3}\ slug/ft^3$	$6.242 \times 10^{-2}\ lbm/ft^3$	$1.123 \times 10^{-6}\ slug/in.^3$	$3.612 \times 10^{-5}\ lbm/in.^3$	
Weight, Force	1 N =	1×10^{5} dyne	0.2248 lbf	7.233 pdl			
Pressure	1 Pa =	$10\ dyne/cm^2$	9.869×10^{-6} atm	4.015×10^{-3} in. H_2O	7.501×10^{-4} cm Hg	$1.450 \times 10^{-4}\ lbf/in.^2$	$2.089 \times 10^{-2}\ lbf/ft^2$
Energy, Work	1 J =	2.778×10^{-7} kW·h	9.481×10^{-4} Btu	1×10^{7} erg	0.7376 ft·lbf	3.725×10^{-7} hp·h	0.2389 cal
Power	1 W =	0.001 kW	3.413 Btu/h	0.7376 ft·lbf/s	1.341×10^{-3} hp	0.2389 cal/s	
Temperature	1 K =	(5/9) × °F + 255.38	°C + 273.15	(5/9) × °R			
Plane Angle	1 rad =	57.30°	3438′	2.063×10^{5}″	0.1592 rev		

UNIT CONVERSIONS

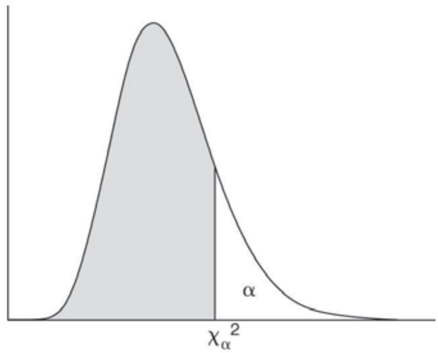
Derived Unit	**Symbol**	**Base Units**
Force	N (newton)	$kg \cdot m \cdot s^{-2}$
Pressure	Pa (pascal)	$kg \cdot m^{-1} \cdot s^{-2}$
Energy, Work, Heat	J (joule)	$kg \cdot m^{2} \cdot s^{-2}$
Power	W (watt)	$kg \cdot m^{2} \cdot s^{-3}$
Electric Charge	C (coulomb)	$A \cdot s$
Electric Potential Difference	V (volt)	$kg \cdot m^{2} \cdot s^{-3} \cdot A^{-1}$
Electric Resistance	Ω (ohm)	$kg \cdot m^{2} \cdot s^{-3} \cdot A^{-2}$
Electric Conductance	S (siemens)	$kg^{-1} \cdot m^{-2} \cdot s^{3} \cdot A^{2}$
Electric Capacitance	F (farad)	$kg^{-1} \cdot m^{-2} \cdot s^{4} \cdot A^{2}$

SI DERIVED UNITS EXPRESSED IN BASE UNITS



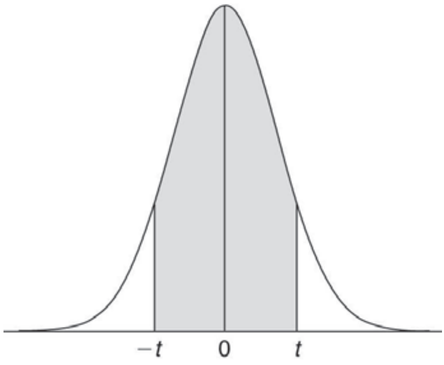
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4758	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4799	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4988	.4989	.4989	.4989	.4990
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
4.0	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

ONE-SIDED NORMAL ERROR FUNCTION VALUES



ν	$\chi^2_{0.99}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.90}$	$\chi^2_{0.50}$	$\chi^2_{0.05}$	$\chi^2_{0.025}$	$\chi^2_{0.01}$
1	0.000	0.000	0.000	0.016	0.455	3.84	5.02	6.63
2	0.020	0.051	0.103	0.211	1.39	5.99	7.38	9.21
3	0.115	0.216	0.352	0.584	2.37	7.81	9.35	11.3
4	0.297	0.484	0.711	1.06	3.36	9.49	11.1	13.3
5	0.554	0.831	1.15	1.61	4.35	11.1	12.8	15.1
6	0.872	1.24	1.64	2.20	5.35	12.6	14.4	16.8
7	1.24	1.69	2.17	2.83	6.35	14.1	16.0	18.5
8	1.65	2.18	2.73	3.49	7.34	15.5	17.5	20.1
9	2.09	2.70	3.33	4.17	8.34	16.9	19.0	21.7
10	2.56	3.25	3.94	4.78	9.34	18.3	20.5	23.2
11	3.05	3.82	4.57	5.58	10.3	19.7	21.9	24.7
12	3.57	4.40	5.23	6.30	11.3	21.0	23.3	26.2
13	4.11	5.01	5.89	7.04	12.3	22.4	24.7	27.7
14	4.66	5.63	6.57	7.79	13.3	23.7	26.1	29.1
15	5.23	6.26	7.26	8.55	14.3	25.0	27.5	30.6
16	5.81	6.91	7.96	9.31	15.3	26.3	28.8	32.0
17	6.41	7.56	8.67	10.1	16.3	27.6	30.2	33.4
18	7.01	8.23	9.39	10.9	17.3	28.9	31.5	34.8
19	7.63	8.91	10.1	11.7	18.3	30.1	32.9	36.2
20	8.26	9.59	10.9	12.4	19.3	31.4	34.2	37.6
30	15.0	16.8	18.5	20.6	29.3	43.8	47.0	50.9
40	22.2	24.4	26.5	29.1	39.3	55.8	59.3	63.7
50	29.7	32.4	34.8	37.7	49.3	67.5	71.4	76.2
60	37.5	40.5	43.2	46.5	59.3	79.1	83.3	88.4
70	45.4	48.8	51.7	55.3	69.3	90.5	95.0	100.4
80	53.5	57.2	60.4	64.3	79.3	101.9	106.6	112.3
90	61.8	65.6	69.1	73.3	89.3	113.1	118.1	124.1
100	70.1	74.2	77.9	82.4	99.3	124.3	129.6	135.8

CHI-SQUARE VARIABLE VALUES



ν	$t_{\nu,P=50\,\%}$	$t_{\nu,P=90\,\%}$	$t_{\nu,P=95\,\%}$	$t_{\nu,P=99\,\%}$
1	1.000	6.341	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.192	5.841
4	0.741	2.132	2.770	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
30	0.683	1.697	2.042	2.750
40	0.681	1.684	2.021	2.704
50	0.680	1.679	2.010	2.679
60	0.679	1.671	2.000	2.660
120	0.677	1.658	1.980	2.617
∞	0.674	1.645	1.960	2.576

TWO-SIDED STUDENT'S t VARIABLE VALUES